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# DYNAMICAL SYSTEMS WITH A LYAPUNOV FUNCTION ON UNBOUNDED SETS

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Abstract. An algorithm for minimizing an objective function on a set can often be considered a self-mapping of the set for which the objective function is a Lyapunov function. In this paper the set is a complete metric space, which is not necessarily bounded, and we study the asymptotic behavior of trajectories of the dynamical system which is induced by the algorithm. To this end, we introduce the notion of a normal mapping and show that the sequence of values of the Lyapunov function tends to the infimum of the Lyapunov function along any sequence of powers of such a mapping. We also show that under certain assumptions, a generic (typical) mapping is indeed normal.

## 1. Introduction and preliminaries

In this paper we study the asymptotic behavior of trajectories of a certain dynamical system which originates in a minimization problem. An algorithm for minimizing an objective function $f : K \to R^1$ on a set $K$ can often be considered a self-mapping $A$ of the set $K$ for which the objective function $f$ is a Lyapunov function. In this paper the set $K$ is a complete metric space which is not necessarily bounded. We introduce the notion of a normal mapping and show that the sequence of values of the Lyapunov function $f$ tends to the infimum of $f$ along any sequence of powers of such a mapping. We also show that under certain assumptions, a generic (typical) self-mapping of $K$, for which $f$ is a Lyapunov function, is indeed normal. More precisely, we consider a complete metric space $\mathcal{A}$ of self-mappings of $K$, which share a common Lyapunov function $f$, equipped with a natural complete metric, and show that there exists a subset which is a countable intersection of open and everywhere dense sets in $\mathcal{A}$ such that each mapping belonging to this subset is indeed normal. Such an approach is common in nonlinear analysis and optimization theory [1, 8, 10, 11, 12, 13]. Thus, instead of considering a certain property for a single operator, we investigate it for a space of all such operators, endowed with some natural complete metric, and show that this property holds for most of these

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operators in the sense of Baire category. This allows us to establish this property without restrictive assumptions on the space and on the operators themselves.

In the present paper we generalize the results which were obtained in [2] and presented in Chapter 4 of [11] in the case where $K$ is a bounded, closed and convex set in a Banach space. In this case our dynamical system was also studied in [8, 9]. In contrast with our previous results, here we no longer assume that $K$ is bounded. In addition, for most of our results, $K$ is, as a matter of fact, a general complete metric space.

Throughout the paper, $(K, \rho)$ is a complete metric space and $f : K \to R^1$ is a lower semicontinuous function, which is bounded from below. Set

$$\inf(f) := \inf\{f(x) : \ x \in K\}$$

and denote by $\mathcal{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(A(x)) \le f(x) \text{ for all } x \in K. \tag{1.1}$$

Denote by $\mathcal{A}_c$ the set of all continuous mappings $A \in \mathcal{A}$, by $\mathcal{A}_u$ the set of all $A \in \mathcal{A}$ which are uniformly continuous on bounded subsets of $K$, and by $\mathcal{A}_b$ the set of all $A \in \mathcal{A}$ which are bounded on bounded subsets of $K$.

From the point of view of the theory of dynamical systems, each element of $\mathcal{A}$ describes a stationary dynamical system with a Lyapunov function $f$. Also, some optimization procedures in Banach spaces can be represented by elements of $\mathcal{A}$ (see examples in Section 4.4 of [11] and in [4, 5]). As we have already mentioned above, we assume that our function $f$ is lower semicontinuous, which is a standard assumption in optimization theory. However, in order to obtain our results, we need to impose on $f$ additional assumptions, which are spelled out in the statements of the theorems.

For each $x \in K$ and each $r > 0$, set

$$B(x, r) := \{y \in K : \ \rho(x, y) \le r\}.$$

Fix a point $\theta \in K$. We equip the set $\mathcal{A}$ with the uniformity determined by the following base:

$$\mathcal{E}(N, \epsilon) := \{(A, B) \in \mathcal{A} \times \mathcal{A} :$$

$$\rho(A(x), B(x)) \le \epsilon \text{ for all } x \in B(\theta, N)\}, \tag{1.2}$$

where $N$ is a natural number and $\epsilon > 0$. Clearly, the uniform space $\mathcal{A}$ is metrizable (by a metric $\sigma$) and complete. The uniform space $\mathcal{A}$ is equipped with the topology generated by its uniformity. It is clear that $\mathcal{A}_b$, $\mathcal{A}_c$ and $\mathcal{A}_u$ are closed subsets of $\mathcal{A}$. The sets $\mathcal{A}_b$, $\mathcal{A}_c$, $\mathcal{A}_u$, $\mathcal{A}_b \cap \mathcal{A}_c \subset \mathcal{A}$ are all equipped with the relative topology (uniformity).

