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NEWTON-TYPE METHODS FOR FRITZ JOHN SYSTEMS OF GENERALIZED NASH EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we derive sufficient conditions for the local fast convergence of certain Newton-type methods applied to a nonsmooth reformulation of the Fritz John (FJ) system associated to a generalized Nash equilibrium problem (GNEP) with shared constraints. To this end, related recent results based on the reformulation of the Karush-Kuhn-Tucker system of a GNEP are extended. Moreover, we show that the sufficient conditions are generically satisfied for GNEPs defined by twice continuously differentiable problem functions. Compared to a local convergence result for an existing Newton-type method applied to another nonsmooth reformulation of the FJ system, no strict complementarity is needed.

1. Introduction

Let us consider a generalized Nash equilibrium problem (GNEP) with $N \geq 2$ players and with shared constraints. Player $\nu \in \{1, \ldots, N\}$ aims at solving the minimization problem

$$\theta_\nu(x^\nu, x^{-\nu}) \to \min_{x^\nu} \quad \text{s.t.} \quad g(x^\nu, x^{-\nu}) \leq 0 \tag{1.1}$$

and controls the variables $x^\nu \in \mathbb{R}^{n_\nu}$, whereas the vector $x^{-\nu}$ concatenates the variables of the rival players. Moreover, with $n := n_1 + \cdots + n_N$, the functions $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are assumed to be twice continuously differentiable. To emphasize the ν-th player's variables within a strategy vector $x \in \mathbb{R}^n$ we sometimes, as in (1.1), use the typical notation $(x^\nu, x^{-\nu})$ instead of x. A strategy vector $x^* \in \mathbb{R}^n$ is called *generalized Nash equilibrium* or simply *solution* of the GNEP if, for every $\nu \in \{1, \ldots, N\}$, $x^{*,\nu}$ is a solution of (1.1) with $x^{-\nu}$ replaced by $x^{*,-\nu}$. For further information on GNEPs, we refer to [10, 12].

It is well-known that a solution of the GNEP, together with some multipliers, satisfies the associated Karush-Kuhn-Tucker (KKT) system if a suitable constraint qualification holds. According to this, several approaches for determining a solution of a GNEP were developed that are based on solving the KKT system, see [5, 6, 7, 11, 17], for example. However, there are meaningful cases, where a solution of a GNEP

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does not provide a solution of its KKT system, see [4, Section 2]. Therefore, the authors of [4] suggest to use the Fritz John (FJ) system instead. Moreover, based on a reformulation of the FJ system as a nonsmooth system of equations, they analyzed properties of the solution set of this system in a generic sense. Furthermore, they suggested the use of the Newton-type method from [1, 2] for solving the nonsmooth system and, based on [21], showed that, generically, this method is well-defined in some neighborhood of any FJ point and converges superlinearly to a solution provided that the FJ point satisfies strict complementarity.

In this paper, we consider another nonsmooth reformulation of the FJ system. As first step, we will derive sufficient conditions for local superlinear convergence of some Newton-type methods applied to this reformulation. In a second step, we will show that these conditions are generically satisfied. In this way, we do not only obtain other methods for solving the FJ system. Rather, since these methods (the LP-Newton method and the constrained Levenberg-Marquardt method, to be precise) are known to possess quite strong local convergence properties [8, 9, 13], local fast convergence is guaranteed under conditions that do neither imply the local uniqueness of solutions nor the differentiability at solutions. Therefore, in contrast to the Newton-type method in [4], no strict complementarity condition is needed.

Let us now provide the basic details needed to fully understand and to accomplish these objectives.

For any fixed $x^{-\nu}$, the FJ conditions of the optimization problem (1.1) are given by

$$\begin{aligned}
&\xi_\nu \nabla_{x^\nu}\theta_\nu(x) + \nabla_{x^\nu} g(x)\lambda^\nu = 0,\\
&\xi_\nu + 1_m^\top \lambda^\nu - 1 = 0,\\
&\lambda^\nu \geq 0,\ g(x) \leq 0,\ (\lambda^\nu)^\top g(x) = 0,\\
&\xi_\nu \geq 0
\end{aligned}$$

with a multiplier vector $\lambda^\nu \in \mathbb{R}^m$, a multiplier $\xi_\nu \in \mathbb{R}$, and $1_m \in \mathbb{R}^m$ denoting the vector consisting of ones only. Concatenating these FJ conditions for all players, yields the FJ system associated to the GNEP

$$\begin{aligned}
&\Psi(x,\xi,\lambda) = 0,\\
&\xi + E\lambda - 1_N = 0,\\
&\lambda \geq 0,\ g(x) \leq 0,\ (\lambda^\nu)^\top g(x) = 0 \quad (\nu = 1,\dots,N),\\
&\xi \geq 0,
\end{aligned} \tag{1.2}$$

where $\lambda \in \mathbb{R}^{Nm}$ denotes the column vector concatenating $\lambda^1,\dots,\lambda^N$, $\xi := (\xi_1,\dots,\xi_N)^\top$, and $E := \operatorname{block}[1_m^\top]_{\nu=1}^N \in \mathbb{R}^{N\times Nm}$. Furthermore, the function $\Psi : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nm} \to \mathbb{R}^n$ is given by

$$\Psi(x,\xi,\lambda) := \Theta(x)\xi + B(x)\lambda$$

with

$$\Theta(x) := \operatorname{block}\left[\nabla_{x^\nu}\theta_\nu(x)\right]_{\nu=1}^N, \quad B(x) := \operatorname{block}\left[\nabla_{x^\nu} g(x)\right]_{\nu=1}^N.$$

In order to shorten the notation, we use block$[\cdot]$ to denote a block diagonal matrix, i.e., block$[A_\nu]_{\nu=1}^N$ is the block diagonal matrix with the (in general rectangular) matrices $A_1,\dots,A_N$ on the main diagonal. In contrast to the KKT system associated to the GNEP, the FJ system uses the additional nonnegative multipliers $\xi_1,\dots,\xi_N$.

In particular, it is well-known that, without a constraint qualification, for any solution x^* of the GNEP, multipliers ξ^* and λ^* exist such that (x^*, ξ^*, λ^*) solves the FJ system (1.2). For further discussion on this subject, we refer to [4].

Let us further briefly explain what is meant by a "generically satisfied condition". Each tuple $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m)$ of functions defines a GNEP. Suppose that the space $C^2(\mathbb{R}^n)$ of all real-valued, twice continuously differentiable functions is endowed with the *Whitney topology* (see [4] and references therein for the definition of this topology). Moreover, let the product space

$$\mathcal{D} := \prod_{\nu=1}^{N} \underbrace{C^2(\mathbb{R}^n)}_{\theta_\nu} \times \prod_{i=1}^{m} \underbrace{C^2(\mathbb{R}^n)}_{g_i} \tag{1.3}$$

of GNEP defining functions be endowed with the product Whitney topology. A condition (on a GNEP or the corresponding FJ system) is said to be *generically satisfied* if there is, regarding the Whitney topology, an open and dense subset $\widehat{\mathcal{D}}$ of $\mathcal{D}$ such that the condition in question holds for any $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m) \in \widehat{\mathcal{D}}$.

