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EQUIVALENCE OF OPTIMALITY CRITERIONS FOR DISCRETE TIME OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper we study solutions of an infinite horizon discrete-time optimal control problem arising in economic dynamics with a compact metric space of states which is a subset of a finite-dimensional Euclidean space. Usually, these problems are studied under assumptions that all their good programs converges to a turnpike which is an interior point of the set of admissible pairs. In this paper we study a large class of control systems for which the turnpike is not necessarily an interior point of the set of admissible pairs and show the equivalence of several optimality criterions.

1. Introduction and the main result

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [3–11, 16, 19, 23, 25, 29–33, 35] and the references mentioned therein. These problems arise in engineering [1, 21, 40, 43], in models of economic growth [9, 12, 17, 18, 24, 27, 28, 34, 36, 38–42], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [2, 37] and in the theory of thermodynamical equilibrium for materials [22, 26]. In this paper we study the infinite horizon problem related to a discrete-time optimal control system describing a general model of economic dynamics [12, 21, 24, 27, 36, 40–42]. Usually, these problems are studied under assumptions that all their good programs converge to a turnpike which is an interior point of the set of admissible pairs. In this paper we study a large class of control systems for which the turnpike is not necessarily an interior point of the set of admissible pairs and show the equivalence of several optimality criterions.

Let the n-dimensional space R^n with the Euclidean norm $\|\cdot\|$ be ordered by the cone $R^n_+ = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}$ and let $x >> y$, $x > y$, $x \geq y$ have their usual meaning.

Let $X \subset R^n_+$ be a compact subset of R^n_+, Ω be a nonempty closed subset of $X \times X$ and let $v : \Omega \to R^1$ be a bounded upper semicontinuous function.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called an (Ω)-program (or just a program if the set Ω is understood) if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \geq 0$. A sequence $\{x_t\}_{t=0}^{T}$, where

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T is a natural number, is called an (Ω)-program (or just a program if the set Ω is understood) if $(x_t, x_{t+1}) \in \Omega$ for all integers $t \in [0, T-1]$.

In models of economic growth the set X is the space of states, v is a utility function and $v(x_t, x_{t+1})$ evaluates consumption at moment t.

We consider the problems

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \to \max, \ \{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega, \ x_0 = z_1$$

and

$$\sum_{i=0}^{T-1} v(x_i, x_{i+1}) \to \max,$$

$$\{(x_i, x_{i+1})\}_{i=0}^{T-1} \subset \Omega, \ x_0 = z_1, \ x_T \geq z_2,$$

where T is a natural number and $z_1, z_2 \in X$.

One of the main topics in the infinite horizon optimal control theory is to study the existence of solutions of problems over an infinite horizon using different optimality criteria. In the present paper, studying infinite horizon problems, we deal with the notion of good programs introduced by D. Gale in [12] which is of great usage in optimal control and economic dynamics (see, for example, [9, 40, 42] and the references mentioned therein), with the notion of agreeable programs introduced and studied in 1973 by P. J. Hammond and the Nobel laureate J. Mirrlees [15] and with the notion of overtaking optimal program [9,12,36,40,42,43]. We also consider the notion of locally maximal programs which is a version of optimality criterion introduced by S. Aubry and P. Y. Le Daeron in their seminal paper on the discrete Frenkel-Kontorova model [2].

Set

$$\|v\| = \sup\{|v(x, y)| : \ (x, y) \in \Omega\}. \tag{1.1}$$

We assume that $\|v\| > 0$.

For each $x, y \in X$ and each integer $T \geq 1$ set

$$\sigma(v, T, x) = \sup\Big\{ \sum_{i=0}^{T-1} v(x_i, x_{i+1}) : \ \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x \Big\}, \tag{1.2}$$

(1.3)

$$\sigma(v, T, x, y) = \sup\Big\{ \sum_{i=0}^{T-1} v(x_i, x_{i+1}) : \ \{x_i\}_{i=0}^{T} \text{ is a program and } x_0 = x, \ x_T \geq y \Big\},$$

$$\sigma(v, T) = \sup\Big\{ \sum_{i=0}^{T-1} v(x_i, x_{i+1}) : \ \{x_i\}_{i=0}^{T} \text{ is a program} \Big\}. \tag{1.4}$$

(Here we use the convention that the supremum over an empty set is $-\infty$).

