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FENCHEL DUALITY FOR CONVEX SET FUNCTIONS

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Abstract. In this paper, we study Fenchel duality for convex set functions. We define convex set functions in a simple way. We introduce Fenchel conjugate and investigate Fenchel duality in terms of convex analysis on an embedding normed space of compact convex subsets. We compare our convex set functions with previous ones. In addition, we introduce robust approach for the uncertain problem as an application.

1. Introduction

In convex analysis, various types of functions have been studied. A typical example is a real-valued convex function on a vector space. This type of functions have been studied widely and have been generalized in several directions. Generalized convexity, for example pseudo-convexity and quasiconvexity, have been investigated extensively, and play an important role in applications. Vector-valued functions are one of the generalization for multi-objective optimization. Additionally, set-valued functions have been studied by various researchers. In [9, 10], Kuroiwa introduced an embedding approach of compact convex subsets of a normed vector space to a specialized embedding normed vector space, $\mathcal{C}^2/\equiv$, as a quotient space. The embedding approach, which was based on the work by Rådström in [25], plays a central and important role in set optimization. As stated above, convexity and generalized convexity have been studied extensively, especially in various types of optimization problems, see [1, 3, 6, 9–11, 21, 22, 24, 26–37].

On the other hand, in [23], Morris introduced set functions, which is defined on the class of measurable subsets of an atomless finite measure space satisfying a certain convexity condition. Although a set-valued function f is defined on a vector space and the value $f(x)$ is a set, a set function F is defined on a class of subsets and the value $F(A)$ is a real number. For this type of set functions, various results in convex analysis have been generalized, for example, the epigraph of convex set function in [4], the subdifferential sum formula in [16], Fenchel-Moreau theorem in [17], and so on, see [5, 7, 8, 18, 20, 23, 38]. However, the domain of a set function is complicated and difficult since the function is defined by Morris sequence. It is expected to study set functions in a simple definition and setting.

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The remainder of the paper is organized as follows. In Section2, we introduce some preliminaries and define convex set functions in a simple way. In Section 3, we study Fenchel conjugate and Fenchel duality for convex set functions in terms of convex analysis of the embedding space. In Section 4, we compare our convex set functions with previous ones. We study applications of our results to uncertain problems. Additionally, we introduce definitions and results for $\mathcal{C}^2/\equiv$ in appendix.

2. Preliminaries and convex set functions

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given nonempty sets A, $B \subset \mathbb{R}^n$, and $\Gamma \subset \mathbb{R}$, we define $A+B$ and ΓA as follows:

$$\begin{aligned} A+B &= \{x+y \in \mathbb{R}^n \mid x \in A, y \in B\}, \\ \Gamma A &= \{\gamma x \in \mathbb{R}^n \mid \gamma \in \Gamma, x \in A\}. \end{aligned}$$

In addition, we define $A+\emptyset = \Gamma\emptyset = \emptyset A = \emptyset$. A set A is said to be convex if for each x, $y \in A$, and $\alpha \in [0,1]$, $(1-\alpha)x + \alpha y \in A$. Let $\mathcal{A}_0$ be the following family of nonempty convex sets:

$$\mathcal{A}_0 = \{A \subset \mathbb{R}^n \mid A : \text{nonempty convex}\}.$$

It is clear that $\mathcal{A}_0$ is closed under addition and multiplication by positive scalars. Let $\mathcal{C} \subset \mathcal{A}_0$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$, that is,

$$\mathcal{C} = \{A \subset \mathbb{R}^n \mid A : \text{nonempty compact convex}\}.$$

Let A, $B \in \mathcal{C}$. We define their Hausdorff distance $d_H(A,B)$ by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\}.$$

Next, we study convexity for families of convex sets. A subfamily $\mathcal{A} \subset \mathcal{A}_0$ is said to be convex if for each A, $B \in \mathcal{A}$, and $\alpha \in [0,1]$, $(1-\alpha)A + \alpha B \in \mathcal{A}$. We introduce the following elementary results without proofs.

Theorem 2.1. *The following statements hold:*

(i) $\mathcal{A}_0$ *is convex.*
(ii) $\mathcal{C}$ *is convex.*
(iii) *If* $\mathcal{A}$, $\mathcal{B}$ *are convex, then* $\mathcal{A}+\mathcal{B} = \{A+B \subset \mathbb{R}^n \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ *is convex.*
(iv) *If* $\mathcal{A}$ *is convex and* $\alpha \in \mathbb{R}$, *then* $\alpha\mathcal{A} = \{\alpha A \mid A \in \mathcal{A}\}$ *is convex.*
(v) *Let* I *be an index set, and* $\mathcal{A}_i$ *a convex subfamily for each* $i \in I$. *Then,* $\bigcap_{i\in I}\mathcal{A}_i$ *is convex.*

