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ON LOCAL PERRON-FROBENIUS THEORY

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Abstract. For $A \in \mathcal{M}_n(\mathbb{C})$, $x \in \mathbb{C}^n$, and any nonnegative integer k, we denote by W_x (respectively, $W_x^{\mathbb{R}}$, $w_k(A,x)$) the A-cyclic subspace (respectively, the real A-cyclic subspace, the convex cone) generated by x (respectively, by x, by $A^i x$ for $i = k, k+1, \ldots$). We relate spectral conditions on $A|_{W_x}$ (such as the Perron-Schaefer condition, or having a positive or nonnegative eigenvalue) to the geometric conditions involving the cones $w_k(A,x)$ $(k = 0,1,2\ldots)$ or their closures (such as being a pointed cone, or being a real subspace). In particular, it is proved that the cone $\mathrm{cl}\, w_0(A,x)$ is pointed if and only if A satisfies the local Perron-Schaefer condition at x (or, equivalently, $A|_{W_x}$ satisfies the Perron-Schaefer condition). By considering the linear map $\mathcal{L}_A$ on $\mathcal{M}_n(\mathbb{C})$ given by $\mathcal{L}_A(X) = AX$, we recover the intrinsic Perron-Frobenius theorems obtained by H. Schneider. Under the assumption that $\mathrm{cl}\, w_0(A,x)$ is a pointed cone, we give two sets of spectral conditions on $A|_{W_x}$ that are equivalent to respectively the condition $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda(A|_{W_x})-1)}(A)x \in w_0(A,x)$ and $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}(A)x \in w_0(A,x)$, where $E_\lambda^{(k)}(A)$ denotes the kth component of A corresponding to λ, $\rho_x(A)$ is the local spectral radius of A at x, and $\Lambda = \{\lambda \in \sigma(A|_{W_x}) : |\lambda| = \rho_x(A), \nu_\lambda(A|_{W_x}) = \nu_{\rho_x(A)}(A|_{W_x})\}$. The latter conditions, in turn, are necessary conditions for the cone $w_0(A,x)$ to be closed, and in the special case when the eigenvalues of $A|_{W_x}$ are all of the same modulus, we characterize when the cone $w_0(A,x)$ is closed. As applications, we characterize when $A|_{W_x^{\mathbb{R}}}$ is irreducible, primitive or strictly positive with respect to $\mathrm{cl}\, w_0(A,x)$. We settle the question of when there exists a closed pointed cone in $\mathbb{R}^n$ (or $\mathbb{C}^n$) that contains some given vectors $x_1,\ldots,x_k$ in $\mathbb{R}^n$ (or in $\mathbb{C}^n$). When A is a real matrix that satisfies the Perron-Schaefer condition, we show that there is an A-invariant proper cone, which is the sum of m cones of the form $\mathrm{cl}\, w_0(A,x)$, where m is the maximum of the geometric multiplicities of the eigenvalues of A, and this m is the best possible lower bound. A complex version of the result is also derived. Some partial results are found for the question of when there exists a proper cone C in $W_x^{\mathbb{R}}$ containing x such that $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(C)$. In particular, we find an equivalent condition for $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(\mathrm{cl}\, w_0(A,x))$. We also characterize when there exists an indecomposable proper polyhedral cone K such that $A \in \mathrm{Aut}(K)$. A treatment of the local Perron-Frobenius theory for cross-positive matrices is also offered. Finally, a number of open questions are posed.

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1. Introduction

This is the seventh of a sequence of papers (namely, [32], [25], [28], [29], [30], [27] and the current paper) on a newly developed subject, the geometric spectral theory of positive linear operators (in finite dimensions), which is concerned with the study of the classical Perron-Frobenius theory of a (square, entrywise) nonnegative matrix and its generalizations from the geometric cone-theoretic viewpoint. For reviews on the subject, see [26] and [31]. For a ramification of the theory in the study of exponents of polyhedral cones, see [14], [15] and [13].

In the previous papers of the sequence, we usually fix a proper (i.e., closed, pointed, full convex) cone K and a (square) matrix A such that A is nonnegative on K (i.e. $AK \subseteq K$) and consider different aspects of A in each paper — the Collatz-Wielandt sets, the distinguished eigenvalues, the core, the invariant faces, linear equations over cones, and the Perron generalized eigenspace and the spectral cone (a unified approach to several topics of interest in combinatorial spectral theory of nonnegative matrices). In this work we change our viewpoint somewhat. We treat the local Perron-Frobenius theory. Here we use the word "local" in the loose sense of "pertaining to a single vector". Given a complex matrix A, we consider closed, pointed (or, proper) cones invariant under A with various properties. In particular, we are interested in closures of A-cyclic cones, i.e., cones of the form $\mathrm{cl}\, w_k(A, x)$, where $w_k(A, x)$ denotes the cone generated by $A^k x, A^{k+1}x, \ldots$ in $\mathbb{C}^n$. One reason for our interest in closures of A-cyclic cones is that, such cones often appeared as illustrative examples in previous papers of this sequence (see [32, Example 3.7], [28, Example 5.2], [29, Example 5.3], [30, Example 4.9], [27, Example 3.10]; [25, Example 5.5] and [29, Example 5.4]). Proper polyhedral cones K for which there exist a K-primitive matrix A such that the digraph associated with A is given by the Wielandt digraph or the near-Wielandt digraph have also played a major role in the recent study of maximal exponents of polyhedral cones ([14], [15] and [13]). As can be readily seen, such cones K are necessarily A-cyclic.

