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APPLICATIONS OF ANALYSIS TO THE DETERMINATION OF THE MINIMUM NUMBER OF DISTINCT EIGENVALUES OF A GRAPH

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Abstract. We establish new bounds on the minimum number of distinct eigenvalues among real symmetric matrices with nonzero off-diagonal pattern described by the edges of a graph and apply these to determine the minimum number of distinct eigenvalues of several families of graphs and small graphs.

1. Introduction

Inverse eigenvalue problems appear in various contexts throughout mathematics and engineering, and refer to determining all possible lists of eigenvalues (spectra) for matrices fitting some description. The *inverse eigenvalue problem of a graph (IEPG)* refers to determining the possible spectra of real symmetric matrices whose pattern of nonzero off-diagonal entries is described by the edges of a given graph. Graphs often describe relationships in a physical system and the eigenvalues of associated matrices govern the behavior of the system. The IEPG and related variants have been of interest for many years. Various parameters have been used to study this problem, most importantly the maximum multiplicity of an eigenvalue of a matrix described by the graph (see, for example, [6]). In [1] the authors introduce the parameter $q(G)$ as the minimum number of eigenvalues among the matrices described by the graph. In this paper we establish additional techniques for bounding q and determine its value for various families of graphs.

The Strong Multiplicity Property (SMP) and the Strong Spectral Property (SSP) are recently developed tools that were introduced in [4] (see also Section 2) and have enabled significant progress on the IEPG. The SMP and SSP have their roots in the implicit function theorem. The SMP allows us to perturb along the intersection of the pattern manifold and the fixed ordered multiplicity list manifold (along the fixed spectrum manifold for SSP) under suitable conditions. In this paper we apply the SMP and SSP and additional matrix tools such as the Kronecker product of matrices (see Section 3) to establish bounds on the minimum number of distinct eigenvalues of a graph. We then apply these results to determine the minimum number of distinct eigenvalues of several families of graphs and of small graphs.

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In this paper, a *graph* is a pair $(V(G), E(G))$ where $V(G) = \{1, 2, \ldots, n\}$ and $E(G)$ is a set of 2-element subsets of $V(G)$, each having the form $\{u, v\}$ where $1 \le u < v \le n$. We also denote an edge $\{u, v\}$ as uv; in this case vertices u and v are *adjacent* and are *neighbors*. The *neighborhood* of vertex u is $N(u) = \{v \in V(G) : uv \in E(G)\}$. A *leaf* is a vertex with only one neighbor. The *order* of G is the number of vertices, $|V(G)|$. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We say that a subgraph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is a spanning subgraph of G if $\widetilde{V} = V$.

Let G be a graph of order n. The set $\mathcal{S}(G)$ of matrices representing G is the set of real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \ne j$, $a_{ij} \ne 0$ if and only if $ij \in E(G)$ (the diagonal is unrestricted). For a matrix A, the number of distinct eigenvalues of A is denoted $q(A)$ and the minimum number of distinct eigenvalues of a graph G is

$$q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}.$$

Section 2 contains a discussion of the SMP and SSP and applies them to the determination of q. Section 3 presents bounds on $q(G)$ for graphs G constructed as Cartesian, tensor, or strong products. Section 4 presents results about $q(G)$ for certain types of block-clique graphs and joins. The ability of these graph operations to raise or lower q is discussed in Section 5. We determine values of $q(G)$ for all graphs of order 6 in Section 6 and then summarize the values of all graphs G for which q is currently known in Section 7. The remainder of this introduction contains additional definitions and results from the literature that will be used.

1.1. **Terminology and notation.** Matrices discussed are real and symmetric, so all eigenvalues are real and each matrix has an orthonormal basis of eigenvectors. Let A be an $n \times n$ matrix. The *spectrum* of A is the multiset of eigenvalues of A (repeated according to multiplicity) and is denoted by $\text{spec}(A)$. The notation $\lambda_k(A)$ denotes the kth eigenvalue of A with $\lambda_1(A) \le \cdots \le \lambda_n(A)$. If the matrix A has distinct eigenvalues $\mu_1 < \mu_2 < \ldots < \mu_q$ with multiplicities $m_1, m_2, \ldots, m_q$, respectively, then the *ordered multiplicity list* of A is $\mathbf{m}(A) = (m_1, m_2, \ldots, m_q)$. In this paper we denote the set of distinct eigenvalues of a matrix A by $\text{dev}(A)$. A *principal submatrix* of A is a submatrix obtained from A by deleting a set of rows and the corresponding set of columns. For $1 \le k \le n$, the principal submatrix $A(k)$ is the $(n-1) \times (n-1)$ matrix obtained from A by deleting row k and column k from A. The formal definitions of the maximum nullity (which is equal to the maximum multiplicity of an eigenvalue) and minimum rank are

$$\text{M}(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G)\} \text{ and } \text{mr}(G) = \min\{\text{rank}(A) : A \in \mathcal{S}(G)\}.$$

It is easy to observe that $\text{mr}(G) + \text{M}(G) = |V(G)|$, so the study of maximum nullity is equivalent to the study of minimum rank.

A *path* of order n is a graph P_n with $V(P_n) = \{v_i : 1 \le i \le n\}$ and $E(P_n) = \{\{v_i, v_{i+1}\} : 1 \le i \le n-1\}$. The *length* of P_n is the number of edges, i.e., $n-1$. A graph G is *connected* if for every pair of distinct vertices u and v, G contains a path from u to v. In a connected graph G, the *distance* from u to v, denoted by $\text{dist}_G(u, v)$, is the minimum length of a path from u to v. If $n \ge 3$, a *cycle* of order n is a graph C_n with $V(C_n) = \{V_i : 1 \le i \le n\}$ and $E(C_n) = \{\{v_i, v_{i+1}\} : 1 \le i \le n-1\} \cup$

$\{\{v_n, v_1\}\}$. A *complete graph* of order n is a graph K_n with $V(K_n) = \{v_i : 1 \le i \le n\}$ and $E(K_n) = \{\{v_i, v_j\} : 1 \le i < j \le n\}$. A *complete bipartite graph* with partite sets X and Y of orders s and t is the graph $K_{s,t}$ with $V(K_{s,t}) = X \cup Y$ where $X = \{x_i : 1 \le i \le s\}$ and $Y = \{y_i : 1 \le i \le t\}$ are disjoint, and $E(K_{s,t}) = \{\{x_i, y_j\} : 1 \le i \le s, 1 \le j \le t\}$.

1.2. **Results cited.**

Theorem 1.1 (Interlacing Theorem). [15, Theorem 8.10]*] For $A \in \mathbb{R}^{n\times n}$ and $1 \le k \le n$,*

$$\lambda_1(A) \le \lambda_1(A(k)) \le \lambda_2(A) \le \cdots \le \lambda_{n-1}(A) \le \lambda_{n-1}(A(k)) \le \lambda_n(A).$$

More generally, if B is a principal submatrix of A obtained by deleting the rows and columns corresponding to a set of m indices, then

$$\lambda_k(A) \le \lambda_k(B) \le \lambda_{k+m}(A) \text{ for } k = 1, \ldots, n-m.$$

Proposition 1.2. [1, Proposition 2.5] *For a graph G, $q(G) \le \mathrm{mr}(G) + 1$.*

Observation 1.3. [4, p. 23] For a graph G on n vertices, $\lceil \frac{n}{\mathrm{M}(G)} \rceil \le q(G)$.

Observation 1.4. [1, p. 678] If $q(G) = 2$, then there exists a symmetric orthogonal matrix $A \in \mathcal{S}(G)$.

Proposition 1.5. *For any $n \ge 2$, $q(K_n) = 2$* [1, Lemma 2.2]. *For any $n \ge 1$, $q(P_n) = n$* [1, p. 676]. *For any $n \ge 3$, $q(C_n) = \lceil \frac{n}{2} \rceil$* [1, Lemma 2.7].

Theorem 1.6. [1, Corollary 6.5] *For any m, n with $1 \le m \le n$,*

$$q(K_{m,n}) = \begin{cases} 2 & m = n \\ 3 & m < n \end{cases}.$$

Theorem 1.7. [11, Theorem 5.2] *Let G and G' be connected graphs of order n. Then $q(G \vee G') = 2$ and there is a matrix $M \in \mathcal{S}(G \vee G')$ with $\mathbf{m}(M) = (n, n)$.*

The next theorem is often referred to as the "unique shortest path theorem."

Theorem 1.8. [1, Theorem 3.2] *If there are vertices u and v in a connected graph G such that $\mathrm{dist}_G(u, v) = d$ and the path of length d is unique, then $q(G) \ge d + 1$.*

Theorem 1.9. [1, Theorem 4.4] *For a connected graph G on n vertices, if $q(G) = 2$, then for any independent set of vertices $\{v_1, \ldots, v_k\}$ we have*

$$\left|\bigcup_{i\neq j}(N(v_i) \cap N(v_j))\right| \ge k \text{ or } \left|\bigcup_{i\neq j}(N(v_i) \cap N(v_j))\right| = 0.$$

Theorem 1.10. [4, Theorem 51] *A graph G has $q(G) \ge |V(G)| - 1$ if and only if G is one of the following: a path; the disjoint union of a path and an isolated vertex; a path with one leaf attached to an interior vertex; a path with an extra edge joining two vertices at distance 2.*

A path on n vertices with one leaf attached to an interior vertex is called a *generalized star* and is denoted by $S(k-1, n-k-1, 1)$, where k is the vertex with the extra leaf with path vertices numbered in path order. An order n path with an extra edge joining the two vertices $k+1$ and $k+3$ $(0 \le k \le n-3)$ is called a *generalized bull* and is denoted by $GB(k, n-k-3)$.

2. Strong properties

The Strong Spectral Property (SSP) and Strong Multiplicity Property (SMP) were introduced in [4] and additional properties and applications are given in [3]. These properties can yield powerful results. In this section we define and apply them.

The entry-wise product of $A, B \in \mathbb{R}^{n\times n}$ is denoted by $A \circ B$ and the trace (sum of the diagonal entries) of A is denoted by $\operatorname{tr} A$. An $n \times n$ symmetric matrix A satisfies the *Strong Spectral Property (SSP)* [4] provided no nonzero symmetric matrix X satisfies

- $A \circ X = 0 = I \circ X$ and
- $AX - XA = 0$.

A $n \times n$ symmetric matrix A satisfies the *Strong Multiplicity Property (SMP)* [4] provided no nonzero symmetric matrix X satisfies

- $A \circ X = 0 = I \circ X$,
- $AX - XA = 0$, and
- $\operatorname{tr}(A^i X) = 0$ for $i = 0, \dots, n-1$.

If a matrix has SSP, then it also has SMP, but not conversely [4]. The definitions of the SMP and SSP just given are linear algebraic conditions that allow the application of the Implicit Function Theorem to perturb one or more pairs of zero entries to nonzero entries while maintaining the nonzero pattern of other entries and preserving the ordered multiplicity list or spectrum (see [4] for more information). The next theorem will be applied to give an upper bound on q.

Theorem 2.1. [4, Theorem 20] *Let G be a graph and let $\widetilde{G}$ be a spanning subgraph of G. If $\widetilde{A} \in \mathcal{S}(\widetilde{G})$ has SMP, then there exists $A \in \mathcal{S}(G)$ with SMP having the same multiplicity list as $\widetilde{A}$.*

The *SMP minimum number of distinct eigenvalues* of a graph H is defined in [4] to be

$$q_M(H) = \min\{q(A) : A \in \mathcal{S}(H), A \text{ has SMP}\}.$$

The next result is clear from the definitions and Theorem 2.1.

Observation 2.2. Let G be a graph and let $\widetilde{G}$ be a spanning subgraph of G. Then $q(G) \leq q_M(G) \leq q_M(\widetilde{G})$.