We suppose that there exist $\bar{x} \in X$ and a constant $\bar{c} > 0$ such that the following assumptions hold.

(A1) $(\bar{x}, \bar{x}) \in \Omega$ and $v : \Omega \to R^1$ is continuous at $(\bar{x}, \bar{x})$.

(A2) $\sigma(v, T) \leq T v(\bar{x}, \bar{x}) + \bar{c}$ for all integers $T \geq 1$.

It is easy to see that for each natural number T and each program $\{x_t\}_{t=0}^T$,

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \le \sigma(v, T) \le Tv(\bar{x}, \bar{x}) + c. \tag{1.5}$$

Inequality (1.5) implies the following result.

Proposition 1.1. *For each program* $\{x_t\}_{t=0}^\infty$ *either the sequence*

$$\Big\{ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \Big\}_{T=1}^\infty$$

is bounded or $\lim_{T\to\infty}[\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x})] = -\infty$.

A program $\{x_t\}_{t=0}^\infty$ is called good [9, 12, 36, 40, 42, 43] if the sequence

$$\Big\{ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \Big\}_{T=1}^\infty$$

is bounded.

We suppose that the following assumption holds.

(A3) (the asymptotic turnpike property) For any good program $\{x_t\}_{t=0}^\infty$ the equality $\lim_{t\to\infty} \|x_t - \bar{x}\| = 0$ holds.

Put

$$v(x, y) = -\|v\| - 1, \ (x, y) \in (X \times X) \setminus \Omega.$$

Clearly, v is a bounded upper semicontinuous function on $X \times X$.

Remark 1.1. In [42] we assume that $(\bar{x}, \bar{x})$ is an interior point of Ω. Namely, we suppose there that there is $\epsilon > 0$ such that $\{(x, y) \in X \times X : \ \|x - \bar{x}\|, \ \|y - \bar{x}\| \le \epsilon\} \subset \Omega$. We show in [42] that this assumption holds for many models of economic dynamics but it does not hold for an important class of Robinson-Solow-Srinivasan models studied in [17, 18, 38, 39] for which the turnpike is not an interior point of Ω. In the present paper instead of this assumption we use another assumptions (A4) and (A5) given below which hold for a large class of control systems containing the Robinson-Solow-Srinivasan model as well as many other models of economic dynamics [41].

We suppose that the following assumptions hold.

(A4) If $(x_0, x_1) \in \Omega$ and if $y_0 \in X$ satisfies $y_0 \ge x_0$, then there exists $y_1 \in X$ such that

$$(y_0, y_1) \in \Omega, \ v(y_0, y_1) \ge v(x_0, x_1) \text{ and } 0 \le y_1 - x_1 \le y_0 - x_0.$$

(A5) There exists $\bar{r} > 0$ such that for each $x, y \in X$ satisfying $\|x - \bar{x}\|, \ \|y - \bar{x}\| \le \bar{r}$ there exists $y' \in X$ such that $y' \ge y$ and $(x, y') \in \Omega$. Moreover, for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x, y \in X$ satisfying $\|x - \bar{x}\|, \ \|y - \bar{x}\| \le \delta$ there is $y' \in X$ such that

$$y' \ge y, \ (x, y') \in \Omega \text{ and } \|y' - \bar{x}\| \le \epsilon.$$

It is known [41] that assumptions (A1)-(A5) hold for many models of economic dynamics. It is clear that assumption (A4) is a natural monotonicity property of the technology set Ω while (A5) is a weakened version of the assumption used in [17, 18, 38, 39].