A mapping $A \in \mathcal{A}$ is called *normal* if for each $\epsilon, M > 0$, there exists a number $\delta > 0$ such that for each point $x \in B(\theta, M)$ satisfying

$$f(x) \ge \inf(f) + \epsilon,$$

the inequality

$$f(A(x)) \le f(x) - \delta$$

holds.

**Example 1.1.** Assume that there exists a point $x_{min} \in K$ such that

$$f(x_{min}) = \inf(f).$$

Then the mapping defined by $A(x) := x_{min}$, $x \in K$, is normal.

For each self-mapping $A : K \to K$, denote by $A^0$ the identity operator $I$, that is, $I(x) = x$ for all $x \in K$.

We begin with a convergence result (Theorem 2.1 below). We then continue with two stability theorems (Theorems 3.1 and 4.2 in Sections 3 and 4, respectively). We conclude our paper with a generic result in complete hyperbolic spaces (Theorem 6.1 in Section 6).

## 2. Convergence result

In this section we use all the notations, definitions and assumptions which were introduced in Section 1.

**Theorem 2.1.** *Let $A \in \mathcal{A}$ be normal, let $f$ be bounded on bounded subsets of $K$ and satisfy*

$$\lim_{\rho(x,\theta)\to\infty} f(x) = \infty, \tag{2.1}$$

*and let $\epsilon, M > 0$ be given. Then there exists a natural number $n_0$ such that for each integer $n \geq n_0$ and each point $x \in B(\theta, M)$, we have*

$$f(A^n(x)) \leq \inf(f) + \epsilon.$$

*Proof.* There exists a number $C_0 > 0$ such that

$$|f(x)| \leq C_0 \text{ for all } x \in B(\theta, M). \tag{2.2}$$

In view of (2.1), there exists $M_1 > 0$ such that

$$\{x \in K : \ f(x) \leq C_0\} \subset B(\theta, M_1). \tag{2.3}$$

Since the mapping $A$ is normal, there exists a number $\delta > 0$ such that

$$\{x \in B(\theta, M_1) : \ f(x) \geq \inf(f) + \epsilon\} \subset \{x \in K : \ f(A(x)) \leq f(x) - \delta\}. \tag{2.4}$$

Choose a natural number

$$n_0 > \delta^{-1}(|\inf(f)| + C_0) \tag{2.5}$$

and assume that

$$x \in B(\theta, M). \tag{2.6}$$

We claim that

$$f(A^n(x)) \leq \inf(f) + \epsilon$$

for every integer $n \geq n_0$. In view of (1.1), it is sufficient to show that there exists an integer $i \in [0, n_0]$ such that

$$f(A^i(x)) \leq \inf(f) + \epsilon.$$

Suppose to the contrary that this does not hold. Then

$$f(A^i(x)) > \inf(f) + \epsilon \text{ for all } 0, \dots, n_0. \tag{2.7}$$

By (1.1),

$$f(A^i(x)) \le f(x) \text{ for all } i = 0, \dots, n_0. \tag{2.8}$$

It follows from (2.2), (2.6) and (2.8) that for all $i = 0, \dots, n_0$,

$$f(A^i(x)) \le C_0. \tag{2.9}$$

Relations (2.3) and (2.9) imply that

$$A^i(x) \in B(\theta, M_1),\ i = 0, \dots, n_0. \tag{2.10}$$

By (2.4), (2.7) and (2.10), for all $i = 0, \dots, n_0$,

$$f(A^{i+1}(x)) \le f(A^i(x)) - \delta. \tag{2.11}$$

It follows from (2.9) and (2.11) that

$$\begin{aligned} |\inf(f)| + C_0 &\ge f(x) - f(A^{n_0}(x)) \\ &= \sum_{i=0}^{n_0-1} (f(A^i(x)) - f(A^{i+1}(x))) \ge n_0\delta \end{aligned}$$