A *Hamilton cycle* in a graph is a cycle that includes every vertex. The next result is a simplified form of [4, Corollary 49] and follows from $q_M(C_n) = \lceil \frac{n}{2} \rceil$ [4, Theorem 48].

Corollary 2.3. [4] *Let G be a graph of order n that has a Hamilton cycle. Then $q(G) \leq \lceil \frac{n}{2} \rceil$.*

It is known (see, for example, [4]) that for any set of distinct eigenvalues $\lambda_1 < \cdots < \lambda_n$ and any graph G of order n there is a matrix $A \in \mathcal{S}(G)$ with $\operatorname{spec}(A) = \{\lambda_1, \dots, \lambda_n\}$. The next result includes the additional requirement that every entry of the diagonal of A is nonzero.

Theorem 2.4. *Let G be a graph of order n. Then any set of n distinct nonzero real numbers can be realized by some matrix $A \in \mathcal{S}(G)$ that has SSP and has all diagonal entries nonzero.*

Proof. Let $\lambda_1, \ldots, \lambda_n$ be distinct nonzero real numbers. As noted in [4, Remark 15], there is a matrix $A \in \mathcal{S}(G)$ that has SSP and $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_n\}$. The matrix A is obtained from the matrix $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ by a perturbation of the entries; note that D has SSP since the diagonal entries are distinct [4, Theorem 34]. Since such perturbation may be chosen arbitrarily small, we may assume the diagonal entries of A are all nonzero. □

The next two results about strong properties appear in [4] and [3] and are used in Section 6. Theorem 2.5 allows verification of the SSP or SMP for $A \in \mathcal{S}(G)$ by computation of the rank of a matrix constructed from A and G. Lemma 2.6 allows us to import results from the solution of the IEPG for graphs of order 5 to determine the value of q for order 6. Some definitions are needed first. The *support* of a vector $\mathbf{x}$ is $\operatorname{supp}(\mathbf{x}) = \{i : x_i \neq 0\}$. Let H be a graph with vertex set $\{1, 2, \ldots, n\}$ and edge-set $\{e_1, \ldots, e_p\}$. We denote the endpoints of e_k by i_k and j_k. For a symmetric $n \times n$ matrix $A = [a_{ij}]$, we denote by $\operatorname{vec}_H(A)$ the $p \times 1$ vector whose kth coordinate is $a_{i_k j_k}$. Thus $\operatorname{vec}_H(A)$ makes a vector out of the elements of A corresponding to the edges in H. The matrix E_{ij} denotes the $n \times n$ matrix with a 1 in the i, j-position and 0 elsewhere, and K_{ij} denotes the $n \times n$ skew-symmetric matrix $E_{ij} - E_{ji}$. The *complement* $\overline{G}$ of G is the graph with the same vertex set as G and edges exactly where G does not have edges. The next theorem is used to determine whether a matrix has SSP.

Theorem 2.5. [4, Theorem 31] *Let G be a graph, let $A \in \mathcal{S}(G)$ and let p be the number of edges in $\overline{G}$. Then A has SSP if and only if the $p \times \binom{n}{2}$ matrix whose columns are $\operatorname{vec}_{\overline{G}}(AK_{ij} - K_{ij}A)$ for $1 \leq i < j \leq n$ has rank p.*

Lemma 2.6 (Augmentation Lemma)**.** [3, Lemma 7.5] *Let G be a graph on vertices $\{1, \ldots, n\}$ and $A \in \mathcal{S}(G)$. Suppose A has SSP and λ is an eigenvalue of A with multiplicity $k \geq 1$. Let α be a subset of $\{1, \ldots, n\}$ of cardinality $k + 1$ with the property that for every eigenvector $\mathbf{x}$ of A corresponding to λ, $|\operatorname{supp}(\mathbf{x}) \cap \alpha| \geq 2$. Construct H from G by appending vertex $n + 1$ adjacent exactly to the vertices in α. Then there exists a matrix $B \in \mathcal{S}(H)$ such that B has SSP, the multiplicity of λ has increased from k to $k + 1$, and other eigenvalues and their multiplicities are unchanged from those of A.*

The Augmentation Lemma is usually applied to a specific matrix where the eigenvectors can be determined (as in Section 6). However, it is also possible to apply it without a specific matrix as is done in the next corollary.

Corollary 2.7. *Suppose G is a graph, each vertex of G has at least two neighbors, and H is constructed from G by adding a new vertex adjacent to every vertex of G. If $A \in \mathcal{S}(G)$ has SSP and $\mathbf{m}(A) = (m_1, \ldots, m_r)$, then for each $j = 1, \ldots, r$ there exists a matrix $B_j \in \mathcal{S}(H)$ such that B_j has SSP, the distinct eigenvalues of B_j are the same as those of A, and $\mathbf{m}(A) = (m_1, \ldots, m_{j-1}, m_j + 1, m_{j+1}, \ldots, m_r)$.*

Proof. We apply the Augmentation Lemma with $\alpha = \{1, \dots, n\}$, so $|\alpha| \geq m_j + 1$. For any vector $\mathbf{x}$, $|\operatorname{supp}(\mathbf{x}) \cap \alpha| = |\operatorname{supp}(\mathbf{x})|$. Suppose $|\operatorname{supp}(\mathbf{x})| = 1$ for some eigenvector $\mathbf{x}$. Let k be the position containing the one nonzero entry of $\mathbf{x}$. Then $A\mathbf{x} = \lambda\mathbf{x}$ implies the kth column of A has at most one nonzero entry, which is impossible since $A \in \mathcal{S}(G)$ and every vertex of G has at least two neighbors. So $|\operatorname{supp}(\mathbf{x}) \cap \alpha| \geq 2$. Then there exists a matrix $B_j \in \mathcal{S}(H)$ with the required properties by the Augmentation Lemma. □

Corollary 2.8. *For $n \geq 4$, $q(K_n - e) = 2$ and there is a matrix $M \in \mathcal{S}(K_n - e)$ with SSP and $\mathbf{m}(M) = \left(\left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n}{2} \right\rfloor\right)$.*

Proof. The graphs $K_4 - e$ and $K_5 - e$ are done in [4], so assume $n \geq 6$. For $n = 2k$, the result follows from joining K_k with $K_k - e$ by Theorem 1.7, which shows there exists a matrix $A \in \mathcal{S}(K_n - e)$ with $\mathbf{m}(A) = (k, k)$. We show that A has SSP, and the result then follows from Corollary 2.7. Note that $A \circ X = O = I \circ X$ implies $X = [x_{ij}]$ has only one symmetrically placed pair of possibly nonzero entries, say $x_{12} = x_{21} = x$. Then $(AX - XA)_{23} = xa_{23}$. Since $a_{23} \neq 0$, $x = 0$ and $X = O$. □

3. Graph products

In this section we compute bounds for q for Cartesian, tensor, and strong products of graphs, and in some cases we determine the value of q for graphs constructed by these products. The Kronecker product of matrices plays a central role in constructing matrices realizing graph parameters for graphs that are products. For $A \in \mathbb{R}^{n \times n}$ and $A' \in \mathbb{R}^{n' \times n'}$, the *Kronecker product* of A and A' is the $nn' \times nn'$ matrix

$$A \otimes A' = \begin{bmatrix} a_{11}A' & a_{12}A' & \cdots & a_{1n}A' \\ a_{21}A' & a_{22}A' & \cdots & a_{2n}A' \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}A' & a_{n2}A' & \cdots & a_{nn}A' \end{bmatrix}.$$

For sets or multisets of real numbers S and T, we define sets or multisets $S + T = \{s + t : s \in S, t \in T\}$ and $ST = \{st : s \in S, t \in T\}$ (for sets duplicates are removed, but for multisets duplicates are left in place). It is well known that $\operatorname{spec}(A \otimes A) = \operatorname{spec}(A) \operatorname{spec}(A')$ (see, for example, [15, Theorem 4.8]); this implies $\operatorname{dev}(A \otimes A) = \operatorname{dev}(A) \operatorname{dev}(A')$.

3.1. Cartesian products.

The Cartesian product of graphs G and G', denoted by $G \,\square\, G'$, has vertex set $V(G) \times V(G')$ and edge set $\{(u, v)(x, y) : u = x \text{ and } vy \in E(G') \text{ or } v = y \text{ and } ux \in E(G)\}$. We present several bounds on the value of q for Cartesian products of graphs that apply when certain hypotheses on the constituent graphs are met.

Proposition 3.1. *Let G_1 and G_2 be graphs. If $q(G_i)$ can be realized by matrices $A_i \in \mathcal{S}(G_i), i = 1, 2$ with $\operatorname{dev}(A_i) = \{1, 2, \dots, q_i(G)\}$, then $q(G_1 \,\square\, G_2) \leq q(G_1) + q(G_2) - 1$.*

Proof. Assume the required A_i exist. We observe that $\operatorname{dev}(A_1 \otimes I + I \otimes A_2) = \operatorname{dev}(A_1) + \operatorname{dev}(A_2) = \{2, \dots, q(G_1) + q(G_2)\}$. Therefore there are $q(G_1) + q(G_2) - 1$ distinct eigenvalues of $(A_1 \otimes I + I \otimes A_2) \in \mathcal{S}(G_1 \,\square\, G_2)$, and so $q(G_1 \,\square\, G_2) \leq q(G_1) + q(G_2) - 1$. □

Since any set of distinct eigenvalues can be realized as the eigenvalues of a path, we have the following result.

Corollary 3.2. *If G is a graph such that $q(G)$ can be realized by a matrix $A \in \mathcal{S}(G)$ with $\operatorname{dev}(A) = \{1, 2, \dots, q(G)\}$, then $q(G \,\square\, P_s) \leq q(G) + s - 1$.*

For $s = 2$, the bound $q(G \,\square\, P_2) \leq 2q(G) - 2$ given in [1, Theorem 6.7] is better than that in Corollary 3.2 when $q(G) = 2$, and the bounds are equal for $q(G) = 3$, but otherwise the bound in Corollary 3.2 is better.

Corollary 3.3. *If G is a graph such that $q(G)$ can be realized by a matrix $A \in \mathcal{S}(G)$ with $\operatorname{dev}(A) = \{1, 2, \dots, q(G)\}$, then $q(G \,\square\, C_n) \leq q(G) + \left\lceil \frac{n}{2} \right\rceil$.*

Proof. Assume the hypotheses. For C_{2k+1} we can realize the ordered multiplicity list (2,...,2,1) with any spectrum by [7]. For C_{2k} we can realize the ordered multiplicity list (2,...,2) with any spectrum by [8] (cited in [1, Lemma 2.7]). □

Proposition 3.4. *Let G and G' be graphs and let d denote the length of the unique shortest path between vertices of distance d in G. If $q(G) = d+1$, then $q(G \,\square\, G') \geq q(G)$.*

Proof. Assume $q(G) = d+1$. Let $v_1, v_{d+1} \in V(G)$ such that $\operatorname{dist}_G(v_1, v_{d+1}) = d$ and let $v_1, v_2, \dots, v_{d+1}$ be the unique shortest path of length d from v_1 to v_{d+1} in G. Then for any $v' \in V(G')$, $(v_1, v'), (v_2, v'), \dots, (v_{d+1}, v')$ is a path of length d in $G \,\square\, H$. It is clear that $\operatorname{dist}_{G \,\square\, H}((v_1, v'), (v_{d+1}, v')) = d$. This path is the unique path of length d since a path involving (v_i, u') for some other $u' \in V(G')$ would be longer and any other path $(v_1, v'), (w_2, v'), \dots, (w_d, v'), (v_{d+1}, v')$ would contradict the uniqueness of the path in G. So by Theorem 1.8, $q(G \,\square\, H) \geq (q(G) - 1) + 1 = q(G)$. □

Corollary 3.5. *For any path P_s on $s \geq 2$ vertices, $q(P_s \,\square\, P_2) = s$.*

Proof. By Proposition 3.4, we have $s \leq q(P_s \,\square\, P_2)$. Observe $P_s \,\square\, P_2$ has a Hamilton cycle of order $2s$, so by Corollary 2.3 we know $q(P_s \,\square\, P_2) \leq s$. Thus, $q(P_s \,\square\, P_2) = s$. □

The matrix $\widehat{C}_s$ obtained from the adjacency matrix of C_s by changing the sign on a pair of symmetrically placed ones is called the *flipped cycle matrix*; note that $\widehat{C}_s$ has every diagonal entry equal to zero. Set $k = \left\lceil \frac{s}{2} \right\rceil$. The distinct eigenvalues of $\widehat{C}_s$ are $\lambda_j = 2\cos\frac{\pi(2j-1)}{s}, j = 1, \dots, k$, each with multiplicity two except that $\lambda_k = -2$ has multiplicity one when s is odd [2].

