Pure and Applied Functional Analysis
Volume 3, Number 2, 2018, 371–391

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

AN IMPLICIT MULTIFUNCTION THEOREM FOR THE HEMIREGULARITY OF MAPPINGS WITH APPLICATION TO CONSTRAINED OPTIMIZATION

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Dedicated to to S.M. Robinson, on the occasion of his 75th birthday

Abstract. The present paper contains some investigations about a uniform variant of the notion of metric hemiregularity, the latter being a less explored property obtained by weakening metric regularity. The introduction of such a quantitative stability property for set-valued mappings is motivated by applications to the penalization of constrained optimization problems, through the notion of problem calmness. As a main result, an implicit multifunction theorem for parameterized inclusion problems is established, which measures the uniform hemiregularity of the related solution mapping in terms of problem data. A consequence on the exactness of penalty functions is discussed.

1. Introduction

The key idea ispiring the Lagrangian approach to constrained optimization is to avoid to determine all elements in the feasible region of a given problem: in fact, solving explicitly a nonlinear equation system is typically a task too hard to be undertaken. Sometimes, it is even superfluous to do it, as far as local optimality is concerned. Instead, the approach proposes to formulate optimality conditions by filling the lack of information about the feasible region with the usage of implicit function theorems. Thus, after such an approach, implicit function theorems became a crucial tool for the constraint system analysis. Historically, constrained optimization acted as a driving force for the development of theorems of this kind. For instance, the celebrated Lyusternik's theorem, one of the earliest implicit function theorems formulated in abstract spaces, which had a remarkable impact on modern variational analysis, was established exactly with this aim (see [17]). That said, it is not surprising that the evolution of optimization conditioned the investigations about implicit function theorems. Essentially, two main facts contributed to shape the evolution process of optimization, stimulated by theoretical and applicational reasons: an increasing complexity of constraint systems and the appearance of nonsmoothness in problem data. Their effect, both in formalizing and solving the

2010 *Mathematics Subject Classification.* 49J52, 49J53, 90C30, 90C31, 90C48.

Key words and phrases. Hemiregularity, constrained optimization, exact penalization, problem calmness, constraint parameterization, strict outer slope, implicit multifunction theorem, Fréchet subdifferential.

resulting optimization problems, was that equations were replaced by more general relations called generalized equations, where set-valued mappings played a fundamental role. In order to devise extensions of the Lagrangian approach suitable to the new context, implicit function theorems had to be adequated. Such a direction of research was soon clearly understood, among the others, by S.M. Robinson, who introduced the term "generalized equation" and provided seminal contributions to the theory coming up around this issue (see [8, 19, 20]). In the large variety of forms taken by the new generation of implicit function theorems that arose with the help of techniques from variational analysis, some common elements can be still recognized: instead of classical functions, they speak of multifunctions, which emerge as a solution mapping of a parameterized generalized equation; instead of differentiability, they establish some kind of Lipschitzian behaviour of the implicitly defined multifunctions, along with related quantitative estimates. Both these features seem to be rather natural within the new context. In particular, notice that differentiability of a mapping can be viewed as a local calmness property of the error resulting from affine approximation of it. Moreover, what is important, they allow to treat effectively a broad spectrum of constraint systems. In the impossibility of providing a comprehensive updated account of all relevant achievements about this theme, the reader is referred to [2, 8, 18, 21, 22] and the bibliographies therein.

The investigations exposed in the present paper proceed along the aforementioned direction of research. In particular, they focus on a property of uniform metric hemiregularity for the solution mapping associated with a parameterized generalized equation, whose interest is motivated by applications to penalty methods in constrained optimization. This property for set-valued mappings can be obtained as a weak variant of the more studied and widely employed property known as metric regularity, which describes a local Lipschitzian behaviour of multifunctions. Even though it made its first appearance in its inverse formulation as Lipschitz lower semicontinuity already in [13], only recently was explicitly formulated and investigated under different names [1] (see [1, 8, 15, 16]).

The contents of the paper are arranged as follows. In Section 2, the basic definitions are introduced, several equivalent reformulations of uniform hemiregularity are provided, along with some examples of uniform hemiregular mappings. This multiple description should help to catch connections with similar properties and then to better understand the main phenomenon under consideration. In Section 3 a motivation for introducing uniform hemiregularity, coming from constrained optimization, is discussed in detail. Section 4 contains the main result of the paper, that is an implicit multifunction theorem. It provides a sufficient condition for the solution mapping, associated with a parameterized inclusion problem, to be uniformly hemiregular at a given point of its graph, along with an estimate of the uniform hemiregularity modulus of it. Such a result is established in a purely metric setting, by means of a variational technique largely employed in this field (see, for instance, [2]). Its impact on constrained optimization in terms of conditions for the exactness of penalty functions and relationships with the existent literature on the

[1] To avoid confusion with another property having the same name (see [22, Definition 10.6.1 (b)]), instead of "semiregularity", which was used in [15, 16], in the present paper the term "hemiregularity", borrowed from [1], is adopted.

subject is then discussed. A specialization of the main result to the Asplund space setting, involving Fréchet coderivatives, is also presented.

Throughout the paper the use of the basic notations is standard. Whenever (P, d) denotes a metric space, given $\bar{p} \in P$ and $r \geq 0$, $\mathrm{B}(\bar{p}, r) = \{p \in P : d(p, \bar{p}) \leq r\}$ indicates the closed ball centred at $\bar{p}$ with radius r. In the same setting, if $S \subseteq P$, $\mathrm{dist}(\bar{p}, S) = \inf_{p \in S} d(\bar{p}, p)$ stands for the distance of $\bar{p}$ from S, with the convention that $\mathrm{dist}(\bar{p}, \varnothing) = +\infty$. By $\mathrm{B}(S, r) = \{p \in P : \mathrm{dist}(p, S) \leq r\}$ the r-enlargement of S is denoted. By $\mathrm{int}\, S$ the topological interior of S is denoted. Whenever $\Theta : P \rightrightarrows X$ is a set-valued mapping, $\mathrm{grph}\, \Theta$ and $\mathrm{dom}\, \Theta$ denote the graph and the domain of Θ, respectively. Throughout the text, the acronyms l.s.c. and u.s.c. stand for lower semicontinuous and upper semicontinuous, respectively. Further special notations will be introduced contextually to their use.

2. Uniform hemiregularity and related notions

In what follows, unless otherwise indicated, all set-valued mappings will be assumed to take closed values. The main property under study is introduced in the following definition.

Definition 2.1. Let $\Theta : P \rightrightarrows X$ be a set-valued mapping between metric spaces and let $(\bar{p}, \bar{x}) \in \mathrm{grph}\, \Theta$. Θ is called:

(i) (*metrically*) *hemiregular at* $(\bar{p}, \bar{x})$ if there exist positive constants κ and r such that

$$\mathrm{dist}\left(\bar{p}, \Theta^{-1}(x)\right) \leq \kappa d(x, \bar{x}), \quad \forall x \in \mathrm{B}(\bar{x}, r);$$

(ii) *uniformly* (*metrically*) *hemiregular at* $(\bar{p}, \bar{x})$ if there exist positive constants κ and r such that

$$\mathrm{dist}\left(\bar{p}, \Theta^{-1}(x)\right) \leq \kappa d(x, z), \quad \forall x \in \mathrm{B}(z, r),\ \forall z \in \Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, r). \tag{2.1}$$

The value

$$\mathrm{u.hreg}(\Theta, (\bar{p}, \bar{x})) = \inf\{\kappa > 0 : \exists r > 0 \text{ for which (2.1) holds }\}$$

is called the *modulus of uniform* (*metric*) *hemiregularity* of Θ at $(\bar{p}, \bar{x})$.

Roughly speaking, the above introduced properties refer to a kind of "quantitative solvability" of the systems

$$x \in \Theta(p),$$

where x is a parameter varying near the reference value $\bar{x}$ and $\bar{p}$ is a solution of the system $\bar{x} \in \Theta(p)$. Notice that, according to the convention made about the value of $\mathrm{dist}(\bar{p}, \varnothing)$, if Θ is hemiregular at $(\bar{p}, \bar{x})$, then each of the perturbed systems must be solvable. Moreover, the distance of the given solution $\bar{p}$ from the varying solution sets must be linearly controlled by the distance of x from $\bar{x}$.

Remark 2.2. The property in Definition 2.1(ii) is clearly a stronger variant than mere hemiregularity, even if the latter holds at each pair $(\bar{p}, z)$, with $z \in \Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, r)$. Indeed, the constants κ and r in Definition 2.1(ii) are postulated to be the same for every $z \in \Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, r)$, whence the term of the resulting property. This uniformity requirement enables one to reformulate such a property in a slightly different way, that will be useful for the purposes of the present analysis: Θ is uniformly hemiregular at $(\bar{p}, \bar{x})$ iff there exist positive κ and δ such that

$$\text{dist}\left(\bar{p}, \Theta^{-1}(x)\right) \leq \kappa \text{dist}\left(x, \Theta(\bar{p})\right), \quad \forall x \in \text{B}(\bar{x}, \delta). \tag{2.2}$$

Indeed, if inequality (2.1) holds true, then for every $x \in \text{B}(\bar{x}, r/2) \backslash \Theta(\bar{p})$ and $\epsilon \in (0,1)$ it is possible to claim the existence of a proper $z_\epsilon \in \Theta(\bar{p})$, such that $d(x, z_\epsilon) < (1+\epsilon)\text{dist}\left(x, \Theta(\bar{p})\right) < r$, where r is as in (2.1). Thus, one obtains

$$\text{dist}\left(\bar{p}, \Theta^{-1}(x)\right) \leq \kappa d(x, z_\epsilon) < \kappa(1+\epsilon)\text{dist}\left(x, \Theta(\bar{p})\right),$$

and hence, by arbitrariness of ϵ, (2.2) is satisfied with $\delta = r/2$. Conversely, since for every $z \in \Theta(\bar{p})$ it is $\text{dist}\left(x, \Theta(\bar{p})\right) \leq d(x, z)$, then from condition (2.2) one gets immediately the validity of (2.1), with $r = \delta$.