and

$$n_0 \le \delta^{-1}(|\inf(f)| + C_0).$$

This contradicts (2.5). The contradiction we have reached shows that there indeed exists an integer $i_0 \in \{0, \dots, n_0\}$ satisfying

$$f(A^{i_0}(x)) \le \inf(f) + \epsilon,$$

as required. This completes the proof of Theorem 2.1. □

## 3. First stability result

We continue to use all the notations, definitions and assumptions introduced in Section 1.

**Theorem 3.1.** *Let $A \in \mathcal{A}$ be normal, let $f$ be bounded and uniformly continuous on bounded subsets of $K$, and satisfy*

$$\lim_{\rho(x,\theta)\to\infty} f(x) = \infty, \tag{3.1}$$

*and let $\epsilon, M > 0$ be given. Then there exist a natural number $n_0$ and a neighborhood $\mathcal{U}$ of $A$ in $\mathcal{A}$ such that for each mapping $B \in \mathcal{U}$, each integer $n \ge n_0$ and each point $x \in B(\theta, M)$, we have*

$$f(B^n(x)) \le \inf(f) + \epsilon.$$

*Proof.* There exists $C_0 > 0$ such that

$$|f(x)| \le C_0 \text{ for all } x \in B(\theta, M). \tag{3.2}$$

In view of (3.1), there exists $M_1 > 0$ such that

$$\{x \in K : \ f(x) \le C_0\} \subset B(\theta, M_1). \tag{3.3}$$

Since the mapping $A$ is normal, there exists a number $\delta > 0$ such that

$$\{x \in B(\theta, M_1) : \ f(x) \ge \inf(f) + \epsilon\} \subset \{x \in K : \ f(A(x)) \le f(x) - \delta\}. \tag{3.4}$$

Since the function $f$ is uniformly continuous on $B(\theta, M_1+1)$, there exists a number $\gamma \in (0,1)$ such that

$$|f(z_1) - f(z_2)| \leq \delta/4$$

$$\text{for all } z_1, z_2 \in B(\theta, M_1+1) \text{ satisfying } \rho(z_1, z_2) \leq \gamma. \tag{3.5}$$

Set

$$\mathcal{U} := \{B \in \mathcal{A} : \ \rho(B(x), A(x)) \leq \gamma \text{ for all } x \in B(\theta, M_1+1)\}. \tag{3.6}$$

Choose a natural number

$$n_0 > 2\delta^{-1}(|\inf(f)| + C_0). \tag{3.7}$$

Assume that

$$B \in \mathcal{U} \tag{3.8}$$

and that

$$x \in B(\theta, M). \tag{3.9}$$

We claim that for each integer $n \geq n_0$, we have

$$f(B^n(x)) \leq \inf(f) + \epsilon.$$

In view of (1.1), it is suffices to show that there exists an integer $i \in [0, n_0]$ such that

$$f(B^i(x)) \leq \inf(f) + \epsilon.$$

Suppose to the contrary that this does not hold. Then

$$f(B^i(x)) > \inf(f) + \epsilon \text{ for all } 0, \ldots, n_0. \tag{3.10}$$

By (1.1) and (3.8), we have

$$f(B^i(x)) \leq f(x) \text{ for all } i = 0, \ldots, n_0. \tag{3.11}$$

It now follows from (3.2), (3.9) and (3.11) that for all $i = 0, \ldots, n_0$,

$$f(B^i(x)) \leq f(x) \leq C_0. \tag{3.12}$$

Relations (3.3) and (3.12) imply that

$$B^i(x) \in B(\theta, M_1), \ i = 0, \ldots, n_0. \tag{3.13}$$

Let $i \in \{0, \ldots, n_0 - 1\}$. By (3.4), (3.10) and (3.13),

$$f(A(B^i(x))) \leq f(B^i(x)) - \delta. \tag{3.14}$$

In view of (3.3), (3.12) and (3.14),

$$A(B^i(x)) \in B(\theta, M_1). \tag{3.15}$$

By (3.6), (3.8) and (3.13),

$$\rho(A(B^i(x)), B^{i+1}(x)) \leq \gamma. \tag{3.16}$$

It follows from (3.5), (3.13), (3.15) and (3.16) that

$$|f(A(B^i(x))) - f(B^{i+1}(x))| \leq \delta/4. \tag{3.17}$$

In view of (3.14) and (3.17),

$$f(B^{i+1}(x)) \leq f(A(B^i(x))) + \delta/4 \leq f(B^i(x)) - \delta + \delta/4$$