Proposition 3.6. *Let G be a graph of order t. If there exists a matrix $A \in \mathcal{S}(G)$ such that $q(A) = q(G)$ and $-\operatorname{dev}(A) = \operatorname{dev}(A)$, then $q(C_4 \,\square\, G) \leq q(G) + 1$. If in addition $0 \notin \operatorname{dev}(A)$, then $q(C_4 \,\square\, G) \leq q(G)$.*

Proof. Assume $A \in \mathcal{S}(G)$, $q(A) = q(G)$, and $-\operatorname{dev}(A) = \operatorname{dev}(A)$. Define

$$M = \begin{bmatrix} A & I_t & 0 & -I_t \\ I_t & -A & I_t & 0 \\ 0 & I_t & A & I_t \\ -I_t & 0 & I_t & -A \end{bmatrix},$$

so

$$M^2 = \begin{bmatrix} A^2+2I_t & 0 & 0 & 0 \\ 0 & A^2+2I_t & 0 & 0 \\ 0 & 0 & A^2+2I_t & 0 \\ 0 & 0 & 0 & A^2+2I_t \end{bmatrix}.$$

This implies $\operatorname{dev}(M) \subseteq S := \{\pm\sqrt{\lambda^2+2} : \lambda \in \operatorname{dev}(A)\}$. If $0 \notin \operatorname{dev}(A)$, then $|\{\lambda^2+2 : \lambda \in \operatorname{dev}(A)\}| = \frac{q(A)}{2}$ and $|S| = q(A)$. If $0 \in \operatorname{dev}(A)$, then $|\{\sqrt{\lambda^2+2} : \lambda \in \operatorname{dev}(A)\}| = \frac{q(A)+1}{2}$ and $|S| = q(A)+1$. Observe that $M = \widehat{C}_4 \otimes I_t + D \otimes A$ for $D = \operatorname{diag}(1,-1,1,-1)$. Thus, $M \in \mathcal{S}(C_4 \,\square\, G)$. □

The next result shows that the bound in Proposition 3.6 is tight.

Corollary 3.7. *For $k \geq 1$, $s \geq 2$ and $s \not\equiv 2 \mod 4$,*

- $q(C_4 \,\square\, P_{2k}) = 2k$, *and*
- $q(C_4 \,\square\, C_s) = \left\lceil \frac{s}{2} \right\rceil$.

Proof. We present upper and lower bounds that are equal to the stated value. For the upper bound we apply Proposition 3.6: Use the adjacency matrix A for $G = P_{2k}$, and note that $-\operatorname{spec}(A) = \operatorname{spec}(A)$ and $0 \notin \operatorname{spec}(A)$. Use the flipped cycle matrix $\widehat{C}_s$ for $G = C_s$, and note that $-\operatorname{spec}(\widehat{C}_s) = \operatorname{spec}(\widehat{C}_s)$, and $0 \notin \operatorname{spec}(A)$ if $s \not\equiv 2 \mod 4$. For P_{2k}, Proposition 3.4 provides the lower bound. Since $\operatorname{M}(C_4 \,\square\, C_s) \leq 8$ (this is well known and is immediate from [2, Proposition 2.4 and Corollary 2.8]), Observation 1.3 provides the lower bound for C_s. □

3.2. **Tensor products.** The *tensor product* of graphs G and G', denoted $G \times G'$, has vertex set $V(G) \times V(G')$ and edge set $\{(u,u')(v,v') : uv \in E(G) \text{ and } u'v' \in E(G')\}$.

Remark 3.8. For $s \geq 2$, the graph $P_s \times P_2$ is two (disjoint) copies of P_s, so $q(P_s \times P_2) = s$.

Proposition 3.9. *Let G and G' be connected graphs. Let $A = [a_{ij}] \in \mathcal{S}(G)$ with a zero diagonal and $A' = [a'_{ij}] \in \mathcal{S}(G')$ with a zero diagonal. Then $A \otimes A' \in \mathcal{S}(G \times G')$.*

Proof. Let $u \in V(G)$ and $u' \in V(G')$. Then, the vertices of $G \times G'$ are (u,u') and the edges are $(u,u')(v,v')$ where uv and $u'v'$ are edges in G and G', respectively. Since $a_{uu} = a'_{u'u'} = 0$, a_{uv} and $a'_{u'v'}$ are both nonzero if and only if $uv \in E(G)$ and $u'v' \in E(G')$. Thus, $(A \otimes A') \in \mathcal{S}(G \times G')$. □

Proposition 3.10. *Let G be a graph. If there exists $A \in \mathcal{S}(G)$ such that the diagonal of A is zero, $q(A) = q(G)$, and $-\operatorname{dev}(A) = \operatorname{dev}(A)$, then $q(C_4 \times G) \leq q(G)$. In particular:*

(1) $q(C_4 \times P_s) = s$.

(2) $q(C_4 \times C_4) = 2$.
(3) $q(C_4 \times C_{2k}) \le k$.

Proof. Assume the hypotheses. Define

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & A & 0 & -A \\ A & 0 & A & 0 \\ 0 & A & 0 & A \\ -A & 0 & A & 0 \end{bmatrix}, \text{ so } M^2 = \begin{bmatrix} A^2 & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & A^2 & 0 \\ 0 & 0 & 0 & A^2 \end{bmatrix}.$$

This implies $\operatorname{dev} M \subseteq \operatorname{dev}(A) \cup (-\operatorname{dev}(A)) = \operatorname{dev}(A)$, so $q(M) = q(A)$. Let $B = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$. Then $M = B \otimes A \in \mathcal{S}(C_4 \times G)$ and $q(C_4 \times G) \le q(G)$.

Since $\operatorname{spec}(A) = -\operatorname{spec}(A)$ for A the adjacency matrix of P_s or C_{2k}, $q(C_4 \times P_s) \le s$ and $q(C_4 \times C_{2k}) \le k$. The specific results then follow from the general upper bound just established, and that $C_4 \times P_s$ has a unique shortest path on s vertices and $q(C_4) = 2$. □

The next result gives a bound on the tensor product of two paths. Since it is known that a path can be realized with any distinct spectrum, it would be reasonable to ask for a spectrum that behaves well under products, e.g., $\{1, 2, 4, \ldots, 2^{k-1}\}$ for $k = s, t$. However, much less is known about what spectra can be realized by paths assuming a zero diagonal. It is not true that a path can be realized with any spectrum and zero diagonal, because the sum of the eigenvalues must be zero.

Proposition 3.11. *For the tensor product of paths,*

$$\min\{s,t\} \le q(P_s \times P_t) \le \begin{cases} \frac{ts}{2} & \text{for } s,t \text{ even} \\ \frac{(t-1)s}{2} + 1 & \text{for } s \text{ even}, t \text{ odd} \\ \frac{(t-1)(s-1)}{2} + 1 & \text{for } s,t \text{ odd} \end{cases}$$

Proof. The lower bound is a direct application of Theorem 1.8.

For the upper bound, note that for paths the adjacency matrix achieves q. We can find the eigenvalues of $P_s \times P_t$ by multiplying all possible pairs of eigenvalues from the adjacency matrices for P_s and P_t. As a path is bipartite, the adjacency eigenvalues of the path are symmetric about zero. We then count the eigenvalues.

If s and t are both even, we have $\frac{t}{2}$ positive eigenvalues of P_t and since the s eigenvalues of P_s are symmetric about zero, we have at most $\frac{ts}{2}$ distinct eigenvalues for $P_s \times P_t$.

If s is even and t is odd, then there are $\frac{t-1}{2}$ distinct positive eigenvalues of P_t and s non-zero eigenvalues of P_s. Thus, we have at most $\frac{(t-1)s}{2}$ distinct nonzero eigenvalues. Since t is odd, P_t contains a zero eigenvalue, and so does $P_s \times P_t$. Therefore we add 1 to our bound.

If s and t are odd, then there are $\frac{t-1}{2}$ distinct positive eigenvalues of P_t and $s-1$ non-zero eigenvalues of P_s. Thus we have at most $\frac{(t-1)(s-1)}{2}$ distinct nonzero eigenvalues. Since t is odd, P_t contains a zero eigenvalue, and so does $P_s \times P_t$. Therefore we add 1 to our bound. □

3.3. **Strong products.** The *strong product* of graphs G and G', denoted $G \boxtimes G'$, has vertex set $V(G \boxtimes G') = V(G) \times V(G')$ and edge set

$$\begin{aligned} E(G \boxtimes G') &= \{(u,u')(v,v') : u = v \text{ and } u'v' \in E(G')\} \\ &\quad \cup \{(u,u')(v,v') : u' = v' \text{ and } uv \in E(G)\} \\ &\quad \cup \{(u,u')(v,v') : u'v' \in E(G') \text{ and } uv \in E(G)\}. \end{aligned}$$

That is, $E(G \boxtimes G') = E(G \times G') \cup E(G \,\square\, G')$. Note that the *strong* in strong product has no connection with the *strong* in Strong Multiplicity Property (or Strong Spectral Property).

Proposition 3.12. *Let $A \in \mathcal{S}(G)$ and $A' \in \mathcal{S}(G')$ with both having every diagonal entry nonzero. Then $A \otimes A' \in \mathcal{S}(G \boxtimes G')$.*

Proof. Let D_A denote the matrix containing the diagonal of A and similarly for $D_{A'}$. We observe that

$$\begin{aligned} A \otimes A' &= (A - D_A + D_A) \otimes (A' - D_{A'} + D_{A'}) \\ &= (A - D_A) \otimes (A' - D_{A'}) + (A - D_A) \otimes D_{A'} + \\ &\quad D_A \otimes (A' - D_{A'}) + D_A \otimes D_{A'}. \end{aligned}$$

Observe that $(A - D_A) \otimes (A' - D_{A'})$ gives the edges of $G \times G'$ by Proposition 3.9. The edges $G \,\square\, G'$ are given by $(A - D_A) \otimes D_{A'} + D_A \otimes (A' - D_{A'})$. We note that the Cartesian and tensor products of graphs have no common edges, so there is no cancellation, and that adding the preceding matrices gives us the off-diagonal nonzero pattern of $G \boxtimes G'$. Adding $D_A \otimes D_{A'}$ will not affect the off-diagonal pattern. Therefore, $A \otimes A' \in \mathcal{S}(G \boxtimes G')$. □

Proposition 3.13. *Let G be a graph. If $A \in \mathcal{S}(G)$, every diagonal entry of A is nonzero, $q(A) = q(G)$, and ($-\operatorname{dev}(A) = \operatorname{dev}(A)$ or $0 \in \operatorname{dev}(A)$), then $q(G \boxtimes P_2) \leq q(G)$.*

