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A GENERAL ITERATIVE METHOD FOR SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES AND APPLICATIONS

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Dedicated to Professor Steve Robinson on the occasion of his 75th birthday with admiration and respect

Abstract. In this paper, using new nonlinear operators called demimetric, we prove a strong convergence theorem for finding a solution of the general split common fixed point problem with zero points of two monotone operators in Hilbert spaces. This solution is the unique solution of the hierarchical variational inequality problem. Using this result, we obtain new and well-known strong convergence theorems in Hilbert spaces.

1. Introduction

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let H_1 and H_2 be two real Hilbert spaces. Given two mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$ and a bounded linear operator $A : H_1 \to H_2$, the split common fixed point problem is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively. Such a problem includes the split feasibility problem and the split common null point problem. In fact, let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $T = P_D$ and $U = P_Q$, where P_D and P_Q are the metric projections of H_1 onto D and H_2 onto Q, respectively, we have that $z \in D \cap A^{-1}Q$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Furthermore, given maximal monotone operators $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the split common null point problem [6] is to find a point $z \in H_1$ such that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Defining $T = J_\lambda$ and $U = Q_\mu$, where J_λ and Q_μ are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively, we have that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$ is equivalent to $z \in$

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$F(T) \cap A^{-1}F(U)$. Thus the split common fixed point problem generalizes the split feasibility problem and the split common null point problem. There are many papers for the split feasibility problem, the split common null point problem and the split common fixed point problem; see, for instance, [6, 7, 20, 23, 35].

Recently, Takahashi [31] introduced a new nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U and J is the duality mapping on E. Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let s be a real number with $0 \leq s < 1$. A mapping $U : C \to H$ is called an s-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + s\|x - Ux - (y - Uy)\|^2 \tag{1.1}$$

for all $x, y \in C$. If $s = 0$ in (1.1), U is nonexpansive. A mapping $T : C \to H$ is called generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.2}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta)$-generalized hybrid mapping. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [15, 16] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [29] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [12]. The class of demimetric mappings in a Hilbert space covers strict pseudo-contractions and generalized hybrid mappings. We also know that the metric resolvent of a maximal monotone operator in a Banach space is a demimetric mapping.

In this paper, using the class of demimetric mappings, we prove a strong convergence theorem for finding a solution of the general split common fixed point problem with zero points of two monotone operators in Hilbert spaces. This solution is the unique solution of the hierarchical variational inequality problem. Using this result, we obtain new and well-known strong convergence theorems in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot\,,\,\cdot\rangle$ and norm $\|\cdot\|$, respectively. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [28] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

If $x = y + z$, then

$$\|x\|^2 \leq \|y\|^2 + 2\langle z, x\rangle. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of H and let $T \colon C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points for T. A mapping $T : C \to H$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [13]. For a nonempty, closed and convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive;

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [10, 26]. Let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha\|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. If a mapping $U : C \to H$ is α-inverse strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda U : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned}
\|(I - \lambda U)x - (I - \lambda U)y\|^2 &= \|x - y - \lambda(Ux - Uy)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ux - Uy\rangle + \lambda^2\|Ux - Uy\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha\|Ux - Uy\|^2 + \lambda^2\|Ux - Uy\|^2 \qquad (2.4)\\
&= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ux - Uy\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus $I - \lambda U$ is nonexpansive; see [1, 22, 28] for more results of inverse strongly monotone mappings.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to D(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [27] that

$$A_r x \in BJ_r x, \quad \forall x \in H,\ r > 0. \tag{2.5}$$

Let B be a maximal monotone operator on H and let

$$B^{-1}0 = \{x \in H : 0 \in Bx\}.$$

Then $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \le \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

We also know the following lemma from [25].

Lemma 2.1 ([25])**.** *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \ge \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have that

$$\|J_\lambda x - J_\mu x\| \le (|\lambda - \mu| / \lambda)\, \|x - J_\lambda x\| \tag{2.7}$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [9, 26].

In case when a Banach space E is a Hilbert space, the definition of a demimetric mapping is as follows: Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\eta \in (-\infty, 1)$. A mapping $S : C \to H$ with $F(S) \ne \emptyset$ is called η-demimetric if, for any $x \in C$ and $q \in F(S)$,

$$\langle x - q, x - Sx\rangle \ge \frac{1-\eta}{2}\|x - Sx\|^2.$$

We give the following examples of demimetric mappings in Hilbert spaces and Banach spaces.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \le k < 1$. If U is a k-strict pseudo-contraction and $F(U) \ne \emptyset$, then U is k-demimetric; see [31].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. If T is generalized hybrid and $F(T) \ne \emptyset$, then T is 0-demimetric. In fact, setting $x = u \in F(T)$ and $y = x \in C$ in (1.2), we have that

$$\alpha\|u - Tx\|^2 + (1-\alpha)\|u - Tx\|^2 \le \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence

$$\|Tx - u\|^2 \le \|x - u\|^2.$$

From $\|Tx - x + x - u\|^2 = \|Tx - x\|^2 + 2\langle Tx - x, x - u\rangle + \|x - u\|^2$, we have that

$$2\langle x - u, x - Tx\rangle \ge \|x - Tx\|^2$$

for all $x \in C$ and $u \in F(T)$. This means that T is 0-demimetric.

(3) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let $U : C \to H$ be an α-inverse strongly monotone mapping with $U^{-1}0 \ne \emptyset$. Then $1 - 2\alpha \in (-\infty, 1)$ and $I - U : C \to H$ is a $(1-2\alpha)$-demimetric mapping. In fact, since $U : C \to H$ is α-inverse strongly monotone, we have that

$$\langle x - y, Ux - Uy\rangle \ge \alpha\|Ux - Uy\|^2, \quad \forall x, y \in C. \tag{2.8}$$

Setting $T = I - U$ and taking $y = z \in F(T) = U^{-1}0$ in (2.8), we have that

$$\langle x - z, x - Tx\rangle \ge \alpha\|x - Tx\|^2, \quad \forall x \in C,\ z \in F(T).$$

This implies that

$$2\langle x - z, x - Tx\rangle \geq (1 - (1 - 2\alpha))\|x - Tx\|^2, \quad \forall x \in C,\ z \in F(T)$$

and hence $T = I - U$ is $(1 - 2\alpha)$-demimetric.

(4) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric; see [31].

(5) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [31].