Assumption (A4) implies that if $\{x_t\}_{t=0}^\infty$ is a program and $y_0 \in X$ satisfies $y_0 \ge x_0$, then there exists a program $\{y_t\}_{t=0}^\infty$ such that for all integers $t \ge 0$, $v(y_t, y_{t+1}) \ge v(x_t, x_{t+1})$, $y_t \ge x_t$ and $y_t - x_t \le y_0 - x_0$. The last two inequalities imply that if all x_t, $t = 0, 1, \dots$ and y_0 are close to $\bar{x}$, then y_t is also close to $\bar{x}$ for all integers $t \ge 0$.

Assumption (A5) means that for each x, y which are close to $\bar{x}$ if a state of the model at time t is x, then at moment $t+1$ a state of the model can be y' which is also close to $\bar{x}$ and satisfies $y' \ge y$.

For each $M > 0$ denote by X_M the set of all $x \in X$ for which there exists a program $\{x_t\}_{t=0}^\infty$ such that $x_0 = x$ and that for all integers $T \ge 1$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) - Tv(\bar{x}, \bar{x}) \ge -M.$$

Clearly, $\cup\{X_M : M > 0\}$ is the set of all points $z \in X$ for which there exists a good program from z.

For each natural number T denote by $\bar{Y}_T$ the set of all $x \in X$ for which there exists a program $\{x_t\}_{t=0}^T$ such that $x_0 = \bar{x}$, $x_T \ge x$.

In the sequel we use a notion of an overtaking optimal program [9,12,36,40,42,43].

A program $\{x_t^*\}_{t=0}^\infty$ is called overtaking optimal if for each program $\{y_t\}_{t=0}^\infty$ satisfying $y_0 = x_0^*$ the inequality

$$\limsup_{T\to\infty} \sum_{t=0}^{T-1} [v(y_t, y_{t+1}) - v(x_t^*, x_{t+1}^*)] \le 0$$

holds.

The following result, which was obtained in [41], establishes the existence of an overtaking optimal program.

Theorem 1.2. *Assume that $x \in X$ and that there exists a good program $\{x_t\}_{t=0}^\infty$ such that $x_0 = x$. Then there exists an overtaking optimal program $\{x_t^*\}_{t=0}^\infty$ such that $x_0^* = x$.*

The following result, also obtained in [41], provides necessary and sufficient conditions for overtaking optimality.

Theorem 1.3. *Let $\{x_t\}_{t=0}^\infty$ be a program such that $x_0 \in \cup\{X_M : M \in (0,\infty)\}$. Then the program $\{x_t\}_{t=0}^\infty$ is overtaking optimal if and only if the following conditions hold: (i) $\lim_{t\to\infty} \|x_t - \bar{x}\| = 0$; (ii) for each natural number T and each program $\{y_t\}_{t=0}^T$ satisfying $y_0 = x_0$, $y_T \ge x_T$ the inequality $\sum_{t=0}^{T-1} v(y_t, y_{t+1}) \le \sum_{t=0}^{T-1} v(x_t, x_{t+1})$ holds.*

A program $\{x_t^*\}_{t=0}^\infty$ is called weakly optimal [9, 40] if for each program $\{y_t\}_{t=0}^\infty$ satisfying $y_0 = x_0^*$ the inequality

$$\liminf_{T\to\infty} \sum_{t=0}^{T-1} [v(y_t, y_{t+1}) - v(x_t^*, x_{t+1}^*)] \le 0$$

holds.

The following optimality notion is a version of the optimality criterion introduced in [2] where it was used for infinite-horizon discrete models of solid-state physics related to dislocations in one-dimensional crystals.

A program $\{x_t^*\}_{t=0}^\infty$ is called locally maximal if for each integer $T > 0$ and each program $\{y_t\}_{t=0}^T$ satisfying

$$y_0 = x_0^*, \ y_T \geq x_T^*$$

the following inequality holds:

$$\sum_{t=0}^{T-1} v(y_t, y_{t+1}) \leq \sum_{t=0}^{T-1} v(x_t^*, x_{t+1}^*).$$

A program $\{x_t^*\}_{t=0}^\infty$ is called agreeable if for any natural number T_0 and any $\epsilon > 0$ there exists an integer $T_\epsilon > T_0$ such that for any integer $T \geq T_\epsilon$ there exists a program $\{x_t\}_{t=0}^T$ which satisfies

$$x_t = x_t^*, \ t = 0, \dots, T_0$$

and

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0^*) - \epsilon.$$

The notion of agreeable programs is well-known in the economic literature [13–15]. In this paper we prove the following result.