In a sense, this work can also be considered as a continuation of the work of Schneider [20] in investigating the relation between the algebraic properties and the geometric properties of a matrix $A \in \mathcal{M}_n(\mathbb{C})$. According to [20], an algebraic property is one which can be determined from the Jordan form of A, while by a geometric property it is meant some association between A and a geometric object, namely a cone.

A natural question to ask is, when a given real matrix leaves invariant a proper cone. The answer is known and is provided by the following:

Theorem A. *For an $n \times n$ real matrix A, there exists a proper cone K in $\mathbb{R}^n$ such that $AK \subseteq K$ if and only if A satisfies the following set of conditions*:

(a) *$\rho(A)$, the spectral radius of A, is an eigenvalue of A.*

(b) *For each eigenvalue λ of A with modulus $\rho(A)$, $\nu_\lambda(A) \leq \nu_{\rho(A)}(A)$, where $\nu_\lambda(A)$ denotes the index of λ as an eigenvalue of A.*

That condition (a) in Theorem A is a necessary condition for the existence of a proper A-invariant cone K was first established by Birkhoff [4], using an elementary argument that makes use of the Jordan basis of $\mathbb{C}^n$ associated with A. (Birkhoff also showed, in addition, that K contains an eigenvector corresponding to $\rho(A)$.) Extending Birkhoff's argument, Vandergraft [33] showed that condition (b) is another necessary condition. In the same paper, Vandergraft also proved that conditions (a) and (b) together is also a sufficient condition for the existence of K. (Elsner [8, Satz 3.1] also established Theorem A in the setting of a compact linear operator on a real Banach space.) Following Schneider [20], we say an $n \times n$ complex (or real) matrix A satisfies the *Perron-Schaefer condition* if conditions (a) and (b) of Theorem A hold. (Rodman et.al. [18] use the term Vandergraft matrices for real matrices that satisfy the Perron-Schaefer condition.)

We would like to add that, with slight modifications, Vandergraft's proof shows that Theorem A still holds if A is an $n \times n$ complex matrix and $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$.

In order to characterize the Perron-Schaefer condition on a complex matrix A by a geometric property directly associated with A, Schneider introduced, for each nonnegative integer k, the *intrinsic cone* $w_k(A)$ of A, which is the cone generated by $A^k, A^{k+1}, \ldots$, and obtained the following result ([20, Theorem 1.4]):

Theorem B. *Let $A \in \mathcal{M}_n(\mathbb{C})$, and let k be a nonnegative integer. Then the cone $\operatorname{cl} w_k(A)$ is pointed if and only if A satisfies the Perron-Schaefer condition.*

Schneider [20, p.255] refers to the above result as an intrinsic Perron-Frobenius theorem and attributes one direction of the result to Schaefer [19]. In [20, the first and second paragraphs on p.265] Schneider also remarked that since $\pi(K)$ is a proper cone in $\mathcal{M}_n(\mathbb{C})$ whenever K is a proper cone in $\mathbb{C}^n$, if $AK \subseteq K$ then $\operatorname{cl} w_0(A)$ is pointed. As a consequence, the "only" if part of Theorem A and the "only if" of Theorem B (for $k = 0$) are "equivalent", and the "if" part of Theorem A, which is due to Vandergraft, implies the "if" part of Theorem B. He also asked whether there is a simple argument to derive the "if" part of Theorem A from the "if" part of Theorem B. We believe the answer to the latter question is in the negative. However, if we use a local version of Theorem B and the concept of cyclic cones (see Theorem 3.6, (a1)⇔(b1)), then there is a natural simple way to construct invariant proper cones for a matrix that satisfies the Perron-Schaefer condition.

Denoting by $\mathcal{L}_A$ the linear map on $\mathcal{M}_n(\mathbb{C})$ given by $\mathcal{L}_A(X) = AX$, one readily shows that $w_0(\mathcal{L}_A, A) = w_1(A)$. Conceivably, the intrinsic Perron-Frobenius theorems obtained by Schnieder [20] can be recovered by proving the corresponding results involving the cones $w_k(A, x)$ first. This is another reason why we are interested in such cones.