Next, we study convex set functions. Let F be a set function from $\mathcal{A}_0$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. A set function F is said to be proper if for all $A \in \mathcal{A}_0$, $F(A) > -\infty$ and there exists $A_0 \in \mathcal{A}_0$ such that $F(A_0) \in \mathbb{R}$. We denote the domain of F by $\mathrm{dom}F$, that is, $\mathrm{dom}F = \{A \in \mathcal{A}_0 \mid F(A) < +\infty\}$. A proper set function F on $\mathcal{A}_0$ is said to be convex if for each A, $B \in \mathrm{dom}F$, and $\alpha \in [0,1]$, $F((1-\alpha)A + \alpha B) \leq (1-\alpha)F(A) + \alpha F(B)$. F is said to be concave if $-F$ is a convex set function. The epigraph of F is defined as $\mathrm{epi}F = \{(A,\alpha) \in \mathcal{A}_0 \times \mathbb{R} \mid F(A) \leq \alpha\}$. We show the following characterization of convex set functions.

Theorem 2.2. *Let F be a proper set function from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$. Then, F is a convex set function if and only if* epi$F$ *is convex.*

Proof. Assume that F is a convex set function and let (A_1, α_1), $(A_2, \alpha_2) \in \text{epi}F$ and $\beta \in (0, 1)$. Then,

$$\begin{aligned} F((1-\beta)A_1 + \beta A_2) &\leq (1-\beta)F(A_1) + \beta F(A_2) \\ &\leq (1-\beta)\alpha_1 + \beta\alpha_2. \end{aligned}$$

This shows that $((1-\beta)A_1 + \beta A_2, (1-\beta)\alpha_1 + \beta\alpha_2) \in \text{epi}F$.

Conversely, let A_1, $A_2 \in \text{dom}F$, and $\beta \in (0, 1)$. Then

$$(A_1, F(A_1)), (A_2, F(A_2)) \in \text{epi}F.$$

By the assumption,

$$F((1-\beta)A_1 + \beta A_2) \leq (1-\beta)F(A_1) + \beta F(A_2).$$

This shows that F is a convex set function. □

We introduce the following elementary results for convex set functions. We leave the proof to the reader.

Theorem 2.3. *The following statements hold:*

(i) *Let F and G be proper convex set functions from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$. Then, $F + G$ is convex.*
(ii) *Let F be a proper convex set function from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$, and $\alpha > 0$. Then, αF is convex.*
(iii) *Let I be an index set, and F_i a proper convex set function from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$ for each $i \in I$. Then, $\sup_{i \in I} F_i$ is convex.*

We introduce some important examples. Let $F(A)$ be the value obtained by integrating the real-valued convex function f on the compact convex set $A \subset \mathbb{R}^n$, then F is a convex set function. Additionally, let F_0 be the following function on $\mathcal{A}_0$: for each $A \in \mathcal{A}_0$,

$$F_0(A) = \sup_{x \in A} f(x),$$

then F_0 is a convex set function.

In the rest of this section, we study affine and linear set functions precisely. Especially, we point out a difference between affine functions on $\mathcal{A}_0$ and $\mathbb{R}^n$.

A set function F is said to be affine if F is a convex and concave set function. Additionally, F is said to be linear if F is an affine set function and $F(\{0\}) = 0$.

We show the following relation between affine and linear set functions.

Theorem 2.4. *Let F be a proper set function on $\mathcal{A}_0$, and assume that $\{0\} \in \text{dom}F$. Then the following statements are equivalent:*

(i) *F is an affine set function,*
(ii) *there exist a proper linear set function V on $\mathcal{A}_0$ and $\beta \in \mathbb{R}$ such that $F = V + \beta$.*

Proof. It is easy to prove by putting $\beta = F(\{0\})$. □

We show the following characterizations of linear set functions.

Theorem 2.5. *Let F be a proper set function on $\mathcal{A}_0$, and assume that $\{0\} \in \mathrm{dom}F$. Then the following statements are equivalent:*

(i) *F is linear,*

(ii) *for each A, $B \in \mathrm{dom}F$, and $\alpha \in [0,1]$,*
$$F((1-\alpha)A + \alpha B) = (1-\alpha)F(A) + \alpha F(B),$$

(iii) *for each A, $B \in \mathrm{dom}F$, and $\lambda \geq 0$,*
$$F(A+B) = F(A) + F(B), F(\lambda A) = \lambda F(A).$$

Proof. By the definition, the statements (i) and (ii) are equivalent.