Of course, whenever Θ is single-valued at $\bar{p}$, uniform hemiregularity reduces to basic hemiregularity.

The property of hemiregularity of Θ at $(\bar{p}, \bar{x})$ is clearly obtained by weakening the well-known notion of metric regularity of Θ at $(\bar{p}, \bar{x})$, which postulates the existence of positive reals κ and r such that

$$\text{dist}\left(p, \Theta^{-1}(x)\right) \leq \kappa \text{dist}\left(x, \Theta(p)\right), \quad \forall p \in \text{B}(\bar{p}, r), \quad \forall x \in \text{B}(\bar{x}, r) \tag{2.3}$$

(see [8, 13, 18, 21]). This is readily done by fixing $p = \bar{p}$ in inequality (2.3). The following example shows that the resulting property is actually weaker than metric regularity.

Example 2.3. (A mapping which is hemiregular, whereas not metrically regular) Let $P = \mathbb{R}^2$ and $X = \mathbb{R}$ be endowed with their usual Euclidean metric structure. Consider the function $\Theta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$\Theta(p_1, p_2) = \begin{cases} p_1 + p_2^2, & \text{if } p_1 \geq 0, \\ p_1 - p_2^2, & \text{if } p_1 < 0, \end{cases}$$

with reference point $\bar{p} = (0,0)$ and $\bar{x} = 0$. Θ is not metrically regular around $((0,0),0)$, inasmuch as, for any fixed $\kappa > 0$ and $r > 0$, by taking $p = (0, \xi)$, with $0 < \xi < \min\{r, \kappa^{-1}\}$, and $x = 0$, the inequality

$$\text{dist}\left(p, \Theta^{-1}(x)\right) = \xi \leq \kappa\xi^2 = \kappa \text{dist}\left(x, \Theta(p)\right)$$

is evidently false. Nevertheless Θ turns out to be hemiregular at the same reference pair. Indeed, for any $\kappa \geq 1$ and $r > 0$, as for every $x \in [-r, r]$ one has $(x, 0) \in \Theta^{-1}(x)$, one obtains

$$\text{dist}\left((0,0), \Theta^{-1}(x)\right) \leq |x| \leq \kappa|x| = \kappa d(x, 0),$$

so that $\text{u.hreg}(\Theta, (0,0)) \leq 1$.

Analogously, uniform hemiregularity of Θ at $(\bar{p}, \bar{x})$ can be obtained by weakening a uniform variant of metric regularity considered in [25], which requires the existence of positive reals κ and δ such that

$$\text{dist}\left(p, \Theta^{-1}(x)\right) \leq \kappa \text{dist}\left(x, \Theta(p)\right), \quad \forall p \in \text{B}(\bar{p}, r), \quad \forall x \in \text{B}(\Theta(\bar{p}), \delta) \tag{2.4}$$

(see Definition 2.2 [25]). To see this, it suffices to fix $p = \bar{p}$ and to notice that, if $(\bar{p}, \bar{x}) \in \text{grph}\, \Theta$, then $\text{B}(\bar{x}, \delta) = \text{B}(\Theta(\bar{p}), \delta) \cap \text{B}(\bar{x}, \delta)$. It follows that any criterion for

(2.4) to hold becomes a sufficient condition for uniform hemiregularity. Some result of this kind can be found in [25]. In particular, as a consequence of Proposition 2.2 in [25], whenever $\Theta : P \rightrightarrows X$ is a convex process with closed graph between Banach spaces, i.e. $\operatorname{grph}\Theta$ is a closed convex cone in $P \times X$, and the following condition holds

$$\|\Theta^{-1}\|^{-} = \sup_{x\in\mathbb{B}} \inf_{p\in\Theta^{-1}(x)} \|p\| = \sup_{x\in\mathbb{B}} \operatorname{dist}\left(\mathbf{0}, \Theta^{-1}(x)\right) < +\infty, \tag{2.5}$$

where $\|\cdot\|$ denotes the norm on P, $\mathbf{0}$ stands for the null vector of P and $\mathbb{B} = \mathrm{B}(\mathbf{0}, 1)$, then Θ is also uniformly hemiregular at any point $(\bar{p}, \bar{x}) \in \operatorname{grph}\Theta$, with the following estimate

$$\text{u.hreg}(\Theta, (\bar{p}, \bar{x})) \leq \|\Theta^{-1}\|^{-}.$$

Remark 2.4. Since any linear bounded operator $\Lambda : P \longrightarrow X$ between Banach spaces, which is onto, is a convex process with closed graph satisfying condition (2.5), then Λ is also uniformly hemiregular at each pair $(\bar{p}, \Lambda\bar{p})$, with

$$\text{u.hreg}(\Lambda, (\bar{p}, \Lambda\bar{p})) \leq \|\Lambda^{-1}\|^{-}.$$

As uniform hemiregularity implies hemiregularity, notice that from the above fact it is possible to derive the sufficient part of Proposition 5.2 in [1].

Convex processes satisfying condition (2.5) and, as a special case, surjective linear bounded operators, provide examples of mappings which are uniformly hemiregular. Below, an example is proposed of a uniformly hemiregular mapping, which fails to be metrically regular in the sense of Definition 2.2 in [25].

Example 2.5. (A mapping failing to be "uniformly metrically regular", yet uniformly hemiregular) Let $P = \mathbb{R}$ and $X = \mathbb{R}^2$ be endowed with their usual Euclidean metric structure. Consider the set-valued mapping $\Theta : \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined by

$$\Theta(p) = \{x = (x_1, x_2) \in \mathbb{R}^2 : \; x_1 x_2 = p\},$$

and $\bar{p} = 0$ and $\bar{x} = (0, 0)$. In Example 2.2 in [25] Θ has been shown to do not satisfy condition (2.4). Nonetheless Θ is uniformly hemiregular at $(0, (0, 0))$, with $\text{u.hreg}(\Theta, (0, (0, 0))) \leq 1$. Indeed, take $\delta = 1$, so that for every $x = (x_1, x_2) \in \mathrm{B}((0, 0), 1)$ one has $|x_1| \leq 1$ and $|x_2| \leq 1$. Since it is $\Theta^{-1}(x) = \{x_1 x_2\}$, one obtains

$$\operatorname{dist}\left(0, \Theta^{-1}(x)\right) = |x_1 x_2| \leq \min\{|x_1|, |x_2|\} = \operatorname{dist}(x, \Theta(0)), \quad \forall x \in \mathrm{B}((0, 0), 1).$$

Therefore, inequality (2.2), and hence Definition 2.1 (ii), are fulfilled with $\delta = \kappa = 1$.

In Section 1 it has been mentioned that the hemiregularity of a set-valued mapping $\Theta : P \rightrightarrows X$ at $(\bar{p}, \bar{x})$ can be characterized as Lipschitz lower semicontinuity property of its inverse $\Theta^{-1} : X \rightrightarrows P$ at $(\bar{x}, \bar{p})$ (see, for instance, [1, 15, 16]). Recall that a set-valued mapping $\Phi : X \rightrightarrows P$ is said to be *Lipschitz l.s.c.* at $(\bar{x}, \bar{p}) \in \operatorname{grph}\Phi$ if there exist positive δ and l such that

$$\Phi(x) \cap \mathrm{B}(\bar{p}, l d(x, \bar{x})) \neq \varnothing, \quad \forall x \in \mathrm{B}(\bar{x}, \delta).$$

An analogous characterization can be established in the case of uniform hemiregularity, provided that the Lipschitz lower semicontinuity of the inverse is enhanced as follows: a set-valued mapping $\Phi : X \rightrightarrows P$ is said to be *uniformly Lipschitz l.s.c.* at $(\bar{x}, \bar{p})$ if there exist positive δ and l such that

$$\Phi(x) \cap \mathrm{B}(\bar{p}, l\mathrm{dist}\left(x, \Phi^{-1}(\bar{p})\right)) \neq \varnothing, \quad \forall x \in \mathrm{B}(\bar{x}, \delta). \tag{2.6}$$

The value

$$\mathrm{u.liplsc}(\Phi, (\bar{x}, \bar{p})) = \inf\{l > 0 : \exists r > 0 \text{ for which (2.6) holds}\}$$

is called the *modulus of uniform Lipschitz lower semicontinuity* of Φ at $(\bar{x}, \bar{p})$.

Proposition 2.6. *Let $\Theta : P \rightrightarrows X$ be a set-valued mapping between metric spaces. Θ is uniformly hemiregular at $(\bar{p}, \bar{x}) \in \mathrm{grph}\,\Theta$ iff Θ^{-1} is uniformly Lipschitz l.s.c. at $(\bar{x}, \bar{p})$. Moreover, it holds*

$$\mathrm{u.hreg}(\Theta, (\bar{p}, \bar{x})) = \mathrm{u.liplsc}(\Theta^{-1}, (\bar{x}, \bar{p})).$$

Proof. The thesis is a straightforward consequence of the above definitions and of inequality (2.2). □

The above characterization will be conveniently employed in the proof of the implicit multifunction theorem presented in Section 4.