The paper is organized as follows. In Section 2, after recalling the LP-Newton method and the constrained Levenberg-Marquardt method for the solution of constrained piecewise continuously differentiable (PC^1) systems of equations, we recapitulate one of the main local convergence results from [13] showing that both methods converge locally quadratically to a solution if some set of local error bound conditions is satisfied and if the pieces are sufficiently smooth. A suitable reformulation of the FJ system (1.2) as a constrained PC^1-system is presented in Section 3. Furthermore, the set of local error bound conditions mentioned above is discussed for the resulting system. In particular, assuming sufficient smoothness of the GNEP defining functions, we will show that the full row rank of a certain matrix at some fixed solution guarantees local quadratic convergence of our methods without requiring strict complementarity or the local uniqueness of solutions. Finally, in Section 4, we will prove that there is an open and dense subset of the set of GNEP defining functions $\mathcal{D}$ such that for every element of that subset the full row rank condition is satisfied at all solutions of (1.2). The latter implies that, generically, local superlinear convergence of the above methods to a solution of (1.2) is guaranteed.

Let us close this section with some words on our notation. Throughout, by $\|\cdot\|$, an arbitrary but fixed vector norm is denoted, where the space it acts on is clear from the context. With respect to this norm, the closed ball with center $y \in \mathbb{R}^l$ and radius $\varepsilon > 0$ is indicated by $\mathcal{B}_\varepsilon(y)$. Further, the distance of a point $y \in \mathbb{R}^l$ to a nonempty set $\mathcal{S} \subseteq \mathbb{R}^l$ is denoted by $\operatorname{dist}[y, \mathcal{S}] := \inf\{\|y - \zeta\| \mid \zeta \in \mathcal{S}\}$. For $\mathcal{J} \subseteq \{1, \ldots, l\}$, the vector $y_\mathcal{J} \in \mathbb{R}^{|\mathcal{J}|}$ consists of all components of y whose indices belong to $\mathcal{J}$. Similarly, for a function $T : \mathbb{R}^l \to \mathbb{R}^\ell$ and $\mathcal{J} \subseteq \{1, \ldots, \ell\}$, the function $T_\mathcal{J} : \mathbb{R}^l \to \mathbb{R}^{|\mathcal{J}|}$ is defined componentwise with components $T_j : \mathbb{R}^l \to \mathbb{R}$ for $j \in \mathcal{J}$. If T is continuously differentiable, we indicate the Jacobian of T at some point y by $T'(y)$ and its transpose by $\nabla T(y)$. In case y is split according to $y = (y_1, y_2)$, then $T'_{y_1}(y)$ (and $\nabla_{y_1} T(y_1, y_2)$) denotes the Jacobian (the transpose of the Jacobian) of T at y with respect to y_1 only. If T is locally Lipschitz continuous, the generalized Jacobian $\partial T(y)$ of T at y is a nonempty compact set given by the convex hull

of $\{\lim_{k\to\infty} T'(y^k) \mid \lim_{k\to\infty} y^k = y,\ T \text{ is differentiable at } y^k \text{ for all } k \in \mathbb{N}\}$, see [3]. Finally, let I_l denote the identity matrix in $\mathbb{R}^{l\times l}$.

2. NEWTON-TYPE METHODS FOR THE SOLUTION OF PC^1-SYSTEMS

Let us first recall the LP-Newton method [9] and the constrained Levenberg-Marquardt method [18] for solving the constrained system of equations

$$F(z) = 0 \quad \text{s.t.} \quad z \in \Omega. \tag{2.1}$$

Here, we assume that $\Omega \subseteq \mathbb{R}^s$ is a nonempty and closed set and that $F : \mathbb{R}^s \to \mathbb{R}^t$ is a PC^1-function. Later in this section, a general complementarity system is reformulated as PC^1-system (2.1). Moreover, a local convergence result from [13] for applying the above Newton-type methods to this constrained PC^1-system is given.

Since the Newton-type methods are based on some generalized derivative of F, let us recall that a function $F : \mathbb{R}^s \to \mathbb{R}^t$ is called *piecewise continuously differentiable* (PC^1) if F is continuous and if there is a finite number of continuously differentiable selection functions $F^1, \ldots, F^\ell : \mathbb{R}^s \to \mathbb{R}^t$ such that

$$F(z) \in \left\{F^1(z), \ldots, F^\ell(z)\right\}$$

holds for all $z \in \mathbb{R}^s$. Based on this, let $G : \mathbb{R}^s \to \mathbb{R}^{t\times s}$ be a function satisfying

$$G(z) \in \left\{(F^j)'(z) \mid j \in \mathcal{A}(z)\right\} \tag{2.2}$$

for all $z \in \mathbb{R}^s$, where $\mathcal{A}(z)$ denotes the index set of all selection functions $F^1, \ldots, F^\ell$ being active at z, i.e.,

$$\mathcal{A}(z) := \{j \in \{1, \ldots, \ell\} \mid F^j(z) = F(z)\}.$$

The LP-Newton method proposed in [9] for the solution of (2.1) is an iterative method. For a given iterate $z^k \in \Omega$ which is not yet a solution of (2.1), the new iterate z^{k+1} has to be determined as the z-part of a solution of the optimization problem

$$\begin{aligned} \gamma \to \min_{z,\gamma} \qquad \text{s.t.} \qquad & \|F(z^k) + G(z^k)(z - z^k)\| \le \gamma \|F(z^k)\|^2, \\ & \|z - z^k\| \le \gamma \|F(z^k)\|,\ z \in \Omega. \end{aligned} \tag{2.3}$$

It is easy to verify that, for any z^k, (2.3) has a solution [9, Proposition 1]. Therefore, the LP-Newton method is well-defined for any starting point $z^0 \in \Omega$. Moreover, if Ω is polyhedral and $\|\cdot\|$ is the infinity norm, the subproblems (2.3) are linear programs.

The constrained Levenberg-Marquardt method for the solution of (2.1) was introduced in [18]. For a given iterate $z^k \in \Omega$ which is not yet a solution of (2.1), the new iterate z^{k+1} has to be determined as a solution of the optimization problem

$$\|F(z^k) + G(z^k)(z - z^k)\|^2 + \|F(z^k)\|^2 \|z - z^k\|^2 \to \min_z \quad \text{s.t.} \quad z \in \Omega. \tag{2.4}$$

By means of the Weierstrass theorem, one can easily check that this subproblem has a solution for any z^k. If Ω is polyhedral and $\|\cdot\|$ is the Euclidean norm, then (2.4) is a quadratic program with a strongly convex objective function and, hence, has a unique solution for any $z^k \in \Omega$. Therefore, the constrained Levenberg-Marquardt method is well-defined for any starting point $z^0 \in \Omega$.