If $S : C \to H$ is η-demimetric and $0 < \lambda \leq 1 - \eta$, then $S_\lambda = (1 - \lambda)I + \lambda S$ is quasi-nonexpansive. In fact, it is obvious that $F(S) = F(S_\lambda)$. We also have that for any $x \in C$ and $z \in F(S_\lambda)$,

$$\begin{aligned} 2\langle x - z, x - S_\lambda x\rangle &= 2\langle x - z, x - (1 - \lambda)x - \lambda Sx\rangle = 2\lambda\langle x - z, x - Sx\rangle \\ &\geq \lambda(1 - \eta)\|x - Sx\|^2 = \lambda^2\frac{1 - \eta}{\lambda}\|x - Sx\|^2 \\ &= \frac{1 - \eta}{\lambda}\|\lambda x - \lambda Sx\|^2 = \frac{1 - \eta}{\lambda}\|x - S_\lambda x\|^2 \\ &\geq \frac{\lambda}{\lambda}\|x - S_\lambda x\|^2 = \|x - S_\lambda x\|^2. \end{aligned} \tag{2.9}$$

Then S_λ is a 0-demimetric mapping. Furthermore, we have from (2.2) that for any $x \in C$ and $z \in F(S_\lambda)$,

$$\begin{aligned} \|x - S_\lambda x\|^2 &\leq 2\langle x - z, x - S_\lambda x\rangle \\ &\Longleftrightarrow \|x - S_\lambda x\|^2 \leq \|x - S_\lambda x\|^2 + \|x - z\|^2 - \|S_\lambda x - z\|^2 \\ &\Longleftrightarrow \|S_\lambda x - z\|^2 \leq \|x - z\|^2 \\ &\Longleftrightarrow \|S_\lambda x - z\| \leq \|x - z\|. \end{aligned} \tag{2.10}$$

Therefore, S_λ is quasi-nonexpansive.

The following lemma which was proved in [31] is important and crucial

Lemma 2.2 ([31])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in (-\infty, 1)$ and let U be a k-demimetric mapping of C into H. Then $F(U)$ is closed and convex.*

To prove our main result, we need the following lemmas.

Lemma 2.3 ([3]; see also [37])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([18]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ satisfies $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \ldots$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Strong convergence theorem

Let H be a Hilbert space. A mapping $g : H \to H$ is a contraction if there exists $k \in (0, 1)$ such that $\|g(x)-g(y)\| \leq k\|x-y\|$ for all $x, y \in H$. We call such a mapping g a k-contraction. A nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\overline{\gamma} > 0$ such that $\langle x - y, Vx - Vy\rangle \geq \overline{\gamma}\|x - y\|^2$ for all $x, y \in H$. Such V is also called $\overline{\gamma}$-strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian continuous if there exists $L > 0$ such that $\|Vx - Vy\| \leq L\|x - y\|$ for all $x, y \in H$. Such V is called L-Lipschitzian continuous.

Let $g : H \to H$ be a k-contraction with $0 < k < 1$. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma} > 0$ and $L > 0$. Let γ be a real number with $0 < \gamma < \frac{\overline{\gamma}}{k}$. According to Lin and Takahashi [17], $V - \gamma g : H \to H$ is a $(\overline{\gamma} - \gamma k)$-strongly monotone and $(L + \gamma k)$-Lipschitzian continuous mapping. Furthermore, take $t > 0$ satisfying

$$2(\overline{\gamma} - \gamma k) > t(L + \gamma k)^2 \quad \text{and} \quad 1 > 2t(\overline{\gamma} - \gamma k).$$

Then $0 < 1 - t\big(2(\overline{\gamma} - \gamma k) - t(L + \gamma k)^2\big) < 1$ and $I - t(V - \gamma g) : H \to H$ is a contraction. In fact, it is obvious that $0 < 1 - t\big(2(\overline{\gamma} - \gamma k) - t(L + \gamma k)^2\big) < 1$. We also have that for any $x, y \in H$,

$$\begin{aligned}
&\|\big(I - t(V - \gamma g)\big)x - \big(I - t(V - \gamma g)\big)y\|^2 \\
&\quad = \|x - y\|^2 - 2t\langle x - y, (V - \gamma g)x - (V - \gamma g)y\rangle + t^2\|(V - \gamma g)x - (V - \gamma g)y\|^2 \\
&\quad \leq \|x - y\|^2 - 2t(\overline{\gamma} - \gamma k)\|x - y\|^2 + t^2(L + \gamma k)^2\|x - y\|^2 \\
&\quad = \big(1 - 2t(\overline{\gamma} - \gamma k) + t^2(L + \gamma k)^2\big)\|x - y\|^2 \\
&\quad = \big(1 - t(2(\overline{\gamma} - \gamma k) - t(L + \gamma k)^2)\big)\|x - y\|^2.
\end{aligned}$$

Therefore $I - t(V - \gamma g)$ is a contraction. Let C be a nonempty, closed and convex subset of H. Then a mapping $P_C(I - t(V - \gamma g)) : C \to C$ is a contraction and hence $P_C(I - t(V - \gamma g))$ has a unique fixed point z_0 in C. This point $z_0 \in C$ is also a unique solution of the variational inequality

$$\langle (V - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in C.$$

Furthermore, this point $z_0 \in C$ is a unique fixed point of $P_C(I - (V - \gamma g))$ in C. In fact, we have that

$$\begin{aligned}
z_0 &= P_C(I - t(V - \gamma g))z_0 \\
&\iff \langle z_0 - t(V - \gamma g)z_0 - z_0, z_0 - y\rangle \geq 0, \quad \forall y \in C \\
&\iff \langle -t(V - \gamma g)z_0, z_0 - y\rangle \geq 0, \quad \forall y \in C \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow \langle (V-\gamma g)z_0, y-z_0\rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow \langle z_0-(V-\gamma g)z_0 - z_0, z_0-y\rangle \geq 0, \quad \forall y \in C \\
&\Longleftrightarrow z_0 = P_C(I-(V-\gamma g))z_0.
\end{aligned}$$

Now we prove a strong convergence theorem of Halpern's type [11] for finding a solution of the general split common fixed point problem with zero points of two monotone operators in Hilbert spaces. For the proof, we follow the ideas of [17, 30, 34]. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \to 0$, $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C into H, then T is demiclosed; see [4] and [28, p. 114].