Proof. Assume $A \in \mathcal{S}(G)$, every diagonal entry of A is nonzero, and $q(A) = q(G)$. If $-\operatorname{dev}(A) = \operatorname{dev}(A)$, choose $B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, so $\operatorname{spec}(B) = \{-1, 1\}$. If $0 \in \operatorname{dev}(A)$, choose $B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so $\operatorname{spec}(B) = \{0, 1\}$. Then, $A \otimes B \in \mathcal{S}(G \boxtimes P_2)$ and $\operatorname{dev}(A \otimes B) = \operatorname{dev}(A)\operatorname{dev}(B) = \operatorname{dev}(A)$. Therefore $q(G \boxtimes P_2) \leq q(A \otimes B) \leq q(G)$. □

Proposition 3.14. *Let $A \in \mathcal{S}(G)$ with every diagonal entry nonzero such that $q(A) = q(G)$ and $\operatorname{dev}(A) = -\operatorname{dev}(A)$. Then*

$$q(G \boxtimes P_3) \leq \begin{cases} q(G) + 1 & \text{if } 0 \notin \operatorname{dev}(A) \\ q(G) & \text{if } 0 \in \operatorname{dev}(A) \end{cases}.$$

Proof. We may realize the spectrum $\{-1, 0, 1\}$ for P_3 with the matrix

$$B = \begin{bmatrix} -\frac{5}{6}\sqrt{\frac{3}{5}} & -\frac{5}{6}\sqrt{\frac{3}{5}} & 0 \\ -\frac{5}{6}\sqrt{\frac{3}{5}} & \frac{1}{2}\sqrt{\frac{3}{5}} & \frac{2}{3}\sqrt{\frac{3}{5}} \\ 0 & \frac{2}{3}\sqrt{\frac{3}{5}} & \frac{1}{3}\sqrt{\frac{3}{5}} \end{bmatrix},$$

which has every diagonal entry nonzero. By similar reasoning as in Proposition 3.13, $\operatorname{dev}(A \otimes B) = \operatorname{dev}(A) \cup 0$. The upper bound follows immediately. □

Corollary 3.15. $q(P_3 \boxtimes P_3) = 3$.

Proof. We observe that $3 \leq q(P_3 \boxtimes P_3)$ since the diagonal vertices in $P_3 \boxtimes P_3$ have a unique shortest path of length 2. Furthermore, $q(P_3 \boxtimes P_3) \leq 3$ by Proposition 3.14. □

The next result is worse for odd paths than Proposition 3.14 because Theorem 2.4 does not apply when a zero eigenvalue is desired.

Proposition 3.16. *For $s' \geq s \geq 2$,*

$$s \leq q(P_s \boxtimes P_{s'}) \leq \begin{cases} s + s' - 2 & \text{for } s, s' \text{ even} \\ s + s' - 1 & \text{otherwise} \end{cases}.$$

Proof. With the vertices of P_s and $P_{s'}$ labeled by $\{1, \ldots, s\}$ and $\{1, \ldots, s'\}$, there is a unique shortest path in $P_s \boxtimes P_{s'}$ between vertices $(1, 1)$ and (s, s), so $s \leq q(P_s \otimes P_{s'})$. By Theorem 2.4, for any $\lambda_1, \ldots, \lambda_n$, there is a matrix $B \in \mathcal{S}(P_n)$ and $\operatorname{spec}(B) = \{\lambda_1, \ldots, \lambda_n\}$. Choose $A \in \mathcal{S}(P_s)$ with $\operatorname{spec}(A) = \{1, 2, \ldots, 2^{s-1}\}$ and $A' \in \mathcal{S}(P_{s'})$ with $\operatorname{spec}(A) = \{1, 2, \ldots, 2^{s'-1}\}$. Then $\operatorname{dev}(A \otimes A') = \{1, 2, \ldots, 2^{s+s'-2}\}$, so $q(A \otimes A') = s + s' - 1$. In the case s and s' are both even, choose $\operatorname{spec}(A) = \{\pm 1, 2, \ldots, \pm 2^{s/2-1}\}$ and $A' \in \mathcal{S}(P_{s'})$ with $\operatorname{spec}(A) = \{\pm 1, \pm 2, \ldots, \pm 2^{s'/2-1}\}$. Then $\operatorname{dev}(A \otimes A') = \{\pm 1, \pm 2, \ldots, \pm 2^{s/2+s'/2-2}\}$, so $q(A \otimes A') \leq s + s' - 2$. □

4. Other graph operations

In this section we present results for block-clique graphs and for joins.

4.1. **Block Clique-Graphs.** Let $G = (V, E)$ and $G' = (V', E')$ be graphs. The *union* of G and G' is the graph $G \cup G' = (V \cup V', E \cup E')$. If $V \cap V' = \emptyset$, then the union is disjoint and can be denoted by $G \dot{\cup} G'$. If $V \cap V' \neq \emptyset$, then the *intersection* of G and G' is the graph $G \cap G' = (V \cap V', E \cap E')$. If $V \cap V' = \{v\}$, then $G \cup G'$ is called the *vertex sum* of G and G' and can be denoted by $G \oplus_v G'$; in this case v is called the *summing vertex.* A block-clique graph is constructed from cliques by a sequence of vertex sums. In this section we establish the value of q for two families of block-clique graphs, clique-paths and clique-stars, which we define below.

Definition 4.1. For $s \geq 2$ and $n_{s_i} \geq 2$ for $i = 1, \ldots, s$, we define a graph $KP(n_1, n_2, \ldots, n_s)$, called a *clique-path*, to be a graph constructed by vertex sums using distinct summing vertices and cliques $K_{n_1}, K_{n_2}, \ldots, K_{n_s}$ in order.

Definition 4.2. For $s \geq 2$ and $n_{s_i} \geq 2$ for $i = 1, \ldots, s$, we define a graph $KS(n_1, n_2, \ldots, n_s)$, called a *clique-star*, to be a graph constructed by vertex sums using only one summing vertex and cliques $K_{n_1}, K_{n_2}, \ldots, K_{n_s}$. The vertex that is in every clique is called the *center* and every other vertex is called *noncentral.*

Of course, $KP(n_1, n_2) = KS(n_1, n_2)$.

Theorem 4.3. *For $s \geq 2$ and $n_i \geq 2, i = 1, \ldots, s$, $q(KP(n_1, n_2, \ldots, n_s)) = s + 1$.*

Proof. We observe that there is a unique shortest path between the first summing vertex and the last summing vertex. We can extend this path by 2 vertices, one in K_{n_1} and one in K_{n_s} to find a unique path of length s. Thus, $q(KP(n_1, n_2, \ldots, n_s)) \geq s+1$ by Theorem 1.8.

For the reverse inequality, number the vertices of K_{n_i} consecutively in order of the cliques, with the first summing vertex in K_{n_i} as first and the second summing vertex in K_{n_i} last among the vertices of K_{n_i} for $i = 2, \ldots, n-1$; the summing vertex of K_{n_1} is last and the summing vertex of K_{n_s} is first among vertices in these cliques. Then the matrix

$$A = \begin{bmatrix} J_{n_1-1} & \mathbb{1}_{n_1-1} & & & & & \\ \mathbb{1}^T_{n_1-1} & 2 & \mathbb{1}^T_{n_2-2} & 1 & & & \\ & \mathbb{1}_{n_2-2} & J_{n_2-2} & \mathbb{1}_{n_2-2} & & & \\ & 1 & \mathbb{1}^T_{n_2-2} & 2 & \ddots & & \\ & & & \ddots & \ddots & & \\ & & & & & 2 & \mathbb{1}^T_{n_s-1} \\ & & & & & \mathbb{1}_{n_s-1} & J_{n_s-1} \end{bmatrix} \in \mathcal{S}(KP(n_1,, \ldots, n_s)).$$

Since $\operatorname{rank} A = s$, $q(KP(n_1, n_2, \ldots, n_s)) \leq s+1$ by Theorem 1.2. ☐

Theorem 4.4. *For all $s \geq 2$ and $n_{s_i} \geq 2, i = 1, \ldots, s$, the clique-star $G := KS(n_1, n_2, \ldots, n_s)$ has $q(G) = 3$.*

Proof. Let $\ell_i = n_i - 1$ (the cardinality of the set of noncentral vertices of K_{n_i}), $n = 1 + \sum_{i=1}^{s} \ell_i$ (the order of G), and number the noncentral vertices of K_{n_i} consecutively in order of the cliques, with the center last (vertex n). There is a unique path of length two from any noncentral vertex in one K_{n_i} to any noncentral vertex in another K_{n_j} $(j \neq i)$ through the center vertex n, so $q(G) \geq 3$ by Theorem 1.8.

Define $\widetilde{J}_k = \frac{1}{k} J_k$, $\widetilde{\mathbb{1}}_k = \frac{1}{k}\mathbb{1}_k$, and

$$A = \begin{bmatrix} \widetilde{J}_{\ell_1} & & & \widetilde{\mathbb{1}}_{\ell_1} \\ & \ddots & & \vdots \\ & & \widetilde{J}_{\ell_s} & \widetilde{\mathbb{1}}_{\ell_s} \\ \widetilde{\mathbb{1}}^T_{\ell_1} & \cdots & \widetilde{\mathbb{1}}^T_{\ell_s} & \sum_{i=1}^{s} \frac{1}{\ell_i} \end{bmatrix} \in \mathcal{S}(G).$$

We show that $q(A) = 3$, implying $q(G) = 3$.

Observe that $A(n) = \widetilde{J}_{\ell_1} \oplus \cdots \oplus \widetilde{J}_{\ell_s}$. We can construct A from $A(n)$ by taking the sum of one row associated with each K_{n_i} to form a new last row, and then adding the corresponding columns of this $n \times (n-1)$ matrix to form a new last column. Thus $\operatorname{rank} A = \operatorname{rank} A(n) = s$, which implies $\operatorname{mult}_A(0) = n - s$. Since $\mathbf{x}_i := [\mathbf{0}^T, \ldots, \mathbf{0}^T, \widetilde{\mathbb{1}}^T_{\ell_i}, \mathbf{0}^T, \ldots, \mathbf{0}^T]^T$ is an eigenvector for eigenvalue 1 of $A(n)$, $\operatorname{mult}_{A(n)}(1) \geq s$. By interlacing (Theorem 1.1), $\operatorname{mult}_A(1) \geq s-1$, so $\operatorname{mult}_A(0) + \operatorname{mult}_A(1) \geq n-1$. Since $q(A) \geq 3$, there is exactly one more eigenvalue (necessarily different from 0 and 1 and of multiplicity one) and $q(A) = 3$. ☐

4.2. **Joins.** The *join* of disjoint graphs $G = (V, E)$ and $G' = (V', E')$, which is denoted by $G \vee G'$, has vertex set $V \cup V'$ and edge set $E \cup E' \cup \{vv' : v \in V, v' \in V'\}$. It was shown in [1] that $q(\overline{K_n} \vee \overline{K_n}) = 2$ and $q(\overline{K_n} \vee \overline{K_m}) = 3$ for $n \neq m$ (see Theorem 1.6). Monfared and Shader showed in [11] that $q(G \vee H) = 2$ for connected graphs G and H of the same order (see Theorem 1.7). The next example shows that a join can require an arbitrarily large number of distinct eigenvalues.

Example 4.5. Since $P_s \vee K_1$ has an induced P_s, $\operatorname{mr}(P_s \vee K_1) \geq \operatorname{mr}(P_s) = s - 1$, and since $P_s \vee K_1$ is not a path, $\operatorname{mr}(P_s \vee K_1) \leq s - 1$. Thus $\operatorname{M}(P_s \vee K_1) = 2$, which implies $q(P_s \vee K_1) \geq \left\lceil \frac{s+1}{2} \right\rceil$ by Observation 1.3. Since $P_s \vee K_1$ has a Hamilton cycle, $q(P_s \vee K_1) \leq \left\lceil \frac{s+1}{2} \right\rceil$ by Corollary 2.3.