We now describe the contents of this paper in some detail. Some necessary definitions, notations, known or preliminary results are given in Section 2. In particular, we provide an equivalent condition for the Perron-Schaefer condition on a complex matrix, given in terms of the roots of the minimal polynomial of the matrix. It is proved that A satisfies the local Perron-Schaefer condition at x if and only if the restriction map $A|_{W_x}$ (or, equivalently, $A|_{W_x^{\mathbb{R}}}$), where W_x (respectively, $W_x^{\mathbb{R}}$) denotes the subspace (respectively, the real subspace) of $\mathbb{C}^n$ generated by $A^i x$ for $i = 0, 1, \ldots$, satisfies the Perron-Schaefer condition. (The definition of the local Perron-Schaefer condition will be given in Section 2.) We offer a direct proof for the known result that if K is a closed, pointed A-invariant cone, then A satisfies the local Perron-Schaefer condition at x for every $x \in K$. In the course of proof, we also take note of the connection between the minimal polynomial of $A|_{W_x}$ and that of $A|_{W_x^{\mathbb{R}}}$ and show that they either both satisfy or both do not satisfy the local Perron-Schaefer condition at x.

In [20] Schneider gave geometric conditions (given in terms of the cones $w_k(A)$ or their closures) for a complex matrix A to satisfy the Perron-Schaefer condition or to have a positive (or nonnegative) eigenvalue. In Section 3 we provide local versions of these results. We examine the connections between spectral conditions on $A|_{W_x}$ or $A|_{W_x^{\mathbb{R}}}$ (such as the Perron-Schaefer condition, or having a positive (or nonnegative) eigenvalue) and geometric conditions involving the cones $w_k(A,x)$ or their closures (such as being a real subspace, or being a pointed cone). In particular, we prove that A satisfies the local Perron-Schaefer condition at x if and only if the cone $\operatorname{cl} w_0(A,x)$ is pointed. The connection between $w_k(A,x)$ (or their closures) for different k's is also noted. We recover the intrinsic Perron-Frobenius theorems obtained by Schneider. Necessary conditions for $\operatorname{cl} w_0(A,x)$ to be a pointed polyhedral cone are found. An equivalent condition for $w_0(A,x)$ to be a pointed polyhedral cone (or a simplicial cone) is also given.

In Section 4, under the assumption that the cone $\operatorname{cl} w_0(A,x)$ is pointed, we examine the following conditions: (α) the cone $w_0(A,x)$ is closed; (β) $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}(A|_{W_x^{\mathbb{R}}})x \in w_0(A,x)$, where $\rho_x(A)$ is the local spectral radius of A at x and $E_\lambda^{(k)}(A)$ is the kth principal component of A corrresponding to λ; and (γ) $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)}(A|_{W_x^{\mathbb{R}}})x \in w_0(A,x)$, where Λ denotes the set of peripheral eigenvalues λ of $A|_{W_x^{\mathbb{R}}}$ for which $\nu_\lambda(A|_{W_x^{\mathbb{R}}}) = \nu_{\rho_x(A)}(A|_{W_x^{\mathbb{R}}})$. According to some known results, we always have the implications $(\alpha) \Rightarrow (\beta)$ and $(\alpha) \Rightarrow (\gamma)$. Here we provide two sets of spectral conditions on $A|_{W_x}$ that are equivalent to conditions (β) and (γ) respectively, and as a consequence we have $(\gamma)\Rightarrow(\beta)$. In the special case when the eigenvalues of $A|_{W_x^{\mathbb{R}}}$ are all of the same modulus, we also provide a spectral characterization of condition (α). As applications, we characterize when $A|_{W_x^{\mathbb{R}}}$ is irreducible, primitive or strictly positive with respect to $\operatorname{cl} w_0(A,x)$.

In Section 5 we settle the question of when there exists a closed pointed A-invariant cone that contains some given vectors $x_1, \dots, x_k$. We also show that if $A \in \mathcal{M}_n(\mathbb{R})$ satisfies the Perron-Schaefer condition, then we can always construct a proper A-invariant cone K in $\mathbb{R}^n$ which is the sum of the closures of finitely many A-cyclic cones. Indeed, we show that the least possible number of A-cyclic cones we need is m, where m is the maximum of the geometric multiplicities of the eigenvalues of A. A complex version of the result is also derived.

In Section 6 we treat the questions of the existence of various kinds of cone automorphisms. For $A \in \mathcal{M}_n(\mathbb{C})$ and $0 \neq x \in \mathbb{C}^n$, it is shown that there exists a proper cone C in $W_x^{\mathbb{R}}$ containing x such that $A|_{W_x^{\mathbb{R}}} \in \operatorname{Aut}(C)$ if and only if $A|_{W_x^{\mathbb{R}}}$ is nonsingular and the cone $\operatorname{cl}(\operatorname{pos}\{(A|_{W_x^{\mathbb{R}}})^i x : 0, \pm1, \pm2, \dots,\}$ is pointed. It is found that the condition that $A|_{W_x^{\mathbb{R}}}$ is nonsingular, and $A|_{W_x^{\mathbb{R}}}$ and its inverse both satisfy the Perron-Schaefer condition is weaker than the preceding equivalent conditions. We also prove that $A|_{W_x^{\mathbb{R}}} \in \operatorname{Aut}(\operatorname{cl} w_0(A,x))$ if and only if $A|_{W_x^{\mathbb{R}}}$ is nonzero, diagonalizable, all eigenvalues of $A|_{W_x^{\mathbb{R}}}$ are of the same modulus and $\rho_x(A)$

is an eigenvalue of $A|_{W_x^{\mathbb{R}}}$. We give some conditions that are either equivalent to or are weaker than the condition that A is nonzero, diagonalizable and all eigenvalues of A are of the same modulus; and when A satisfies the Perron-Schaefer condition, all these conditions are equivalent and are also equivalent to the condition that there exists a proper cone K such that $A \in \mathrm{Aut}(K)$ and A has an eigenvector in $\mathrm{int}\, K$. Finally, we characterize when there exists an (indecomposable) proper polyhedral cone K such that $A \in \mathrm{Aut}(K)$.