Remark 2.7. It is useful to observe that the Lipschitz lower semicontinuity of a mapping $\Phi : X \rightrightarrows P$, which is single-valued in a neighbourhood of a point $\bar{x} \in X$, reduces to calmness at that point, i.e. there exist positive δ and l such that

$$\Phi(x) \in \mathrm{B}(\Phi(\bar{x}), ld(x, \bar{x})), \quad \forall x \in \mathrm{B}(\bar{x}, \delta).$$

Therefore, whenever a hemiregular set-valued mapping admits an inverse which is locally single-valued, the latter turns out to be calm.

Metric regularity as well as many of its variants are known to admit also characterization in terms of local surjection (openness) properties. This is true also for uniform hemiregularity, whose surjective behaviour is described in the next proposition.

Proposition 2.8. *Let $\Theta : P \rightrightarrows X$ be a set-valued mapping between metric spaces and let $(\bar{p}, \bar{x}) \in \mathrm{grph}\,\Theta$.*

(i) If Θ is uniformly hemiregular at $(\bar{p}, \bar{x})$ with modulus $\mathrm{u.hreg}(\Theta, (\bar{p}, \bar{x})) < +\infty$, *then for any $0 < a < \frac{1}{\mathrm{u.hreg}(\Theta,(\bar{p},\bar{x}))}$, there exists $\tilde{\delta} > 0$ such that*

$$\Theta(\mathrm{B}(\bar{p}, r)) \supseteq \mathrm{B}(\Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, \tilde{\delta}), ar), \quad \forall r \in [0, \tilde{\delta}). \tag{2.7}$$

(ii) If there exist positive reals a and $\tilde{\delta}$ such that inclusion (2.7) *is satisfied, then Θ is uniformly hemiregular at $(\bar{p}, \bar{x})$ with modulus* $\mathrm{u.hreg}(\Theta, (\bar{p}, \bar{x})) \leq 1/a$.

Proof. (i) According to the equivalent reformulation of uniform hemiregularity given in Remark 2.2, for any fixed κ such that $\mathrm{u.hreg}(\Theta, (\bar{p}, \bar{x})) < \kappa < 1/a$, there exists $\delta > 0$ such that inequality (2.2) holds. Then, set $\tilde{\delta} = \frac{\delta\kappa}{1+\kappa}$ and take arbitrary $r \in [0, \tilde{\delta})$ and $x \in \mathrm{B}(\Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, \tilde{\delta}), ar)$. Notice that, with that choice of constants, one has

$$\mathrm{dist}\left(x, \mathrm{B}(\bar{x}, \tilde{\delta})\right) \leq \mathrm{dist}\left(x, \Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, \tilde{\delta})\right) \leq ar,$$

whence

$$d(x, \bar{x}) \leq ar + \tilde{\delta} \leq \tilde{\delta}(a+1) < \tilde{\delta}\left(\frac{1}{\kappa} + 1\right) = \delta.$$

Thus, inequality (2.2) applies, that is

$$\mathrm{dist}\left(\bar{p}, \Theta^{-1}(x)\right) \leq \kappa \mathrm{dist}\left(x, \Theta(\bar{p})\right) \leq \kappa ar < r.$$

This entails that there exists $p \in P$ such that $x \in \Theta(p)$ and $p \in \mathrm{B}(\bar{p}, r)$, what gives that $x \in \Theta(\mathrm{B}(\bar{p}, r))$. This shows the first assertion in the thesis.

(ii) Let us consider an arbitrary $\tilde{a} < a$ and fix a positive δ such that $\delta < \min\{\tilde{a}\tilde{\delta}, \frac{\tilde{a}\tilde{\delta}}{a+\tilde{a}}\}$. In order to prove that inequality (2.2) holds, take an arbitrary $x \in \mathrm{B}(\bar{x}, \delta) \backslash \Theta(\bar{p})$ (if $x \in \Theta(\bar{p})$ the inequality to be proved becomes trivial) and define $r = \mathrm{dist}\left(x, \Theta(\bar{p})\right)$. Notice that $r > 0$ because $\Theta(\bar{p})$ is a closed set. Moreover, as $r \leq d(x, \bar{x}) \leq \delta$, it is $\dfrac{r}{\tilde{a}} \leq \dfrac{\delta}{\tilde{a}} < \tilde{\delta}$. Therefore, by hypothesis, one has

$$\mathrm{B}\left(\Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, \tilde{\delta}), a\,\frac{r}{\tilde{a}}\right) \subseteq \Theta\left(\mathrm{B}\left(\bar{p}, \frac{r}{\tilde{a}}\right)\right). \tag{2.8}$$

Now, let $z_x \in \Theta(\bar{p})$ such that $d(x, z_x) < \dfrac{a}{\tilde{a}} r$. Then, by recalling the choice of δ, one obtains

$$d(z_x, \bar{x}) \leq d(z_x, x) + d(x, \bar{x}) < \frac{a}{\tilde{a}}\, r + \delta \leq \left(\frac{a}{\tilde{a}} + 1\right)\delta < \tilde{\delta},$$

so that $z_x \in \mathrm{B}(\bar{x}, \tilde{\delta}) \cap \Theta(\bar{p})$. This implies

$$\mathrm{dist}\left(x, \Theta(\bar{p}) \cap \mathrm{B}(\bar{x}, \tilde{\delta})\right) \leq d(x, z_x) < \frac{a}{\tilde{a}} r.$$

Thus, by virtue of inclusion (2.8), one obtains that $x \in \Theta\left(\mathrm{B}\left(\bar{p}, \frac{r}{\tilde{a}}\right)\right)$. This means that there exists $p \in \mathrm{B}(\bar{p}, r/\tilde{a})$ such that $x \in \Theta(p)$, that is $p \in \Theta^{-1}(x)$. It follows

$$\mathrm{dist}\left(\bar{p}, \Theta^{-1}(x)\right) \leq d(\bar{p}, p) \leq \frac{r}{\tilde{a}} = \frac{1}{\tilde{a}}\,\mathrm{dist}\left(x, \Theta(\bar{p})\right),$$

namely Θ satisfies (2.2), with δ as above and $\kappa = 1/\tilde{a}$. By arbitrariness of $\tilde{a} \in (0, a)$, the last inequality leads to the estimate of $\mathrm{u.hreg}(\Theta, (\bar{p}, \bar{x}))$ to be proved, thereby completing the proof. □

3. Uniform hemiregularity and exact penalization

Let us consider a constrained optimization problem of the general form

$$(\mathcal{P}) \qquad \min \varphi(x) \quad \text{subject to} \quad x \in R,$$

where $\varphi : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is the objective function and R denotes the feasible region, that throughout the paper is assumed to be a nonempty closed set. The basic idea of penalty methods consists in seeking solutions to $(\mathcal{P})$ by solving unconstrained optimization problems, whose objective function is formed by adding to φ a term measuring the constraint violation (see [10, 26, 27]). Since R is closed, one possible representation of the geometric constraint set R is as $R = \{x \in X : \mathrm{dist}\,(x, R) \leq 0\}$. Consequently, one way of implementing penalty methods is to consider the unconstrained problems

$$(\mathcal{P}_l) \qquad \min_{x \in X}\, [\varphi(x) + l\mathrm{dist}\,(x, R)],$$

with $l > 0$. Letting $\varphi_l = \varphi + l\text{dist}(\cdot, R)$, function φ_l is said to be exact at a local solution $\bar{x} \in R$ to $(\mathcal{P})$ provided that $\bar{x}$ is also a local solution to problem $(\mathcal{P}_l)$. Thus, one is interested in establishing conditions under which φ_l is exact, for some l. It is well know that, whenever φ is locally Lipschitz with constant κ at $\bar{x}$, then φ_l is exact at the same point, for every $l > \kappa$ (see, for instance [5]). This fact can be taken as a starting point for developing applicable optimality conditions for $(\mathcal{P})$, especially with the aid of nonsmooth analysis tools. When, as it often happens in concrete applications, R is defined by specific constraints (such as inequality/equality constraints, variational/equilibrium conditions, and so on) some further conditions are employed to replace the geometric penalty term $\text{dist}(x, R)$ by verifiable measures of the constraint violation, called error bounds, which are expressed in terms of problem data.

The aforementioned exactness condition comes quite expected, inasmuch as it links the behaviour of φ with that of the function $x \mapsto \text{dist}(x, R)$, which is Lipschitz continuous, indeed. If φ fails to be locally Lipschitz the above approach must be modified, but its spirit can be somehow maintained by introducing an additional assumption called problem calmness (see [3, 4, 23]). This notion requires to embed the given problem $(\mathcal{P})$ in a class of parametric optimization problems, whose feasible region comes to depend on a parameter p varying in a metric space (P, d), and then to postulate a controlled behaviour for the variations of φ near $\bar{x}$, with respect to parameter (and hence feasible region) variations. Here, fixed a reference element $\bar{p} \in P$, a set-valued mapping $\mathcal{R} : P \rightrightarrows X$ is meant to be a parameterization of R near $(\bar{p}, \bar{x})$ provided that it fulfills the following two requirements

(i) $\mathcal{R}(\bar{p}) = R$;
(ii) there exist $r > 0$ and $\tau_0 > 0$ such that

$$\forall \tau \in (0, \tau_0)\ \exists p_\tau \in \mathrm{B}(\bar{p}, \tau) \backslash \{\bar{p}\} \text{ such that } \mathcal{R}(p_\tau) \cap \mathrm{B}(\bar{x}, r) \neq \varnothing.$$

A given parameterization $\mathcal{R} : P \rightrightarrows X$ of R near $(\bar{p}, \bar{x})$ enables one to define the related family of parametric optimization problems

$$(\mathcal{P}_p) \qquad \min \varphi(x) \quad \text{subject to} \quad x \in \mathcal{R}(p)$$

embedding $(\mathcal{P})$, in the sense that for $p = \bar{p}$ one obtains $(\mathcal{P})$ as a special case.