Theorem 3.1. *Let H_1 and H_2 be Hilbert spaces. Let B and G be maximal monotone operators on H_1. Let $J_{\lambda_n} = (I+\lambda_n B)^{-1}$ and $T_{r_n} = (I + r_n G)^{-1}$ be the resolvents of B and G for $\lambda_n > 0$ and $r_n > 0$, respectively. Let $\eta, \tau \in (-\infty, 1)$. Let S be an η-demimetric and demiclosed mapping of H_1 into H_1 and let T be a τ-demimetric and demiclosed mapping of H_2 into H_2. Define $S_\lambda = (1-\lambda)I + \lambda S$ for some λ with $0 < \lambda \leq 1-\eta$. Let $k \in (0,1)$ and let g be a k-contraction of H_1 into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ be a bounded linear operator such that $\|A\| \neq 0$. Suppose $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0 \neq \emptyset$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)\{\beta_n x_n + (1-\beta_n) S_\lambda J_{\lambda_n}(I - \lambda_n A^*(I-T)A)T_{r_n} x_n\}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$, $\{\lambda_n\}, \{r_n\} \subset (0,\infty)$ and $a, b, c \in \mathbb{R}$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

$$0 < a \leq \lambda_n \leq b < \frac{1-\tau}{\|A\|^2} \quad \text{and} \quad 0 < c \leq r_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0$, where z_0 is a unique fixed point of $P_{F(S)\cap B^{-1}0 \cap A^{-1}F(T)\cap G^{-1}0}(I - V + \gamma g)$ in the set $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0$.

Proof. Since S is η-demimetric and $0 < \lambda \leq 1-\eta$, we have from (2.9) and (2.10) that S_λ is 0-demimetric and quasi-nonexpansive. Let $z \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0$. We have that $z \in F(S) = F(S_\lambda)$, $z = J_{\lambda_n} z$, $z = T_{r_n} z$ and $(I-T)Az = 0$. Put $z_n = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)T_{r_n} x_n$ and $u_n = T_{r_n} x_n$. Since J_{λ_n} and T_{r_n} are nonexpansive and T is τ-demimetric, we obtain that

$$\begin{aligned}
\|z_n - z\|^2 &= \|J_{\lambda_n}(u_n - \lambda_n A^*(I-T)Au_n) - J_{\lambda_n} z\|^2 \\
&\leq \|u_n - \lambda_n A^*(I-T)Au_n - z\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|u_n - z - \lambda_n A^*(I-T)Au_n\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n \langle u_n - z, A^*(I-T)Au_n\rangle + \lambda_n^2 \|A^*(I-T)Au_n\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n \langle Au_n - Az, (I-T)Au_n\rangle + \lambda_n^2 \|A^*(I-T)Au_n\|^2 \qquad (3.2)\\
&\leq \|u_n - z\|^2 - \lambda_n(1-\tau) \|(I-T)Au_n\|^2 + \lambda_n^2\|A\|^2 \|(I-T)Au_n\|^2 \\
&= \|u_n - z\|^2 + \lambda_n(\lambda_n\|A\|^2 - (1-\tau)) \|(I-T)Au_n\|^2 \\
&\leq \|x_n - z\|^2 + \lambda_n(\lambda_n\|A\|^2 - (1-\tau)) \|(I-T)Au_n\|^2 \\
&\leq \|x_n - z\|^2 .
\end{aligned}
$$

Put $y_n = \beta_n x_n + (I-\beta_n)S_\lambda z_n$. Since S_λ is quasi-nonexpansive, we have that

$$
\begin{aligned}
\|y_n - z\| &= \|\beta_n(x_n - z) + (1-\beta_n)(S_\lambda z_n - z)\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n) \|S_\lambda z_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n) \|z_n - z\| \\
&\leq \beta_n \|x_n - z\| + (1-\beta_n) \|x_n - z\| \\
&= \|x_n - z\| .
\end{aligned}
$$

Furthermore, put $s = \overline{\gamma} - \frac{L^2\mu}{2}$. We have $s > 0$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha_n s > 0$ and $\alpha_n < \mu$ for all $n \geq n_0$. Then we have that for any $x, y \in H_1$ and $n \geq n_0$,

$$
\begin{aligned}
\|(I-\alpha_n V)x - (I-\alpha_n V)y\|^2 &= \|x - y - \alpha_n(Vx - Vy)\|^2 \\
&= \|x-y\|^2 - 2\alpha_n\langle x-y, Vx - Vy\rangle + \alpha_n^2\|Vx - Vy\|^2 \\
&\leq \|x-y\|^2 - 2\alpha_n\overline{\gamma}\|x-y\|^2 + \alpha_n^2 L^2\|x-y\|^2 \\
&= (1 - 2\alpha_n\overline{\gamma} + \alpha_n^2L^2)\|x-y\|^2 \qquad (3.3)\\
&= (1 - 2\alpha_n s - \alpha_n L^2\mu + \alpha_n^2L^2)\|x-y\|^2 \\
&\leq \big(1 - 2\alpha_n s - \alpha_n(L^2\mu - \alpha_n L^2) + \alpha_n^2 s^2\big)\|x-y\|^2 \\
&\leq (1 - 2\alpha_n s + \alpha_n^2 s^2)\|x-y\|^2 \\
&= (1-\alpha_n s)^2\|x-y\|^2.
\end{aligned}
$$

Since $1 - \alpha_n s > 0$, we obtain that for any $x, y \in H_1$ and $n \geq n_0$,

$$
\|(I-\alpha_n V)x - (I-\alpha_n V)y\| \leq (1-\alpha_n s)\|x-y\|. \qquad (3.4)
$$

Since $x_{n+1} = \alpha_n\gamma g(x_n) + (I-\alpha_n V)y_n$ and $z = \alpha_n Vz + z - \alpha_n Vz$, we have from (3.4) and $s - \gamma\ k > 0$ that

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(\gamma g(x_n) - Vz) + (I-\alpha_n V)y_n - (I-\alpha_n V)z\| \\
&\leq \alpha_n\gamma\ k \|x_n - z\| + \alpha_n\|\gamma g(z) - Vz\| + (1-\alpha_n s)\|y_n - z\| \\
&\leq \alpha_n\gamma\ k \|x_n - z\| + \alpha_n\|\gamma g(z) - Vz\| + (1-\alpha_n s) \|x_n - z\| \qquad (3.5)\\
&= \{1 - \alpha_n(s - \gamma\ k)\} \|x_n - z\| + \alpha_n\|\gamma g(z) - Vz\| \\
&= \{1 - \alpha_n(s - \gamma\ k)\} \|x_n - z\|
\end{aligned}
$$

$$+\alpha_n(s-\gamma\ k)\frac{\|\gamma g(z)-Vz\|}{s-\gamma\ k}.$$

Putting $K=\max\{\|x_1-z\|,\frac{\|\gamma g(z)-Vz\|}{s-\gamma\ k}\}$, we have from (3.5) that $\|x_n-z\|\le K$ for all $n\in\mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. We know from Lemma 2.2 that $F(S)$ and $F(T)$ are closed and convex. Then we have that $F(S)\cap B^{-1}0\cap A^{-1}F(T)\cap G^{-1}0$ is closed and convex. Using (3.1), we can take a unique $z_0\in F(S)\cap B^{-1}0\cap A^{-1}F(T)\cap G^{-1}0$ such that

$$z_0=P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)\cap G^{-1}0}(I-V+\gamma g)z_0.$$