A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is cross-positive on a proper cone K in $\mathbb{C}^n$ if for all $x \in K, z \in K^*$, where K^* denotes the dual cone of K, $\mathrm{Re}(z^*x) = 0$ implies $\mathrm{Re}(z^*Ax) \geq 0$. It is readily shown that the class of matrices cross-positive on K includes the extension, by multiples of the identity matrix, of the class of matrices nonnegative on K.

In Section 7 we treat the local Perron-Frobenius theory for cross-positive matrices and derive the parallel results. Instead of the A-cyclic cones $w_k(A, x)$ (respectively, the principal component $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}$) we work with the cones $\mathrm{pos}\{e^{tA}x : t \geq 0\}$ (respectively, the principal component $E_{\xi_x(A)}^{(\nu_{\xi_x(A)}-1)}$), where $\xi_x(A)$ is the local spectral abscissa of A at x. Also, we introduce the new concept of the real spectral pair of a matrix relative to a vector, the ESV (Elsner-Schneider-Vidyasagar) condition and the local ESV condition.

To motivate further work, a number of open questions are posed in Section 8, the final section.

In previous papers of this sequence, we usually formulate our results in the setting of a real matrix acting on a cone in a (finite-dimensional) real vector space, because "cone" is a real concept. It is explained in [28, Section 8] how one can obtain the corresponding results for a complex matrix acting on a cone in a complex vector space. However, for a complex matrix, sometimes it is more natural to give results directly in the complex setting. So in this paper we formulate our results mostly in the setting when the underlying matrix is a complex matrix. The results in the real setting either follow from the corresponding results in the complex setting or have a parallel proof.

This research work began more than twenty eight years ago and was carried out off and on. The forthcoming of this paper was announced in the reference list of [28] under the tentative title "On matrices with invariant closed pointed cones", and a few results in this paper have appeared, without proofs, in the review papers [26] and [31]. Due to a shift of research interest and other reasons, this research work was suspended about fifteen years ago. Most of the results in the present paper were obtained and some new ideas were found when the research work was resumed in the past few months.

2. Preliminaries

A familiarity with convex cones, convex sets and cone-preserving maps is assumed. For references, see [1], [3], [17], [24], [26], [31]. For convenience and to fix notation, we collect in this section some of the necessary definitions, notations and known results that are used throughout the paper. A few more definitions and notations will be introduced in later sections.

A (*convex*) *cone* K is a nonempty subset of a finite-dimensional real or complex vector space V such that $K+K \subseteq K$ and $\alpha K \subseteq K$ for all $\alpha \geq 0$. The cone K is said to be *pointed* if $K \cap (-K) = \{0\}$; K is *closed* if K is closed relative to the usual topology of V and *full* if $\operatorname{int} K \neq \emptyset$ or, equivalently, $K - K (= \text{real span}\, K) = V$. A cone is *proper* if it is pointed, closed and full.

We give our definitions and formulate our results in the setting when the underlying space is $\mathbb{R}^n$ or $\mathbb{C}^n$. As can be seen, our definitions or results can also be stated in the slightly more general setting of a finite-dimensional real or complex vector space.

The following notation will be adopted.

$\mathcal{M}_n(\mathbb{F})$: the set of all $n \times n$ matrices with entries from the field F.

$\mathbb{R}^n_+$: the nonnegative orthant of $\mathbb{R}^n$.

$F[t]$: the set of all polynomials with coefficients from the field F.

$\mathbb{R}_+[t]$: the set of all polynomials with nonnegative coefficients.

$J_k(\lambda)$: the $k \times k$ upper triangular elementary Jordan block associated with the eigenvalue λ.

$\operatorname{pos}(S)$: the positive hull of S, i.e., the set of all nonnegative linear combinations of vectors taken from S.

$\sigma(A)$: the spectrum of the (square) matrix A, i.e., the set of eigenvalues of A.

$\rho(A)$: the spectral radius of the matrix A.

$\nu_\lambda(A)$ (or ν_λ): the index of λ relative to A, i.e., the least nonnegative integer k such that $\operatorname{rank}(A-\lambda I)^{k+1} = \operatorname{rank}(A-\lambda I)^k$.

$\mathcal{N}(A)$: the null space of the matrix A.

$\operatorname{nullity}(A)$: the nullity of the matrix A, i.e., $\dim \mathcal{N}(A)$.