We show that (ii) implies (iii). Let A, $B \in \mathrm{dom}F$, and $\lambda \geq 0$. If $\lambda \geq 1$, then $\frac{1}{\lambda} \in (0,1]$. Hence
$$\begin{aligned} F(A) &= F\left(\frac{1}{\lambda}\lambda A + \left(1 - \frac{1}{\lambda}\right)\{0\}\right) \\ &= \frac{1}{\lambda}F(\lambda A) + \left(1 - \frac{1}{\lambda}\right)F(\{0\}) \\ &= \frac{1}{\lambda}F(\lambda A). \end{aligned}$$
This shows that $F(\lambda A) = \lambda F(A)$. If $\lambda \in [0,1)$, then
$$F(\lambda A) = F(\lambda A + (1-\lambda)\{0\}) = \lambda F(A) + (1-\lambda)F(\{0\}) = \lambda F(A).$$
Additionally,
$$\begin{aligned} F(A+B) &= F\left(\frac{1}{2}2A + \frac{1}{2}2B\right) \\ &= \frac{1}{2}F(2A) + \frac{1}{2}F(2B) \\ &= \frac{1}{2}2F(A) + \frac{1}{2}2F(B) \\ &= F(A) + F(B). \end{aligned}$$
This shows that the statement (iii) holds.

Finally, we show that (iii) implies (ii). Let A, $B \in \mathrm{dom}F$, and $\alpha \in [0,1]$. Then,
$$\begin{aligned} F((1-\alpha)A + \alpha B) &= F((1-\alpha)A) + F(\alpha B) \\ &= (1-\alpha)F(A) + \alpha F(B). \end{aligned}$$
This completes the proof. □

Let f be a real-valued function on $\mathbb{R}^n$. Then, the following statements are equivalent:

(i) f is convex and concave,

(ii) for each x, $y \in \mathbb{R}^n$, and $t \in \mathbb{R}$,
$$f((1-t)x + ty) = (1-t)f(x) + tf(y),$$

(iii) for each x, $y \in \mathbb{R}^n$, and $t \in \mathbb{R}$,
$$f(x+y) = f(x) + f(y), f(tx) = tf(x).$$

We can show the above equivalence relations by '$x + (-x) = 0$' on $\mathbb{R}^n$. However, $A + (-A) \neq \{0\}$ in general. Hence, the statement (ii) in Theorem 2.5 is slightly different to the above statement (ii), see the following example.

Example 2.6. Let F be the following set function on $\mathbb{R}$: for each $A \subset \mathbb{R}$,

$$F(A) = \sup A.$$

We can easily show that F is a proper, convex and concave set function on $\mathcal{A}_0$. By the statement (ii) of Theorem 2.5, for each A, $B \in \mathrm{dom}F$, and $\alpha \in [0, 1]$,

$$F((1-\alpha)A + \alpha B) = (1-\alpha)F(A) + \alpha F(B).$$

However, there exist A_0, $B_0 \in \mathrm{dom}F$, and $\alpha_0 \in \mathbb{R} \setminus [0, 1]$ such that

$$F((1-\alpha_0)A_0 + \alpha_0 B_0) \neq (1-\alpha_0)F(A_0) + \alpha_0 F(B_0).$$

Actually, let $A_0 = [0, 1]$, $B_0 = [2, 3]$, and $\alpha_0 = -1$, then,

$$\begin{aligned} F((1-\alpha_0)A_0 + \alpha_0 B_0) &= F(-2[0,1] - [2,3]) \\ &= F([-5,-2]) \\ &= -2 \\ &\neq -1 \\ &= 2F([0,1]) + (-1)F([2,3]) \\ &= (1-\alpha_0)F(A_0) + \alpha_0 F(B_0). \end{aligned}$$

Hence, the statement (ii) in Theorem 2.5 is slightly different to the above statement (ii) for f.

Remark 2.7. Let $v \in \mathbb{R}^n$, then the following set function V is linear: for each $A \in \mathcal{A}_0$,

$$V(A) = \sup_{x \in A} \langle v, x \rangle .$$

Hence,

$$\{V : \mathcal{A}_0 \to \overline{\mathbb{R}} \mid v \in \mathbb{R}^n, V(A) = \sup_{x \in A} \langle v, x \rangle\} \subsetneq \{V : \mathcal{A}_0 \to \overline{\mathbb{R}} \mid V : \text{ linear } \}.$$

The converse inclusion does not hold. Actually, the area of A is a linear set function, and there does not exist $v \in \mathbb{R}^n$ such that the function is defined by v.

3. Fenchel duality for convex set functions

We study Fenchel duality for convex set functions in terms of convex analysis on the embedding normed space $\mathcal{C}^2/\equiv$. By embedding method, $\mathcal{C}^2/\equiv$ is a normed space, in detail, see appendix.