Theorem 1.4. *Let $\{x_t^*\}_{t=0}^\infty$ be a program and assume that there exists a good program $\{x_t'\}_{t=0}^\infty$ satisfying $x_0^* = x_0'$. Then the following properties are equivalent:*

(i) *the program $\{x_t^*\}_{t=0}^\infty$ is overtaking optimal;*
(ii) *the program $\{x_t^*\}_{t=0}^\infty$ is weakly optimal;*
(iii) *the program $\{x_t^*\}_{t=0}^\infty$ is locally maximal and good;*
(iv) *the program $\{x_t^*\}_{t=0}^\infty$ is locally maximal and satisfies $\lim_{t\to\infty} x_t = \bar{x}$;*
(v) *the program $\{x_t^*\}_{t=0}^\infty$ is locally maximal and satisfies $\liminf_{t\to\infty} \|x_t - \bar{x}\| = 0$;*
(vi) *the program $\{x_t^*\}_{t=0}^\infty$ is agreeable.*

2. A turnpike result

The proof of Theorem 1.4 is based on the following turnpike result obtained in [41].

Theorem 2.1. *Let ϵ, M be positive numbers. Then there exist a natural number L and a positive number δ such that for each integer $T > 2L$ and each program $\{x_t\}_{t=0}^T$ which satisfies*

$$x_0 \in X_M, \ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0) - \delta$$

there exist nonnegative integers $\tau_1, \ \tau_2 \in [0, L]$ such that $\|x_t - \bar{x}\| \leq \epsilon$ for all $t = \tau_1, \dots, T - \tau_2$.

3. Proof of Theorem 1.4

Clearly, (i) implies (ii). We show that (ii) implies (iii). Assume that the program $\{x_t^*\}_{t=0}^\infty$ is weakly optimal. Then

$$\liminf_{T\to\infty} \sum_{t=0}^{T-1}[v(x'_t, x'_{t+1}) - v(x_t^*, x_{t+1}^*)] \le 0. \tag{3.1}$$

Since the program $\{x'_t\}_{t=0}^\infty$ is good we have

$$\sup\Big\{\Big|\sum_{t=0}^{T-1} v(x'_t, x'_{t+1}) - Tv(\bar{x},\bar{x})\Big| : \ T = 1, 2, \dots\Big\} < \infty. \tag{3.2}$$

By (3.1) and (3.2),

$$\limsup_{T\to\infty} \Big[\sum_{t=0}^{T-1} v(x_t^*, x_{t+1}^*) - Tv(\bar{x},\bar{x})\Big] > -\infty. \tag{3.3}$$

Proposition 1.1 and (3.3) imply that $\{x_t^*\}_{t=0}^\infty$ is good. In view of (A3),

$$\lim_{t\to\infty} x_t^* = \bar{x}. \tag{3.4}$$

We claim that $\{x_t^*\}_{t=0}^\infty$ is locally maximal. Assume the contrary. Then there exist $\Delta > 0$, a natural number T_0 and a program $\{y_t\}_{t=0}^{T_0}$ such that

$$y_0 = x_0^*, \ y_{T_0} \ge x_{T_0}^*, \tag{3.5}$$

$$\sum_{t=0}^{T_0-1} v(y_t, y_{t+1}) \ge \sum_{t=0}^{T_0-1} v(x_t^*, x_{t+1}^*) + \Delta. \tag{3.6}$$