Definition 3.1. Given a problem $(\mathcal{P})$, let $\mathcal{R} : P \rightrightarrows X$ be a parameterization of $R = \mathcal{R}(\bar{p})$ near $(\bar{p}, \bar{x})$, where $\bar{x}$ is a local minimizer of $(\mathcal{P})$. Problem $(\mathcal{P})$ is called *calm* at $\bar{x}$ with respect to $\mathcal{R}$ if there exist positive r and ζ such that

$$\varphi(x) \geq \varphi(\bar{x}) - \zeta d(p, \bar{p}), \quad \forall x \in \mathrm{B}(\bar{x}, r) \cap \mathcal{R}(p),\ \forall p \in \mathrm{B}(\bar{p}, r). \tag{3.1}$$

The value

$$\mathrm{clm}(\mathcal{P}, \mathcal{R}, \bar{x}) = \inf\{\zeta > 0 : \exists r > 0 \text{ for which (3.1) holds }\}$$

is called *modulus of problem calmness* of $(\mathcal{P})$ at $\bar{x}$, with respect to $\mathcal{R}$.

Roughly speaking, the concept of problem calmness captures a suitable interplay that intertwines the "not optimal behaviour" of φ out from the feasible region of $(\mathcal{P})$ and the perturbation behaviour of a parameterization of R near $\bar{x}$, as p approaches $\bar{p}$. This fact is illustrated in a very simple case through the next example.

Example 3.2. Consider a problem $(\mathcal{P})$ defined by $X = \mathbb{R}$, $R = (-\infty, 0]$, and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, given by

$$\varphi(x) = \begin{cases} \sqrt{-x}, & \text{if } x \le 0, \\ -\sqrt{x}, & \text{if } x > 0. \end{cases}$$

It is evident that $\bar{x} = 0$ is a (global) solution to $(\mathcal{P})$. Letting $P = \mathbb{R}$ equipped with its usual Euclidean metric and let $\bar{p} = 0$, consider the parameterization $\mathcal{R}_\beta : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$\mathcal{R}_\beta(p) = (-\infty, |p|^\beta], \quad \beta > 0.$$

Taking $x_p = |p|^\beta$, with $|p| < r$, one easily finds

$$\frac{\inf_{x \in \mathcal{R}_\beta(p)} \varphi(x) - \varphi(\bar{x})}{|p|} = \frac{\varphi(x_p) - \varphi(\bar{x})}{|p|} = -|p|^{\frac{\beta}{2} - 1}, \quad \forall p \in \mathbb{R} \backslash \{0\}.$$

Therefore, according to Definition 3.1, $(\mathcal{P})$ turns out to be calm at $\bar{x}$ with respect to $\mathcal{R}_\beta$ iff $\beta \ge 2$. Notice that φ is not locally Lipschitz at 0.

Once a parameterization of R has been defined, the related notion of problem calmness allows one to establish an exact penalization result by introducing the following penalty functions $\varphi_l : P \times X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$

$$\varphi_l(p, x) = \varphi(x) + l \operatorname{dist}(x, \mathcal{R}(p)).$$

In this concern, the property of uniform hemiregularity of $\mathcal{R}$ at $\bar{p}$ plays an essential role, as it appears from the below result.

Theorem 3.3. *Let $\bar{x} \in R$ be a local solution to $(\mathcal{P})$ and let $\mathcal{R} : P \rightrightarrows X$ be a parameterization of R at $(\bar{p}, \bar{x})$, with $\bar{p} \in P$ being a reference value. If*

(i) *$\mathcal{R} : P \rightrightarrows X$ is uniformly hemiregular at $(\bar{p}, \bar{x})$;*
(ii) *$(\mathcal{P})$ is calm at $\bar{x}$ with respect to $\mathcal{R}$;*

then, function $\varphi_l(\bar{p}, \cdot)$ is exact at $\bar{x}$ for every $l >$ u.hreg$(\mathcal{R}, \bar{p}) \cdot$ clm$(\mathcal{P}, \mathcal{R}, \bar{x})$.

Proof. Fix an abitrary l, with $l >$ u.hreg$(\mathcal{R}, \bar{p}) \cdot$ clm$(\mathcal{P}, \mathcal{R}, \bar{x})$. Then, according to Remark 2.2 and Definition 3.1, it is possible to pick $\kappa >$ u.hreg$(\mathcal{R}, \bar{p})$, $\zeta >$ clm$(\mathcal{P}, \mathcal{R}, \bar{x})$ and $\epsilon > 0$ such that:

- for some $r_1 > 0$ it holds

$$\operatorname{dist}\left(\bar{p}, \mathcal{R}^{-1}(x)\right) \le \kappa \operatorname{dist}(x, \mathcal{R}(\bar{p})), \quad \forall x \in \mathrm{B}(\bar{x}, r_1); \tag{3.2}$$

- for some $r_2 > 0$ it holds

$$\varphi(x) \ge \varphi(\bar{x}) - \zeta d(p, \bar{p}), \quad \forall x \in \mathrm{B}(\bar{x}, r_2) \cap \mathcal{R}(p), \ \forall p \in \mathrm{B}(\bar{p}, r_2); \tag{3.3}$$

- it is

$$l > \kappa(1 + \epsilon)\zeta. \tag{3.4}$$

Ab absurdo, let us suppose that $\varphi_l(\bar{p}, \cdot)$ fails to be exact. This means that for every $n \in \mathbb{N}$ there exists $x_n \in \mathrm{B}(\bar{x}, 1/n)$ such that

$$\varphi(x_n) + l \operatorname{dist}(x_n, \mathcal{R}(\bar{p})) < \varphi(\bar{x}). \tag{3.5}$$

Since $\bar{x}$ is a local solution to $(\mathcal{P})$ and $x_n \to \bar{x}$ as $n \to \infty$, there must exist $\bar{n} \in \mathbb{N}$ such that $x_n \notin \mathcal{R}(\bar{p}) = R$ for every $n \in \mathbb{N}$, with $n \geq \bar{n}$. Consequently, as $\mathcal{R}(\bar{p})$ is a closed set, one has

$$\operatorname{dist}(x_n, \mathcal{R}(\bar{p})) > 0, \quad \forall n \in \mathbb{N},\ n \geq \bar{n}.$$

On the other hand, as $\bar{x} \in \mathcal{R}(\bar{p})$, one has

$$\operatorname{dist}(x_n, \mathcal{R}(\bar{p})) \leq d(x_n, \bar{x}) \leq \frac{1}{n},$$

whence

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{R}(\bar{p})) = 0.$$

Thus, by increasing the value of $\bar{n}$ if needed, one obtains $x_n \in \mathrm{B}(\bar{x}, r_1)\backslash\mathcal{R}(\bar{p})$ and hence, according to (3.2), it must be

$$\operatorname{dist}\left(\bar{p}, \mathcal{R}^{-1}(x_n)\right) \leq \kappa \operatorname{dist}(x_n, \mathcal{R}(\bar{p})), \quad \forall n \in \mathbb{N},\ n \geq \bar{n}.$$

This means that for every $n \in \mathbb{N}$, with $n \geq \bar{n}$, there exists $p_n \in \mathcal{R}^{-1}(x_n)$ such that

$$d(p_n, \bar{p}) < \kappa(1+\epsilon)\operatorname{dist}(x_n, \mathcal{R}(\bar{p})), \tag{3.6}$$

where ϵ is as in inequality (3.4). Notice that, as $x_n \in \mathcal{R}(p_n)$ and $x_n \notin \mathcal{R}(\bar{p})$, it has to be $p_n \neq \bar{p}$. From inequalities (3.5) and (3.6), it follows

$$\frac{\kappa(1+\epsilon)}{d(p_n, \bar{p})}[\varphi(x_n) - \varphi(\bar{x})] < \frac{\varphi(x_n) - \varphi(\bar{x})}{\operatorname{dist}(x_n, \mathcal{R}(\bar{p}))} < -l,$$

whence, on account of inequality (3.4), one obtains

$$\varphi(x_n) < \varphi(\bar{x}) - \frac{l}{\kappa(1+\epsilon)} d(p_n, \bar{p}) < \varphi(\bar{x}) - \zeta d(p_n, \bar{p}).$$

Since $p_n \to \bar{p}$ as $n \to \infty$ because of (3.6), by increasing further the value of $\bar{n} \in \mathbb{N}$, if needed, one finds that $x_n \in \mathrm{B}(\bar{x}, r_2) \cap \mathcal{R}(p_n)$ and $p_n \in \mathrm{B}(\bar{p}, r_2)$. Therefore, the last inequality contradicts inequality (3.3). This completes the proof. □

Remark 3.4. It is to be noted that the above theorem can be also derived as a special case from a more general theorem, which was recently established within a unifying approach to the theory of exactness in penalization methods (see [7, Theorem 2.12]). Nevertheless, in formulating that theorem, the uniform hemicontinuity is not mentioned and its role remains hidden, because the mere topological space setting, where optimization problems are considered, does not allow to do so. Moreover, some extra assumptions enter the statement of that result. Theorem 3.3 is therefore a refinement of a special case of Theorem 2.12, whose self-contained proof here proposed emphasizes the role of the main property under study.