We define the following set $\mathcal{F}_L$ as follows:

$$\mathcal{F}_L = \{V : \mathcal{A}_0 \to \mathbb{R} \cup \{+\infty\}, \text{linear}\}.$$

Let F be a proper set function on $\mathcal{A}_0$. Then, we define the Fenchel conjugate of F as follows: $F^* : \mathcal{F}_L \to \overline{\mathbb{R}}$,

$$F^*(V) = \sup_{A \in \mathrm{dom}F} \{V(A) - F(A)\}.$$

Since F is a proper set function, $F^*(\mathcal{F}_L) \subset (-\infty, \infty]$. We define the Fenchel biconjugate as follows: $F^{**} : \mathcal{A}_0 \to \overline{\mathbb{R}}$,

$$F^{**}(A) = \sup_{V \in \mathrm{dom} F^*} \{V(A) - F^*(V)\}.$$

For each $A \in \mathrm{dom} F$ and $V \in \mathrm{dom} F^*$,

$$\infty > F^*(V) = \sup_{B \in \mathrm{dom} F} \{V(B) - F(B)\} \geq V(A) - F(A).$$

This shows that $A \in \mathrm{dom} V$. Hence,

$$\begin{aligned} V(A) - F^*(V) &= V(A) - \sup_{B \in \mathrm{dom} F} \{V(B) - F(B)\} \\ &\leq V(A) - V(A) + F(A) \\ &= F(A). \end{aligned}$$

This shows that for each $A \in \mathcal{A}_0$,

$$F(A) \geq F^{**}(A).$$

A set function F is said to be lower semicontinuous (lsc) on $\mathcal{C}$ in terms of Hausdorff distance if for each $\{B_k\} \subset \mathcal{C}$ and $B \in \mathcal{C}$ with $H(B_k, B)$ converges to 0,

$$\liminf_{k \to \infty} F(B_k) \geq F(B).$$

In addition, F is said to be continuous on $\mathcal{C}$ in terms of Hausdorff distance if F and $-F$ are lsc in terms of Hausdorff distance. These definitions are equivalent to the usual continuity and lower semicontinuity of real-valued functions in a metric space. Hence, the level set and the epigraph of a lsc function F are closed.

We need the following lemma.

Lemma 3.1. *Let $\{B_k\} \subset \mathcal{C}$, C, $D \in \mathcal{C}$, assume that $H(B_k + D, C)$ converges to 0. Then there exists $B \in \mathcal{C}$ such that $C = B + D$ and $H(B_k, B)$ converges to 0.*

Proof. Let $\{B_k\} \subset \mathcal{C}$, C, $D \in \mathcal{C}$ satisfying $H(B_k + D, C)$ converges to 0. Let $B = \{b \in \mathbb{R}^n \mid b + D \subset C\}$. Clearly, B is compact convex and $B + D \subset C$. Now we show that $B + D \supset C$. Let $x \in C$. Since $H(B_k + D, C)$ converges to 0, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $B_{n_k} + D \subset C + \frac{1}{k}B(0,1)$ and $C \subset B_{n_k} + D + \frac{1}{k}B(0,1)$, where $B(0,1) = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Hence there exists $b_k \in B_{n_k}$ such that $x \in b_k + D + \frac{1}{k}B(0,1)$ and $b_k + D \subset C + \frac{1}{k}B(0,1)$ for each $k \in \mathbb{N}$. Since $\{b_k\}$ is bounded, there exists a subsequence $\{b_{k_i}\} \subset \{b_k\}$ such that $\{b_{k_i}\}$ converges to some $b_0 \in B$. Then, we can prove that $x \in b_0 + D$ and $b_0 + D \subset C$. Therefore, $x \in B + D$. This shows that $C = B + D$ and $H(B_k, B)$ converges to 0. □

Next, we show the following theorem concerned with Fenchel conjugate.

Theorem 3.2. *Let F be a proper, lsc and convex set function from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$. Assume that* $\mathrm{dom} F \subset \mathcal{C}$. *Then*

$$F = F^{**}.$$

Proof. As stated above, we already show that $F \geq F^{**}$. Hence we assume that there exist $A \in \mathcal{A}_0$ and $\alpha \in \mathbb{R}$ such that

$$F(A) > \alpha > F^{**}(A).$$

We define the following function $\bar{F}$ on $(\mathcal{C}^2/\equiv)$ as follows: for each $[B, \{0\}] \in \{[S, \{0\}] \in (\mathcal{C}^2/\equiv) \mid S \in \mathrm{dom}F\}$,

$$\bar{F}([B, \{0\}]) = F(B),$$

and for each $[B, C] \notin \{[S, \{0\}] \in (\mathcal{C}^2/\equiv) \mid S \in \mathrm{dom}F\}$, $\bar{F}([B, C]) = \infty$. Clearly, $\bar{F}$ is a proper convex function on $(\mathcal{C}^2/\equiv)$, $\mathrm{dom}\bar{F} = \{[S, \{0\}] \in (\mathcal{C}^2/\equiv) \mid S \in \mathrm{dom}F\}$, and $\bar{F}([A, \{0\}]) > \alpha$.