$N^k_\lambda(A) = \{x \in \mathbb{C}^n : (A-\lambda I)^k x = 0\}$.

$N^{\nu_\lambda(A)}_\lambda(A)$ (or $N^{\nu_\lambda}_\lambda$): generalized eigenspace of A corresponding to the eigenvalue λ.

$\operatorname{span}(S)$: (linear) subspace of $\mathbb{C}^n$ spanned by the subset S.

$\operatorname{span}_{\mathbb{R}}(S)$: real subspace of $\mathbb{C}^n$ spanned by the subset S.

$\operatorname{ri} C$: relative interior of the convex set C.

$E^{(0)}_\lambda(A)$ (or $E^{(0)}_\lambda$): the projection of $\mathbb{C}^n$ onto the generalized eigenspace $N^{\nu_\lambda(A)}_\lambda(A)$ along the direct sum of other generalized eigenspaces of A.

$E^{(k)}_\lambda(A)$ (or $E^{(k)}_\lambda$): kth principal component of A corresponding to λ, i.e., $(A-\lambda I)^k E^{(0)}_\lambda(A)$.

A cone K is said to be *polyhedral* if $K = \operatorname{pos}(S)$ for some nonempty set S; if, in addition, S is a set of linearly independent vectors, then K is said to be *simplicial*.

A (not necessarily closed, pointed) cone K is said to be the *direct sum* of cones $K_1, \ldots, K_p$, and we write $K = K_1 \oplus \cdots \oplus K_p$, if each vector in K can be expressed uniquely as $x_1 + \cdots + x_p$, where $x_i \in K_i, 1 \leq i \leq p$. K is called *decomposable* if it is the direct sum of two nonzero cones; otherwise, it is said to be *indecomposable*.

A linear transformation T on an n-dimensional vector space U is said to be a *cyclic transformation* if there exists a vector $b \in U$ for which $\{b, Tb, \ldots, T^{n-1}b\}$ is a basis for U. A matrix $A \in \mathcal{M}_n(\mathbb{C})$ is said to be *nonderogatory* if every eigenvalue of A has geometric multilplicity 1. It is known that a matrix $A \in \mathcal{M}_n(\mathbb{C})$ is nonderogratory if and only if A acts as a cyclic linear transformation on $\mathbb{C}^n$. For references, see [10, Chapter 6] or [11, Section 3.3].

Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $x \in \mathbb{C}^n$. We denote by W_x (respectively, $W_x^{\mathbb{R}}$), or $W_x(A)$ (respectively, $W_x^{\mathbb{R}}(A)$) if there is the need to indicate the dependence on A, the A-cyclic subspace (respectively, the real A-cyclic subspace) of $\mathbb{C}^n$ generated by x, i.e. $W_x = \{p(A)x : p(t) \in \mathbb{C}[t]\}$ (respectively, $W_x^{\mathbb{R}} = \{p(A)x : p(t) \in \mathbb{R}[t]\}$). When $A \in \mathcal{M}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$, to follow the notation of our earlier papers, sometimes we write $W_x^{\mathbb{R}}$ simply as W_x.

Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in W$, where W is an A-invariant subspace of $\mathbb{C}^n$. Since the representation of x as a sum of generalized eigenvectors of $A|_W$ and as a sum of generalized eigenvectors of A are the same, for any eigenvalue λ of $A|_W$ and any nonnegative integer k, we have $E_\lambda^{(k)}(A|_W) = E_\lambda^{(k)}(A)|_W$. As a consequence, we may write $E_\lambda^{(k)}(A|_{W_x})x$ simply as $E_\lambda^{(k}(A)x$.

For $A \in \mathcal{M}_n(\mathbb{C})$, by the *peripheral spectrum of* A we mean the set $\{\lambda \in \sigma(A) : |\lambda| = \rho(A)\}$. We also use terms peripheral eigenvalues and non-peripheral eigenvalues with the obvious meanings.

For $A \in \mathcal{M}_n(\mathbb{C})$ and $x \in \mathbb{C}^n$, $\rho_x(A)$ (or simply ρ_x), the *local spectral radius of* A *at* x is given by $\rho_x(A) = \lim\ \sup_{m\to\infty} \|A^m x\|^{1/m}$, where $\|\cdot\|$ is any norm of $\mathbb{C}^n$ or, equivalently, $\rho_x(A) = \rho(A|_{W_x})$. If x is nonzero and $x = x_1 + \cdots + x_k$ is the representation of x as a sum of generalized eigenvectors of A corresponding, respectively, to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\rho_x(A)$ is also equal to $\max_{1\leq i\leq k}|\lambda_i|$. (See, for instance, [32, Theorem 2.3].) We also define and denote the *order of* x *relative to* A by $\mathrm{ord}_A(x) = \max\{\mathrm{ord}_A(x_i) : |\lambda_i| = \rho_x(A)\}$, where $\mathrm{ord}_A(x_i)$ is the order of the generalized eigenvector x_i, i.e., the least positive integer l such that $(A - \lambda_i I)^l x_i = 0$. The ordered pair $(\rho_x(A), \mathrm{ord}_A(x))$, denoted by $\mathrm{sp}_A(x)$, is called the *spectral pair of* x *relative to* A. It was first introduced in [29] and has proved to be a useful concept.