In [9], a set of assumptions is provided under which local quadratic convergence of the LP-Newton method to some solution of (2.1) is shown for any starting point in a sufficiently small neighborhood of a fixed solution z^*, see [9, Theorem 1]. The same assumptions guarantee local quadratic convergence for the constrained Levenberg-Marquardt method, see [8, Theorem 3]. In [13], the set of assumptions from [9] is discussed in detail for some suitable reformulation of complementarity systems

$$a(z) = 0, \quad b(z) \geq 0, \quad c(z) \geq 0,\ d(z) \geq 0,\ c(z)^\top d(z) = 0, \tag{2.5}$$

where the functions $a : \mathbb{R}^s \to \mathbb{R}^p$, $b : \mathbb{R}^s \to \mathbb{R}^q$, and $c, d : \mathbb{R}^s \to \mathbb{R}^r$ are assumed to be differentiable with locally Lipschitz continuous derivatives. More precisely, the following equivalent reformulation of (2.5) as a constrained PC^1-system of equations is considered in [13]:

$$F(z) := \begin{pmatrix} a(z) \\ \min\{c(z), d(z)\} \end{pmatrix} = 0 \quad \text{s.t.} \quad z \in \Omega, \tag{2.6}$$

where the minimum is taken componentwise and Ω is given by

$$\Omega := \{z \in \mathbb{R}^s \mid b(z) \geq 0,\ c(z) \geq 0,\ d(z) \geq 0\}. \tag{2.7}$$

Then, one of the main results in [13] shows that the convergence assumptions from [9] are satisfied at some arbitrary but fixed solution z^* of (2.6) if a set of local error bound conditions, called piecewise error bound condition in [13], is valid at z^*. Therefore, taking into account the local convergence analysis of the LP-Newton method and the constrained Levenberg-Marquardt method in [9] and [8], the piecewise error bound condition is sufficient for local quadratic convergence of these methods. The latter condition neither implies the local uniqueness nor the strict complementarity of solutions of (2.5). Recall that strict complementarity is violated at some solution $\bar{z}$ of the complementarity system (2.5) if there is at least one $i \in I := \{1, \ldots, r\}$ with $c_i(\bar{z}) = d_i(\bar{z}) = 0$. In that case, F from (2.6) is not differentiable at $\bar{z}$.

Let us review the result from [13] mentioned above including consequences for the local convergence of the LP-Newton method and the constrained Levenberg-Marquardt method applied to (2.6) with Ω defined in (2.7). To this end, suppose that z^* is an arbitrary but fixed solution of (2.5) and let us introduce the following index sets:

$$\begin{aligned} I_c &:= \{i \in I \mid 0 = c_i(z^*) < d_i(z^*)\}, \\ I_d &:= \{i \in I \mid 0 = d_i(z^*) < c_i(z^*)\}, \\ I_0 &:= \{i \in I \mid 0 = c_i(z^*) = d_i(z^*)\}. \end{aligned}$$

Furthermore, for every partition $\mathcal{I} := (I_1, I_2)$ of I_0, we denote by $Z_{\mathcal{I}}$ the solution set of the following system of equations and inequalities:

$$\begin{aligned} a(z) = 0, \qquad & b(z) \geq 0, \\ c_{I_c \cup I_1}(z) = 0, \quad & c_{I_d \cup I_2}(z) \geq 0, \quad d_{I_d \cup I_2}(z) = 0, \quad d_{I_c \cup I_1}(z) \geq 0. \end{aligned} \tag{2.8}$$

If the solution set of (2.5) is denoted by Z, then, for every partition $\mathcal{I}$ of I_0, we have

$$\{z^*\} \subseteq Z_{\mathcal{I}} \subseteq Z,$$

in particular the set $Z_{\mathcal{I}}$ is nonempty. Condition 1 below is [13, Condition 5] and requires that, for every partition of I_0, a local error bound condition for system (2.8) is satisfied at z^*.

Condition 1. *There are $\delta_1 > 0$ and $\omega_1 > 0$ such that*

$$\begin{aligned}\operatorname{dist}[z, Z_{\mathcal{I}}] \;\leq\; & \omega_1 \left(\|a(z)\| + \|\min\{0, b(z)\}\| + \|c_{I_c \cup I_1}(z)\| + \|d_{I_d \cup I_2}(z)\| \right. \\ & \left. + \|\min\{0, c_{I_d \cup I_2}(z)\}\| + \|\min\{0, d_{I_c \cup I_1}(z)\}\|\right)\end{aligned}$$

holds for all partitions $\mathcal{I} = (I_1, I_2)$ of I_0 and all $z \in \mathcal{B}_{\delta_1}(z^*)$.

Now, the subsequent theorem follows by [13, Theorem 3] together with [9, Theorem 1] and [8, Theorem 3].

Theorem 2.1. *Let Condition 1 be satisfied. Then, there is $\rho > 0$ such that $z^0 \in \mathcal{B}_\rho(z^*) \cap \Omega$ implies that any sequence generated by the LP-Newton method with subproblems* (2.3) *or by the constrained Levenberg-Marquardt method with subproblems* (2.4) *either terminates after finitely many steps with a solution of* (2.6) *or converges quadratically to a solution of* (2.6).

Remark 2.2. If any of the functions b, c, d is nonlinear, the set Ω given by (2.7) is generally not a polyhedron so that determining a solution of the subproblems (2.3) and (2.4) of the LP-Newton method and the constrained Levenberg-Marquardt method can be difficult from a computational point of view. Therefore, the introduction of slack variables and the application of the methods to the following system is advisable:

$$\widetilde{F}(z, u, v, w) := \begin{pmatrix} a(z) \\ \min\{v, w\} \\ b(z) - u \\ c(z) - v \\ d(z) - w \end{pmatrix} \quad \text{s.t.} \quad (z, u, v, w) \in \widetilde{\Omega} \tag{2.9}$$

with $\widetilde{\Omega} := \mathbb{R}^s \times \mathbb{R}^q_+ \times \mathbb{R}^{2r}_+$. The local quadratic convergence of the LP-Newton method and the constrained Levenberg-Marquardt method applied to (2.9) is preserved if Condition 1 is satisfied. This follows by [13, Proposition 8] and the discussion at the end of [13, Section 5]. In fact, the local quadratic convergence is even kept if slacks are introduced for the nonlinear components of the functions b, c, d only.

3. NEWTON-TYPE METHODS FOR THE SOLUTION OF FJ SYSTEMS

In this section, we first present a suitable reformulation of the FJ system (1.2) associated to the GNEP as constrained PC^1-system of equations. Afterwards, we rephrase Condition 1 and Theorem 2.1 in the context of the FJ system and discuss conditions implying Condition 1. The results of this section are in analogy to those in [13, Section 7], where Condition 1 is discussed for KKT systems associated to GNEPs.