It is worth mentioning that in the original definition of problem calmness the parameter p was supposed to perturb linearly the constraining mappings (see [3, 4]). In that special case, it was possible to fully characterize the exactness of penalty functions by means of the resulting notion of problem calmness, what does not remain true for perturbations of more general type (see [23]). Thus, the above result is complemented here with a result providing a sufficient condition, upon which problem $(\mathcal{P}_{\bar{p}})$ turns out to be calm with respect to a given parameterization.

Proposition 3.5. *With reference to a problem parameterization* $(\mathcal{P}_p)$, *let* $\bar{x} \in \mathcal{R}(\bar{p})$ *be a local minimizer of* $(\mathcal{P}_{\bar{p}})$, *with* $\bar{p} \in P$. *Suppose that*

(i) $\mathcal{R}$ *is calm at* $(\bar{p}, \bar{x})$, *i.e. there exist positive reals* ζ *and* r *such that*

$$\mathcal{R}(p) \cap \mathrm{B}(\bar{x}, r) \subseteq \mathrm{B}(\mathcal{R}(\bar{p}), \zeta \mathrm{dist}\,(p, \bar{p})), \quad \forall p \in \mathrm{B}(\bar{p}, r). \tag{3.7}$$

(ii) *there exists* $l > 0$ *such that* $\varphi_l(\bar{p}, \cdot)$ *is exact at* $\bar{x}$.

Then, problem $(\mathcal{P}_{\bar{p}})$ *is calm at* $\bar{p}$ *with respect to* $\mathcal{R}$.

Proof. Assume, ab absurdo, that for every $n \in \mathbb{N}$ there exist $p_n \in \mathrm{B}(\bar{p}, 1/n)\backslash\{\bar{p}\}$ and $x_n \in \mathcal{R}(p_n) \cap \mathrm{B}(\bar{x}, 1/n)$ such that

$$\varphi(x_n) < \varphi(\bar{x}) - n\mathrm{dist}\,(p_n, \bar{p})\,. \tag{3.8}$$

Since $\mathcal{R}$ is supposed to be calm at $(\bar{p}, \bar{x})$, there exist positive reals ζ and r such that inclusion (3.7) holds true. By virtue of this inclusion, the fact that p_n converges to $\bar{p}$ and x_n converges to $\bar{x}$ as $n \to +\infty$ implies that $x_n \in \mathrm{B}(\mathcal{R}(\bar{p}), \zeta \mathrm{dist}\,(p_n, \bar{p}))$, so that one obtains

$$\mathrm{dist}\,(x_n, \mathcal{R}(\bar{p})) \leq \zeta \mathrm{dist}\,(p_n, \bar{p})\,.$$

Consequently, from inequality (3.8) it follows

$$\varphi(x_n) < \varphi(\bar{x}) - \frac{n}{\zeta}\mathrm{dist}\,(x_n, \mathcal{R}(\bar{p}))\,,$$

which evidently contradicts hypothesis (ii). □

4. An implicit function theorem for uniform hemiregularity

In the main result of the previous section, the exact penalization of a constrained optimization problem is obtained upon a uniform hemiregularity assumption on a parameterization of its feasible region. In order to make viable such an approach, conditions are needed, which can guarantee a given parameterization to be uniformly hemiregular. This issue is considered in the present section in the case of feasible regions defined by an abstract equilibrium constraint, namely by constraints of the form

$$\omega \in \Phi(x), \tag{$\mathcal{E}$}$$

where $\Phi : X \rightrightarrows Y$ is a given set-valued mapping between metric spaces and ω is a given element of Y. The format of problem $(\mathcal{E})$ is general enough to cover the constraint systems mostly occurring in the mainly investigated optimization problems, such as equality/inequality systems, cone constraints, equilibrium conditions, generalized equations, lower level optimality in hierarchic optimization problems, and so on. In order to define a parameterization of the solution set of problem $(\mathcal{E})$, one may consider the following problem perturbation, which is defined via any set-valued mapping $F : P \times X \rightrightarrows Y$, such that $F(\bar{p}, x) = \Phi(x)$ for every $x \in X$:

$$\omega \in F(p, x). \tag{$\mathcal{E}_p$}$$

The solution mapping associated with $(\mathcal{E}_p)$ is therefore given by

$$\mathcal{R}(p) = F^{-1}(p, \cdot)(\omega) = \{x \in X : \ \omega \in F(p, x)\}.$$

It is clear that an analytical expression of the (generally) set-valued mapping $\mathcal{R}$ can be hardly derived from $(\mathcal{E}_p)$ by direct computations, because of the severe difficulties in solving explicitly each problem $(\mathcal{E}_p)$. Therefore, it is convenient to investigate the hemiregularity property of $\mathcal{R}$ via an implicit multifunction theorem. Such a task is carried out in what follows by a variational technique. To this aim, let us denote by $|[F]| : P \times X \longrightarrow [0, +\infty]$ the following functional quantifying the the displacement of F from ω:

$$|[F]|(p, x) = \operatorname{dist}(\omega, F(p, x)) .$$

In view of a subsequent employment, a first semicontinuity property of $|[F]|(\cdot, x)$ is stated in the next technical lemma, that can be easily obtained as a special case of [24, Lemma 3.2].

Lemma 4.1. *Let $F : P \rightrightarrows Y$ be a set-valued mapping between metric spaces and let $\omega \in Y$. If F is Hausdorff u.s.c. at $\bar{p} \in \operatorname{dom} F$, i.e. for every $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that*

$$F(p) \subseteq \mathrm{B}(F(\bar{p}), \epsilon), \quad \forall p \in \mathrm{B}(\bar{p}, \delta_\epsilon),$$

then the function $p \mapsto \operatorname{dist}(\omega, F(p))$ is l.s.c. at $\bar{p}$.

Remark 4.2. In the sequel, it will be exploited the fact that the thesis of Lemma 4.1 is true a fortiori if F is u.s.c.. Indeed, the (merely topological) notion of upper semicontinuity at a point implies Hausdorff upper semicontinuity at the same point.

For the purposes of the present analysis, the continuity properties of the function $|[F]|$ are not enough. Derivative-like tools, that enable one to formulate conditions generalizing the nonsingularity requirement in the classical implicit function theorem, are actually needed. In a purely metric space setting, such tools are mainly based on the notion of strong slope (see [6]). More precisely, a more robust variant of it, called strict outer slope, will be employed here in connection with the displacement function, which is defined as follows:

$$\overline{|\nabla_p|[F]||}^{>}(\bar{p}) = \lim_{\epsilon \to 0^+} \inf \{|\nabla_p|[F]||(p, x) : \ p \in \mathrm{B}(\bar{p}, \epsilon), \ x \in \mathrm{B}(\bar{x}, \epsilon), \\ |[F]|(\bar{p}, \bar{x}) < |[F]|(p, x) < |[F]|(\bar{p}, \bar{x}) + \epsilon\},$$

where

$$|\nabla_p|[F]||(p, x) = \begin{cases} 0, & \text{if } p \text{ is a local minimizer to } |[F]|(\cdot, x), \\ \displaystyle\limsup_{q \to p} \frac{|[F]|(p, x) - |[F]|(q, x)}{d(q, p)}, & \text{otherwise,} \end{cases}$$

is the partial strong slope of function $|[F]|$ with respect to the variable p, calculated at $(p, x) \in P \times X$. For more details on this slope as well as on other variations on this theme, the reader is refereed, for instance, to [11].

Now, all the needed elements having been introduced, the main result of the paper can be formulated.

Theorem 4.3. *Let $F : P \times X \rightrightarrows Y$ be a set-valued mapping defining a problem perturbation $(\mathcal{E}_p)$, with solution mapping $\mathcal{R} : P \rightrightarrows X$. Given $\bar{p} \in P$, let $\bar{x} \in \mathcal{R}(\bar{p})$. Suppose that:*

(i) *(P, d) is metrically complete;*

(ii) *there exists* $\delta_0 > 0$ *such that* $F(\cdot, x) : P \rightrightarrows Y$ *is Hausdorff u.s.c. on* $\mathrm{B}(\bar{p}, \delta_0)$, *for every* $x \in \mathrm{B}(\bar{x}, \delta_0)$*;*

(iii) $F(\bar{p}, \cdot) : X \rightrightarrows Y$ *is uniformly Lipschitz l.s.c. at* $(\bar{x}, \omega)$*;*

(iv) $\overline{|\nabla_p|[F]||}^{>}(\bar{p}) > 0$.