We show that $\mathrm{epi}\bar{F}$ is closed. Let $\{([B_k, \{0\}], \beta_k)\} \subset \mathrm{epi}\bar{F}$ and assume that $\{([B_k, \{0\}], \beta_k)\}$ converges to $([C, D], \beta) \in (\mathcal{C}^2/\equiv) \times \mathbb{R}$. By Lemma 3.1, there exists $B \in \mathcal{C}$ such that $[C, D] = [B, \{0\}]$ and $H(B_k, B)$ converges to 0. Since F is lsc,

$$\begin{aligned}
\bar{F}([C, D]) &= \bar{F}([B, \{0\}]) \\
&= F(B) \\
&\leq \liminf_{k\to\infty} F(B_k) \\
&= \liminf_{k\to\infty} \bar{F}([B_k, \{0\}]) \\
&\leq \liminf_{k\to\infty} \beta_k \\
&= \beta.
\end{aligned}$$

This shows that $([C, D], \beta) \in \mathrm{epi}\bar{F}$. Hence $\bar{F}$ is lsc in $\mathcal{C}^2/\equiv$.

Since $\bar{F}$ is proper, lsc and convex, there exist a continuous, real-valued linear function v on $\mathcal{C}^2/\equiv$ and $\beta \in \mathbb{R}$ such that

$$\bar{F} \geq v + \beta \text{ and } v([A, \{0\}]) + \beta > \alpha.$$

Let V be the following proper function on $\mathcal{A}_0$: for each $B \in \mathcal{C}$,

$$V(B) = v([B, \{0\}]),$$

and for each $B \notin \mathcal{C}$, $V(B) = \infty$. We can check that V is a linear set function. In addition, for each $B \in \mathrm{dom}F$,

$$F(B) \geq V(B) + \beta \text{ and } V(A) + \beta > \alpha.$$

Since $F(B) \geq V(B) + \beta$ for each $B \in \mathrm{dom}F$, $-\beta \geq F^*(V) > -\infty$, that is, $V \in \mathrm{dom}F^*$. Hence,

$$\begin{aligned}
F^{**}(A) &\geq V(A) - F^*(V) \\
&\geq V(A) + \beta \\
&> \alpha.
\end{aligned}$$

This is a contradiction. Hence $F = F^{**}$. □

Finally, we show Fenchel duality for convex set functions. In convex analysis, the following Fenchel duality for real-valued convex functions plays an important role.

Theorem 3.3 ([3]). *Let X be a normed space, f and g proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$. Assume that there exists $x_0 \in \mathrm{dom} f \cap \mathrm{dom} g$ such that f is continuous at x_0. Then*

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{v \in X^*} \{-f^*(v) - g^*(-v)\},$$

where X^* is the dual space of X, and f^* is the Fenchel conjugate of f, that is, $f^*(v) = \sup_{x \in X} \{\langle v, x \rangle - f(x)\}$.

In the following theorem, we show Fenchel duality for convex set functions in terms of convex analysis on the embedding normed space $(\mathcal{C}^2/\equiv)$.

Theorem 3.4. *Let F and G be proper convex set functions from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$. Assume that $\mathrm{dom}F \cup \mathrm{dom}G \subset \mathcal{C}$, $\mathrm{dom}F \cap \mathrm{dom}G$ is nonempty, and F is continuous on $\mathcal{C}$. Then*

$$\inf_{A \in \mathcal{A}_0} \{F(A) + G(A)\} = \max_{V \in \mathcal{F}_L} \{-F^*(V) - G^*(-V)\}.$$

Proof. Let $\mu = \inf_{A \in \mathcal{A}_0} \{F(A) + G(A)\}$. At first, we show the following Fenchel weak duality:

$$\mu \geq \sup_{V \in \mathcal{F}_L} \{-F^*(V) - G^*(-V)\}.$$

Actually, for each $A \in \mathrm{dom}F \cap \mathrm{dom}G$ and $V \in \mathrm{dom}F^* \cap \mathrm{dom}G^*$, we can check that $A \in \mathrm{dom}V$. Hence,

$$\begin{aligned} -F^*(V) - G^*(-V) &= -\sup_{B \in \mathrm{dom}F} \{V(B) - F(B)\} - \sup_{B \in \mathrm{dom}G} \{-V(B) - G(B)\} \\ &\leq -V(A) + F(A) + V(A) + G(A) \\ &= F(A) + G(A). \end{aligned}$$

Since F and G be proper, $F^*(\mathcal{F}_L) \cup G^*(\mathcal{F}_L) \subset (-\infty, \infty]$. Therefore,

$$\begin{aligned} \sup_{V \in \mathcal{F}_L} \{-F^*(V) - G^*(-V)\} &= \sup_{V \in \mathrm{dom}F^* \cap \mathrm{dom}G^*} \{-F^*(V) - G^*(-V)\} \\ &\leq \inf_{A \in \mathrm{dom}F \cap \mathrm{dom}G} \{F(A) + G(A)\} \\ &= \mu. \end{aligned}$$

If $\mu = -\infty$, then Fenchel strong duality holds.