Obviously, the FJ system (1.2) is a complementarity system, i.e., it can be written in the form (2.5) with $p := n+N$, $q := N$, $r := Nm$, $s := n+N+Nm$, $z := (x, \xi, \lambda)$,

and

$$a(z) := \begin{pmatrix} \Psi(z) \\ \xi + E\lambda - 1_N \end{pmatrix}, \quad b(z) := \xi,$$
$$c(z) := \big(-g(x)\big)_{\nu=1}^{N}, \qquad d(z) := \lambda = \big(\lambda^{\nu}\big)_{\nu=1}^{N}.$$

Hence, the corresponding PC^1-system (2.6) reads as

$$F(z) = \begin{pmatrix} \Psi(z) \\ \xi + E\lambda - 1_N \\ \min\{-g(x), \lambda^1\} \\ \vdots \\ \min\{-g(x), \lambda^N\} \end{pmatrix} = 0 \quad \text{s.t.} \quad z \in \Omega \tag{3.1}$$

with

$$\Omega = \{z = (x, \xi, \lambda) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nm} \mid \xi \geq 0,\ \lambda \geq 0,\ g(x) \leq 0\}. \tag{3.2}$$

To rephrase Condition 1 within the special context of the FJ system (1.2) let $z^* = (x^*, \xi^*, \lambda^*)$ be an arbitrary but fixed solution of (1.2). Based on this, let us introduce several index sets. At first, we define

$$\mathcal{A} := \{i \in \{1, \ldots, m\} \mid g_i(x^*) = 0\}, \quad \mathcal{N} := \{i \in \{1, \ldots, m\} \mid g_i(x^*) < 0\}.$$

The set $\mathcal{A}$ consists of the indices of those constraints being active at x^*. Of course, the sets $\mathcal{A}$ and $\mathcal{N}$ partition the set $\{1, \ldots, m\}$. Moreover, we define

$$\mathcal{A}_+ := \{i \in \mathcal{A} \mid \exists \nu \in \{1, \ldots, N\} : \lambda_i^{*,\nu} > 0\} \quad \text{and} \quad \mathcal{A}_0 := \mathcal{A} \setminus \mathcal{A}_+.$$

An index $i \in \{1, \ldots, m\}$ belongs to $\mathcal{A}_+$ if and only if g_i is active at x^* and the corresponding multiplier $\lambda_i^{*,\nu}$ of at least one player ν is strictly positive. The set $\mathcal{A}_0$ consists of the indices of those active constraints where the corresponding multipliers of all players are equal to zero. For any $\nu \in \{1, \ldots, N\}$, let us define the index sets

$$\mathcal{A}_+^{\nu} := \{i \in \mathcal{A} \mid \lambda_i^{*,\nu} > 0\} \quad \text{and} \quad \mathcal{A}_0^{\nu} := \{i \in \mathcal{A} \mid \lambda_i^{*,\nu} = 0\}.$$

Obviously, for every ν, the sets $\mathcal{A}_+^{\nu}$ and $\mathcal{A}_0^{\nu}$ partition the set $\mathcal{A}$. Moreover, the union of all sets $\mathcal{A}_+^{\nu}$ $(\nu = 1, \ldots, N)$ equals $\mathcal{A}_+$, and the intersection of all sets $\mathcal{A}_0^{\nu}$ $(\nu = 1, \ldots, N)$ is equal to $\mathcal{A}_0$.

In analogy to the definition of an arbitrary partition $\mathcal{I} = (I_1, I_2)$ of I_0 in Section 2, we introduce the partitions

$$\mathcal{I} := \{(\mathcal{I}_1^{\nu}, \mathcal{I}_2^{\nu})\}_{\nu=1}^{N}, \tag{3.3}$$

where $(\mathcal{I}_1^{\nu}, \mathcal{I}_2^{\nu})$ denotes an arbitrary partition of $\mathcal{A}_0^{\nu}$ $(\nu = 1, \ldots, N)$. Moreover, the set $Z_{\mathcal{I}}$ denotes the solution set of the following system of equations and inequalities (in analogy to system (2.8)):

$$\begin{aligned} \Psi(z) = 0, \quad \xi + E\lambda - 1_N = 0, \quad \xi \geq 0 & \\ g_{\mathcal{A}_+ \cup \mathcal{I}_1}(x) = 0, \qquad g_{\mathcal{N} \cup \mathcal{I}_2}(x) \leq 0, & \\ \lambda^{\nu}_{\mathcal{N} \cup \mathcal{I}_2^{\nu}} = 0, \qquad \lambda^{\nu}_{\mathcal{A}_+^{\nu} \cup \mathcal{I}_1^{\nu}} \geq 0 & \qquad (\nu = 1, \ldots, N), \end{aligned} \tag{3.4}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined according to

$$\mathcal{I}_1 := \left(\bigcup_{\nu=1}^{N} \mathcal{I}_1^{\nu} \right) \setminus \mathcal{A}_+ \quad \text{and} \quad \mathcal{I}_2 := \left(\bigcap_{\nu=1}^{N} \mathcal{I}_2^{\nu} \right) \setminus (\mathcal{A}_+ \cup \mathcal{I}_1). \tag{3.5}$$

Thus, $(\mathcal{I}_1, \mathcal{I}_2)$ is a partition of the index set $\mathcal{A}_0$. Note that $Z_{\mathcal{I}}$ is nonempty since it contains z^*. Furthermore, $Z_{\mathcal{I}}$ is a subset of the solution set of (1.2) and (3.1).

Now, it is not difficult to see that Condition 1 is satisfied for (1.2) if and only if the following Condition 2 holds which is the analogon of (49) in [13] developed for KKT systems of GNEPs.

Condition 2. *There are $\delta_2 > 0$ and $\omega_2 > 0$ such that, for every partition $\mathcal{I}$ according to* (3.3)*,*

$$\begin{aligned} \operatorname{dist}[z, Z_{\mathcal{I}}] \;\leq\; & \omega_2 \Big(\|\Psi(z)\| + \|\xi + E\lambda - 1_N\| + \|g_{\mathcal{A}_+ \cup \mathcal{I}_1}(x)\| \\ & + \|\min\{0, -g_{\mathcal{N} \cup \mathcal{I}_2}(x)\}\| + \|\min\{0, \xi\}\| \\ & + \sum_{\nu=1}^{N} \Big(\|\lambda^\nu_{\mathcal{N} \cup \mathcal{I}_2^\nu}\| + \|\min\{0, \lambda^\nu_{\mathcal{A}_+^\nu \cup \mathcal{I}_1^\nu}\}\| \Big) \Big) \end{aligned} \tag{3.6}$$

holds for all $z = (x, \xi, \lambda) \in \mathcal{B}_{\delta_2}(z^*)$, where $\mathcal{I}_1$ and $\mathcal{I}_2$ are defined by (3.5)*.*

In other words, Condition 2 requires that, for every partition $\mathcal{I}$ according to (3.3), system (3.4) satisfies a local error bound condition at $z^* = (x^*, \xi^*, \lambda^*)$.

The considerations of this section and Theorem 2.1 lead to the following local convergence result for the LP-Newton method and the constrained Levenberg-Marquardt method if they are applied to the constrained system (3.1) with Ω defined according to (3.2).