and

$$f(B^{i+1}(x)) \le f(B^i(x)) - \delta/2,\ i = 0, \dots, n_0 - 1. \tag{3.18}$$

By (3.12) and (3.18),

$$\begin{aligned} |\inf(f)| + C_0 &\ge f(x) - f(B^{n_0}(x)) \\ &= \sum_{i=0}^{n_0-1} (f(B^i(x)) - f(B^{i+1}(x))) \ge n_0\delta/2 \end{aligned}$$

and

$$n_0 \le 2\delta^{-1}(|\inf(f)| + C_0).$$

This contradicts (3.7). The contradiction we have reached shows that Theorem 3.1 is indeed true. □

## 4. Second stability result

In this section we continue to use all the notations, definitions and assumptions introduced in Section 1.

Suppose that there exists a point $x_* \in K$ such that

$$f(x_*) = \inf(f) \tag{4.1}$$

and that the following property holds:

(P1) for each $\epsilon > 0$, there exists $\delta > 0$ such that for each point $x \in K$ which satisfies $f(x) \le \inf(f) + \delta$, the inequality $\rho(x, x_*) \le \epsilon$ holds.

In other words, the minimization problem $f(x) \to \min,\ x \in K$, is well posed [12, 13].

Theorem 2.1 and property (P1) imply the following result.

**Theorem 4.1.** *Let $A \in \mathcal{A}$ be normal, let $f$ be bounded on bounded subsets of $K$ and satisfy*

$$\lim_{\rho(x,\theta)\to\infty} f(x) = \infty,$$

*and let $\epsilon, M > 0$ be given. Then there exists a natural number $n_0$ such that for each integer $n \ge n_0$ and each point $x \in B(\theta, M)$, we have*

$$\rho(A^n(x), x_*) \le \epsilon.$$

In this section we establish the following stability result.

**Theorem 4.2.** *Let $A \in \mathcal{A}$ be normal and uniformly continuous on bounded subsets of $K$, let $f$ be bounded on bounded subsets of $K$, continuous at $x_*$ and satisfy*

$$\lim_{\rho(x,\theta)\to\infty} f(x) = \infty, \tag{4.2}$$

*and let $\epsilon, M > 0$ be given. Then there exist a natural number $n_0$ and a neighborhood $\mathcal{U}$ of $A$ in $\mathcal{A}$ such that for each $B \in \mathcal{U}$, each integer $n \ge n_0$ and each point $x \in B(\theta, M)$, we have*

$$\rho(B^n(x), x_*) \le \epsilon.$$

*Proof.* Property (P1) implies that there exists $\epsilon_0 \in (0, \epsilon)$ such that for each $x \in K$ which satisfies $f(x) \le \inf(f) + \epsilon_0$, we have

$$\rho(x, x_*) \le \epsilon. \tag{4.3}$$

Since the function $f$ is continuous at $x_*$, there exists $\epsilon_1 \in (0, \min\{\epsilon_0, 1\})$ such that

$$f(x) \le \inf(f) + \epsilon_0 \text{ for all } x \in B(x_*, \epsilon_1). \tag{4.4}$$

Theorem 4.1 implies that there exists a natural number $n_0$ such that for each integer $n \ge n_0$ and each point $x \in B(\theta, M)$,

$$\rho(A^n(x), x_*) \le \epsilon_1/4. \tag{4.5}$$

Since the mapping $A$ is bounded on bounded sets, there exists $M_0 > M + 1$ such that

$$\{A^i(x) : \ x \in B(\theta, M), \ i = 0, \dots, n_0\} \subset B(\theta, M_0 - 1). \tag{4.6}$$