Assume that $\mu > -\infty$. Since $\mathrm{dom}F \cap \mathrm{dom}G$ is nonempty, $\mu \in \mathbb{R}$. Clearly, $F \geq -G + \mu$. We define $\bar{F}$ and $\bar{G}$ by the similar way in the proof of Theorem 3.2, then $\bar{F} \geq -\bar{G} + \mu$. We can easily show that $\bar{F}$ and $\bar{G}$ are proper convex functions, and $\mathrm{dom}\bar{F} \cap \mathrm{dom}\bar{G}$ is nonempty. Additionally, we can show that $\bar{F}$ and $-\bar{F}$ is lsc on $\mathcal{C}^2/\equiv$ by the similar way in the proof of Theorem 3.2. This shows that $\bar{F}$ is continuous on $\mathcal{C}^2/\equiv$.

By Theorem 3.3, Fenchel duality on the embedding normed space $(\mathcal{C}^2/\equiv)$ holds. Hence, there exist a continuous, real-valued linear function v on $\mathcal{C}^2/\equiv$ and $\beta \in \mathbb{R}$ such that

$$\bar{F} \geq v + \beta \geq -\bar{G} + \mu.$$

Let V_0 be the following proper function on $\mathcal{A}_0$: for each $B \in \mathcal{C}$,

$$V_0(B) = v([B, \{0\}]),$$

and for each $B \notin \mathcal{C}$, $V_0(B) = \infty$. We can check that V_0 is a linear set function. In addition,

$$F \geq V_0 + \beta \geq -G + \mu.$$

This shows that $-\beta \leq F^*(V_0)$ and $-\mu + \beta \leq G^*(-V_0)$. Hence,

$$\sup_{V \in \mathcal{F}_L} \{-F^*(V) - G^*(-V)\} \geq -F^*(V_0) - G^*(-V_0) \geq \beta + \mu - \beta = \mu.$$

This completes the proof. □

By the similar way, we can show the sandwich theorem: for each proper convex set functions F, G satisfying $F \geq -G$. Assume that there exists $A_0 \in \mathrm{dom}F \cap \mathrm{dom}G$ such that F is continuous at A_0, then there exists an affine set function V such that

$$F \geq V \geq -G.$$

In addition, we can show the following Toland duality: for each proper, lsc convex set functions F, G, the following equation holds:

$$\sup_{A \in \mathcal{A}_0} \{F(A) - G(A)\} = \sup_{V \in \mathcal{F}_L} \{G^*(V) - F^*(V)\}.$$

4. Discussions and applications

We compare our convex set functions with Morris's set functions. In addition, we study applications of our results to uncertain problems with motion uncertainty. We regard a decision variable set as an error caused by a motion, and introduce robust approach for the uncertain problem.

Let $(X, \mathcal{A}, m)$ be an atomless finite measure space with $L_1 := L_1(X, \mathcal{A}, m)$ separable. For $\Omega \in \mathcal{A}$, χ_Ω denotes the characteristic function of Ω. For $\mathcal{S} \subset \mathcal{A}$, we denote $\chi_{\mathcal{S}} = \{\chi_\Omega \mid \Omega \in \mathcal{S}\}$ and define $\mathrm{cl}\mathcal{S}$ is the w^*-closure of $\chi_{\mathcal{S}}$ in L_∞. In [23], Morris proved that for each Ω, $\Lambda \in \mathcal{A}$ and $\alpha \in [0,1]$, there exist L_∞-sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} (1-\alpha)\chi_{\Omega \setminus \Lambda}, \quad \chi_{\Lambda_n} \xrightarrow{w^*} \alpha \chi_{\Lambda \setminus \Omega},$$

and

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} (1-\alpha)\chi_\Omega + \alpha \chi_\Lambda,$$

consequently, $\mathrm{cl}\chi_{\mathcal{A}}$ contains the convex hull of $\chi_{\mathcal{A}}$. We call the sequence $\{\Gamma_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ a Morris sequence associated with $(\alpha, \Omega, \Lambda)$. A subfamily $\mathcal{S} \subset \mathcal{A}$ is said to be convex if for every $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{S}$. Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. A set function $F : \mathcal{S} \to \mathbb{R}$ is said to be convex if for every $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{S}$ and

$$\limsup_{k \to \infty} F(\Gamma_{n_k}) \leq (1-\alpha)F(\Omega) + \alpha F(\Lambda).$$

Clearly, our definition and Morris's definition are different. Our definition is implied by convex analysis in a natural way.

Finally, we study applications of our results to uncertain problems with motion uncertainty. Mathematical programming problems with data uncertainty are becoming important in optimization due to the reality of uncertainty in many real-world optimization problems. Various researchers study duality theory for mathematical programming problems under uncertainty with the worst-case approach, see [2,12–15,19,32]. In mathematical programming, we often regard a decision variable as the level of activity or the amount of a resource to use. However, in many cases, such a decision may have an error caused by motion and/or data uncertainty. In this paper, we regard a decision variable set as an error caused by a motion, and introduce the following robust approach for the problem with motion uncertainty.