Theorem 3.1. *Let Condition 2 be satisfied and suppose that the problem functions $\theta_1, \ldots, \theta_N, g_1, \ldots, g_m$ have locally Lipschitz continuous second-order derivatives. Then, there is $\rho > 0$ such that $z^0 = (x^0, \xi^0, \lambda^0) \in \mathcal{B}_\rho(z^*) \cap \Omega$ implies that any sequence generated by the LP-Newton method with subproblems* (2.3) *or the constrained Levenberg-Marquardt method with subproblems* (2.4) *applied to* (3.1) *either terminates after finitely many steps with a solution of* (1.2) *or converges quadratically to a solution of* (1.2)*.*

Remark 3.2. It was shown in Remark 2.2 that the use of additional slack variables is very helpful from a computational point of view if the function g is nonlinear. In the context of FJ systems, we accordingly suggest to apply the LP-Newton method or the constrained Levenberg-Marquardt method to the following system with slack variables w.

$$\widetilde{F}(x, \xi, \lambda, w) := \begin{pmatrix} \Psi(z) \\ \xi + E\lambda - 1_N \\ g(x) + w \\ \min\{w, \lambda^1\} \\ \vdots \\ \min\{w, \lambda^N\} \end{pmatrix} = 0 \quad \text{s.t.} \quad \xi \geq 0,\ w \geq 0,\ \lambda \geq 0.$$

If the assumptions of Theorem 3.1 hold, the local quadratic convergence is kept if one of these methods is applied to the latter system, cf. Remark 2.2.

The rest of this section is devoted to conditions implying Condition 2. Our first aim is to show that some constant rank condition, more precisely the constant rank

of certain matrices in a neighborhood of $z^* = (x^*, \xi^*, \lambda^*)$, is sufficient for Condition 2 to hold. In order to formulate that constant rank condition, we need some notation.

First, we introduce the following index sets depending on the ξ-part of the fixed solution $z^* = (x^*, \xi^*, \lambda^*)$:

$$\Xi_+ := \{\nu \in \{1, \ldots, N\} \mid \xi^*_\nu > 0\} \quad \text{and} \quad \Xi_0 := \{\nu \in \{1, \ldots, N\} \mid \xi^*_\nu = 0\}.$$

It is obvious that Ξ_+ and Ξ_0 partition the set $\{1, \ldots, N\}$. Moreover, with $P(\xi, \lambda) := \xi + E\lambda - 1_N$, for any index set $\mathcal{M} \subseteq \{1, \ldots, N\}$, the matrices

$$D^\Psi_\mathcal{M}(x) := \Psi'_{\xi_\mathcal{M}}(x, \xi, \lambda) \quad \text{and} \quad D^P_\mathcal{M} := P'_{\xi_\mathcal{M}}(\xi, \lambda)$$

are used to abbreviate the Jacobians of Ψ and P (at (x, ξ, λ)) with respect to $\xi_\mathcal{M}$. Subsequently, the latter matrices are used for a compact notation of some parts of $G(z)$, where $G(z)$ is given according to (2.2) with F from (3.1). In addition, to simplify formulas within some of the following matrices, we use the notation shown in Section 1, where

$$\text{block}\left[A_\nu\right]_{\nu=1}^N \quad \text{is replaced by} \quad \text{block}\left[A_\nu\right]$$

with certain matrices A_ν for $\nu = 1, \ldots, N$.

Now we are in the position to formulate Condition 3 which requires the constant rank of certain matrices near the fixed solution z^* and which will turn out to be sufficient for Condition 2 to hold. Condition 3 is the analogon of [13, Condition 7] which was developed for KKT systems of GNEPs.

Condition 3. *There is $\delta_3 > 0$ such that, for every tuple $(\mathcal{K}, \mathcal{K}^0, \mathcal{K}^1, \ldots, \mathcal{K}^N)$ of index sets $\mathcal{K} \subseteq \mathcal{A}_0$, $\mathcal{K}^0 \subseteq \Xi_0$, and $\mathcal{K}^\nu \subseteq \mathcal{A}^\nu_0$ $(\nu = 1, \ldots, N)$, the matrices*

$$\begin{pmatrix} \Psi'_x(z) & D^\Psi_{\Xi_+ \cup \mathcal{K}^0}(x) & \text{block}\left[\nabla_{x^\nu} g_{\mathcal{A}^\nu_+ \cup \mathcal{K}^\nu}(x)\right] \\ 0 & D^P_{\Xi_+ \cup \mathcal{K}^0} & \text{block}\left[1^\top_{|\mathcal{A}^\nu_+ \cup \mathcal{K}^\nu|}\right] \\ g'_{\mathcal{A}_+ \cup \mathcal{K}}(x) & 0 & 0 \end{pmatrix}$$

have the same rank for all $z = (x, \xi, \lambda) \in \mathcal{B}_{\delta_3}(z^*)$.

In order to prove that Condition 3 implies Condition 2, we will show that, for every partition $\mathcal{I}$ according to (3.3), system (3.4) satisfies the relaxed constant rank constraint qualification (RCRCQ) at z^* if Condition 3 holds. Since the RCRCQ implies the local error bound condition (see [20] and [19]), we will deduce later on that the local error bound conditions (3.6) hold implying that Condition 2 is satisfied.

Let us briefly recall the RCRCQ for a general system of equations and inequalities

$$R(z) = 0, \quad S(z) \geq 0.$$

Let z^* be an arbitrary but fixed solution of this system and let $\mathcal{J}_0$ denote the index set of those components of S being active at z^*. The RCRCQ is satisfied at z^* if there is $\varepsilon > 0$ such that, for each index set $\mathcal{J} \subseteq \mathcal{J}_0$, the matrices

$$\begin{pmatrix} R'(z) \\ S'_\mathcal{J}(z) \end{pmatrix}$$

have the same rank for all $z \in \mathcal{B}_\varepsilon(z^*)$.

The proof of the subsequent theorem is similar to the proof of [13, Theorem 5], where the sufficiency of some constant rank condition for some analogon of Condition 2 is shown for KKT systems of GNEPs.

Theorem 3.3. *Let Condition 3 be valid. Then, Condition 2 holds.*

Proof. Let $\mathcal{I}$ be an arbitrary but fixed partition according to (3.3). Moreover, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be defined according to (3.5). We are going to show that system (3.4) satisfies the RCRCQ at $z^* = (x^*, \xi^*, \lambda^*)$. It is not difficult to see that the RCRCQ holds for system (3.4) at z^* if and only if, for each $(N+2)$-tuple $(\mathcal{L}, \mathcal{L}^0, \mathcal{L}^1, \ldots, \mathcal{L}^N)$ of index sets with $\mathcal{L} \subseteq \mathcal{I}_2$, $\mathcal{L}^0 \subseteq \Xi_0$, and $\mathcal{N} \cup \mathcal{I}_2^\nu \subseteq \mathcal{L}^\nu \subseteq \mathcal{I}_1^\nu \cup \mathcal{N} \cup \mathcal{I}_2^\nu$ $(\nu = 1, \ldots, N)$, the large matrices