(A4) and (3.5) imply that there exist $y_t \in X$ for all integers $t > T_0$ such that $\{y_t\}_{t=0}^\infty$ is a program, $y_t \ge x_t^*$ for all integers $t \ge T_0$ and that for all integers $t \ge T_0$, we have

$$v(y_t, y_{t+1}) \ge v(x_t^*, x_{t+1}^*).$$

Together with (3.6) this implies that

$$\liminf_{T\to\infty} \Big[\sum_{t=0}^{T-1} v(y_t, y_{t+1}) - \sum_{t=0}^{T-1} v(x_t^*, x_{t+1}^*)\Big] \ge \sum_{t=0}^{T_0-1} v(y_t, y_{t+1}) - \sum_{t=0}^{T_0-1} v(x_t^*, x_{t+1}^*) \ge \Delta > 0$$

and $\{x_t^*\}_{t=0}^\infty$ is not weakly optimal. The contradiction we have reached proves that $\{x_t^*\}_{t=0}^\infty$ is locally maximal.

By (A3), (iii) implies (iv). Clearly, (iv) implies (v). We show that (v) implies (iii).

Assume that $\{x_t^*\}_{t=0}^\infty$ is a locally maximal program and that

$$\liminf_{t\to\infty} \|x_t^* - \bar{x}\| = 0. \tag{3.7}$$

We show that $\{x_t^*\}_{t=0}^\infty$ is a good program. (A1) implies that there exists $\epsilon_0 \in (0, 1)$ such that

$$|v(x, y) - v(\bar{x},\bar{x})| \le 1 \text{ for each } (x, y) \in \Omega \text{ satisfying } \|x - \bar{x}\|, \|y - \bar{x}\| \le \epsilon_0. \tag{3.8}$$

(A5) implies that there exists $\delta_0 \in (0, \epsilon)$ such that the following property holds:

(a) for each $(x, y) \in X$ satisfying $\|x - \bar{x}\|,\ \|y - \bar{x}\| \le \delta_0$ there exists $y' \in X$ such that

$$y' \ge y,\ (x, y') \in \Omega,\ \|y' - \bar{x}\| \le \epsilon_0.$$

There exists $M_0 > 0$ such that for each natural number T,

$$\Big| \sum_{t=0}^{T-1} v(x'_t, x'_{t+1}) - T v(\bar{x}, \bar{x}) \Big| \le M_0. \tag{3.9}$$

It follows from (A3) that

$$\lim_{t\to\infty} x'_t = \bar{x}. \tag{3.10}$$

By (3.7) and (3.10), there exists a strictly increasing sequence of natural numbers $\{T_k\}_{k=1}^{\infty}$ such that $T_1 \ge 10$,

$$\|x^*_{T_k} - \bar{x}\| \le \delta_0,\ k = 1, 2, \dots, \tag{3.11}$$

$$\|x'_t - \bar{x}\| \le \delta_0 \text{ for all integers } t \ge T_1 - 1. \tag{3.12}$$

Let $k \ge 1$ be an integer. It follows from (3.11) and (3.12) that

$$\|x^*_{T_k} - \bar{x}\| \le \delta_0,\ \|x'_{T_k - 1} - \bar{x}\| \le \delta_0. \tag{3.13}$$

Property (a) and (3.13) imply that there exists $z \in X$ such that

$$z \ge x^*_{T_k},\ (x'_{T_k-1}, z) \in \Omega,\ \|z - \bar{x}\| \le \epsilon_0. \tag{3.14}$$

Define

$$y_t = x'_t,\ t = 0, \dots, T_k - 1,\ y_{T_k} = z. \tag{3.15}$$

By (3.14) and (3.15), $\{y_t\}_{t=0}^{T_k}$ is a program satisfying

$$y_0 = x^*_0,\ y_{T_k} \ge x^*_{T_k}. \tag{3.16}$$

It follows from (3.9), (3.15) and (3.16) that

$$\sum_{t=0}^{T_k-1} v(x^*_t, x^*_{t+1}) \ge \sum_{t=0}^{T_k-1} v(y_t, y_{t+1}) \ge \sum_{t=0}^{T_k-1} v(x'_t, x'_{t+1}) - 2\|v\|$$

$$\ge -2\|v\| - M_0 + T_k v(\bar{x}, \bar{x}). \tag{3.17}$$

Theorem 1.1 and (3.17) imply that $\{x^*_t\}_{t=0}^{\infty}$ is a good program and (iii) holds. In view of Theorem 1.3, (iv) implies (i). Therefore properties (i)-(v) are equivalent.