Let I be an index set, f an extended real-valued convex function on $\mathbb{R}^n$, g_i an extended real-valued convex function on $\mathbb{R}^n$ for each $i \in I$. The following problem (P) is a convex programming problem on $\mathbb{R}^n$ without uncertainty:

$$(P) \begin{cases} \text{minimize } f(x), \\ \text{subject to } g_i(x) \leq 0, \forall i \in I. \end{cases}$$

For such a problem, we may not be able to choose an exact vector because of an error by a motion. Hence, we introduce a worst case approach with motion uncertainty. Let F be the following function on $\mathcal{A}_0$: for each $A \in \mathcal{A}_0$,

$$F(A) = \sup_{x \in A} f(x).$$

For constraint functions, we define G_i similarly, that is, $G_i(A) = \sup_{x \in A} g_i(x)$. We consider the following robust problem (RP) with motion uncertainty:

$$(RP) \begin{cases} \text{minimize } F(A), \\ \text{subject to } G_i(A) \leq 0, \forall i \in I. \end{cases}$$

In (RP), F and G_i are set functions, and A means an error caused by a motion. Since $F(A)$ is the supremum of the value of f at $x \in A$, (RP) is one of the worst-case approach. Additionally, we can easily prove that F and G_i are convex set functions. Hence we can solve the problem (RP) by using our results, for example, Fenchel duality.

Appendix

We introduce an embedding vector space $\mathcal{C}^2/\equiv$ and an embedding function ψ. All definitions and results are based on the previous literatures, see [9–11]. Let $\equiv$ be a binary relation on $\mathcal{C}^2$ defined by

$$(A, B) \equiv (C, D) \text{ if and only if } A + D = B + C,$$

then $\equiv$ is an equivalence relation on $\mathcal{C}^2$. To show this, the following cancellation law is used: for each A, B, $C \in \mathcal{C}$,

$$A + C = B + C \Longrightarrow A = B.$$

Denote the equivalence class of $(A, B) \in \mathcal{C}^2$ as $[A, B] = \{(C, D) \in \mathcal{C}^2 \mid (A, B) \equiv (C, D)\}$, and the quotient space of $\mathcal{C}^2$ by $\equiv$ as $C^2/\equiv = \{[A, B] \mid (A, B) \in \mathcal{C}^2\}$. On the quotient space, we define addition and scalar multiplication as follows:

$$[A, B] + [C, D] = [A + C, B + D],$$

$$\lambda \cdot [A, B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases}$$

Then $(\mathcal{C}^2/\equiv, +, \cdot)$ becomes a vector space over $\mathbb{R}$ with the null vector $[\{0\}, \{0\}](=: \theta)$. Clearly, $[A, A] = \theta$ for each $A \in \mathcal{C}$ by using the cancellation law. Next we can define a norm on $\mathcal{C}^2/\equiv$. Define

$$\|[A, B]\| = H(A, B),$$

for every $[A, B] \in \mathcal{C}^2/\equiv$, then $\|\cdot\|$ is a norm on $\mathcal{C}^2/\equiv$, and we equip the vector space $\mathcal{C}^2/\equiv$ with the topology which is induced by the norm. Define an embedding function $\psi : \mathcal{C} \to \mathcal{C}^2/\equiv$ by

$$\psi(A) = [A, \{0\}]$$

for all $A \in \mathcal{C}$. The embedding space $\mathcal{C}^2/\equiv$ and the embedding function ψ play very important role to study set optimization problems.

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References

[1] M. Avriel, W. E. Diewert, S. Schaible and I. Zang, *Generalized concavity*, Math. Concepts Methods Sci. Engrg. Plenum Press, New York, 1988.

[2] A. Beck and A. Ben-Tal, *Duality in robust optimization: Primal worst equals dual best*, Oper. Res. Lett. **37** (2009), 1–6.

[3] R. I. Boţ, *Conjugate duality in convex optimization*, Lecture Notes in Economics and Mathematical Systems, 637. Springer-Verlag, Berlin. 2010.

[4] J. H. Chou, W. S Hsia and T. Y. Lee, *Epigraphs of convex set functions*, J. Math. Anal. Appl. **118** (1986), 247–254.

[5] J. H. Chou, W. S Hsia and T. Y. Lee, *Convex programming with set functions*, Rocky Mountain J. Math. **17** (1987), 535–543.

[6] H. J. Greenberg and W. P. Pierskalla, *Quasi-conjugate functions and surrogate duality*, Cah. Cent. Étud. Rech. Opér **15** (1973), 437–448.

[7] W. S. Hsia, J. H. Lee and T. Y. Lee, *Convolution of set functions*, Rocky Mountain J. Math. **21** (1991), 1317–1325.