$$\begin{pmatrix} \Psi'_x(z) & D^\Psi_{\mathcal{L}^0}(x) & D^\Psi_{\Xi_+ \cup \mathcal{K}^0}(x) & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{L}^\nu}(x)\right] & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}_+^\nu \cup \mathcal{K}^\nu}(x)\right] \\ 0 & D^P_{\mathcal{L}^0} & D^P_{\Xi_+ \cup \mathcal{K}^0} & \operatorname{block}\left[1^\top_{|\mathcal{L}^\nu|}\right] & \operatorname{block}\left[1^\top_{|\mathcal{A}_+^\nu \cup \mathcal{K}^\nu|}\right] \\ g'_{\mathcal{A}_+ \cup \mathcal{K}}(x) & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I}_{|\mathcal{L}^0|} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{block}\left[\mathrm{I}_{|\mathcal{L}^\nu|}\right] & 0 \end{pmatrix}$$

have the same rank for all $z = (x, \xi, \lambda)$ in a sufficiently small neighborhood of z^*, where the index sets $\mathcal{K}, \mathcal{K}^0, \mathcal{K}^1, \ldots, \mathcal{K}^N$ are defined by $\mathcal{K} := \mathcal{I}_1 \cup \mathcal{L}$, $\mathcal{K}^0 := \Xi_0 \setminus \mathcal{L}^0$, and $\mathcal{K}^\nu := \mathcal{I}_1^\nu \setminus \mathcal{L}^\nu$ $(\nu = 1, \ldots, N)$.

The rank of the above large matrix is equal to the sum of the rank of the reduced matrix

$$\begin{pmatrix} \Psi'_x(z) & D^\Psi_{\Xi_+ \cup \mathcal{K}^0}(x) & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}_+^\nu \cup \mathcal{K}^\nu}(x)\right] \\ 0 & D^P_{\Xi_+ \cup \mathcal{K}^0} & \operatorname{block}\left[1^\top_{|\mathcal{A}_+^\nu \cup \mathcal{K}^\nu|}\right] \\ g'_{\mathcal{A}_+ \cup \mathcal{K}}(x) & 0 & 0 \end{pmatrix} \tag{3.7}$$

and $|\mathcal{L}^0| + \sum_{\nu=1}^N |\mathcal{L}^\nu|$, see the proof of [13, Theorem 5] for a more detailed justification of this assertion. Thus, for any fixed $(N+2)$-tuple $(\mathcal{L}, \mathcal{L}^0, \mathcal{L}^1, \ldots, \mathcal{L}^N)$, the large matrices above have the same rank for all z near z^* if and only if the reduced matrices in (3.7) have this property. The matrices in (3.7) actually have the same rank for all $z \in \mathcal{B}_{\delta_3}(z^*)$ due to Condition 3. Therefore, system (3.4) satisfies the RCRCQ at z^*. Hence, the local error bound condition for this system at z^* holds as well. Since the partition $\mathcal{I}$ according to (3.3) was arbitrarily chosen and since there is only a finite number of such partitions, the validity of Condition 2 is shown. □

In analogy to [13, Condition 6] (for KKT systems of a GNEP), Condition 4 below requires the full row rank of a certain matrix at $z^* = (x^*, \xi^*, \lambda^*)$. We will prove that this full row rank condition is sufficient for Condition 3 to hold and therefore implies Condition 2 as well.

Condition 4. The following matrix has full row rank:

$$\begin{pmatrix} \Psi'_x(z^*) & D^{\Psi}_{\Xi_+}(x^*) & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}^\nu_+}(x^*)\right] \\ 0 & D^P_{\Xi_+} & \operatorname{block}\left[1^\top_{|\mathcal{A}^\nu_+|}\right] \\ g'_{\mathcal{A}}(x^*) & 0 & 0 \end{pmatrix}.$$

Theorem 3.4. *Let Condition 4 be satisfied. Then, Condition 3 holds.*

Proof. Let $(\mathcal{K}, \mathcal{K}^0, \mathcal{K}^1, \ldots, \mathcal{K}^N)$ be an arbitrary but fixed $(N+2)$-tuple of index sets $\mathcal{K} \subseteq \mathcal{A}_0$, $\mathcal{K}^0 \subseteq \Xi_0$, and $\mathcal{K}^\nu \subseteq \mathcal{A}^\nu_0$ $(\nu = 1, \ldots, N)$. Due to $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_0$, we have $\mathcal{A}_+ \cup \mathcal{K} \subseteq \mathcal{A}$. Therefore, by Condition 4, the matrix

$$\begin{pmatrix} \Psi'_x(z^*) & D^{\Psi}_{\Xi_+}(x^*) & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}^\nu_+}(x^*)\right] \\ 0 & D^P_{\Xi_+} & \operatorname{block}\left[1^\top_{|\mathcal{A}^\nu_+|}\right] \\ g'_{\mathcal{A}_+\cup\mathcal{K}}(x^*) & 0 & 0 \end{pmatrix} \tag{3.8}$$

has full row rank. The rows of the matrix

$$\begin{pmatrix} \Psi'_x(z^*) & D^{\Psi}_{\Xi_+\cup\mathcal{K}^0}(x^*) & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}^\nu_+\cup\mathcal{K}^\nu}(x^*)\right] \\ 0 & D^P_{\Xi_+\cup\mathcal{K}^0} & \operatorname{block}\left[1^\top_{|\mathcal{A}^\nu_+\cup\mathcal{K}^\nu|}\right] \\ g'_{\mathcal{A}_+\cup\mathcal{K}}(x^*) & 0 & 0 \end{pmatrix}$$

are still linearly independent since, compared to (3.8), at most the number of columns has increased. Since all functions involved in the latter matrix are continuous, the rows stay linearly independent for all $z = (x, \xi, \lambda)$ in a sufficiently small neighborhood of z^*. Hence, taking into account that the tuple $(\mathcal{K}, \mathcal{K}^0, \mathcal{K}^1, \ldots, \mathcal{K}^N)$ was arbitrarily chosen and that there is only a finite number of such tuples, the validity of Condition 3 is shown. □

The following corollary is an immediate consequence of Theorems 3.3 and 3.4.

Corollary 3.5. *Let Condition 4 be satisfied. Then, Condition 2 holds.*

Let us briefly summarize the main contributions of this section. From the latter results and Theorem 3.1 we can deduce that Condition 4 and even Condition 3 imply local quadratic convergence of the LP-Newton method as well as of the constrained Levenberg-Marquardt method to a solution of the constrained system (3.1) provided that the problem functions $\theta_1, \ldots, \theta_N$, $g_1, \ldots, g_m$ have locally Lipschitz continuous second-order derivatives. The local convergence rate is kept if the methods are applied to a suitable reformulation of (3.1) by means of slack variables, cf. Remark 3.2.

4. GENERIC SATISFACTION OF THE FULL ROW RANK CONDITION

Let us first recall that the space $\mathcal{D}$ of all twice continuously differentiable functions which define a GNEP is given in (1.3). The aim of this section is to prove that there is an open and dense subset $\widehat{\mathcal{D}}$ of $\mathcal{D}$ such that, for any GNEP given by $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m) \in \widehat{\mathcal{D}}$ and any solution of the corresponding FJ system (1.2), Condition 4 is satisfied. To this end, we will exploit a result from [4].