Let us show that (vi) implies (iv). Assume that $\{x^*_t\}_{t=0}^{\infty}$ is agreeable. Since $\{x'_t\}_{t=0}^{\infty}$ is good and $x'_0 = x^*_0$ there exists $M > 0$ such that

$$x^*_0 \in X_M.$$

We show that $\{x^*_t\}_{t=0}^{\infty}$ is locally maximal. Assume the contrary. Then there exist $\Delta > 0$, a natural number T_0 and a program $\{y_t\}_{t=0}^{T_0}$ such that

$$y_0 = x^*_0,\ y_{T_0} \ge x^*_{T_0}, \tag{3.18}$$

$$\sum_{t=0}^{T_0-1} v(y_t, y_{t+1}) \geq \sum_{t=0}^{T_0-1} v(x^*_t, x^*_{t+1}) + \Delta. \tag{3.19}$$

Since $\{x^*_t\}_{t=0}^{\infty}$ is an agreeable program there exist a natural number $T_1 > T_0 + 8$ and a program $\{x_t\}_{t=0}^{T_1}$ such that

$$x_t = x^*_t, \ t = 0, \dots, T_0, \tag{3.20}$$

$$\sum_{t=0}^{T_1-1} v(x_t, x_{t+1}) \geq \sigma(v, T_1, x^*_0) - \Delta/4. \tag{3.21}$$

(A4) and (3.18) imply that there exist y_t, $t = T_0 + 1, \dots, T_1$ such that $\{y_t\}_{t=0}^{T_1}$ is a program,

$$y_t \geq x_t, \ t = T_0, \dots, T_1, \tag{3.22}$$

$$v(y_t, y_{t+1}) \geq v(x_t, x_{t+1}), \ t = T_0, \dots, T_1 - 1. \tag{3.23}$$

By (3.18), (3.19), (3.21) and (3.23),

$$\begin{aligned} \Delta/4 &\geq \sigma(v, T_1, x^*_0) - \sum_{t=0}^{T_1-1} v(x_t, x_{t+1}) \\ &\geq \sum_{t=0}^{T_1-1} v(y_t, y_{t+1}) - \sum_{t=0}^{T_1-1} v(x_t, x_{t+1}) \\ &\geq \sum_{t=0}^{T_0-1} v(y_t, y_{t+1}) - \sum_{t=0}^{T_0-1} v(x_t, x_{t+1}) > \Delta, \end{aligned}$$

a contradiction. The contradiction we have reached proves that $\{x^*_t\}_{t=0}^{\infty}$ is a locally maximal program.

Let us show that $\lim_{t\to\infty} \|x^*_t - \bar{x}\| = 0$. Let $\epsilon > 0$. Theorem 2.1 implies that there exist a natural number L_0 and a positive number $\delta \in (0, \epsilon)$ such that the following property holds:

(b) for each integer $T > 2L_0$ and each program $\{x_t\}_{t=0}^{T}$ which satisfies

$$x_0 = x^*_0, \ \sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T, x_0) - \delta$$

we have $\|x_t - \bar{x}\| \leq \epsilon$ for all $t = L_0, \dots, T - L_0$.

Let a natural number $S > 2L_0$. Since $\{x^*_t\}_{t=0}^{\infty}$ is an agreeable program there exist a natural number $T > S + L_0$ and a program $\{x_t\}_{t=0}^{T}$ such that

$$x_t = x^*_t, \ t = 0, \dots, S, \tag{3.24}$$

$$\sum_{t=0}^{T-1} v(x_t, x_{t+1}) \geq \sigma(v, T_1, x^*_0) - \delta. \tag{3.25}$$

Property (b), (3.24) and (3.25) imply that

$$\|x_t - \bar{x}\| \leq \epsilon, \ t = L_0, \dots, T - L_0. \tag{3.26}$$

In view of (3.24) and (3.26),

$$\|x_t - \bar{x}\| \le \epsilon,\ t = L_0, \dots, S.$$

This implies that

$$\|x_t - \bar{x}\| \le \epsilon \text{ for all interges } t \ge L_0$$

and $\lim_{t\to\infty} \|x_t^* - \bar{x}\| = 0$. Thus (iv) holds.