Theorem 4.6. *Let G and G' be connected graphs such that $|V(G)| = n$ and $|V(G')| = n - \ell$ for some $1 \leq \ell < n$. Then $q(G \vee G') \leq 2 + \ell$.*

Proof. Create a graph G'' by adding new vertices $v_1, \ldots, v_\ell$ to G' and adding some combination of possible edges involving these vertices to make G'' connected. Then $G \vee G'$ is a subgraph of $G \vee G''$. By Theorem 1.7 we have $q(G \vee G'') = 2$ and there is a matrix $A \in \mathcal{S}(G \vee G'')$ with two eigenvalues each of multiplicity n. Then $\lambda_1(A) = \cdots = \lambda_n(A) < \lambda_{n+1}(A) = \cdots = \lambda_{2n}(A)$. By deleting rows and columns of A corresponding to the new vertices $v_1, \ldots, v_\ell$, we obtain a principal submatrix $B \in \mathcal{S}(G \vee G')$. Then by eigenvalue interlacing (Theorem 1.1), we have

$$\lambda_1(A) \leq \lambda_1(B) \leq \cdots \leq \lambda_{n-\ell}(B) \leq \lambda_n(A) = \lambda_1(A)$$

$$\lambda_1(A) = \lambda_{n-\ell+1}(A) \leq \lambda_{n-\ell+1}(B) \leq \cdots \leq \lambda_n(B) \leq \lambda_{n+\ell}(A) = \lambda_{2n}(A)$$

$$\lambda_{n+1}(A) \leq \lambda_{n+1}(B) \leq \cdots \leq \lambda_{2n-\ell}(B) \leq \lambda_{2n}(A) = \lambda_{n+1}(A).$$

This gives us $\lambda_1(A) = \lambda_1(B) = \cdots = \lambda_{n-\ell}(B)$ and $\lambda_{n+1}(A) = \lambda_{n+1}(B) = \cdots = \lambda_{2n-\ell}(B)$. The remaining ℓ eigenvalues are bounded such that $\lambda_1(A) \leq \lambda_{n-\ell+1}(B) \leq \cdots \leq \lambda_n(B) \leq \lambda_{2n}(A)$. Therefore $q(G \vee G') \leq 2 + \ell$. □

5. Summary of the impact of graph operations

In this section we provide some new examples illustrating the impact of graph operations on q and summarize what is known about the impact of other operations. If we say that an operation $\cdot$ on two graphs G and H raises q, this means that $q(G \cdot H) > \max\{q(G), q(H)\}$. Saying that $\cdot$ on G and H lowers q means that $q(G \cdot H) < \min\{q(G), q(H)\}$, whereas saying $\cdot$ maintains q means that $q(G \cdot H) = q(G) = q(H)$. The meaning of *raises, lowers*, and *maintains* is clear when the operation is on a single graph.

It is clear from Theorem 1.7 that the join operation is capable of decreasing q; for example, $q(P_n \vee P_n) = 2$ but $q(P_n) = n$. Of course, the join can also maintain q. To see that the join can raise q, define the *dth hypercube* recursively by $Q_1 = P_2$ and $Q_d = Q_{d-1} \,\square\, P_2$. The vertices of Q_d are written as strings of zeros and ones of length d, and two vertices are adjacent if and only if they differ in exactly one place.

Proposition 5.1. *The join can raise the value of q, because $q(Q_5 \vee P_2) \geq 3$.*

Proof. The set of vertices $S = \{00000, 00111, 11110\}$ in Q_5 is an independent set of $Q_5 \vee P_2$. The only common neighbors of these vertices are $v_1, v_2 \in V(P_2)$. That is,

$$\left| \bigcup_{u,w \in S, u \neq w} (N(u) \cap N(w)) \right| = |\{v_1, v_2\}| = 2 < 3 = |S|.$$

By the contrapositive of Theorem 1.9, we have $q(Q_5 \vee P_2) \neq 2$ and therefore $q(Q_5 \vee P_2) \geq 3$. Note that $q(Q_5) = 2$ [1, Corollary 6.9] and $q(P_2) = 2$. □

Let $G = (V, E)$ be a graph. For $e \in E$, the notation $G - e$ means the result of deleting edge e from G. For $v \in V$, the notation $G - v$ means the result of deleting v and all edges incident with v. The *contraction* of edge $e = uv$ of G, denoted by G/e, is obtained from G by identifying the vertices u and v, deleting a loop if one arises in this process, and replacing any multiple edges by a single edge. The subdivision of edge $e = uv$ of G, denoted by G_e, is the graph obtained from G by deleting e and inserting a new vertex w adjacent exactly to u and v.

Examples are given in [1] showing that the difference between $q(G)$ and $q(G - v)$ and the difference between $q(G)$ and $q(G - e)$ can grow arbitrarily large in either direction as a function of the number of vertices. The construction of a main example can be done with vertex sums. Let x and y be two nonadjacent vertices of C_4, and denote the other two vertices by w and z. Suppose also that x is an endpoint of one P_{k+1} and y is an endpoint of another P_{k+1}. The graph $P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$ is denoted by $S_{k,k}$ in [1] and it is shown there that $q(S_{k,k}) = k + 2$ [1, Lemma 6.6].

Remark 5.2. Deleting the midpoint of P_{2k+1} creates $2P_k$ and lowers q. Deleting a vertex from K_n creates K_{n-1} and maintains q. Deleting the vertex z from $G = P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$ results in a path on $2k + 3$ vertices. Since $q(G) = k + 2$ and $q(P_{2n+3}) = 2k + 3$, the deletion of z has raised q.

Remark 5.3. Deleting the middle edge from P_{2k} creates $2P_k$ and lowers q. Deleting an edge from K_n creates $K_n - e$ and maintains q for $n \geq 4$ (see Proposition 2.8). Deleting the edge xz from $G = P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$ results in $S(k + 2, k, 1)$, a path with an extra leaf. Since $q(G) = k + 2$ and $q(S(k + 2, k, 1)) = 2k + 3$, the deletion of xz has raised q.

Remark 5.4. Contracting an edge of P_n creates P_{n-1} and lowers q. Contracting an edge of K_n creates K_{n-1} and maintains q (for $n \geq 3$). Contracting the edge $e = xz$ of $G = P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$ results in a P_{2k+2} with 3 cycle. Thus $q(G) = k + 2$ and $q(G/e) = 2k + 2$, raising q.

Remark 5.5. Subdividing the edge $\{k + 1, k + 3\}$ of $GB(k, n - k - 3)$ creates $P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$ and lowers q. Subdividing an edge of C_{2k+1} maintains q because $q(C_{2k+1}) = k+1 = q(C_{2k+2})$. Subdividing a cycle edge of $P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$ creates a unique shortest path on $2k + 3$ vertices and raises q.

Table 1 summarizes the possible effect on q of various graph operations.

TABLE 1. Possible effects on q of various graph operations. A column headed # gives the result # that describes the example illustrating lowering, maintaining, or raising q.

Operation	Lower q	#	Maintain q	#	Raise q	#
Join	$P_n \vee P_n, n \geq 3$	1.7	$P_2 \vee P_2$	1.7	$Q_5 \vee P_2$	5.1
Cartesian Product			$P_s \,\square\, P_2$	3.5		
Tensor Product			$C_4 \times P_s$	3.10	$K_3 \times P_2 = C_6$	1.5
Strong Product			$P_3 \boxtimes P_3$	3.15		
Vertex Sum			$KS(n,n,n)$	4.4	$KP(3,3)$	4.3
Vertex Deletion	P_n	5.2	K_n	5.2	C_n	1.5
Edge Deletion	P_n	5.3	K_n	5.3	C_n	1.5
Edge Contract	P_n	5.4	K_n	5.4	$P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$	5.4
Edge Subdivide	$GB(k, n-k-3)$	5.5	C_{2k+1}	5.5	$P_{k+1} \oplus_x C_4 \oplus_y P_{k+1}$	5.5

TABLE 2. Values of q for graphs of order at most 5 (using the graph numbering in [14]). All values of q can be determined from the information in [3, Figure 1].

$G\#$	$q(G\#)$	$G\#$	$q(G\#)$	$G\#$	$q(G\#)$	$G\#$	$q(G\#)$	$G\#$	$q(G\#)$
$G1$	1	$G3$	2	$G6$	3	$G7$	2	$G13$	3
$G14$	4	$G15$	3	$G16$	2	$G17$	2	$G18$	2
$G29$	3	$G30$	4	$G31$	5	$G34$	3	$G35$	4
$G36$	4	$G37$	3	$G38$	3	$G40$	3	$G41$	3
$G42$	3	$G43$	3	$G44$	3	$G45$	3	$G46$	3
$G47$	3	$G48$	2	$G49$	2	$G50$	2	$G51$	2
$G52$	2								

6. VALUES OF q FOR GRAPHS OF ORDER AT MOST 6

The IEPG has been solved for all connected graphs of order at most 4 in [5] and order 5 in [3]. Solution of the IEPG establishes the value of q; the results for all connected graphs of order at most 5 are summarized in Table 2. In this section we apply our previous results and additional ideas to determine q for all connected graphs of order 6 (see Table 3). Note that Ahn, Alar, Bjorkman, Butler, Carlson, Goodnight, Harris, Knox, Monroe, and Wigal have recently determined all possible ordered multiplicity lists for graphs of order 6; most of their work is independent but in a few cases they cite results from this paper.

Note that if a graph G is disconnected with connected components $G_i, i = 1, \ldots, c$ then $q(G) = \max\{q(G_i) : i = 1, \ldots, c\}$ and the solution to the IEPG for G can be deduced immediately from the solutions for each G_i, so data is customarily provided only for connected graphs. All graphs are numbered using the notation in *Atlas of Graphs* [14].

We begin by establishing ordered multiplicity lists attaining the minimum value of q that are attainable with SSP or SMP for some specific graphs. We then apply those results to determine q for other graphs by using Observation 2.2. In many cases there is more than one way to establish the result, and in a few cases (most notably $K_6 = G208$) the result is already known. However, we have grouped graphs by a subgraph having a matrix with SMP (or SSP, which implies SMP) for efficiency. We begin with graphs having $q(G) = 3$. Oblak and Šmigoc [12, Example 4.8] provide the matrix M_{96} in the next lemma and state its spectrum $\{-1, -1, 0, 0, 2, 2\}$.

Lemma 6.1. *The matrix*

$$M_{96} = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -1 & 0 \end{bmatrix}$$

has SSP, $\mathbf{m}(M_{96}) = (2, 2, 2)$*, and* $M_{96} \in \mathcal{S}(G96)$*. Furthermore,* $q(G96) = q_M(G96) = 3$*.*

Proof. It can be verified by computation that M_{96} has SSP (see [10], where Theorem 2.5 is applied to M_{96}). Since there is a unique shortest path on three vertices, $q(G96) = q_M(G96) = 3$. □

Corollary 6.2. *The following graphs G have $q(G) = q_M(G) = 3$: $G111$, $G114$, $G118$, $G121$, $G126$, $G133$, $G135$, $G136$, $G137$, $G140$, $G141$, $G144$, $G145$, $G149$, $G150$, $G156 - G159$, $G161 - 167$, $G169 - 173$, $G177 - 180$, $G182 - 185$, $G193$.*

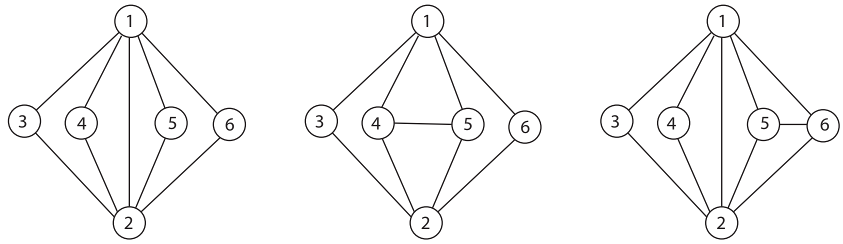
Proof. Each graph G has $G96$ as a spanning subgraph, so by Lemma 6.1 and Observation 2.2, $q(G) \le q_M(G96) = 3$. With three exceptions, each G has a unique shortest path on three vertices, and so has $q(G) = 3$ by Theorem 1.8.