[8] W. S. Hsia and T. Y. Lee, *Some minimax theorems on set functions*, Bull. Inst. Math. Acad. Sinica **25** (1997), 29–33.

[9] D. Kuroiwa, *On derivatives of set-valued maps and optimality conditions for set optimization*, J. Nonlinear Convex Anal. **10** (2009), 41–50.

[10] D. Kuroiwa, *Generalized Minimality in Set Optimization*, in: Set Optimization and Applications - The State of the Art, Springer Proceedings in Mathematics & Statistics, vol. 151, Springer, Heidelberg, 2015, pp. 293–311.

[11] D. Kuroiwa and T. Nuriya, *A generalized embedding vector space in set optimization*, in: Nonlinear Analysis and Convex Analysis, Yokohama Publisher, Yokohama, 2007, pp. 297–303.

[12] V. Jeyakumar and G. Y. Li, *Strong duality in robust convex programming: complete characterizations*, SIAM J. Optim. **20** (2010), 3384–3407.

[13] V. Jeyakumar and G. Y. Li, *Characterizing robust set containments and solutions of uncertain linear programs without qualifications*, Oper. Res. Lett. **38** (2010), 188–194.

[14] V. Jeyakumar and G. Y. Li, *Robust Farkas' lemma for uncertain linear systems with applications*, Positivity **15** (2011), 331–342.

[15] V. Jeyakumar and G. Y. Li, *Strong duality in robust semi-definite linear programming under data uncertainty*, Optimization **63** (2014), 713–733.

[16] H. C. Lai and L. J. Lin, *Moreau-Rockafellar type theorem for convex set functions*, J. Math. Anal. Appl. **132** (1988), 558–571.

[17] H. C. Lai and L. J. Lin, *The Fenchel-Moreau theorem for set functions*, Proc. Amer. Math. Soc. **103** (1988), 85–90.

[18] T. Y. Lee, *Generalized convex set functions*, J. Math. Anal. Appl. **141** (1989), 278–290.

[19] G. Y. Li, V. Jeyakumar and G. M. Lee, *Robust conjugate duality for convex optimization under uncertainty with application to data classification*, Nonlinear Anal. **74** (2011), 2327–2341.

[20] L. J. Lin, *On the optimality of differentiable nonconvex n-set functions*, J. Math. Anal. Appl. **168** (1992), 351–366.

[21] D. G. Luenberger, *Quasi-convex programming*, SIAM J. Appl. Math. **16** (1968), 1090–1095.

[22] J. J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. **49** (1970), 109–154.

[23] R. J. T. Morris, *Optimal constrained selection of a measurable subset*, J. Math. Anal. Appl. **70** (1979), 546–562.

[24] J. P. Penot and M. Volle, *On quasi-convex duality*, Math. Oper. Res. **15** (1990), 597–625.

[25] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.

[26] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J. 1970.

[27] S. Suzuki and D. Kuroiwa, *On set containment characterization and constraint qualification for quasiconvex programming*, J. Optim. Theory Appl. **149** (2011), 554–563.

[28] S. Suzuki and D. Kuroiwa, *Optimality conditions and the basic constraint qualification for quasiconvex programming*, Nonlinear Anal. **74** (2011), 1279–1285.

[29] S. Suzuki and D. Kuroiwa, *Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming*, Nonlinear Anal. **75** (2012), 2851–2858.

[30] S. Suzuki and D. Kuroiwa, *Necessary and sufficient constraint qualification for surrogate duality*, J. Optim. Theory Appl. **152** (2012), 366–377.

[31] S. Suzuki and D. Kuroiwa, *Some constraint qualifications for quasiconvex vector-valued systems*, J. Global Optim. **55** (2013), 539–548.

[32] S. Suzuki, D. Kuroiwa and G. M. Lee, *Surrogate duality for robust optimization*, European J. Oper. Res. **231** (2013), 257–262.

[33] S. Suzuki and D. Kuroiwa, *Characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential*, J. Global Optim. **62** (2015), 431–441.

[34] S. Suzuki and D. Kuroiwa, *A constraint qualification characterizing surrogate duality for quasiconvex programming*, Pacific J. Optim. **12** (2016), 87–100.

[35] S. Suzuki and D. Kuroiwa, *Nonlinear error bounds for quasiconvex inequality systems*, Optim. Lett. **11** (2017), 107–120.

[36] S. Suzuki and D. Kuroiwa, *Duality theorems for separable convex programming without qualifications*, J. Optim. Theory Appl. **172** (2017), 669–683.

[37] S. Suzuki and D. Kuroiwa, *Generators and constraint qualifications for quasiconvex inequality systems*, J. Nonlinear Convex Anal. **18** (2017), 2101–2121

[38] C. Zălinescu, *On several results about convex set functions*, J. Math. Anal. Appl. **328** (2007), 1451–1470.

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