We show that (iv) implies (vi). Assume that $\{x_t^*\}_{t=0}^{\infty}$ i a locally maximal program and

$$\|x_t^* - \bar{x}\| = 0. \tag{3.27}$$

Let T_0 be a natural number and $\epsilon \in (0,1)$. (A1) implies that there exists $\delta_0 \in (0, \epsilon/4)$ such that

$$|v(x,y) - v(\bar{x},\bar{x})| \le \epsilon/8 \tag{3.28}$$

for each $(x,y) \in \Omega$ satisfying $\|x - \bar{x}\|,\ \|y - \bar{x}\| \le 2\delta_0$. (A5) implies that there exists $\delta \in (0, \delta_0)$ such that the following property holds:

(c) for each $x, y \in X$ satisfying

$$\|x - \bar{x}\|,\ \|y - \bar{x}\| \le \delta$$

there exists $y' \in X$ such that

$$y' \ge y,\ (x, y') \in \Omega,\ \|y' - \bar{x}\| \le \delta_0.$$

Clearly, there exists $M > 0$ such that

$$x_0^* \in X_M \tag{3.29}$$

and there exists a natural number T_1 such that

$$\|x_t' - \bar{x}\| \le \delta \text{ for all integers } t \ge T_1. \tag{3.30}$$

Theorem 2.1 and (3.29) imply that there exists a natural number L_0 such that the following property holds:

(d) for each integer $T > 2L_0$ and each program $\{z_t\}_{t=0}^{T}$ which satisfies

$$z_0 = x_0^*,\ \sum_{t=0}^{T-1} v(z_t, z_{t+1}) = \sigma(v, T, x_0^*)$$

we have $\|z_t - \bar{x}\| \le \delta$ for all $t = L_0, \dots, T - L_0$.

Let a natural number

$$T > T_0 + 2L_0 + T_1 + 4. \tag{3.31}$$

There exists a program $\{y_t\}_{t=0}^{T}$ such that

$$y_0 = x_0^*,\ \sum_{t=0}^{T-1} v(y_t, y_{t+1}) = \sigma(v, T, x_0^*). \tag{3.32}$$

Property (d), (3.31) and (3.32) imply that

$$\|y_t - \bar{x}\|\ \le \delta,\ t = L_0, \dots, T - L_0. \tag{3.33}$$

In view of (3.30) and (3.33),

$$\|y_{T-L_0-1} - \bar{x}\| \le \delta,\ \|y_{T-L_0} - \bar{x}\| \le \delta, \tag{3.34}$$

$$\|x^*_{T-L_0-1}-\bar{x}\| \le \delta,\ \|x^*_{T-L_0}-\bar{x}\| \le \delta. \tag{3.35}$$

Property (c), (3.34) and (3.35) imply that there exists $y' \in X$ such that

$$y' \ge y_{T-L_0},\ \|y'-\bar{x}\| \le \delta_0,\ (x^*_{T-L_0-1}, y') \in \Omega. \tag{3.36}$$

It follows from (A4) and (3.36) that there exists a program $\{\widehat{y}_t\}_{t=0}^T$ such that

$$\widehat{y}_t = x^*_t,\ t = 0, \dots, T-L_0-1, \tag{3.37}$$

$$\widehat{y}_{T-L_0} = y',\ \widehat{y}_t \ge y_t,\ t = T-L_0, \dots, T, \tag{3.38}$$

$$v(\widehat{y}_t, \widehat{y}_{t+1}) \ge v(y_t, y_{t+1}),\ t = T-L_0, \dots, T-1. \tag{3.39}$$