Let A be an $n \times n$ complex (or real) matrix. A set K is said to be *invariant under* A, (or A *leaves invariant* K or A is K*-nonnegative*) if $AK \subseteq K$. When K is a proper cone, we denote by $\pi(K)$ the set of all such matrices A. Matrices in $\pi(K)$ are often referred to as *cone-preserving maps* or as *positive operators* on K.

A matrix $A \in \pi(K)$ is said to be *irreducible with respect to* K or K*-irreducible* if A has no eigenvectors in the boundary of K or, equivalently, the only faces of K that A leaves invariant are $\{0\}$ and K itself; A is *strictly positive with respect to* K or *strictly* K*-positive* if $A(K \setminus \{0\}) \subseteq \mathrm{int}K$; A is *primitive with respect to* K or K*-primitive* if A^p is strictly K-positive for some positive integer p.

If $A \in \pi(K)$ and $x \in K$ is an eigenvector (respectively, generalized eigenvector), then x is called a *distinguished eigenvector* (respectively, *distinguished generalized eigenvector) of* A *for* K, and the corresponding eigenvalue is known as a *distinguished eigenvalue of* A *for* K. When there is no danger of confusion, we simply use the terms distinguished eigenvector, distinguished generalized eigenvector and distinguished eigenvalue (of A).

Let K be a proper cone in $\mathbb{C}^n$ (or $\mathbb{R}^n$). A matrix $A \in \pi(K)$ is said to be an *automorphism of* K if its inverse A^{-1} exists and belongs to $\pi(K)$. The set of all automorphisms of K forms a group under matrix multiplication and is denoted by

Aut(K). It is clear that for any $A \in \mathcal{M}_n(\mathbb{C})$ (or $\mathcal{M}_n(\mathbb{R})$), $A \in \text{Aut}(K)$ if and only if $AK = K$.

An $n \times n$ complex matrix A is said to *satisfy the Perron-Schaefer condition* if $\rho(A) \in \sigma(A)$ and $\nu_\lambda(A) \le \nu_{\rho(A)}(A)$ for any peripheral eigenvalue λ of A.

To follow the common practice, by the spectrum of an $n \times n$ real matrix A we mean its spectrum as a complex matrix or the spectrum of the extension of A (as a linear operator) in the complexification $\mathbb{C}^n$ of $\mathbb{R}^n$. To say a real matrix (respectively, a linear transformation on a finite-dimensional real vector space V) satisfies the Perron-Schaefer condition, we really mean the real matrix, regarded as a complex matrix (respectively, the complex extension of the linear transformation to the complexification of V), satisfies the Perron-Schaefer condition. One can also formulate an equivalent definition for the Perron-Schaefer condition on a real matrix that corresponds to its real Jordan form, but in this paper we do not pursue such approach.

It is well-known that for a complex matrix A, $\sigma(A)$ equals the set of roots of the minimal polynomial of A, and for any $\lambda \in \sigma(A)$, $\nu_\lambda(A)$ equals the multiplicity of λ as a root of the minimal polynomial of A. So the Perron-Schaefer condition on a complex matrix (or a complex linear transformation) can also be restated in an equivalent form in terms of its minimal polynomial.

Remark 2.1. Let $A \in \mathcal{M}_n(\mathbb{C})$ (or $T \in L(V)$, where V is a finite-dimensional complex vector space). Then A (or T) satisfies the Perron-Schaefer condition if and only if $\rho(A)$ (or, $\rho(T)$) is a root of the minimal polynomial of A (or, of T) and with multiplicity not less than that of any other root with the same modulus.

If A is an $n \times n$ real matrix (or a linear transformation on a finite-dimensional real vector space), then the minimal polynomial of A over $\mathbb{C}$ and the minimal polynomial of A over $\mathbb{R}$ are the same. However, if A is complex, then the said minimal polynomials are usually different but are closely related.

Lemma 2.2. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Denote the minimal polynomial of A over $\mathbb{C}$ and the minimal polynomial of A over $\mathbb{R}$ by $\phi(t)$ and $\phi^{\mathbb{R}}(t)$ respectively.*

(i) *$\phi^{\mathbb{R}}(t)$ is the unique monic polynomial determined from $\phi(t)$ by the following properties: $\phi^{\mathbb{R}}(t)$ has the same real roots as $\phi(t)$ and with the same multiplicities; the set of non-real complex roots of $\phi^{\mathbb{R}}(t)$ equals the set of non-real complex roots of $\phi(t)$ together with their complex conjugates; and for each non-real complex root λ of $\phi(t)$, the multiplicities of λ and $\bar{\lambda}$ as roots of $\phi^{\mathbb{R}}(t)$ are both equal to the maximum of the multiplicities of λ and $\bar{\lambda}$ as roots of $\phi(t)$. (If $\bar{\lambda}$ is not a root of $\phi(t)$, its multiplicity is taken to be zero.)*