From the definition of $\{x_n\}$, we have that

$$x_{n+1}-x_n=\alpha_n\gamma g(x_n)+(I-\alpha_nV)y_n-x_n.$$

Thus we have that

$$\begin{aligned}&\langle x_{n+1}-x_n-\alpha_n\gamma g(x_n),x_n-z_0\rangle\\&\qquad=\langle y_n-x_n-\alpha_nVy_n,x_n-z_0\rangle\\&\qquad=\langle y_n-x_n,x_n-z_0\rangle-\alpha_n\langle Vy_n,x_n-z_0\rangle.\end{aligned}\tag{3.6}$$

Using (2.1), we have that

$$\begin{aligned}\|y_n-z_0\|^2&=\|\beta_nx_n+(1-\beta_n)S_\lambda z_n-z_0\|^2\\&=\beta_n\|x_n-z_0\|^2+(1-\beta_n)\|S_\lambda z_n-z_0\|^2-\beta_n(1-\beta_n)\|S_\lambda z_n-x_n\|^2\\&\le\beta_n\|x_n-z_0\|^2+(1-\beta_n)\|x_n-z_0\|^2-\beta_n(1-\beta_n)\|S_\lambda z_n-x_n\|^2\\&=\|x_n-z_0\|^2-\beta_n(1-\beta_n)\|S_\lambda z_n-x_n\|^2.\end{aligned}\tag{3.7}$$

From (2.2) and (3.7), we have that

$$\begin{aligned}&2\langle y_n-x_n,x_n-z_0\rangle\\&\qquad=\|y_n-z_0\|^2-\|y_n-x_n\|^2-\|x_n-z_0\|^2\\&\qquad\le\|x_n-z_0\|^2-\beta_n(1-\beta_n)\|S_\lambda z_n-x_n\|^2\\&\qquad\qquad-(1-\beta_n)^2\|S_\lambda z_n-x_n\|^2-\|x_n-z_0\|^2\\&\qquad=-\beta_n(1-\beta_n)\|S_\lambda z_n-x_n\|^2-(1-\beta_n)^2\|S_\lambda z_n-x_n\|^2\\&\qquad=-(1-\beta_n)\|S_\lambda z_n-x_n\|^2.\end{aligned}\tag{3.8}$$

From (3.6) and (3.8), we also have that

$$\begin{aligned}&2\langle x_{n+1}-x_n,x_n-z_0\rangle\\&\qquad=2\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\&\qquad\qquad+2\langle y_n-x_n,x_n-z_0\rangle-2\alpha_n\langle Vy_n,x_n-z_0\rangle\\&\qquad\le2\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\\&\qquad\qquad-(1-\beta_n)\|S_\lambda z_n-x_n\|^2-2\alpha_n\langle Vy_n,x_n-z_0\rangle.\end{aligned}\tag{3.9}$$

Furthermore, using (2.2) and (3.9), we have that

$$\begin{aligned}&\|x_{n+1}-z_0\|^2-\|x_n-x_{n+1}\|^2-\|x_n-z_0\|^2\\&\qquad\le2\alpha_n\langle\gamma g(x_n),x_n-z_0\rangle\end{aligned}$$

$$- (1-\beta_n)\|S_\lambda z_n - x_n\|^2 - 2\alpha_n\langle Vy_n, x_n - z_0\rangle.$$

Setting $\Gamma_n = \|x_n - z_0\|^2$, we have that

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n - \|x_n - x_{n+1}\|^2 \\
&\qquad \le 2\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\
&\qquad\quad - (1-\beta_n)\|S_\lambda z_n - x_n\|^2 - 2\alpha_n\langle Vy_n, x_n - z_0\rangle.
\end{aligned} \tag{3.10}$$

We also have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n\gamma g(x_n) + (I - \alpha_n V)y_n - x_n\| \\
&= \|\alpha_n(\gamma g(x_n) - Vy_n) + y_n - x_n\| \\
&\le \|y_n - x_n\| + \alpha_n\|\gamma g(x_n) - Vy_n\|
\end{aligned} \tag{3.11}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &\le \big(\|y_n - x_n\| + \alpha_n\|\gamma g(x_n) - Vy_n\|\big)^2 \\
&= \|y_n - x_n\|^2 + 2\alpha_n\|y_n - x_n\|\|\gamma g(x_n) - Vy_n\| \\
&\quad + \alpha_n^2\|\gamma g(x_n) - Vy_n\|^2 \\
&= (1-\beta_n)^2\|S_\lambda z_n - x_n\|^2 + 2(1-\beta_n)\alpha_n\|S_\lambda z_n - x_n\|\|\gamma g(x_n) - Vy_n\| \\
&\quad + \alpha_n^2\|\gamma g(x_n) - Vy_n\|^2.
\end{aligned} \tag{3.12}$$

We have from (3.10) and (3.12) that

$$\begin{aligned}
\Gamma_{n+1} - \Gamma_n &\le \|x_n - x_{n+1}\|^2 + 2\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|S_\lambda z_n - x_n\|^2 - 2\alpha_n\langle Vy_n, x_n - z_0\rangle \\
&\le (1-\beta_n)^2\|S_\lambda z_n - x_n\|^2 + 2(1-\beta_n)\alpha_n\|S_\lambda z_n - x_n\|\|\gamma g(x_n) - Vy_n\| \\
&\quad + \alpha_n^2\|\gamma g(x_n) - Vy_n\|^2 + 2\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\
&\quad - (1-\beta_n)\|S_\lambda z_n - x_n\|^2 - 2\alpha_n\langle Vy_n, x_n - z_0\rangle
\end{aligned}$$

and hence

$$\begin{aligned}
&\Gamma_{n+1} - \Gamma_n + \beta_n(1-\beta_n)\|S_\lambda z_n - x_n\|^2 \\
&\qquad \le 2(1-\beta_n)\alpha_n\|S_\lambda z_n - x_n\|\|\gamma g(x_n) - Vy_n\| \\
&\qquad\quad + \alpha_n^2\|\gamma g(x_n) - Vy_n\|^2 + 2\alpha_n\langle \gamma g(x_n), x_n - z_0\rangle \\
&\qquad\quad - 2\alpha_n\langle Vy_n, x_n - z_0\rangle.
\end{aligned} \tag{3.13}$$

We divide the proof into two cases.