The exceptions are $G161$, $G170$, and $G179$. In each of these cases we exhibit a set of independent vertices without enough common neighbors, so $q(G) \ne 2$ by Theorem 1.9. The vertices are numbered as in Figure 1.
$G161$: The set $\{3, 4, 5, 6\}$ is an independent set of four vertices, but the union of neighborhood intersections is $\{1, 2\}$.
$G170$: The set $\{3, 4, 6\}$ is an independent set of three vertices, but the union of neighborhood intersections is $\{1, 2\}$.
$G179$: The set $\{3, 4, 5\}$ is an independent set of three vertices, but the union of neighborhood intersections is $\{1, 2\}$. □

Remark 6.3. Since each of the graphs $G = G105, G127, G147, G148, G151 - G153$ has a Hamilton cycle and each has a unique shortest path on three vertices, $q(G) = q_M(G) = 3$.

Lemma 6.4. *The graph $G125$ has a matrix with SSP and ordered multiplicity list* $(2, 2, 2)$*.*

Proof. Observe that graph $G125$ can be constructed by adding a new vertex 6 adjacent to vertices 2 and 3 of the Banner $= G37$ (see Figure 2). It can be verified



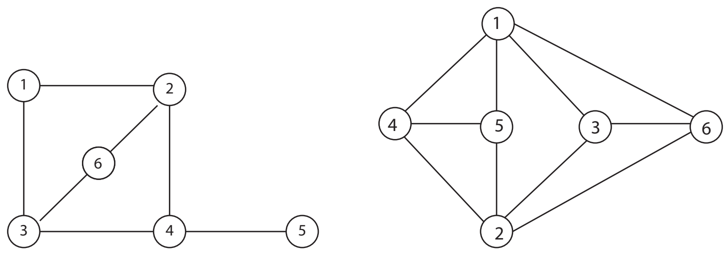
$G161$ $G170$ $G179$

FIGURE 1. Graphs to which the Theorem 1.9 is applied in the proof of Corollary 6.2

by computation (see [10]) that Goodnight's matrix [9]

$$M = \begin{bmatrix} \frac{4}{3} & -\sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 & 0 & \frac{2}{3} & 0 \\ \sqrt{\frac{2}{3}} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\ 0 & 0 & 0 & \frac{2}{3} & 0 \end{bmatrix} \in \mathcal{S}(G37)$$

has SSP and eigenvalues $\mu_1 = -2/3$, $\mu_2 = 0$, and $\mu_3 = 2$ with multiplicities 2, 1, 2, respectively, so the ordered multiplicity list of M is $(2,1,2)$. Furthermore, the vector $\mathbf{x} = [0, -\frac{1}{2}, -\frac{1}{2}, 0, 1]^T$ is a basis for the eigenspace of eigenvalue $\mu_2 = 0$. Since $\operatorname{supp}(\mathbf{x}) = \{2,3,5\}$, $|\operatorname{supp}(\mathbf{x}) \cap \{2,3\}| = 2$. Therefore, we can apply the Augmentation Lemma (Lemma 2.6) to obtain a matrix having eigenvalue $\mu_2 = 0$ with multiplicity 2 and also eigenvalues μ_1 and μ_3 each with multiplicity 2. Thus the graph $G125$ has a matrix with SSP and ordered multiplicity list $(2,2,2)$. $\square$



$G125$ $G190$

FIGURE 2. Graphs to which the Augmentation Lemma is applied

Corollary 6.5. *The following graphs G have $q(G) = q_M(G) = 3$: $G138, G143, G160$.*

Proof. Each graph G has $G125$ as a spanning subgraph, so by Lemma 6.4 and Observation 2.2, $q(G) \leq q_M(G96) = 3$. Since each has a unique shortest path on three vertices, each has $q(G) = q_M(G) = 3$ by Theorem 1.8. $\square$

Lemma 6.6. *The graph $G129$ has a matrix with SMP and ordered multiplicity list $(2,2,2)$. Furthermore, $q(G129) = q_M(G129) = 3$.*

Proof. Observe that graph $G129$ can be constructed by adding a new vertex adjacent to two nonadjacent vertices v and w of $C_5 = G38$. In [4, Theorem 48] it was shown that $\widehat{C}_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix}$ has SMP. The eigenvalues of $\widehat{C}_5$ are $\mu_1 = -2$, $\mu_2 = \frac{1}{2}\left(1-\sqrt{5}\right)$, and $\mu_3 = \frac{1}{2}\left(1+\sqrt{5}\right)$ with ordered multiplicity list $(1,2,2)$. Furthermore, the vector $[1,-1,1,-1,1]^T$ is a basis for the eigenspace of eigenvalue $\mu_1 = -2$. Thus it is not possible for an eigenvector $\mathbf{x}$ for μ_1 to have a zero entry, so $|\operatorname{supp}(\mathbf{x}) \cap \{v,w\}| = 2$. Therefore, we can apply the Augmentation Lemma to obtain a matrix having eigenvalue $\mu_1 = -2$ with multiplicity 2 and also eigenvalues μ_2 and μ_3 each with multiplicity 2. Thus the graph $G129$ has a matrix with SMP and ordered multiplicity list $(2,2,2)$. Since $G129$ has a unique shortest path on three vertices, $q(G129) = q_M(G129) = 3$. □

Remark 6.7. Oblak and Šmigoc show that $G99$ has a matrix with every eigenvalue of even multiplicity [12, Example 3.1] and give a form to construct such a matrix in [12, Theorem 3.1]. One such matrix is M_{99} below. They also provided the matrix M_{115} [13], which they found in their research in preparation for [12].

$$M_{99} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad M_{115} = \begin{bmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ 2 & -3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & -2 & -2 & \sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 \end{bmatrix}.$$

It is straightforward to verify that $\operatorname{spec}(M_{99}) = \left\{-\sqrt{3}, -\sqrt{3}, 0, 0, \sqrt{3}, \sqrt{3}\right\}$ and $\operatorname{spec}(M_{115}) = \left\{-1-2\sqrt{3}, -1-2\sqrt{3}, 0, 0, -1+2\sqrt{3}, -1+2\sqrt{3}\right\}$. Since each of $G99$ and $G115$ has a unique shortest path on three vertices, $q(G99) = 3 = q(G115)$ by Theorem 1.8. Note that no matrix $A \in \mathcal{S}(G99)$ or $A \in \mathcal{S}(G115)$ that has $q(A) = 3$ can have SMP because each is a spanning subgraph of $G134$, which has a unique shortest path on four vertices.

Next we establish that $q(G) = 2$ for various graphs G, starting with some that have SSP. The statement that $q(G)$ can be realized by a matrix with SSP implies $q_M(G) = q(G)$, because SSP implies SMP.

Lemma 6.8. *Each matrix $M_{\#}$ below is orthogonal with SSP, $\mathbf{m}(M_{G\#}) = (3,3)$, and $M_{\#} \in \mathcal{S}(G\#)$ for the graphs $G\# = G174, G186$. Thus $q(G\#) = 2$.*

$$M_{174} = \begin{bmatrix} -\frac{1}{\sqrt{10}} & \sqrt{\frac{2}{5}} & \sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} & 0 & 0 \\ \sqrt{\frac{2}{5}} & -\frac{1}{\sqrt{10}} & \sqrt{\frac{2}{5}} & 0 & \frac{1}{\sqrt{10}} & 0 \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{2}{5}} & -\frac{1}{\sqrt{10}} & 0 & 0 & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & 0 & 0 & \frac{1}{\sqrt{10}} & -\sqrt{\frac{2}{5}} & -\sqrt{\frac{2}{5}} \\ 0 & \frac{1}{\sqrt{10}} & 0 & -\sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} & -\sqrt{\frac{2}{5}} \\ 0 & 0 & \frac{1}{\sqrt{10}} & -\sqrt{\frac{2}{5}} & -\sqrt{\frac{2}{5}} & \frac{1}{\sqrt{10}} \end{bmatrix}.$$

$$M_{186} = \begin{bmatrix} \frac{1}{9}\left(-3-\sqrt{3}\right) & \frac{1}{18}\left(2\sqrt{3}-3\right) & 0 & \frac{1}{3}\sqrt{\frac{23}{6}-\frac{1}{\sqrt{3}}} & -\frac{1}{3} & -\frac{1}{2} \\ \frac{1}{18}\left(2\sqrt{3}-3\right) & \frac{1}{18}\left(-3-4\sqrt{3}\right) & \frac{1}{18}\left(3-2\sqrt{3}\right) & 0 & \frac{2}{3} & -\frac{1}{2} \\ 0 & \frac{1}{18}\left(3-2\sqrt{3}\right) & \frac{1}{9}\left(-3-\sqrt{3}\right) & \frac{1}{3}\sqrt{\frac{23}{6}-\frac{1}{\sqrt{3}}} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3}\sqrt{\frac{23}{6}-\frac{1}{\sqrt{3}}} & 0 & \frac{1}{3}\sqrt{\frac{23}{6}-\frac{1}{\sqrt{3}}} & \frac{1}{9}\left(3+\sqrt{3}\right) & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Corollary 6.9. *The graphs $G = G188, G192, G194, G196 - G208$ have $q(G) = 2$ and the ordered multiplicity list $(3, 3)$ can be realized by a matrix with SSP.*

Proof. Each of the graphs $G188, G196, G198, G199, G202{-}G208$ has $G174$ as a spanning subgraph and each of $G192, G194, G197, G200$, $G201$ has $G186$ as a spanning subgraph. So by Lemma 6.8 and Observation 2.2, $q(G) = 2$. □

Lemma 6.10. *The graph $G190$ has a matrix with SSP and ordered multiplicity list $(3, 3)$.*

Proof. The graph $G190$ is constructed by adding vertex 6 adjacent to $\{1, 2, 3\}$ of $G48$ (see Figure 2). The ordered multiplicity list $(3, 2)$ of $G48$ is realized by the matrix $M = \begin{bmatrix} 1 & 0 & \sqrt{2} & 1 & 1 \\ 0 & 1 & -\sqrt{2} & 1 & 1 \\ \sqrt{2} & -\sqrt{2} & 4 & 0 & 0 \\ 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 \end{bmatrix}$, which has SSP [3, Lemma 3.5]. Furthermore, the vectors $[\frac{1}{2}, \frac{1}{2}, 0, 1, 1]^T$ and $[\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, 1, 0, 0]^T$ are a basis for the eigenspace of eigenvalue 5. Thus, it is not possible for an eigenvector for $\lambda = 5$ to have more than two zero entries, and the only way to achieve two zeros in an eigenvalue for $\lambda = 5$ is to have the zeros in positions 4 and 5. Therefore, $|\operatorname{supp}(\mathbf{x}) \cap \{1, 2, 3\}| \geq 2$, and we can apply the Augmentation Lemma to conclude there is a matrix $B \in \mathcal{S}(G190)$ which has SSP and has eigenvalues $\lambda = 5$ and $\lambda = 0$ each with multiplicity 3. □

Corollary 6.11. *For $G = G195$, $q(G) = 2$ and ordered multiplicity list $(3, 3)$ can be realized by a matrix with SSP.*

Proof. The graph $G195$ has $G190$ as a spanning subgraph, so by Lemma 6.10 and Observation 2.2, $q(G195)$ and ordered multiplicity list $(3, 3)$ can be realized by a matrix with SSP. □

The next result can be verified by computation.