Then, $\mathcal{R}$ *is uniformly hemiregular at* $(\bar{p}, \bar{x})$ *and the following estimate holds*

$$\text{u.hreg}(\mathcal{R}, (\bar{p}, \bar{x})) \le \frac{\text{u.liplsc}(F, (\bar{x}, \omega))}{\overline{|\nabla_p|[F]||}^{>}(\bar{p})}. \tag{4.1}$$

Proof. According to hypothesis (iv), it is possible to pick a constant α such that

$$0 < \alpha < \overline{|\nabla_p|[F]||}^{>}(\bar{p}). \tag{4.2}$$

As established in Proposition 2.6, hypothesis (iii) is equivalent to suppose the mapping $F^{-1}(\bar{p}, \cdot) : Y \rightrightarrows X$ to be uniformly hemiregular at $(\omega, \bar{x})$. Since the related moduli coincide, this means that, corresponding to any $\kappa > \text{u.liplsc}(F, (\bar{x}, \omega))$, there exists $r_\kappa > 0$ such that

$$\text{dist}\,(\omega, F(\bar{p}, x)) \le \kappa \text{dist}\left(x, F^{-1}(\bar{p}, \cdot)(\omega)\right), \ \forall x \in \mathrm{B}(\bar{x}, r_\kappa), \tag{4.3}$$

where it is to be recalled that $F^{-1}(\bar{p}, \cdot)(\omega) = \mathcal{R}(\bar{p})$. Define $\tilde{\delta} = \min\{\delta_0, r_\kappa\}$. Observe that inequality (4.2) means that, corresponding to α, it is possible to find $\delta_* \in (0, \tilde{\delta})$ such that

$$|\nabla_p|[F]||(p, x) > \alpha, \tag{4.4}$$

$$\forall p \in \mathrm{B}(\bar{p}, \delta_*), \ \forall x \in \mathrm{B}(\bar{x}, \delta_*), \quad \text{with } 0 = |[F]|(\bar{p}, \bar{x}) < |[F]|(p, x) < \delta_*.$$

In turn, the inequality (4.4) implies that, whenever $(p, x) \in \mathrm{B}(\bar{p}, \delta_*) \times \mathrm{B}(\bar{x}, \delta_*)$, with $0 < |[F]|(p, x) < \delta_*$, then for every $\eta > 0$ there exists $p_\eta \in \mathrm{B}(p, \eta)$ such that

$$|[F]|(p, x) > |[F]|(p_\eta, x) + \alpha d(p_\eta, p). \tag{4.5}$$

Now, choose a positive real r_* satisfying the following condition

$$r_* < \min\left\{\frac{\delta_*}{2}, \frac{\delta_*}{\kappa}\right\},$$

and fix an arbitrary $x \in \mathrm{B}(\bar{x}, r) \backslash \mathcal{R}(\bar{p})$, with

$$0 < r < \min\left\{r_*, \frac{\alpha r_*}{3\kappa}\right\}. \tag{4.6}$$

Let us consider the function $|[F]|(\cdot, x) : \mathrm{B}(\bar{p}, r_*) \longrightarrow [0, +\infty]$. It is obviously bounded from below and, since $r < r_* < \delta_* < \tilde{\delta} \le \delta_0$, then, by virtue of hypothesis (ii) and Lemma 4.1, function $|[F]|(\cdot, x)$ is l.s.c. on $\mathrm{B}(\bar{p}, r_*)$. Owing to hypothesis (i), $\mathrm{B}(\bar{p}, r_*)$ turns out to be a complete metric space. Furthermore, notice that, since $r < r_* < \delta_* < \tilde{\delta} \le r_\kappa$ and hence $x \in \mathrm{B}(\bar{x}, r_\kappa)$, then according to (4.3) it holds

$$|[F]|(\bar{p}, x) \le \kappa \text{dist}\,(x, \mathcal{R}(\bar{p})) \le \inf_{p \in \mathrm{B}(\bar{p}, r_*)} |[F]|(p, x) + \kappa \text{dist}\,(x, \mathcal{R}(\bar{p}))\,.$$

By applying the Ekeland's variational principle (see [9]), one obtains the existence of an element $p_0 \in \mathrm{B}(\bar{p}, r_*)$ such that

$$|[F]|(p_0, x) \le |[F]|(\bar{p}, x); \tag{4.7}$$

$$d(p_0, \bar{p}) \leq \frac{\kappa \text{dist}\,(x, \mathcal{R}(\bar{p}))}{\alpha}; \tag{4.8}$$

$$|[F]|(p_0, x) < |[F]|(p, x) + \alpha d(p, p_0), \quad \forall p \in \mathrm{B}(\bar{p}, r_*) \backslash \{p_0\}. \tag{4.9}$$

Let us show that the last inequalities entail that

$$|[F]|(p_0, x) = 0,$$

so that, as F takes closed values, $x \in \mathcal{R}(p_0)$. Assume, ab absurdo, that $|[F]|(p_0, x) > 0$. Since it is $x \in \mathrm{B}(\bar{x}, r_*)$ and

$$|[F]|(p_0, x) \leq |[F]|(\bar{p}, x) \leq \kappa \text{dist}\,(x, \mathcal{R}(\bar{p})) \leq \kappa r_* < \delta_*,$$

one has

$$p_0 \in \mathrm{B}(\bar{p}, \delta_*) \quad \text{and} \quad x \in \mathrm{B}(\bar{x}, \delta_*), \quad \text{with } 0 < |[F]|(p_0, x) < \delta_*.$$

Thus, if taking $\eta = r_*/2$, according to inequality (4.5), an element p_η must exist in $\mathrm{B}(p_0, r_*/2)$, with $p_\eta \neq p_0$, such that

$$|[F]|(p_0, x) > |[F]|(p_\eta, x) + \alpha d(p_\eta, p_0). \tag{4.10}$$

Observe that, by virtue of inequalities (4.8) and (4.6), it results in

$$d(p_0, \bar{p}) \leq \frac{\kappa}{\alpha} \cdot \frac{\alpha r_*}{3\kappa} < \frac{r_*}{2}.$$

As a consequence, p_η must belong to $\mathrm{B}(\bar{p}, r_*) \backslash \{p_0\}$, because it holds

$$d(p_\eta, \bar{p}) \leq d(p_\eta, p_0) + d(p_0, \bar{p}) < \frac{r_*}{2} + \frac{r_*}{2}.$$

Therefore, inequality (4.9) is found to be evidently contradicted by inequality (4.10). From the fact that $x \in \mathcal{R}(p_0)$, by recalling once again inequality (4.8), one obtains that

$$\text{dist}\left(\bar{p}, \mathcal{R}^{-1}(x)\right) \leq d(\bar{p}, p_0) \leq \frac{\kappa}{\alpha} \text{dist}\,(x, \mathcal{R}(\bar{p}))\,.$$

Since by arbitrariness of x the last inequality remains true all over $\mathrm{B}(\bar{x}, r)$, the set-valued mapping $\mathcal{R}$ is shown to be uniformly hemiregular at $\bar{p}$, with $\text{u.hreg}(\mathcal{R}, \bar{p}) \leq \kappa/\alpha$. Since α and κ can be taken arbitrarily closed to the value of $\overline{|\nabla_p |[F]||}^{>}(\bar{p})$ and $\text{u.hreg}(F^{-1}(\bar{p}, \cdot), \omega)$, respectively, then from the last inequality it is possible to derive the estimate appearing in the thesis. This completes the proof. □

As a comment to Theorem 4.3, it is to be noted that its thesis combines solvability and sensitivity information, according to the spirit of implicit function theorems. Indeed, problems $(\mathcal{E}_p)$ turn out to be solvable for every p in a neighbourhood of $\bar{p}$, as a direct consequence of the hemiregularity of $\mathcal{R}$ at $(\bar{p}, \bar{x})$. The sensitivity part comes from the estimation of $\text{u.hreg}(\mathcal{R}, (\bar{p}, \bar{x}))$, which is fully expressed in terms of problem data.

To the best of the author's knowledge, the only existing implicit multifunction theorem involving hemiregularity is [1, Theorem 5.4]. A direct comparison of Theorem 4.3 with this result can not be accomplished for several reasons. First, even if restated in a common setting ([1, Theorem 5.4] is valid in Banach spaces), they consider solution mappings associated with different problems (an inclusion problem involving a set-valued mapping versus an equation with a perturbing term). Besides, assuming to consider a single-valued mapping F in Theorem 4.3 and a null perturbation term $g \equiv \mathbf{0}$ in Theorem 5.4, the former considers uniform hemiregularity, whereas the latter deals with a mere hemiregularity with respect to one variable, which is uniform with respect to the other variable. Nevertheless, with all that, a common pattern can be traced: Theorem 4.3 assumes the uniform Lipschitz lower semicontinuity with respect to x of the problem data to gain the uniform hemiregularity of the solution mapping, while Theorem 5.4 assumes the hemiregularity with respect to x of the problem data to achieve the Lipschitz lower semicontinuity [2] of the solution mapping. The condition enabling this phenomenon is the nondegeneracy of the strict outer slope with respect to p of the displacement functional in the first case, which is replaced by a calmness condition with respect to p in Theorem 5.4.

To assess the impact of the above result on constrained optimization, let us consider problems of the form

$$(\mathcal{P}_{\mathcal{E}}) \qquad \min \varphi(x) \quad \text{subject to} \quad x \in R = \Phi^{-1}(\omega),$$

that is with constraints in the abstract form $(\mathcal{E})$. The reader should notice that, even though inequality (2.2) involves the set $\mathcal{R}(\bar{p})$, which seems to require the knowledge of the feasible region of $(\mathcal{P}_{\mathcal{E}})$, nonetheless Theorem 4.3 can be effectively exploited for achieving the exactness of penalty functions, if combined with problem calmness, as stated next. Below, by penalty function $\varphi_l : P \times X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ associated with problem $(\mathcal{P}_{\mathcal{E}})$, the following functional is meant:

$$\varphi_l(p, x) = \varphi(x) + l|[F(p, x)]|.$$

Observe that in order to evaluate φ_l one needs only the problem data.