Case 1: Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Since $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\lim_{n\to\infty}\alpha_n = 0$ and $\{S_\lambda z_n\}$, $\{g(x_n)\}$ and $\{Vy_n\}$ are bounded, we have from (3.13) that

$$\lim_{n\to\infty}\|S_\lambda z_n - x_n\| = 0. \tag{3.14}$$

Using (3.12), we also have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.15}$$

Since $y_n - x_n = \beta_n x_n + (1-\beta_n)S_\lambda z_n - x_n = (1-\beta_n)(S_\lambda z_n - x_n)$, we have from (3.14) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.16}$$

We show $\lim_{n\to\infty} \|S_\lambda z_n - z_n\| = 0$. Since $z_0 = \alpha_n V z_0 + z_0 - \alpha_n V z_0$, we also have from (2.3), (3.2), (3.4) and (3.7) that

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &= \|\alpha_n(\gamma g(x_n) - Vz_0) + (I-\alpha_n V)y_n - (I-\alpha_n V)z_0\|^2 \\
&\le (1-\alpha_n s)^2 \|y_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle \\
&\le \|y_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle \\
&\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|S_\lambda z_n - z_0\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle \\
&\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|z_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle \qquad (3.17)\\
&\le \beta_n\|x_n - z_0\|^2 + (1-\beta_n)\|x_n - z_0\|^2 \\
&\quad + (1-\beta_n)\lambda_n(\lambda_n\|A\|^2 - (1-\tau))\,\|(I-T)Au_n\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle. \\
&\le \|x_n - z_0\|^2 + (1-\beta_n)\lambda_n(\lambda_n\|A\|^2 - (1-\tau))\,\|(I-T)Au_n\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&(1-\beta_n)\lambda_n(1-\tau-\lambda_n\|A\|^2)\,\|(I-T)Au_n\|^2 \\
&\qquad \le \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle. \qquad (3.18)
\end{aligned}$$

Using the conditions of $\{\Gamma_n\}$, $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$, we have that

$$\lim_{n\to\infty} \|(I-T)Au_n\| = 0. \tag{3.19}$$

Since J_{λ_n} is firmly nonexpansive, we have from (3.2) that

$$\begin{aligned}
2\|z_n - z_0\|^2 &= 2\,\|J_{\lambda_n}(u_n - \lambda_n A^*(I-T)Au_n) - J_{\lambda_n} z_0\|^2 \\
&\le 2\,\langle u_n - \lambda_n A^*(I-T)Au_n - z_0, z_n - z_0\rangle \\
&= \|u_n - \lambda_n A^*(I-T)Au_n - z_0\|^2 + \|z_n - z_0\|^2 \\
&\qquad - \|u_n - \lambda_n A^*(I-T)Au_n - z_n\|^2 \\
&\le \|x_n - z_0\|^2 + \|z_n - z_0\|^2 \\
&\qquad - \|u_n - z_n - \lambda_n A^*(I-T)Au_n\|^2 \\
&\le \|x_n - z_0\|^2 + \|z_n - z_0\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\,\langle u_n - z_n, A^*(I-T)Au_n\rangle - \lambda_n^2\,\|A^*(I-T)Au_n\|^2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\|z_n - z_0\|^2 &\le \|x_n - z_0\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\,\langle u_n - z_n, A^*(I-T)Au_n\rangle - \lambda_n^2\,\|A^*(I-T)Au_n\|^2.
\end{aligned} \tag{3.20}$$

Using (3.17) and (3.20), we obtain

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &\le \|y_n-z_0\|^2+2\alpha_n\langle \gamma g(x_n)-Vz_0,x_{n+1}-z_0\rangle\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|z_n-z_0\|^2+2\alpha_n\langle \gamma g(x_n)-Vz_0,x_{n+1}-z_0\rangle)\\ &\le \beta_n\|x_n-z_0\|^2+(1-\beta_n)\|x_n-z_0\|^2\\ &\quad-(1-\beta_n)\|u_n-z_n\|^2+2(1-\beta_n)\lambda_n\langle u_n-z_n,A^*(I-T)Au_n\rangle\\ &\quad-(1-\beta_n)\lambda_n^2\|A^*(I-T)Au_n\|^2+2\alpha_n\langle \gamma g(x_n)-Vz_0,x_{n+1}-z_0\rangle\\ &= \|x_n-z_0\|^2-(1-\beta_n)\|u_n-z_n\|^2\\ &\quad+2(1-\beta_n)\lambda_n\langle u_n-z_n,A^*(I-T)Au_n\rangle-(1-\beta_n)\lambda_n^2\|A^*(I-T)Au_n\|^2\\ &\quad+2\alpha_n\langle \gamma g(x_n)-Vz_0,x_{n+1}-z_0\rangle,\end{aligned}$$

from which it follows that

$$\begin{aligned}(1-\beta_n)\|u_n-z_n\|^2 &\le \|x_n-z_0\|^2-\|x_{n+1}-z_0\|^2\\ &\quad+2(1-\beta_n)\lambda_n\langle u_n-z_n,A^*(I-T)Au_n\rangle\\ &\quad-(1-\beta_n)\lambda_n^2\|A^*(I-T)Au_n\|^2+2\alpha_n\langle \gamma g(x_n)-Vz_0,x_{n+1}-z_0\rangle.\end{aligned}$$

Using (3.19) and the conditions of $\{\Gamma_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$, we have that

$$\lim_{n\to\infty}\|u_n-z_n\|=0. \tag{3.21}$$

We also have from (2.6) that

$$\begin{aligned}2\|u_n-z_0\|^2 &= 2\|T_{r_n}x_n-T_{r_n}z_0\|^2\\ &\le 2\langle x_n-z_0,u_n-z_0\rangle\\ &= \|x_n-z_0\|^2+\|u_n-z_0\|^2-\|u_n-x_n\|^2\end{aligned}$$

and hence

$$\|u_n-z_0\|^2\le\|x_n-z_0\|^2-\|u_n-x_n\|^2. \tag{3.22}$$

From (3.2) and (3.22) we have that

$$\|S_\lambda z_n-z_0\|^2\le\|u_n-z_0\|^2\le\|x_n-z_0\|^2-\|u_n-x_n\|^2$$

and hence

$$\begin{aligned}\|u_n-x_n\|^2 &\le \|x_n-z_0\|^2-\|S_\lambda z_n-z_0\|^2\\ &= \|x_n-S_\lambda z_n\|^2+2\langle x_n-S_\lambda z_n,S_\lambda z_n-z_0\rangle+\|S_\lambda z_n-z_0\|^2-\|S_\lambda z_n-z_0\|^2\\ &= \|x_n-S_\lambda z_n\|^2+2\langle x_n-S_\lambda z_n,S_\lambda z_n-z_0\rangle.\end{aligned}$$