In view of (3.32) and (3.37),

$$\sum_{t=0}^{T-1} v(\widehat{y}_t, \widehat{y}_{t+1}) \le \sigma(v, T, x^*_0) = \sum_{t=0}^{T-1} v(y_t, y_{t+1}).$$

By (3.28), (3.30) and (3.35)-(3.39),

$$\begin{aligned}
\sum_{t=0}^{T-1} v(\widehat{y}_t, \widehat{y}_{t+1}) - \sum_{t=0}^{T-1} v(y_t, y_{t+1}) &\ge \sum_{t=0}^{T-L_0-1} v(\widehat{y}_t, \widehat{y}_{t+1}) - \sum_{t=0}^{T-L_0-1} v(y_t, y_{t+1}) \\
&= \sum_{t=0}^{T-L_0-1} v(x^*_t, x^*_{t+1}) - \sum_{t=0}^{T-L_0-1} v(y_t, y_{t+1}) \\
&\quad - v(x^*_{T-L_0-1}, x^*_{T-L_0}) + v(x^*_{T-L_0-1}, y') \\
&\ge \sum_{t=0}^{T-L_0-1} v(x^*_t, x^*_{t+1}) - \sum_{t=0}^{T-L_0-1} v(y_t, y_{t+1}) \\
&\quad - \epsilon/4.
\end{aligned} \tag{3.40}$$

Property (c), (3.34) and (3.35) imply that there exists $z' \in X$ such that

$$z' \ge x^*_{T-L_0},\ \|z'-\bar{x}\| \le \delta_0,\ (y^*_{T-L_0-1}, z') \in \Omega. \tag{3.41}$$

By (3.31) and (3.41) there exists a program $\{\widehat{z}_t\}_{t=0}^{T-L_0}$ such that

$$\widehat{z}_t = y_t,\ t = 0, \dots, T-L_0-1,\ \widehat{z}_{T-L_0} = z'. \tag{3.42}$$

In view of (3.32), (3.41) and (3.42),

$$\sum_{t=0}^{T-L_0-1} v(x^*_t, x^*_{t+1}) \ge \sum_{t=0}^{T-L_0-1} v(\widehat{z}_t, \widehat{z}_{t+1}). \tag{3.43}$$

By (3.28), (3.34) and (3.41)-(3.43),

$$
\begin{aligned}
0 &\le \sum_{t=0}^{T-L_0-1} v(x_t^*, x_{t+1}^*) - \sum_{t=0}^{T-L_0-1} v(\widehat{z}_t, \widehat{z}_{t+1}) \\
&= \sum_{t=0}^{T-L_0-1} v(x_t^*, x_{t+1}^*) - \sum_{t=0}^{T-L_0-1} v(y_t, y_{t+1}) \\
&\quad + v(y_{T-L_0-1}, y_{T-L_0}) - v(y_{T-L_0-1}, z') \\
&\le \sum_{t=0}^{T-L_0-1} v(x_t^*, x_{t+1}^*) - \sum_{t=0}^{T-L_0-1} v(y_t, y_{t+1}) + \epsilon/4. \qquad (3.44)
\end{aligned}
$$

In view of (3.44),

$$
\sum_{t=0}^{T-L_0-1} v(x_t^*, x_{t+1}^*) - \sum_{t=0}^{T-L_0-1} v(y_t, y_{t+1}) \ge -\epsilon/4. \qquad (3.45)
$$

It follows from (3.40) and (3.45) that

$$
\sum_{t=0}^{T-1} v(\widehat{y}_t, \widehat{y}_{t+1}) - \sum_{t=0}^{T-1} v(y_t, y_{t+1}) \ge -\epsilon/2.
$$

By the relation above, (3.31), (3.32) and (3.37),

$$
\sum_{t=0}^{T-1} v(\widehat{y}_t, \widehat{y}_{t+1}) \ge \sigma(v, T, x_0^*) - \epsilon/2,
$$

$$
\widehat{y}_t = x_t^*, \ t = 0, \dots, T_0.
$$

Thus $\{x_t^*\}_{t=0}^{\infty}$ is an agreeable program, and (iv) implies (vi). This completes the proof of Theorem 1.4.

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