Lemma 6.12. *Each matrix $M_{\#}$ below is orthogonal, $\mathbf{m}(M_{G\#}) = (3,3)$, and $M_{\#} \in \mathcal{S}(G\#)$ for the graphs $G\#$. Thus $q(G\#) = 2$.*

$M_{154} =$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{4}\sqrt{\frac{1}{2}\left(7-\sqrt{33}\right)} & 0 & -\frac{1}{2} & 0 & -\frac{1}{4}\sqrt{\frac{15-\sqrt{33}}{7-\sqrt{33}}} \\ -\frac{1}{4}\sqrt{\frac{1}{2}\left(7-\sqrt{33}\right)} & -\frac{1}{2} & \frac{1}{2}\sqrt{\frac{9-\sqrt{33}}{7-\sqrt{33}}} & 0 & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2}\sqrt{\frac{9-\sqrt{33}}{7-\sqrt{33}}} & \frac{1}{2} & -\frac{1}{4}\sqrt{\frac{1}{2}\left(9-\sqrt{33}\right)} & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{4}\sqrt{\frac{1}{2}\left(9-\sqrt{33}\right)} & -\frac{1}{2} & -\frac{1}{\sqrt{2\left(7-\sqrt{33}\right)}} & 0 \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{\sqrt{2\left(7-\sqrt{33}\right)}} & \frac{1}{2} & \frac{1}{4}\sqrt{\frac{1}{2}\left(15-\sqrt{33}\right)} \\ -\frac{1}{4}\sqrt{\frac{15-\sqrt{33}}{7-\sqrt{33}}} & 0 & 0 & 0 & \frac{1}{4}\sqrt{\frac{1}{2}\left(15-\sqrt{33}\right)} & -\frac{1}{2} \end{bmatrix}$$

$$M_{168} = \begin{bmatrix} -\frac{7}{12} & 0 & \frac{1}{12} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{\sqrt{\frac{5}{6}}}{3} & -\frac{\sqrt{\frac{5}{3}}}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{\sqrt{\frac{10}{3}}}{3} & -\frac{\sqrt{\frac{5}{3}}}{3} \\ \frac{1}{12} & 0 & -\frac{7}{12} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{\sqrt{\frac{5}{6}}}{3} & \frac{\sqrt{\frac{5}{3}}}{3} \\ \frac{\sqrt{\frac{3}{2}}}{2} & 0 & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{\sqrt{\frac{5}{6}}}{3} & \frac{\sqrt{\frac{10}{3}}}{3} & \frac{\sqrt{\frac{5}{6}}}{3} & 0 & \frac{2}{3} & 0 \\ -\frac{\sqrt{\frac{5}{3}}}{3} & -\frac{\sqrt{\frac{5}{3}}}{3} & \frac{\sqrt{\frac{5}{3}}}{3} & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$$M_{181} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{\sqrt{23}}{8} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{8} & -\frac{\sqrt{23}}{8} & 0 \\ 0 & \frac{1}{8} & -\frac{1}{8} & \frac{1}{8}\left(4-\sqrt{23}\right) & \frac{1}{8} & \frac{1}{4}\sqrt{\frac{11}{2}+2\sqrt{23}} \\ 0 & \frac{\sqrt{23}}{8} & -\frac{\sqrt{23}}{8} & \frac{1}{8} & \frac{1}{2}-\frac{1}{8\sqrt{23}} & -\frac{1}{4}\sqrt{\frac{11}{46}+\frac{2}{\sqrt{23}}} \\ 0 & 0 & 0 & \frac{1}{4}\sqrt{\frac{11}{2}+2\sqrt{23}} & -\frac{1}{4}\sqrt{\frac{11}{46}+\frac{2}{\sqrt{23}}} & \frac{3}{\sqrt{23}}-\frac{1}{2} \end{bmatrix}$$

For graphs $G = G154, G168, G181$ and matrix $A \in \mathcal{S}(G)$, if $q(A) = 2$, then A does not have SMP, because in each case it is possible to add an edge to G and obtain a unique shortest path on 3 vertices.

Remark 6.13. As it may be useful for future research, here we briefly describe the method that was used to find the matrices M_{186} and M_{168}. The graph $G186$ has three independent vertices, which we label 4, 5, and 6. Vertices 1, 2, and 3 form a clique missing one edge. All but one of the possible edges between vertices in $\{1,2,3\}$ and $\{4,5,6\}$ are present. Thus we have the form $M_{186} = \begin{bmatrix} A & C \\ C^T & D \end{bmatrix}$ where D is diagonal, C has one zero, $A^T = A$, and A has one pair of symmetrically placed zeros. In order for M_{186} to be orthogonal, we must have $C^TC + D^2 = I$, so the columns of C are orthogonal (but may have different lengths). Then $AC + CD = 0$, so $A = -CDC^{-1}$. The conditions

(i) D is diagonal with distinct diagonal entries strictly between zero and one,
(ii) the columns of C are orthogonal and scaled so that $C^TC + D^2 = I$, and
(iii) $A = -CDC^{-1}$

suffice to ensure $\begin{bmatrix} A & C \\ C^T & D \end{bmatrix}$ is orthogonal. The columns of C can be chosen with a zero in the first column, and one diagonal entry of D can be used as a parameter that is set to achieve the desired pair of zeros in A. The case of M_{168} is similar except that now there are two pairs of zeros in A, and some care must be taken in the choice of the vectors for C.

Next we show the two graphs $G187$ and $G189$ have $q(G) = 3$ by showing they do not allow an orthogonal realization.

Lemma 6.14. *The graph $G187$, the wheel on 6 vertices, does not allow an orthogonal matrix and $q(G) = 3$.*

Proof. Since $G197$ has a Hamilton cycle, $q(G187) \leq 3$ by Corollary 2.3. Showing that $G187$ does not allow an orthogonal matrix completes the proof because $q(G) = 2$ implies G allows an orthogonal matrix by Observation 1.4. We have the following matrix:

$$M = \begin{bmatrix} a & w & 0 & 0 & v & q \\ w & b & x & 0 & 0 & r \\ 0 & x & c & y & 0 & s \\ 0 & 0 & y & d & z & t \\ v & 0 & 0 & z & e & u \\ q & r & s & t & u & f \end{bmatrix}$$

Suppose M is orthogonal, so $M^2 = I$ where $M^2 =$

$$\begin{bmatrix} a^2+q^2+v^2+w^2 & qr+aw+bw & qs+wx & qt+vz & qu+av+ev & aq+fq+uv+rw \\ qr+aw+bw & b^2+r^2+w^2+x^2 & rs+bx+cx & rt+xy & ru+vw & br+fr+qw+sx \\ qs+wx & rs+bx+cx & c^2+s^2+x^2+y^2 & st+cy+dy & su+yz & cs+fs+rx+ty \\ qt+vz & rt+xy & st+cy+dy & d^2+t^2+y^2+z^2 & tu+dz+ez & dt+ft+sy+uz \\ qu+av+ev & ru+vw & su+yz & tu+dz+ez & e^2+u^2+v^2+z^2 & eu+fu+qv+tz \\ aq+fq+uv+rw & br+fr+qw+sx & cs+fs+rx+ty & dt+ft+sy+uz & eu+fu+qv+tz & f^2+q^2+r^2+s^2+t^2+u^2 \end{bmatrix}.$$

We denote the i, j-entry of M^2 by h_{ij}, we know $h_{ij} = 0$ for $i \neq j$, and we apply this repeatedly to specific entries.

$$0 = h_{1,3} = qs + wx \quad \Rightarrow \quad x = -\frac{qs}{w} \tag{6.1}$$

$$0 = h_{2,5} = ru + vw \quad \Rightarrow \quad v = -\frac{ru}{w} \tag{6.2}$$

$$0 = h_{3,5} = su + yz \quad \Rightarrow \quad z = -\frac{su}{y} \tag{6.3}$$

$$0 = h_{3,6} = qs + wx \quad \Rightarrow \quad y = -\frac{qs}{w} \tag{6.4}$$

$$(6.5)\quad 0 = h_{15} = qu + av + ev = 0 \;\Rightarrow\; w = -\frac{(a+e)r}{q} \text{ and } a+e \neq 0$$

$$(6.6)\quad (6.1) \text{ and } (6.5) \Rightarrow x = -\frac{qs}{(a+e)r}$$

$$(6.7)\quad (6.2) \text{ and } (6.5) \;\Rightarrow\; v = -\frac{qu}{a+e}$$

$$(6.8)\quad (6.6) \text{ and } (6.4) \;\Rightarrow\; y = \frac{s}{t}\left(\frac{q^2-(c+f)(a+e)}{a+e}\right)$$

$$(6.9)\quad (6.8) \text{ and } (6.3) \;\Rightarrow\; z = \frac{-tu(a+e)}{q^2-(c+f)(a+e)}$$

$$(6.10)\quad (6.5) \text{ and } 0 = h_{1,2} = qr + aw + bw \;\Rightarrow\; q^2 = -(a+b)(a+e)$$

$$(6.11)\quad (6.5), (6.10), \text{ and } 0 = h_{2,3} = rs + bx + cx \;\Rightarrow\; r^2 = -(a+b)(b+c)$$

$$(6.12)\quad (6.8), (6.10), \text{ and } 0 = h_{3,4} = st + cy + dy \;\Rightarrow\; t^2 = (c+d)(a+b+c+f)$$

$$(6.13)\quad (6.9) \text{ and } 0 = h_{4,5} = tu + dz + ez \;\Rightarrow\; q^2 = (a+e)(c+d+e+f)$$

$$(6.14)\quad (6.10), (6.13), \text{ and } a+e \neq 0 \;\Rightarrow\; a+b+c+d+e+f = 0$$

$$(6.15)\quad (6.5), (6.7), (6.10), (6.12), (6.15), \text{ and } 0 = h_{1,6} = aq + fq + uv + rw! \Rightarrow u^2 = -(a+e)(d+e)$$

$$(6.16)\quad (6.5), (6.6), (6.10), (6.12), (6.15), \text{ and } 0 = h_{2,6} = br + fr + qw + sx! \Rightarrow s^2 = -(c+d)(b+c)$$

We then consider the following chart, which begins with two possible cases for equation (6.10) using $q^2 > 0$. Each of these cases is then applied successively to other equations that require positive values.

#	equation	Case 1	Case 2
(6.10)	$q^2 = -(a+e)(a+b) > 0$	$(a+e) > 0$ and $(a+b) < 0$	$(a+e) < 0$ and $(a+b) > 0$
(6.12)	$r^2 = -(a+b)(b+c) > 0$	$(b+c) > 0$	$(b+c) < 0$
(6.17)	$s^2 = -(c+d)(b+c) > 0$	$(c+d) < 0$	$(c+d) > 0$
(6.13)	$t^2 = -(d+e)(c+d) > 0$	$(d+e) > 0$	$(d+e) < 0$
(6.16)	$u^2 = -(a+e)(d+e) > 0$	$(a+e) < 0$	$(a+e) > 0$

In each case, we find the contradiction that $(a+e) < 0$ and $(a+e) > 0$. □

Lemma 6.15. *Graph $G189$ does not allow an orthogonal matrix and $q(G) = 3$.*

Proof. Since $G96$ is a subgraph of $G189$, $q(G189) \leq 3$ by Observation 2.2. Showing that $G189$ does not allow an orthogonal matrix completes the proof because $q(G) = 2$ implies G allows an orthogonal matrix by Observation 1.4. We have the following matrix:

$$M = \begin{bmatrix} a & q & r & t & 0 & 0 \\ q & b & s & 0 & u & v \\ r & s & c & 0 & w & x \\ t & 0 & 0 & d & y & z \\ 0 & u & w & y & e & 0 \\ 0 & v & x & z & 0 & f \end{bmatrix}.$$

Suppose M is orthogonal, so M^2 is the identity matrix. Observe that M^2 is and

$$\begin{bmatrix} a^2+q^2+r^2+t^2 & aq+bq+rs & ar+cr+qs & at+dt & qu+rw+ty & qv+rx+tz \\ aq+bq+rs & b^2+q^2+s^2+u^2+v^2 & qr+bs+cs+uw+vx & qt+uy+vz & bu+eu+sw & bv+fv+sx \\ ar+cr+qs & qr+bs+cs+uw+vx & c^2+r^2+s^2+w^2+x^2 & rt+wy+xz & su+cw+ew & sv+cx+fx \\ at+dt & qt+uy+vz & rt+wy+xz & d^2+t^2+y^2+z^2 & dy+ey & dz+fz \\ qu+rw+ty & bu+eu+sw & su+cw+ew & dy+ey & e^2+u^2+w^2+y^2 & uv+wx+yz \\ qv+rx+tz & bv+fv+sx & sv+cx+fx & dz+fz & uv+wx+yz & f^2+v^2+x^2+z^2 \end{bmatrix}$$

. .

denote the i,j-entry of M^2 by h_{ij}.