The following nonsmooth system of equations shows another way of reformulating the FJ system (1.2) associated to the GNEP. It stems from a more general reformulation suggested in [4] which allows not only shared constraints.

$$\mathcal{F}(x,\beta,\gamma) := \begin{pmatrix} \Theta(x)\beta_+ + B(x)\gamma_+ \\ g(x) + \gamma^{\Pi} \\ \beta_+ + E\gamma_+ - 1_N \end{pmatrix} = 0 \tag{4.1}$$

with $\beta := (\beta_1, \ldots, \beta_N)^\top \in \mathbb{R}^N$, $\beta_+ := (\max\{0, \beta_1\}, \ldots, \max\{0, \beta_N\})^\top$ and similar for γ_+, where

$$\gamma := \begin{pmatrix} \gamma^1 \\ \vdots \\ \gamma^N \end{pmatrix} \in \mathbb{R}^{Nm} \quad \text{and} \quad \gamma^\nu \in \mathbb{R}^m \text{ for } \nu = 1, \ldots, N.$$

Moreover, $\gamma^{\Pi} \in \mathbb{R}^m$ is a vector defined by

$$\begin{aligned} \gamma_i^{\Pi} &:= \begin{cases} \prod_{\nu=1}^{N} |\gamma_i^\nu| & \text{if } \gamma_i^\nu < 0 \text{ for all } \nu = 1, \ldots, N, \\ 0 & \text{else} \end{cases} \\ &= (-1)^N \prod_{\nu=1}^{N} \min\{0, \gamma_i^\nu\} \end{aligned}$$

for $i = 1, \ldots, m$.

By [4, Lemma 2.1], the nonsmooth system (4.1) is an equivalent reformulation of the FJ system (1.2) in the following sense. If (x^*, β^*, γ^*) is a solution of (4.1), then (x^*, ξ^*, λ^*) with

$$\xi^* := \beta_+^* \quad \text{and} \quad \lambda^* := \gamma_+^*$$

solves (1.2). Conversely, if (x^*, ξ^*, λ^*) is a solution of the FJ system (1.2), then (x^*, β^*, γ^*) with $\beta^* := \xi^*$ and, recalling the definitions of the index sets $\mathcal{A}$ and $\mathcal{N}$ in Section 3,

$$\gamma_i^{*,\nu} := \begin{cases} \lambda_i^{*,\nu} & \text{if } i \in \mathcal{A}, \\ -|g_i(x^*)|^{\frac{1}{N}} & \text{if } i \in \mathcal{N} \end{cases} \qquad (\nu = 1, \ldots, N,\ i = 1, \ldots, m) \tag{4.2}$$

solves the nonsmooth system (4.1).

Obviously, the function $\mathcal{F}$ in (4.1) is not everywhere differentiable, but a PC^1-function and, thus, locally Lipschitz continuous. Therefore, for any $(x, \beta, \gamma) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nm}$, Clarke's generalized Jacobian $\partial\mathcal{F}(x, \beta, \gamma)$ is well-defined, see Section 1. According to [4, Theorem 2.2], there is an open and dense subset of the space $\mathcal{D}$ of all GNEP defining twice continuously differentiable functions such that, for any $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m)$ in this subset and any corresponding solution $(\bar{x}, \bar{\beta}, \bar{\gamma})$ of (4.1), all elements of $\partial\mathcal{F}(\bar{x}, \bar{\beta}, \bar{\gamma})$ have full row rank. We will use this result to prove that there is an open and dense subset of $\mathcal{D}$ such that, for any $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m)$ in this subset, Condition 4 is satisfied at all corresponding solutions of (1.2).

Lemma 4.1 below provides a sufficient condition for the validity of Condition 4 at a solution (x^*, ξ^*, λ^*) of the FJ system (1.2).

Lemma 4.1. *Let $z^* = (x^*, \xi^*, \lambda^*)$ be a solution of the FJ system (1.2). Moreover, suppose that $\beta^* = \xi^*$ and that γ^* is defined according to (4.2). If all matrices belonging to $\partial\mathcal{F}(x^*, \beta^*, \gamma^*)$ have full row rank, then Condition 4 is satisfied.*

Proof. Let $\{(x^k, \beta^k, \gamma^k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{Nm}$ be a sequence converging to (x^*, β^*, γ^*) such that the following relations are satisfied for all $k \in \mathbb{N}$:

$$\begin{array}{lcll} \beta_\nu^k & > & 0 & \text{for } \nu \in \Xi_+, \\ \beta_\nu^k & < & 0 & \text{for } \nu \in \Xi_0, \\ \gamma_i^{k,\nu} & > & 0 & \text{for } \nu \in \{1, \ldots, N\},\ i \in \mathcal{A}_+^\nu, \\ \gamma_i^{k,\nu} & < & 0 & \text{for } \nu \in \{1, \ldots, N\},\ i \in \mathcal{A}_0^\nu \cup \mathcal{N}. \end{array} \tag{4.3}$$

Taking into account that (x^*, β^*, γ^*) is obtained from $z^* = (x^*, \xi^*, \lambda^*)$ by $\beta^* := \xi^*$ and (4.2), the properties of z^* indeed guarantee that a sequence $\{(x^k, \beta^k, \gamma^k)\}_{k\in\mathbb{N}}$ with the above properties exists. Let us fix k for the moment. The inequalities in (4.3) stay true for all points (x, β, γ) in a sufficiently small neighborhood of (x^k, β^k, γ^k). Thus, for all such points sufficiently close to (x^k, β^k, γ^k), the function value of $\mathcal{F}$ is given by

$$\mathcal{F}(x,\beta,\gamma) = \begin{pmatrix} \left(\beta_\nu \nabla_{x^\nu}\theta_\nu(x) + \sum_{i\in\mathcal{A}_+^\nu} \gamma_i^\nu \nabla_{x^\nu} g_i(x)\right)_{\nu\in\Xi_+} \\ \left(\sum_{i\in\mathcal{A}_+^\nu} \gamma_i^\nu \nabla_{x^\nu} g_i(x)\right)_{\nu\in\Xi_0} \\ (g_i(x))_{i\in\mathcal{A}_+} \\ \left(g_i(x) + (-1)^N \gamma_i^1 \cdot \ldots \cdot \gamma_i^N\right)_{i\in\mathcal{A}_0\cup\mathcal{N}} \\ \left(\beta_\nu + \sum_{i\in\mathcal{A}_+^\nu} \gamma_i^\nu - 1\right)_{\nu\in\Xi_+} \\ \left(\sum_{i\in\mathcal{A}_+^\nu} \gamma_i^\nu - 1\right)_{\nu\in\Xi_0} \end{pmatrix}.$$

Clearly, by (4.3), it follows that $\mathcal{F}$ is differentiable at (x^k, β^k, γ^k) and the Jacobian $\mathcal{F}'(x^k, \beta^k, \gamma^k)$ coincides, after some row and column permutations if necessary, with the matrix

$$J_k := \begin{pmatrix} \Psi'_x(x^k, \beta_+^k, \gamma_+^k) & D_{\Xi_+}^\Psi(x^k) & 0 & 0 & 0 & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}_+^\nu}(x^k)\right] \\ g'_{\mathcal{A}_+}(x^k) & 0 & 0 & 0 & 0 & 0 \\ g'_{\mathcal{A}_0}(x^k) & 0 & 0 & 0 & M_2(\gamma^k) & 0 \\ g'_{\mathcal{N}}(x^k) & 0 & 0 & M_1(\gamma^k) & 0 & 0 \\ 0 & D_{\Xi_+}^P & 0 & 0 & 0 & \operatorname{block}\left[1_{|\mathcal{A}_+^\nu|}^\top\right] \end{pmatrix}.$$