(ii) *$\phi^{\mathbb{R}}(t)$ equals the minimal polynomial (over $\mathbb{R}$) of A when treated as a real linear transformation acting on $\mathbb{C}^n$ (as a real vector space).*

Proof. (i) Let $\psi(t)$ denote the monic polynomial with the described properties. Clearly, $\psi(t)$ is a real polynomial, divisible by $\phi(t)$. As $\phi(t)$ annihilates A, so does $\psi(t)$. If $f(t)$ is a real annihilating polynomial for A, then $f(t)$ is divisible by $\phi(t)$. So each real root of $\phi(t)$ is a root of $f(t)$ and with multiplicity not less, and the non-real complex roots of $f(t)$ occur in conjugate pair and with a common multiplicity not less than the maximum of the corresponding multiplicities of these numbers as roots of $\phi(t)$. Thus $f(t)$ is divisible by $\psi(t)$. This proves that $\psi(t)$ is the minimal polynomial of A over $\mathbb{R}$, as desired.

(ii) By [28, Lemma 8.1] when A is treated as a real linear transformation acting on $\mathbb{C}^n$, it can be represented by a matrix similar to $\mathrm{diag}(A, \bar{A})$, where $\bar{A}$ denotes the conjugate matrix of A. So its minimal polynomial (over $\mathbb{R}$) is equal to the polynomial $\psi(t)$, which, in turn, is $\phi^{\mathbb{R}}(t)$. □

Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. We say A *satisfies the local Perron-Schaefer condition at* x if in the representation of x as a sum of generalized eigenvectors of A, there is a generalized eigenvector corresponding to $\rho_x(A)$ and the order of this generalized eigenvector is not less than that of any other generalized eigenvector that appears in the representation and corresponds to an eigenvalue with modulus $\rho_x(A)$. The local Perron-Schaefer condition was first introduced in [30] in the setting when $A \in \mathcal{M}_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. Here we extend the concept to the complex case. (Certainly, we can also formulate the concept in the slightly more general setting when $A \in L(V)$, where V is a finite dimensional real or complex vector space.) For convenience, we adopt the convention that A satisfies the local Perron-Schaefer condition at the zero vector.

It is clear that A satisfies the local Perron-Schaefer condition at x if and only if $\mathrm{sp}_A(x) = (\rho_x(A), \nu_{\rho_x(A)}(A|_{W_x}))$.

We denote by $\preceq$ the lexicographic ordering between ordered pairs of real numbers given by: $(\xi_1, \xi_2) \preceq (\eta_1, \eta_2)$ if either $\xi_1 < \eta_1$ or $\xi_1 = \eta_1$ and $\xi_2 \leq \eta_2$. We also write $(\xi_1, \xi_2) \prec (\eta_1, \eta_2)$ if $(\xi_1, \xi_2) \preceq (\eta_1, \eta_2)$ but the equality (in the usual sense) does not hold.

In [29, Theorem 4.7], using the concept of the spectral pair of a vector relative to a matrix, it is proved that if $A \in \pi(K)$, where K is a proper cone in $\mathbb{R}^n$, then for every $x \in K$, A satisfies the local Perron-Schaefer condition at x. The latter result has been applied in the work of [30] and [27]. We will give a direct proof for the complex version (and hence also the real version) of the latter result.

Lemma 2.3. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding respectively to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. For $j = 1, \ldots, k$, let n_j be the order of the generalized eigenvector x_j. Then $\beta = \bigcup_{j=1}^{k} \{(A - \lambda_j I)^{n_j - 1} x_j, (A - \lambda_j I)^{n_j - 2} x_j, \ldots, (A - \lambda_j I) x_j, x_j\}$ is an ordered basis for W_x, the matrix of $A|_{W_x}$ relative to β is $J_{n_1}(\lambda_1) \oplus \cdots J_{n_k}(\lambda_k)$, and the minimal polynomial of $A|_{W_x}$ is $\prod_{j=1}^{k} (t - \lambda_j)^{n_j}$.*

Proof. As can be readily checked, for each $j = 1, \ldots, k$, $\beta_j = \{(A - \lambda_j I)^{n_j - 1} x_j, (A - \lambda_j I)^{n_j - 2} x_j, \ldots, (A - \lambda_j I) x_j, x_j\}$ is an ordered basis for $A|_{W_{x_j}}$ and the matrix of $A|_{W_{x_j}}$ relative to β_j equals $J_{n_j}(\lambda_j)$. Since $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, the sum $W_{x_1} + \cdots + W_{x_k}$ is a direct sum and the matrix of the restriction of A to this direct sum relative to the ordered basis $\beta_1 \cup \cdots \cup \beta_k$ is $\bigoplus_{j=1}^{k} J_{n_j}(\lambda_j)$. We are going to show that $W_x = W_{x_1} \oplus \cdots \oplus W_{x_k}$. Once this is proved, the matrix of $A|_{W_x}$ relative to β is clearly $\bigoplus_{j=1}^{k} J_{n_j}(\lambda_j)$ and the assertion about the minimal polynomial of $A|_{W_x}$ will also follow. (We would like to add that the assertion about the minimal polynomial of $A|_{W_x}$ can also be established directly by showing that

$\prod_{j=1}^{k}(t-\lambda_j)^{n_j}$ is an annihilating polynomial for $A|_{W_x}$ but none of its proper divisor annihilates $A|_{W_x}$.)