Thus from (3.14) we have that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.23}$$

Since $\|S_\lambda z_n-z_n\|\le\|S_\lambda z_n-x_n\|+\|x_n-u_n\|+\|u_n-z_n\|$, we have from (3.14), (3.21) and (3.23) that

$$\lim_{n\to\infty}\|S_\lambda z_n-z_n\|=0. \tag{3.24}$$

Take $\lambda_0 \in \mathbb{R}$ with $0 < a \leq \lambda_0 \leq b < \frac{1-\tau}{\|A\|^2}$. Put $s_n = (I - \lambda_n A^*(I-T)A)u_n$. Using $z_n = J_{\lambda_n}(I - \lambda_n A^*(I-T)A)u_n$, we have from Lemma 2.1 that

$$\begin{aligned} &\|J_{\lambda_0} s_n - z_n\| \\ &= \|J_{\lambda_0}(I - \lambda_n A^*(I-T)A)u_n - J_{\lambda_n}(I - \lambda_n A^*(I-T)A)u_n\| \\ &= \|J_{\lambda_0} s_n - J_{\lambda_n} s_n\| \\ &\leq \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0} s_n - s_n\|. \end{aligned} \tag{3.25}$$

We also have that

$$\begin{aligned} \|s_n - J_{\lambda_0} s_n\| &\leq \|s_n - u_n\| + \|u_n - z_n\| + \|z_n - J_{\lambda_0} s_n\| \\ &= \|\lambda_n A^*(I-T)Au_n\| + \|u_n - z_n\| + \|z_n - J_{\lambda_0} s_n\|. \end{aligned} \tag{3.26}$$

We will use (3.25) and (3.26) later.

Let us show that $\limsup_{n\to\infty} \langle (V - \gamma g)z_0, x_n - z_0\rangle \geq 0$. Put

$$A = \limsup_{n\to\infty} \langle (V - \gamma g)z_0, x_n - z_0\rangle.$$

Without loss of generality, we may assume that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $A = \lim_{i\to\infty} \langle (V-\gamma g)z_0, x_{n_i} - z_0\rangle$ and $\{x_{n_i}\}$ converges weakly to some point $w \in H_1$. Since $\|x_n - z_n\| \to 0$, we also have that $\{z_{n_i}\}$ converges weakly to $w \in H_1$. On the other hand, from $\{\lambda_{n_i}\} \subset [a,b]$ there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in [a,b]$. Without loss of generality, we assume that $z_{n_i} \rightharpoonup w$, $x_{n_i} \rightharpoonup w$ and $\lambda_{n_i} \to \lambda_0$. We know from (3.24) that $\lim_{n\to\infty} \|S_\lambda z_n - z_n\| = 0$. Since

$$\|S_\lambda z_n - z_n\| = \|(1-\lambda)z_n + \lambda S z_n - z_n\| = \lambda\|Sz_n - z_n\| \to 0,$$

we have $Sz_n - z_n \to 0$. Thus, we have $w \in F(S) = F(S_\lambda)$ because S is demiclosed. Since $\lambda_{n_i} \to \lambda_0$, we have from (3.25) that

$$\|J_{\lambda_0} s_{n_i} - z_{n_i}\| \to 0.$$

Furthermore, we have from (3.26) that

$$\|s_{n_i} - J_{\lambda_0} s_{n_i}\| \to 0.$$

Since $s_{n_i} \rightharpoonup w$ and J_{λ_0} is nonexpansive, we have that $w = J_{\lambda_0} w$. Furthermore, since $u_{n_i} \rightharpoonup w$ and A is bounded and linear, we have that $Au_{n_i} \rightharpoonup Aw$. We also know from (3.19) that $\lim_{n\to\infty} \|(I-T)Au_n\| = 0$. Since T is demiclosed, we have $Aw = TAw$, i.e., $w \in A^{-1}F(T)$. Finally, since G is a monotone operator and $\frac{x_{n_i} - u_{n_i}}{r_{n_i}} \in Gu_{n_i}$, we have that for any $(u,v) \in G$,

$$\left\langle u - u_{n_i}, v - \frac{x_{n_i} - u_{n_i}}{r_{n_i}} \right\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_i} \rightharpoonup w$ and $x_{n_i} - u_{n_i} \to 0$, we have

$$\langle u - w, v\rangle \geq 0.$$

Since G is a maximal monotone operator, we have $0 \in Gw$ and hence $w \in G^{-1}0$. Thus we have

$$w \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0.$$

Since $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)\cap G^{-1}0}(I - V + \gamma g)z_0$, we have from (3.1) that

$$A = \lim_{i\to\infty} \langle (V - \gamma g)z_0, x_{n_i} - z_0\rangle = \langle (V - \gamma g)z_0, w - z_0\rangle \geq 0. \tag{3.27}$$

Since $x_{n+1} - z_0 = \alpha_n(\gamma g(x_n) - Vz_0) + (I - \alpha_n V)y_n - (I - \alpha_n V)z_0$, we have from (2.3) that

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &\leq (1 - \alpha_n s)^2 \|y_n - z_0\|^2 + 2\langle \alpha_n(\gamma g(x_n) - Vz_0), x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n s)^2 \|x_n - z_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz_0, x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n s)^2 \|x_n - z_0\|^2 + 2\alpha_n\gamma\ k\|x_n - z_0\|\|x_{n+1} - z_0\| \\
&\qquad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n s)^2 \|x_n - z_0\|^2 + \alpha_n\gamma\ k(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\
&\qquad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&\leq (1 - \alpha_n s)^2 \|x_n - z_0\|^2 + 2\alpha_n\gamma\ k\|x_n - z_0\|^2 \\
&\qquad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&= \{(1 - \alpha_n s)^2 + 2\alpha_n\gamma\ k\} \|x_n - z_0\|^2 \\
&\quad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&= (1 - 2\alpha_n s + (\alpha_n s)^2 + 2\alpha_n\gamma\ k) \|x_n - z_0\|^2 \\
&\qquad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&= (1 - 2(s - \gamma\ k)\alpha_n) \|x_n - z_0\|^2 + (\alpha_n s)^2 \|x_n - z_0\|^2 \\
&\qquad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&= (1 - 2(s - \gamma\ k)\alpha_n) \|x_n - z_0\|^2 + \alpha_n \cdot \alpha_n s^2 \|x_n - z_0\|^2 \\
&\qquad + 2\alpha_n\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle \\
&= (1 - \beta_n) \|x_n - z_0\|^2 \\
&\qquad + \beta_n \left(\frac{\alpha_n s^2 \|x_n - z_0\|^2}{2(s - \gamma\ k)} + \frac{\langle \gamma g(z_0) - Vz_0, x_{n+1} - z_0\rangle}{s - \gamma\ k} \right),
\end{aligned}$$

where $\beta_n = 2(s - \gamma\ k)\alpha_n$. Since $\sum_{n=1}^{\infty} \beta_n = \infty$, we have from Lemma 2.3 that $x_n \to z_0$, where $z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)\cap G^{-1}0}(I - V + \gamma g)z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