Set

$$\delta_{n_0} := \epsilon_1/16. \tag{4.7}$$

Using induction, we now define a sequence of positive numbers $\{\delta_i\}_{i=0}^{n_0}$ such that for every $i = 0, \dots, n_0 - 1$,

$$\delta_{i-1} < \delta_i/4 \tag{4.8}$$

and such that for each $x, y \in B(\theta, M_0)$ satisfying $\rho(x, y) \le \delta_{i-1}$, we have

$$\rho(A(x), A(y)) \le \delta_i/4. \tag{4.9}$$

Set

$$\mathcal{U} := \{B \in \mathcal{A} : \ \rho(B(x), A(x)) \le \delta_0 \text{ for all } x \in B(\theta, M_0)\}. \tag{4.10}$$

Let

$$B \in \mathcal{U}, \tag{4.11}$$

$$x \in B(\theta, M) \tag{4.12}$$

and let $n \ge n_0$ be an integer. We claim that

$$\rho(B^n(x), x_*) \le \epsilon.$$

In view of the choice of $\epsilon_0$ (see (4.3)), it suffices to show that

$$f(B^n(x)) \le \inf(f) + \epsilon_0.$$

By (1.1) and (4.11), in order to meet this goal it is sufficient to show that

$$f(B^{n_0}(x)) \le \inf(f) + \epsilon_0. \tag{4.13}$$

It follows from (4.4) that in order to complete the proof, it is sufficient to show that

$$\rho(B^{n_0}(x), x_*) \le \epsilon_1. \tag{4.14}$$

Relations (4.5) and (4.12) imply that

$$\rho(A^{n_0}(x), x_*) \le \epsilon_1/4. \tag{4.15}$$

Next we show by induction that for all $i = 0, \dots, n_0$,

$$\rho(A^i(x), B^i(x)) \le \delta_i. \tag{4.16}$$

Clearly, (4.16) holds when $i = 0$. Assume that $k \in \{0, \dots, n_0 - 1\}$ and that (4.16) holds for all $i = 0, \dots, k$. In view of (4.6), we have

$$A^k(x) \in B(\theta, M_0 - 1). \tag{4.17}$$

It follows from (4.7), (4.8), (4.16) with $i = k$ and (4.17) that

$$B^k(x) \in B(\theta, M_0). \tag{4.18}$$

By (4.9), (4.16) with $i = k$, (4.17) and (4.18),

$$\rho(A(A^k(x)), A(B^k(x))) \le \delta_{k+1}/4. \tag{4.19}$$

Relations (4.11) and (4.18) imply that

$$\rho(A(B^k(x)), B(B^k(x))) \le \delta_0 \le \delta_{k+1}/4. \tag{4.20}$$

In view of (4.19) and (4.20),

$$\rho(B^{k+1}(x), A^{k+1}(x)) \le \delta_{k+1}$$

and so (4.16) holds for $i = k + 1$ too. Therefore we have shown by induction that (4.16) indeed holds for all $i = 0, \dots, n_0$. In particular, in view of (4.7), we have

$$\rho(A^{n_0}(x), B^{n_0}(x)) \le \delta_{n_0} = \epsilon_1/16.$$

When combined with (4.15), this inequality implies that

$$\rho(B^{n_0}(x), x_*) \le \epsilon_1/2,$$

as requited. This completes the proof of Theorem 4.2. □

## 5. Hyperbolic spaces

It turns out that our final result, which is proved in the next section, holds in complete hyperbolic spaces, an important class of metric spaces the definition of which we now recall.

Let $(X, \rho)$ be a metric space and let $R^1$ denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of $R^1$ into $X$ if $\rho(c(s), c(t)) = |s - t|$ for all real $s$ and $t$. The image of $R^1$ under a metric embedding is called a *metric line*. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping is called a *metric segment*.

Assume that $(X, \rho)$ contains a family $\mathcal{M}$ of metric lines such that for each pair of distinct points $x$ and $y$ in $X$, there is a unique metric line in $\mathcal{M}$ which passes through $x$ and $y$. This metric line determines a unique metric segment joining $x$ and $y$. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point $z$ in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point is denoted by $(1 - t)x \oplus ty$. We say that $X$, or more precisely $(X, \rho, \mathcal{M})$, is a *hyperbolic space* if

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\Big) \le \frac{1}{2}\rho(y, z)$$

for all $x, y$ and $z$ in $X$. An equivalent requirement is that

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\Big) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all $x, y, z$ and $w$ in $X$. This inequality, in its turn, implies that

$$\rho((1-t)x \oplus ty, (1-t)w \oplus tz) \le (1-t)\rho(x,w) + t\rho(y,z)$$

for all points $x, y, z$ and $w$ in $X$, and all numbers $0 \le t \le 1$.