Corollary 4.4. *Let $F : P \times X \rightrightarrows Y$ be a perturbation of Φ defining a parameterization of the feasible reagion of problem $(\mathcal{P}_{\mathcal{E}})$, let $\bar{p} \in P$ such that $F(\bar{p}, \cdot) = \Phi$, and let $\bar{x}$ be a local solution of $(\mathcal{P}_{\mathcal{E}})$. Suppose that*

(i) *(P, d) is metrically complete;*
(ii) *there exists $\delta_0 > 0$ such that $F(\cdot, x) : P \rightrightarrows Y$ is Hausdorff u.s.c. on $\mathrm{B}(\bar{p}, \delta_0)$, for every $x \in \mathrm{B}(\bar{x}, \delta_0)$;*
(iii) *$F(\bar{p}, \cdot) : X \rightrightarrows Y$ is uniformly Lipschitz l.s.c. at $(\bar{x}, \omega)$;*
(iv) *$\overline{|\nabla_p|[F]||}^{>}(\bar{p}) > 0$;*
(v) *$(\mathcal{P}_{\mathcal{E}})$ is calm at $\bar{x}$ with respect to the parameterization $\mathcal{R}$ defined by F.*

Then, for every

$$l > \frac{\mathrm{u.liplsc}(F, (\bar{x}, \omega)) \cdot \mathrm{clm}(\mathcal{P}_{\mathcal{E}}, \mathcal{R}, \bar{x})}{\overline{|\nabla_p|[F]||}^{>}(\bar{p})} \tag{4.11}$$

function $\varphi_l(\bar{p}, \cdot)$ is exact at $\bar{x}$.

Proof. It suffices to apply Theorem 4.3 and Theorem 3.3. The estimate (4.11) can be immediately obtained by inequality (4.1) and the condition on the penalty term appearing in the thesis of Theorem 3.3. □

[2] Actually, in the statement of Theorem 5.4 this property is not mentioned, but is expressed as hemiregularity of the inverse multifunction.

To guide a comparison of Theorem 4.3 with other similar implicit multifunction theorems of new generation, it must be pointed out that, often, along with the local solvability of the parameterized system $(\mathcal{E}_p)$, a local error bound of the form

$$\operatorname{dist}(x, \mathcal{R}(p)) \leq \kappa \operatorname{dist}(\omega, F(p,x)),$$

is also established, with $\kappa > 0$ and with p varying around $\bar{p}$, or $p = \bar{p}$ (let us mention here [2, Theorem 5.5.5], which served as a paradigm for many epigones in the subsequent literature). Such distance estimates, stemming from the Lyusternik's theorem, are useful for deriving optimality conditions for problems with Lipschitz objective functions. Of course, they can be generalized obtaining Hölder type estimates in order to treat problems with corresponding Hölder objective functions. In contrast to this, in Corollary 4.4 no assumption is made on the objective function of problem $(\mathcal{P}_{\mathcal{E}})$, apart problem calmness (hypothesis (v)), which relates to both φ and $\mathcal{R}$. Thus, the present approach to implicit multifunction theorem reveals that Lipschitz/Hölder assumptions on the objective function can be dropped out at the price of introducing a suitable interplay between the parameterization of the feasible region and the objective function.

The rest of the current section is devoted to establish a version of Theorem 4.3 working in Banach spaces. Such a setting, which is more structured than purely metric spaces, enables one to reformulate the condition on the strict outer slope of the displacement functional in terms of derivative-like objects. Since the displacement functional is rarely expected to be differentiable, this will be done by employing tools of nonsmooth analysis. More precisely, the partial Fréchet coderivative of the set-valued mapping F will be used. In order to recall this generalized derivative construction, some basic elements of the Fréchet subdifferential calculus and the related geometry are needed. In what follows, whenever $(X, \|\cdot\|)$ denotes a Banach space, its continuous dual and the related unit ball are indicated by X^* and $\mathbb{B}^*$, respectively. Given a function $\varphi : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ defined on a Banach space and $\bar{x} \in \operatorname{dom} \varphi = \{x \in \mathbb{X} : \ |\varphi(x)| < +\infty\}$, the Fréchet (alias, regular) subdifferential of φ at $\bar{x}$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) = \left\{ x^* \in X^* : \ \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

Given a subset $S \subseteq X$ and $\bar{x} \in S$, the Fréchet (alias, regular) normal cone of S at $\bar{x}$ is defined by

$$\widehat{\mathrm{N}}(\bar{x}, S) = \left\{ x^* \in X^* : \ \limsup_{x \overset{S}{\to} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

Notice that the two aforementioned notions are linked through the set indicator function $\iota_S : X \longrightarrow \{0, +\infty\}$, in the sense that

$$\widehat{\mathrm{N}}(\bar{x}, S) = \widehat{\partial}\iota_S(\bar{x}), \quad \bar{x} \in X.$$

Given a set-valued mapping $\Phi : X \rightrightarrows Y$ between Banach spaces and $(\bar{x}, \bar{y}) \in \operatorname{grph} \Phi$, the Fréchet coderivative of Φ at $(\bar{x}, \bar{y})$ is the set-valued mapping $\widehat{\mathrm{D}}^*\Phi(\bar{x}, \bar{y}) : Y^* \rightrightarrows$

X^* defined through the Fréchet normal cone to its graph as follows

$$\widehat{\mathrm{D}}^*\Phi(\bar{x},\bar{y})(y^*) = \{x^* \in X^* : \ (x^*, -y^*) \in \widehat{\mathrm{N}}((\bar{x},\bar{y}), \operatorname{grph}\Phi)\}, \quad y^* \in Y^*.$$

The Fréchet subdifferential, the Fréchet normal cone and the Fréchet coderivative are the basic pillars of the nonsmooth calculus here employed (see, for more details, [2, 14, 18, 22]). It is well known that the natural environment where to handle the aforementioned Fréchet constructions are Asplund spaces. Recall that a Banach space $(X, \|\cdot\|)$ is said to be Asplund if every continuous convex function defined on a nonempty open convex subset C of X is Fréchet differentiable on a dense G_δ subset of C. It has been proved that the Asplund property for a Banach space can be characterized by the fact that each of its separable subspaces admits a separable dual (see [2, 18]). The class of Asplund spaces, including all weakly compactly generated spaces and, hence, all reflexive Banach spaces, is wide enough for many applications. Moreover, every Banach space having Fréchet smooth bump functions (in particular, every space admitting a Fréchet smooth renorm) is Asplund. One of reasons why Asplund spaces are the natural environment for the Fréchet nonsmooth calculus deals with the fact that the Asplund property can be also characterized in terms of the validity of the following Lipschitz local approximate Fréchet subdifferential (for short, fuzzy) sum rule. Such a rule plays a key role in many circumstances arising in optimization and variational analysis.

Definition 4.5. Let $(X, \|\cdot\|)$ be a Banach space. X is said to satisfy the *Fréchet fuzzy sum rule* if for any l.s.c. function $\varphi_1 : X \longrightarrow \mathbb{R}\cup\{+\infty\}$, any Lipschitz function $\varphi_2 : X \longrightarrow \mathbb{R}$, and any $\epsilon > 0$, whenever $\bar{x} \in X$ is a local minimizer of $\varphi_1 + \varphi_2$, there exist $x_i \in X$ and $x_i^* \in \widehat{\partial}\varphi_i(x_i)$, $i = 1, 2$, such that

$$(x_i, \varphi_i(x_i)) \in \mathrm{B}((\bar{x}, \varphi_i(\bar{x})), \epsilon), \quad i = 1, 2,$$

and

$$\|x_1^* + x_2^*\| < \epsilon.$$

Among the notable achievements of nonlinear functional analysis, there is the understanding that a Banach space is Asplund iff it satisfies the Fréchet fuzzy sum rule (see [2, 18]). In other words, any Asplund space is $\widehat{\partial}$-trustworthy in the sense of [12]. The next lemma adapts [12, Proposition 1, Ch. 3] to the specific need of the present analysis.