Note that $h_{1,4} = h_{4,5} = h_{4,6} = 0$ implies $a = -d = e = f$. We make these substitutions in M and M^2 becomes Denote the i,j-entry of this matrix by k_{ij}. We

$$\begin{bmatrix} a^2+q^2+r^2+t^2 & aq+bq+rs & ar+cr+qs & 0 & qu+rw+ty & qv+rx+tz \\ aq+bq+rs & b^2+q^2+s^2+u^2+v^2 & qr+bs+cs+uw+vx & qt+uy+vz & au+bu+sw & av+bv+sx \\ ar+cr+qs & qr+bs+cs+uw+vx & c^2+r^2+s^2+w^2+x^2 & rt+wy+xz & su+aw+cw & sv+ax+cx \\ 0 & qt+uy+vz & rt+wy+xz & a^2+t^2+y^2+z^2 & 0 & 0 \\ qu+rw+ty & au+bu+sw & su+aw+cw & 0 & a^2+u^2+w^2+y^2 & uv+wx+yz \\ qv+rx+tz & av+bv+sx & sv+ax+cx & 0 & uv+wx+yz & a^2+v^2+x^2+z^2 \end{bmatrix}.$$

. . . . . .

know $k_{ij} = 0$ for $i \neq j$, and we apply this repeatedly to specific entries.

$$\text{(6.17)}\quad 0 = k_{1,2} = aq+bq+rs \text{ and } 0 = k_{1,3} = ar+cr+qs \;\Rightarrow\; q^2 = \frac{(a+c)}{(a+b)}r^2$$

$$\text{(6.18)}\quad 0 = k_{1,2} = aq+bq+rs \text{ and } 0 = k_{1,3} = ar+cr+qs \;\Rightarrow\; s^2 = (a+b)(a+c)$$

$$\text{(6.19)}\quad 0 = k_{2,6} = av+bv+sx \text{ and } 0 = k_{3,6} = sv+ax+cx \;\Rightarrow\; v^2 = \frac{(a+c)}{(a+b)}x^2$$

$$\text{(6.20)}\quad 0 = k_{2,5} = au+bu+sw \text{ and } 0 = k_{3,5} = su+aw+cw \;\Rightarrow\; u^2 = \frac{(a+c)}{(a+b)}w^2$$

¿From (6.17) and (6.18), $q = \pm\sqrt{\frac{(a+c)}{(a+b)}}r$ and $s = \pm\sqrt{(a+b)(a+c)}$. If q and s are both positive roots or both negative roots,

$$0 = k_{1,2} = aq+bq+rs \quad\Rightarrow\quad \sqrt{2(a+b)\sqrt{a+c}} = 0,$$

which is a contradiction. Therefore q and s must be roots of opposite sign. Similarly, we can see v and u must be the opposite sign of s as well. Therefore we have the following two cases.

Case 1: For the first case, we let $s = \sqrt{(a+b)(a+c)}$, $q = -\sqrt{\frac{(a+c)}{(a+b)}}r$, $v = -\sqrt{\frac{(a+c)}{(a+b)}}x$, and $u = -\sqrt{\frac{(a+c)}{(a+b)}}w$.

size

$$0 = k_{1,5} = qu + rw + ty, \tag{6.21}$$

$$(6.17),\ \text{and}\ (6.20) \quad \Rightarrow \quad y = \frac{-rw(2a+b+c)}{t(a+b)}$$

$$0 = k_{1,6} = qv + rx + tz, \tag{6.22}$$

$$(6.17),\ \text{and}\ (6.19) \quad \Rightarrow \quad z = \frac{-rx(2a+b+c)}{t(a+b)}$$

$$0 = k_{5,6} = uv + wx + yz, \tag{6.23}$$

$$(6.22),\ \text{and}\ (6.23) \quad \Rightarrow \quad t = \sqrt{-\frac{(2a+b+c)^2 r^2}{(a+c)(a+b)+(a+b)^2}}$$

Equation (6.24) yields a contradiction since t is imaginary.

Case 2: For the second case, we let $s = -\sqrt{(a+b)(a+c)}$, $q = \sqrt{\frac{(a+c)}{(a+b)}}r$, $v = \sqrt{\frac{(a+c)}{(a+b)}}x$, and $u = \sqrt{\frac{(a+c)}{(a+b)}}w$.

We observe that the same equations result from case 2 as in case 1 and we obtain the same contradiction. □

Finally we establish the value of q for the few remaining graphs.

Remark 6.16. It is well known that the path is the only graph for which $q(G) = |V(G)|$ (see [1, Proposition 3.1]). Thus $q(G83 = P_6) = 6$. It is shown in [1] that $G = S(k-1, n-k-1, 1)$ and $G = GB(k, n-k-3)$ have $q(G) = n-1$. Since $G80 = S(2,2,1)$ and $G81 = S(3,1,1)$, $q(G80) = 5 = q(G81)$. Since $G97 = GB(2,1)$ and $G102 = GB(3,0)$, $q(G97) = 5 = q(G102)$. By Theorem 1.10, every graph other than $G80, G81, G83, G97, G102$ has $q(G) \leq 4$. For $G = G78$, $G79$, $G93$, $G94$, $G95$, $G98$, $G100$, $G103$, $G104$, $G112$, $G113$, $G119$, $G120$, $G122 - G124$, $G130, G134, G139, G142$, G has a unique shortest path on 4 vertices, so $q(G) = 4$.

We have now established q for all graphs of order six. For each graph, a reason is given.

Theorem 6.17. *Tables 3 lists the value of q for each connected graph of order six.*

TABLE 3. Values of q for graphs of order 6 using the graph numbering in [14]. A column headed # gives the result # that justifies the corresponding $q(G\#)$.

$G\#$	$q(G\#)$	#	$G\#$	$q(G\#)$	#	$G\#$	$q(G\#)$	#
$G77$	3	4.4	$G78$	4	6.16	$G79$	4	6.16
$G80$	5	6.16	$G81$	5	6.16	$G83$	6	6.16
$G92$	3	4.4	$G93$	4	6.16	$G94$	4	6.16
$G95$	4	6.16	$G96$	3	6.1	$G97$	5	6.16
$G98$	4	6.16	$G99$	3	6.7	$G100$	4	6.16
$G102$	5	6.16	$G103$	4	6.16	$G104$	4	6.16
$G105$	3	6.3	$G111$	3	6.2	$G112$	4	6.16
$G113$	4	6.16	$G114$	3	6.2	$G115$	3	6.7
$G117$	3	4.4	$G118$	3	6.2	$G119$	4	6.16
$G120$	4	6.16	$G121$	3	6.2	$G122$	4	6.16
$G123$	4	6.16	$G124$	4	6.16	$G125$	3	6.4
$G126$	3	6.2	$G127$	3	6.3	$G128$	3	3.5
$G129$	3	6.6	$G130$	4	6.16	$G133$	3	6.2
$G134$	4	6.16	$G135$	3	6.2	$G136$	3	6.2
$G137$	3	6.2	$G138$	3	6.5	$G139$	4	6.16
$G140$	3	6.2	$G141$	3	6.2	$G142$	4	6.16
$G143$	3	6.5	$G144$	3	6.2	$G145$	3	6.2
$G146$	3	1.6	$G147$	3	6.3	$G148$	3	6.3
$G149$	3	6.2	$G150$	3	6.2	$G151$	3	6.3
$G152$	3	6.3	$G153$	3	6.3	$G154$	2	6.12
$G156$	3	6.2	$G157$	3	6.2	$G158$	3	6.2
$G159$	3	6.2	$G160$	3	6.5	$G161$	3	6.2
$G162$	3	6.2	$G163$	3	6.2	$G164$	3	6.2
$G165$	3	6.2	$G166$	3	6.2	$G167$	3	6.2
$G168$	2	6.12	$G169$	3	6.2	$G170$	3	6.2
$G171$	3	6.2	$G172$	3	6.2	$G173$	3	6.2
$G174$	2	6.8	$G175$	2	1.6	$G177$	3	6.2
$G178$	3	6.2	$G179$	3	6.2	$G180$	3	6.2
$G181$	2	6.12	$G182$	3	6.2	$G183$	3	6.2
$G184$	3	6.2	$G185$	3	6.2	$G186$	2	6.8
$G187$	3	6.14	$G188$	2	6.9	$G189$	3	6.15
$G190$	2	6.10	$G191$	3	4.3	$G192$	2	6.9
$G193$	3	6.2	$G194$	2	6.9	$G195$	2	6.11
$G196$	2	6.9	$G197$	2	6.9	$G198$	2	6.9
$G199$	2	6.9	$G200$	2	6.9	$G201$	2	6.9
$G202$	2	6.9	$G203$	2	6.9	$G204$	2	6.9
$G205$	2	6.9	$G206$	2	6.9	$G207$	2	6.9
$G208$	2	6.9						

7. Values of q for families of graphs

The next table summarizes known values of $q(G)$.

Table 4. Values of q for families of graphs

Graph G	$q(G)$	Reason
K_n	2	[1, Lemma 2.2]
C_n	$\lceil \frac{n}{2} \rceil$	[1, Lemma 2.7]
P_n	n	[1, Proposition 3.1]
$K_{n,m}$	$\begin{cases} 2, & \text{if } m = n \\ 3, & \text{if } m < n \end{cases}$	[1, Corollary 6.5]
Q_n	2	[1, Corollary 6.9]
$GB(k, n-k-3)$	$n-1$	[1, Proposition 7.1]
$S(k-1, n-k-1, 1)$	$n-1$	[1, Proposition 7.2]
$\lvert V(G) \rvert \leq 6$		Tables 2 and 3
$KP(n_1, n_2, \dots, n_s)$ for $s \geq 2, n_i \geq 2$	$s+1$	Theorem 4.3
$KS(n_1, n_2, \dots, n_s)$ for $s \geq 2, n_i \geq 2$	3	Theorem 4.4
$P_s \,\square\, P_2$	s	Corollary 3.5
$C_4 \,\square\, P_{2s}$	$2s$	Corollary 3.7
$C_4 \,\square\, C_s$ for $s \geq 4$ & $s \not\equiv 2 \mod 4$	$\lceil \frac{s}{2} \rceil$	Corollary 3.7
$P_s \times P_2$	s	Corollary 3.8
$C_4 \times P_s$	s	Proposition 3.10
$P_3 \boxtimes P_3$	3	Corollary 3.15
$P_s \vee K_1$	$\lceil \frac{s+1}{2} \rceil$	Example 4.5
$K_n - e$	2	Corollary 2.8

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