A set $K \subset X$ is called *$\rho$-convex* if $[x, y] \subset K$ for all $x$ and $y$ in $K$.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in [3, 6, 7].

Let $(X, \rho, \mathcal{M})$ be a complete hyperbolic space and let $K$ be a nonempty, closed and $\rho$-convex subset of $X$. We consider the metric space $(K, \rho)$ and use all the notations, definitions and assumptions introduced in Section 1.

Assume that

$$\lim_{\rho(x,\theta)\to\infty} f(x) = \infty, \tag{5.1}$$

there exists a point $x_* \in K$ satisfying

$$f(x_*) = \inf(f), \tag{5.2}$$

and that for each $x \in K$ and each $\lambda \in [0, 1]$,

$$f(\lambda x \oplus (1-\lambda)x_*) \le \lambda f(x) + (1-\lambda) f(x_*). \tag{5.3}$$

For each $A \in \mathcal{A}$ and each $\gamma \in (0, 1)$, define

$$A_\gamma(x) := \gamma x_* \oplus (1-\gamma)A(x),\ x \in K. \tag{5.4}$$

Let $A \in \mathcal{A}$ and $\gamma \in (0, 1)$. By (1.1) and (5.2)–(5.4),

$$\begin{aligned} f(A_\gamma(x)) = f(\gamma x_* \oplus (1-\gamma)A(x)) &\le \gamma f(x_*) + (1-\gamma) f(A(x)) \\ &\le \gamma \inf(f) + (1-\gamma) f(x) \\ &= f(x) - \gamma(f(x) - \inf(f)). \end{aligned} \tag{5.5}$$

It is clear that $A_\gamma \in \mathcal{A}$. In view of (5.5), for all points $x \in K$, we have

$$f(x) - f(A_\gamma(x)) \ge \gamma(f(x) - \inf(f)). \tag{5.6}$$

It follows from (5.6) that $A_\gamma$ is normal. Clearly, if $A \in \mathcal{A}_b$, then $A_\gamma \in \mathcal{A}_b$, if $A \in \mathcal{A}_c$, then $A_\gamma \in \mathcal{A}_c$, and if $A \in \mathcal{A}_u$, then $A_\gamma \in \mathcal{A}_u$. Note that $\mathcal{A}_u \subset \mathcal{A}_b$.

## 6. Generic result

In this section we use all the notations, definitions and assumptions introduced in Sections 1 and 5.

**Theorem 6.1.** *Let the function $f$ be uniformly continuous on bounded subsets of $K$ and let $(\mathcal{B}, \sigma)$ be one of the following spaces: $(\mathcal{A}_b, \sigma)$, $(\mathcal{A}_c \cap \mathcal{A}_b, \sigma)$ or $(\mathcal{A}_u, \sigma)$. Then there exists a set $\mathcal{F} \subset \mathcal{B}$, which is a countable intersection of open and everywhere dense subsets of $(\mathcal{B}, \sigma)$, such that each $A \in \mathcal{F}$ is normal.*

*Proof.* Let $A \in \mathcal{B}$ and $\gamma \in (0, 1)$. Clearly, $A_\gamma \in \mathcal{B}$. In view of (5.4), we have

$$\rho(A_\gamma(x), A(x)) = \rho(\gamma x_* \oplus (1-\gamma)A(x), A(x)) \le \gamma\rho(x_*, A(x)). \tag{6.1}$$

In view of (6.1),

$$\lim_{\gamma \to 0^+} A_\gamma = A. \tag{6.2}$$

By (6.2),

$$\{A_\gamma : \ A \in \mathcal{B}, \ \gamma \in (0,1)\}$$

is an everywhere dense subset of $\mathcal{B}$. Let $A \in \mathcal{B}$, $\gamma \in (0,1)$ and let $k$ be a natural number. Since the mapping $A_\gamma$ is normal, there exists $\delta(A, \gamma, k) > 0$ such that the following property holds:

(P2) if $x \in B(\theta, k)$ satisfies $f(x) \geq \inf(f) + 1/k$, then

$$f(A_\gamma(x)) \leq f(x) - 2\delta(A, \gamma, k).$$

Clearly, there exists $M_1 > 1$ such that

$$A_\gamma(B(\theta, k)) \subset B(\theta, M_1 - 1). \tag{6.3}$$

Since the function $f$ is uniformly continuous on $B(\theta, M_1)$, there exists a number $\epsilon \in (0,1)$ such that

$$|f(z_1) - f(z_2)| \leq \delta(A, \gamma, k) \text{ for all } z_1, z_2 \in B(\theta, M_1) \tag{6.4}$$
$$\text{satisfying } \rho(z_1, z_2) \leq \epsilon.$$

In view of (1.2), there exists an open neighborhood $\mathcal{U}(A, \gamma, k)$ of $A_\gamma$ in $\mathcal{B}$ such that

$$\mathcal{U}(A, \gamma, k) \subset \{B \in \mathcal{B} : \ \rho(A_\gamma(z), B(z)) \leq \epsilon \text{ for all } z \in B(\theta, k)\}. \tag{6.5}$$

Let

$$x \in B(\theta, k), \tag{6.6}$$

$$f(x) \geq \inf(f) + 1/k \tag{6.7}$$

and

$$B \in \mathcal{U}(A, \gamma, k). \tag{6.8}$$

Property (P2), (6.6) and (6.7) imply that

$$f(A_\gamma(x)) \leq f(x) - 2\delta(A, \gamma, k). \tag{6.9}$$

By (6.5), (6.6) and (6.8),

$$\rho(B(x), A_\gamma(x)) \leq \epsilon. \tag{6.10}$$

It follows from (6.3), (6.6) and (6.10) that

$$A_\gamma(x) \in B(\theta, M_1 - 1) \tag{6.11}$$

and

$$B(x) \in B(\theta, M_1). \tag{6.12}$$

By (6.4) and (6.10)–(6.12),

$$|f(B(x)) - f(A_\gamma(x))| \leq \delta(A, \gamma, k). \tag{6.13}$$

In view of (6.9) and (6.13),

$$f(B(x)) \leq f(A_\gamma(x)) + \delta(A, \gamma, k) \leq f(x) - \delta(A, \gamma, k).$$

Thus we have shown that the following property holds:

(P3) for each mapping $B \in \mathcal{U}(A, \gamma, k)$ and each point $x \in B(\theta, k)$ which satisfies $f(x) \geq \inf(f) + 1/k$, we have

$$f(B(x)) \leq f(x) - \delta(A, \gamma, k).$$

Now Define

$$\mathcal{F} := \cap_{p=1}^{\infty} \cup \{\mathcal{U}(A, \gamma, k) : \ A \in \mathcal{B},\ \gamma \in (0,1) \text{ and } k \geq p \text{ is an integer } \}. \tag{6.14}$$

Evidently, $\mathcal{F}$ is a countable intersection of open and everywhere dense sets in $\mathcal{B}$.

Let

$$B \in \mathcal{F}. \tag{6.15}$$

We claim that the mapping $B$ is normal. To see this, let $M, \epsilon > 0$ be given. Choose a natural number

$$p > M + 1/\epsilon. \tag{6.16}$$

By (6.14) and (6.15), there exist $A \in \mathcal{B}$, $\gamma \in (0,1)$ and an integer $p \geq k$ such that

$$B \in \mathcal{U}(A, \gamma, k). \tag{6.17}$$

Property (P3), (6.16) and (6.17) imply that for each $x \in B(\theta, M) \subset B(\theta, k)$ satisfying

$$f(x) \geq \inf(f) + \epsilon \geq \inf(f) + p^{-1} \geq \inf(f) + k^{-1},$$

we have

$$f(B(x)) \leq f(x) - \delta(A, \gamma, k).$$

Therefore the mapping $B$ is indeed normal, as claimed. This completes the proof of Theorem 6.1. $\square$

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