According to Lemma 2.4, the function τ is defined on $\{n_0, n_0 + 1, \dots\}$ for some $n_0 \in \mathbb{N}$. Then we have from Lemma 2.4 that $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Thus we have from (3.13) that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\beta_{\tau(n)}(1-\beta_{\tau(n)})\|S_\lambda z_{\tau(n)} - x_{\tau(n)}\|^2 \\
&\qquad \leq (1 - \beta_{\tau(n)})2\alpha_{\tau(n)}\|S_\lambda z_{\tau(n)} - x_{\tau(n)}\|\|\gamma g(x_{\tau(n)}) - Vy_{\tau(n)}\| \\
&\qquad\quad + \alpha_{\tau(n)}^2\|\gamma g(x_{\tau(n)}) - Vy_{\tau(n)}\|^2 \\
&\qquad\quad + 2\alpha_{\tau(n)}\langle \gamma g(x_{\tau(n)}), x_{\tau(n)} - z_0\rangle
\end{aligned} \tag{3.28}$$

$$- 2\alpha_{\tau(n)}\langle Vy_{\tau(n)}, x_{\tau(n)} - z_0\rangle.$$

Using $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, we have from (3.28) that

$$\lim_{n\to\infty}\|S_\lambda z_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.29}$$

As in the proof of Case 1 we also have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \tag{3.30}$$

and $\lim_{n\to\infty}\|y_{\tau(n)} - x_{\tau(n)}\| = 0$. Furthermore, we have $\lim_{n\to\infty}\|(I-T)Au_{\tau(n)}\| = 0$, $\lim_{n\to\infty}\left\|u_{\tau(n)} - z_{\tau(n)}\right\| = 0$ and $\lim_{n\to\infty}\left\|u_{\tau(n)} - x_{\tau(n)}\right\| = 0$. From these we have that

$$\lim_{n\to\infty}\|S_\lambda z_{\tau(n)} - z_{\tau(n)}\| = 0.$$

As in the proof of Case 1, we can show that

$$\limsup_{n\to\infty}\left\langle (V-\gamma g)z_0, x_{\tau(n)} - z_0\right\rangle \ge 0.$$

From $x_{n+1} - z_0 = \alpha_n(\gamma g(x_n) - Vz_0) + (I - \alpha_n V)y_n - (I - \alpha_n V)z_0$ and (2.3), we also have that

$$\begin{aligned}\left\|x_{\tau(n)+1} - z_0\right\|^2 &\le (1-\alpha_{\tau(n)}s)^2\left\|y_{\tau(n)} - z_0\right\|^2\\ &\quad + 2\alpha_{\tau(n)}\langle\gamma g(x_{\tau(n)}) - Vz_0, x_{\tau(n)+1} - z_0\rangle\\ &\le (1-\alpha_{\tau(n)}s)^2\left\|x_{\tau(n)} - z_0\right\|^2\\ &\quad + 2\alpha_{\tau(n)}\langle\gamma g(x_{\tau(n)}) - Vz_0, x_{\tau(n)+1} - z_0\rangle\\ &\le (1-\alpha_{\tau(n)}s)^2\left\|x_{\tau(n)} - z_0\right\|^2\\ &\quad + 2\alpha_{\tau(n)}\gamma\, k\|x_{\tau(n)} - z_0\|\|x_{\tau(n)+1} - z_0\|\\ &\quad + 2\alpha_{\tau(n)}\langle\gamma g(z_0) - Vz_0, x_{\tau(n)+1} - z_0\rangle\\ &\le (1-\alpha_{\tau(n)}s)^2\left\|x_{\tau(n)} - z_0\right\|^2\\ &\quad + \alpha_{\tau(n)}\gamma\, k(\|x_{\tau(n)} - z_0\|^2 + \|x_{\tau(n)+1} - z_0\|^2)\\ &\quad + 2\alpha_{\tau(n)}\langle\gamma g(z_0) - Vz_0, x_{\tau(n)+1} - z_0\rangle\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_{\tau(n)}\gamma\, k)\left\|x_{\tau(n)+1} - z_0\right\|^2 &\le (1-\alpha_{\tau(n)}\gamma\, k)\left\|x_{\tau(n)} - z_0\right\|^2\\ &\quad + \{-2\alpha_{\tau(n)}s + 2\alpha_{\tau(n)}\gamma\, k + (\alpha_{\tau(n)}s)^2\}\|x_{\tau(n)} - z_0\|^2\\ &\quad + 2\alpha_{\tau(n)}\langle\gamma g(z_0) - Vz_0, x_{\tau(n)+1} - z_0\rangle.\end{aligned}$$

From $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we have that

$$\begin{aligned}&2\alpha_{\tau(n)}(s-\gamma\, k)\|x_{\tau(n)} - z_0\|^2\\ &\qquad \le (\alpha_{\tau(n)}s)^2\|x_{\tau(n)} - z_0\|^2 + 2\alpha_{\tau(n)}\langle\gamma g(z_0) - Vz_0, x_{\tau(n)+1} - z_0\rangle.\end{aligned}$$

Since $\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}&2(s-\gamma\, k)\|x_{\tau(n)} - z_0\|^2\\ &\qquad \le \alpha_{\tau(n)}s^2\|x_{\tau(n)} - z_0\|^2 + 2\langle\gamma g(z_0) - Vz_0, x_{\tau(n)+1} - z_0\rangle.\end{aligned}$$

Then we have that

$$\limsup_{n\to\infty} 2(\tau - \gamma\ k) \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0,$$

We have from $s - \gamma\ k > 0$ that $\|x_{\tau(n)} - z_0\| \to 0$. Since $x_{\tau(n)} - x_{\tau(n)+1} \to 0$, we have $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.4 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Applications