Lemma 4.6. *Let $F : W \rightrightarrows Y$ be a set-valued mapping between Banach spaces and let $W \subseteq P \times X$ be an open set. Suppose that:*

(i) *$(P, \|\cdot\|)$ is Asplund;*

(ii) *the set-valued mapping $F(\cdot, x)$ is Hausdorff u.s.c. on $\Pi_P(W) = \{p \in P : \exists x \in X : \ (p, x) \in W\}$, for each $x \in \Pi_X(W) = \{x \in X : \ \exists p \in P : \ (p, x) \in W\}$.*

Then, it holds

$$\inf_{(p,x)\in W} |\nabla_p|[F]||(p, x) \geq \inf\{\|p^*\| : \ p^* \in \widehat{\partial}_p|[F]|(p, x), \ (p, x) \in W\}.$$

Proof. Observe that, by virtue of Lemma 4.1, each function $p \mapsto |[F]|(p, x)$, with $x \in \Pi_X(W)$, is l.s.c. on $\Pi_P(W)$. Set $\mu = \inf_{(p,x)\in W} |\nabla_p|[F]||(p, x)$ and take an arbitrary

$\epsilon > 0$. Corresponding to $\epsilon/2$, there exists $(p_\epsilon, x_\epsilon) \in W$ such that $|\nabla_p|[F]||(p_\epsilon, x_\epsilon) < \mu + \frac{\epsilon}{2}$. According to the definition of partial strong slope, the last inequality implies the existence of $\delta > 0$ such that

$$|[F]|(p, x_\epsilon) + \left(\mu + \frac{\epsilon}{2}\right) \|p - p_\epsilon\| \geq |[F]|(p_\epsilon, x_\epsilon), \quad \forall p \in \mathrm{B}(p_\epsilon, \delta).$$

Notice that the function $p \mapsto \left(\mu + \frac{\epsilon}{2}\right) \|p - p_\epsilon\|$ is convex and Lipschitz continuous, so by well-known properties of the Fréchet subdifferential (see, for instance, [14, Proposition 1.2]) and well-known formulas of convex analysis, one has

$$\widehat{\partial} \left(\mu + \frac{\epsilon}{2}\right) \| \cdot - p_\epsilon\|(p) \subseteq \left(\mu + \frac{\epsilon}{2}\right) \mathbb{B}^*, \quad \forall p \in P.$$

Since the function $|[F]|(\cdot, x_\epsilon) + \left(\mu + \frac{\epsilon}{2}\right) \| \cdot - p_\epsilon\|$, which is the sum of a l.s.c. and a Lipschitz function, attains a local minimum at p_ϵ, it is possible to apply the Fréchet fuzzy sum rule in Definition 4.5, in force of hypothesis (i). Accordingly, taken $\eta \in (0, \epsilon/2)$ in such a way that $\mathrm{B}(p_\epsilon, \eta) \times \{x_\epsilon\} \in W$, one gets consequent (p_i, x_ϵ) and $p_i^* \in P^*$, $i = 1, 2$, such that

$$\|p_i - p_\epsilon\| < \eta, \qquad i = 1, 2,$$

$$p_1^* \in \widehat{\partial}_p |[F]|(p_1, x_\epsilon), \qquad p_2^* \in \widehat{\partial} \left(\mu + \frac{\epsilon}{2}\right) \| \cdot - p_\epsilon\|(p_2),$$

and

$$\|p_1^* + p_2^*\| < \eta.$$

As it is $\eta < \epsilon/2$, one can deduce that $p_1^* \in (\mu + \epsilon)\mathbb{B}^*$, and hence, since it is $(p_1, x_\epsilon) \in \mathrm{B}(p_\epsilon, \eta) \times \{x_\epsilon\} \subseteq W$, one obtains

$$\inf\{\|p^*\| : \ p^* \in \widehat{\partial}_p |[F]|(p, x), \ (p, x) \in W\} \leq \mu + \epsilon.$$

The thesis follows by arbitrariness of ϵ. □

Now, for formulating the next technical lemma, some further notations are needed. Given a set-valued mapping $F : P \times X \rightrightarrows Y$ and $(p, x) \in P \times X$, let us set

$$\sigma(p, x) = \lim_{\epsilon \to 0^+} \inf\{\|p^*\| \ : \ p^* \in \widehat{\mathrm{D}}^* F(\cdot, x)(p', y')(y^*), \ \|y^*\| = 1, \ p' \in \mathrm{B}(p, \epsilon), \ y' \in Y : \|y'\| \leq |[F]|(p', x) + \epsilon\},$$

where $\widehat{\mathrm{D}}^* F(\cdot, x)(p', y') : Y^* \rightrightarrows P^*$ denotes the Fréchet coderivative of the set-valued mapping $F(\cdot, x) : P \rightrightarrows Y$ (hence, the partial coderivative of F with respect to p), calculated at $(p', y') \in \operatorname{grph} F(\cdot, x)$. Furthermore, set

$$V_\eta = \operatorname{int} [\mathrm{B}(\bar{p}, \eta) \times \mathrm{B}(\bar{x}, \eta)] \backslash F^{-1}(\mathbf{0}).$$

Remark 4.7. Notice that, since $F : P \times X \rightrightarrows Y$ is closed valued, one has

$$(P \times X) \backslash F^{-1}(\mathbf{0}) = \{(p, x) \in P \times X : \ |[F]|(p, x) > 0\}.$$

Therefore, whenever F is u.s.c. on a set $\operatorname{int} [\mathrm{B}(\bar{p}, \delta_0) \times \mathrm{B}(\bar{x}, \delta_0)]$, so that the function $|[F]| : P \times X \longrightarrow [0, +\infty]$ is l.s.c. on the same set (remember Remark 4.2), each set V_η, with $\eta < \delta_0$, turns out to be open.

Lemma 4.8. *Let $F : W \rightrightarrows Y$ be a set-valued mapping between Banach spaces and let $W \subseteq P \times X$ be an open set. Suppose that:*

(i) $(P, \| \cdot \|)$ *and* $(Y, \| \cdot \|)$ *are Asplund;*

(ii) *there exists* $\delta_0 > 0$ *such that* $F(\cdot, x)$ *is u.s.c. on* $\mathrm{B}(\bar{p}, \delta_0)$*, for each* $x \in \mathrm{B}(\bar{x}, \delta_0)$*;*

(iii) *it is* $W \subseteq [\mathrm{B}(\bar{p}, \delta_0) \times \mathrm{B}(\bar{x}, \delta_0)] \backslash F^{-1}(\mathbf{0})$ *and there exists a constant* $\sigma > 0$ *such that*

$$\inf_{(p,x)\in W} \sigma(p, x) \geq \sigma.$$

Then, it holds

$$\inf\{\|p^*\| : \ p^* \in \widehat{\partial}_p|[F]|(p, x), \ (p, x) \in W\} \geq \sigma.$$

Proof. The thesis follows at once from [2, Lemma 5.5.4]. Indeed, it suffices to replace the Fréchet subdifferential and coderivative with their partial counterparts and to observe that, in order to apply the Fréchet fuzzy sum rule, the hypothesis about the Fréchet smoothness assumed in [2, Lemma 5.5.4] can be replaced with the Asplund property of P and Y. Recall that the Cartesian product of Asplund spaces is still Asplund (see [18]). □

By means of the above constructions, it is possible to establish the following coderivative condition for the uniform hemiregularity of the multifunction implicitly defined by a problem $(\mathcal{E})$, in a Banach space setting.

Theorem 4.9. *Let* $F : P \times X \rightrightarrows Y$ *be a set-valued mapping between Banach spaces defining a parameterization* $\mathcal{R} : P \rightrightarrows X$ *for the solution set* R *of a problem* $(\mathcal{E})$*. Given* $\bar{p} \in P$*, let* $\bar{x} \in \mathcal{R}(\bar{p})$*. Suppose that:*

(i) $(P, \|\cdot\|)$ *and* $(Y, \|\cdot\|)$ *are Asplund;*

(ii) *there exists* $\delta_0 > 0$ *such that* F *is u.s.c. on* $\mathrm{B}(\bar{p}, \delta_0) \times \mathrm{B}(\bar{x}, \delta_0)$*;*

(iii) $F(\bar{p}, \cdot) : X \rightrightarrows Y$ *is uniformly Lipschitz l.s.c. at* $(\bar{x}, \mathbf{0})$*;*

(iv) *it is*

$$\sigma = \lim_{\eta\to 0^+} \inf_{(p,x)\in V_\eta} \sigma(p, x) > 0. \tag{4.12}$$

Then, $\mathcal{R}$ *is uniformly hemiregular at* $(\bar{p}, \bar{x})$ *and the following estimate holds*

$$\mathrm{u.hreg}(\mathcal{R}, (\bar{p}, \bar{x})) \leq \frac{\mathrm{u.liplsc}(F, (\bar{x}, \mathbf{0}))}{\sigma}.$$

Proof. The proof clearly relies on the application of Theorem 4.3. Let us check that all hypotheses of that theorem are actually fulfilled under the current assumptions.

Hypothesis (i) takes trivially place in a Banach space setting. As to hypothesis (ii), it suffices to recall Remark 4.2. It remains to show that condition (4.12) guarantees the validity of hypothesis (iv). To this aim, let us start by observing that, fixed an arbitrary $\zeta > 0$, inequality (4.12) implies that it is possible to find $\eta \in (0, \delta_0/2)$ such that

$$\inf_{(p,x)\in V_\eta} \sigma(p, x) \geq \sigma - \zeta.$$

Thus, by applying Lemma 4.8 with $W = V_\eta$ (note that, under the current hypotheses, it is an open set according to Remark 4.7), one finds

$$\inf\{\|p^*\| : \ p^* \in \widehat{\partial}_p|[F]|(p, x), \ (p, x) \in V_\eta\} \geq \sigma - \zeta.$$

In turn, on account of Lemma 4.6, the last inequality gives

$$\inf_{(p,x)\in V_\eta} |\nabla_p|[F]||(p,x) \geq \sigma - \zeta.$$

By recalling the definition of $\overline{|\nabla_p|[F]||}^{>}(\bar{p})$, since for a proper $\epsilon > 0$ it happens that $\{(p,x) \in \mathrm{B}(\bar{p},\epsilon) \times \mathrm{B}(\bar{x},\epsilon) :\ 0 < |[F]|(p,x) < \epsilon\} \subseteq V_\eta$, one obtains

$$\overline{|\nabla_p|[F]||}^{>}(\bar{p}) \geq \inf_{(p,x)\in V_\eta} |\nabla_p|[F]||(p,x) \geq \sigma - \zeta.$$

As ζ has been arbitrarily taken, the above inequality completes the proof. □

Acknowledgments The author thanks an anonymous referee for very helpful suggestions and comments.

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