The six column blocks of this matrix contain the derivatives of $\mathcal{F}$ at point (x^k, β^k, γ^k) with respect to the variables

$$x,\ \beta_{\Xi_+},\ \beta_{\Xi_0},\ (\gamma_{\mathcal{N}}^\nu)_{\nu=1}^N,\ (\gamma_{\mathcal{A}_0^\nu}^\nu)_{\nu=1}^N,\ (\gamma_{\mathcal{A}_+^\nu}^\nu)_{\nu=1}^N$$

in this order. The matrices $M_1(\gamma^k)$ and $M_2(\gamma^k)$ are defined as follows. For any vector $\gamma \in \mathbb{R}^{Nm}$, the matrix $M_1(\gamma)$ is given by

$$M_1(\gamma) := \big(M_1^1(\gamma), \ldots, M_1^N(\gamma)\big) \in \mathbb{R}^{|\mathcal{N}| \times (N \cdot |\mathcal{N}|)}$$

with

$$(M_1^\nu(\gamma))_{ij} := \begin{cases} (-1)^N \prod_{\mu \neq \nu} \gamma_i^\mu & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\nu \in \{1, \ldots, N\}$ and $i, j \in \mathcal{N}$. The matrix $M_2(\gamma)$ is set to

$$M_2(\gamma) := \big(M_2^1(\gamma), \ldots, M_2^N(\gamma)\big) \in \mathbb{R}^{|\mathcal{A}_0| \times \sum_{\nu=1}^N |\mathcal{A}_0^\nu|}$$

with

$$(M_2^\nu(\gamma))_{ij} := \begin{cases} (-1)^N \prod_{\mu \neq \nu} \gamma_i^\mu & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for $\nu \in \{1, \ldots, N\}$, $i \in \mathcal{A}_0$, and $j \in \mathcal{A}_0^\nu$.

Since the sequence $\{(x^k, \beta^k, \gamma^k)\}$ converges to (x^*, β^*, γ^*), the matrices $M_1(\gamma^k)$ and $M_2(\gamma^k)$ converge to $M_1(\gamma^*)$ and $M_2(\gamma^*)$, respectively. The matrix $M_2(\gamma^*)$ consists of zeros only because $\gamma_i^{*,\nu} = 0$ holds for $\nu \in \{1, \ldots, N\}$ and $i \in \mathcal{A}_0$. Therefore, the matrices J_k converge to the matrix

$$J := \begin{pmatrix} \Psi'_x(x^*, \beta_+^*, \gamma_+^*) & D_{\Xi_+}^{\Psi}(x^*) & 0 & 0 & 0 & \operatorname{block}\left[\nabla_{x^\nu} g_{\mathcal{A}_+^\nu}(x^*)\right] \\ g'_{\mathcal{A}_+}(x^*) & 0 & 0 & 0 & 0 & 0 \\ g'_{\mathcal{A}_0}(x^*) & 0 & 0 & 0 & 0 & 0 \\ g'_{\mathcal{N}}(x^*) & 0 & 0 & M_1(\gamma^*) & 0 & 0 \\ 0 & D_{\Xi_+}^{P} & 0 & 0 & 0 & \operatorname{block}\left[1_{|\mathcal{A}_+^\nu|}^\top\right] \end{pmatrix}.$$

After some column and row permutations of the matrix J, it becomes an element of Clarke's generalized Jacobian $\partial \mathcal{F}(x^*, \beta^*, \gamma^*)$. Therefore, due to the assumption of the lemma, J has full row rank. Consider the matrix $\hat{J}$ arising from J by firstly deleting the row block containing $g'_{\mathcal{N}}(x^*)$ and $M_1(\gamma^*)$ and, subsequently, those columns of the obtained matrix consisting of zeros only. The resulting matrix $\hat{J}$ has full row rank as well. Taking into account $\xi^* = \beta^* = \beta_+^*$, $\lambda^* = \gamma_+^*$, and $\mathcal{A} = \mathcal{A}_+ \cup \mathcal{A}_0$, the matrix $\hat{J}$ is precisely the matrix from Condition 4 (after some row permutations). Therefore, Condition 4 is satisfied at z^* so that the assertion is proved. □

Using Lemma 4.1 together with [4, Theorem 2.2], we obtain the following corollary.

Corollary 4.2. *There is an open and dense subset $\widehat{\mathcal{D}} \subseteq \mathcal{D}$ such that, for any $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m) \in \widehat{\mathcal{D}}$ and any solution $z^* = (x^*, \xi^*, \lambda^*)$ of the FJ system (1.2) of the corresponding GNEP, Condition 4 is satisfied.*

Taking into account the results from Section 3, Corollary 4.2 implies that, for any tuple $(\theta_1, \ldots, \theta_N, g_1, \ldots, g_m)$ of GNEP defining functions belonging to the set $\widehat{\mathcal{D}}$, the LP-Newton method as well as the constrained Levenberg-Marquardt method applied to (3.1) generate sequences converging quadratically to a solution of (1.2) if the

starting point belongs to a sufficiently small neighborhood of some solution of (1.2) and if the second-order derivatives of the problem functions $\theta_1, \ldots, \theta_N, g_1, \ldots, g_m$ are locally Lipschitz continuous. Thus, unlike the Newton-type method proposed in [4], the two Newton-type methods above provide local quadratic convergence even if strict complementarity is violated. For globalization techniques for the latter two methods, we refer to [5, 14, 15, 18].

Of course, there might be elements of $\widehat{\mathcal{D}}$ such that the second-order derivative of at least one of the functions $\theta_1, \ldots, \theta_N, g_1, \ldots, g_m$ is not locally Lipschitz continuous. In that case, local quadratic convergence of our methods cannot be expected. However, we could still prove local superlinear convergence of the LP-Newton method and the constrained Levenberg-Marquardt method applied to (3.1) to a solution of (1.2). Note that local quadratic convergence of the Newton-type method used in [4] cannot be expected, too, if one of the functions $\theta_1, \ldots, \theta_N, g_1, \ldots, g_m$ does not have a locally Lipschitz continuous second-order derivative, cf. the local convergence analysis in [21].

Besides, it is known that the space $C^3(\mathbb{R}^n)$ of all real-valued, three times continuously differentiable functions is dense in $C^2(\mathbb{R}^n)$ regarding the Whitney topology, see for example [16, Theorem 2.6 of Chapter 2]. Consequently, the space of all real-valued, twice differentiable functions with locally Lipschitz continuous second-order derivatives is, regarding the Whitney topology, dense in $C^2(\mathbb{R}^n)$ as well.

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