In this section, using Theorem 3.1, we can obtain well-known and new strong convergence theorems in Hilbert spaces. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let $U : C \to H$ be an α-inverse strongly monotone mapping, that is,

$$\alpha\|Ux - Uy\|^2 \leq \langle x - y, Ux - Uy\rangle, \quad \forall x, y \in C. \tag{4.1}$$

Putting $T = I - U$ in (4.1), we have from (3) in Examples that T is $(1 - 2\alpha)$-demimetric. Furthermore, since

$$I - 2\alpha U = I - 2\alpha(I - T) = (1 - 2\alpha)I + 2\alpha T,$$

we have from (2.4) that $(1 - 2\alpha)I + 2\alpha T$ is nonexpansive; see [33]. Using this, T is demiclosed. In fact, let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Then

$$x_n - ((1 - 2\alpha)I + 2\alpha T)x_n = 2\alpha(I - T)x_n \to 0.$$

Since $(1-2\alpha)I+2\alpha T$ is nonexpansive, we have $z \in F((1-2\alpha)I+2\alpha T) = F(T)$. This implies that T is demiclosed. If T is a k-strict pseudo-contraction, then $U = I - T$ is $\frac{1-k}{2}$-inverse strongly monotone. So, we have the following lemma obtained by Marino and Xu [19]; see also [32].

Lemma 4.1 ([19, 32])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and $T : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.*

We also know the following lemma from Kocourek, Takahashi and Yao [14]; see also [36].

Lemma 4.2 ([14, 36])**.** *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $S : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Sx_n \to 0$, then $z \in F(S)$.*

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let U be an α-inverse strongly monoton mapping of H_2 into H_2. Let B be a maximal monotone operator on H_1. Let $J_{\lambda_n} = (I + \lambda_n B)^{-1}$ be the resolvents of B for $\lambda_n > 0$. Let S be a nonexpansive mapping of H_1 into H_1. Let $k \in (0, 1)$ and let g be a k-contraction of H_1 into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ be a bounded linear operator such that $\|A\| \neq 0$. Suppose $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)\{\beta_n x_n + (1 - \beta_n) S J_{\lambda_n}(I - \lambda_n A^* U A)x_n\}$$

for all $n \in \mathbb{N}$, where $a, b \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$$

$$\text{and} \quad 0 < a \leq \lambda_n \leq b < \frac{2\alpha}{\|A\|^2}.$$

Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$, where z_0 is a unique fixed point of $P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)}(I - V + \gamma g)$ in $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0)$.

Proof. Put $I - T = U$ in Theorem 3.1, where U is an α-inverse strongly monoton mapping. Then, T is $(1 - 2\alpha)$-demimetric. In Theorem 3.1, we also have

$$1 - \tau = 1 - (1 - 2\alpha) = 2\alpha.$$

Since S is nonexpansive, it is 0-demimetric. Furthermore, since S and T are demiclosed, we obtain the desired result by Theorem 3.1. □

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces. Let T be a generalized hybrid mapping of H_2 into H_2. Let $s \in [0, 1)$ and let S be an s-strict pseudo-contraction of H_1 into H_1. Define $S_\lambda = (1-\lambda)I + \lambda S$ for some λ with $0 < \lambda \leq 1-s$. Let $k \in (0, 1)$ and let g be a k-contraction of H_1 into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ be a bounded linear operator such that $\|A\| \neq 0$. Suppose $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)\{\beta_n x_n + (1 - \beta_n) S_\lambda(I - \lambda_n A^*(I - T)A)x_n\}$$

for all $n \in \mathbb{N}$, where $a, b \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$$

$$\text{and} \quad 0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}.$$

Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap A^{-1}F(T)$, where z_0 is a unique fixed point of $P_{F(S)\cap A^{-1}F(T)}(I - V + \gamma g)$ in $F(S) \cap A^{-1}F(T)$.

Proof. Since T is generalized hybrid, it is 0-demimetric. Furthermore, since S is an s-strict pseudo-contraction, it is s-demimetric and $S_\lambda = (1 - \lambda)I + \lambda S$ for some λ with $0 < \lambda \leq 1 - s$. Since S and T are demiclosed from Lemma 4.1 and 4.2, respectively, we obtain the desired result by Theorem 3.1. □

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemmas were given in Combettes and Hirstoaga [8] and Takahashi, Takahashi and Toyoda [25]; see also [2].

Lemma 4.5 ([8])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Assume that $f : C \times C \to \mathbb{R}$ satisfies (A1) $-$ (A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\}.$$

Then, the following hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*
(4) *$EP(f)$ is closed and convex.*

We call such T_r the resolvent of f for $r > 0$.

Lemma 4.6 ([25])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy (A1) $-$ (A4). Let A_f be a set-valued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $D(A_f) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = (I + rA_f)^{-1}x.$$

Using Lemmas 4.5, 4.6 and Theorem 3.1, we also obtain the following result with equilibrium problem in Hilbert spaces; see also [21, 24].

Theorem 4.7. *Let H_1 and H_2 be Hilbert spaces. Let T be a nonexpansive mapping of H_2 into H_2. Let C be a nonempty, closed and convex subset of H_1. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 3.12. Let $\eta \in (-\infty, 1)$ and let S be an η-demimetric and demiclosed mapping of H_1 into H_1. Define $S_\lambda = (1-\lambda)I + \lambda S$ for some λ with $0 < \lambda \le 1-\eta$. Let $k \in (0,1)$ and let g be a k-contraction of H_1 into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ be a bounded linear operator such that $\|A\| \neq 0$. Suppose $F(S) \cap A^{-1}F(T) \cap E(f) \neq \emptyset$. Let $x_1 = x \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence generated by

$$x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)\{\beta_n x_n + (1-\beta_n) S_\lambda (I - \lambda_n A^*(I-T)A) T_{r_n} x_n\}$$

for all $n \in \mathbb{N}$, where $a, b, c \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$$

$$0 < a \le \lambda_n \le b < \frac{1}{\|A\|^2} \quad \text{and} \quad 0 < c \le r_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap A^{-1}F(T) \cap E(f)$, where z_0 is a unique fixed point of $P_{F(S)\cap A^{-1}F(T)\cap E(f)}(I - V + \gamma g)$ in $F(S) \cap A^{-1}F(T) \cap E(f)$.

Proof. For the bifunction $f : C \times C \to \mathbb{R}$, define A_f as in Lemma 4.6. Let T_{r_n} be the resolvent of A_f for $r_n > 0$. Thus, we obtain the desired result by Theorem 3.1. □

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