The inclusion $W_x \subseteq \bigoplus_{j=1}^{k} W_{x_j}$ is obvious because W_x and $\bigoplus_{j=1}^{k} W_{x_j}$ are both A-invariant and $x \in \bigoplus_{j=1}^{k} W_{x_j}$. For the polynomial $p(t) := \prod_{j=1}^{k}(t-\lambda_j)^{n_j}$, we have $p(A)x = 0$, but $f(A)x \neq 0$ for any proper divisor $f(t)$ of $p(t)$. So $p(t)$ is the minimal polynomial of $A|_{W_x}$. Hence we have

$$\dim W_x = \deg p(t) = n_1 + \cdots + n_k = \dim(W_{x_1} \oplus \cdots \oplus W_{x_k}).$$

This establishes the equality relation $W_x = \bigoplus_{j=1}^{k} W_{x_j}$ and completes the proof. □

Corollary 2.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. The following conditions are equivalent*:

(a) *A satisfies the local Perron-Schaefer condition at x.*
(b) *$A|_{W_x}$ satisfies the Perron-Schaefer condition.*
(c) *$A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition.*

Proof. The equivalence of (a) and (b) clearly follows from Lemma 2.3.

(b)⇔(c): It is readily checked that the minimal polynomial of $A|_{W_x}$ over $\mathbb{C}$ (respectively, of $A|_{W_x^{\mathbb{R}}}$ over $\mathbb{R}$) equals the minimal polynomial of A with respect to x over $\mathbb{C}$ (respectively, over $\mathbb{R}$), i.e. the unique monic complex (respectively, real) polynomial $\psi_{A,x}(t)$ (respectively, $\psi^{\mathbb{R}}_{A,x}(t)$ of least degree such that $\psi_{A,x}(A)x = 0$ (respectively, $\psi^{\mathbb{R}}_{A,x}(A)x = 0$). So $\psi_{A,x}(t)$ divides $\psi^{\mathbb{R}}_{A,x}(t)$ and, in fact, $\psi^{\mathbb{R}}_{A,x}(t)$ is the unique (necessarily real) monic polynomial obtained from $\psi_{A,x}(t)$ as follows: $\psi^{\mathbb{R}}_{A,x}(t)$ has the same real roots as $\psi_{A,x}(t)$ and with the same multiplicities; the set of non-real complex roots of $\phi^{\mathbb{R}}(t)$ equals the set of non-real complex roots of $\psi_{A,x}(t)$ together with their complex conjugates; and for each non-real complex root λ of $\psi_{A,x}(t)$, the multiplicities of λ and $\bar{\lambda}$ as roots of $\psi^{\mathbb{R}}_{A,x}(t)$ are both equal to the maximum of the multiplicities of λ and $\bar{\lambda}$ as roots of $\psi_{A,x}(t)$. (If $\bar{\lambda}$ is not a root of $\psi_{A,x}(t)$, its multiplicity is taken to be zero.) Hence $\rho(A|_{W_x})$ equals $\rho(A|_{W_x^{\mathbb{R}}})$ and $\rho(A|_{W_x})$ is a root of $\psi_{A,x}(t)$ if and only if $\rho(A|_{W_x^{\mathbb{R}}}$ is a root of $\phi^{\mathbb{R}}(t)$ and when it happens, the multiplicity of $\rho(A|_{W_x})$ as a root of $\psi_{A,x}(t)$ is equal to that of $\rho(A|_{W_x^{\mathbb{R}}})$ as a root of $\psi^{\mathbb{R}}_{A,x}(t)$.

Now by Remark 2.1 $A|_{W_x}$ satisfies the Perron-Schaefer condition if and only if $\rho(A|_{W_x})$ is a root of $\psi_{A,x}(t)$ and with multiplicity not less than that of other root with the same modulus. In view of what we have done above, the latter is equivalent to the condition that $\rho(A|_{W_x^{\mathbb{R}}})$ is a root of $\psi^{\mathbb{R}}_{A,x}(t)$ and with multiplicity not less than that of other root with the same modulus. On the other hand, since $A|_{W_x^{\mathbb{R}}}$ is a real linear transformation, by Lemma 2.2(i) (or rather by its linear transformation version), its minimal polynomials over $\mathbb{C}$ and over $\mathbb{R}$ are the same. By Remark 2.1 again it follows that $A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition if and only if $\rho(A|_{W_x^{\mathbb{R}}})$ is a root of the minimal polynomial of $A|_{W_x^{\mathbb{R}}}$ over $\mathbb{R}$, i.e., the polynomial $\psi^{\mathbb{R}}_{A,x}(t)$, and with multiplicity not less than that of any other root with the same modulus. We can now conclude that $A|_{W_x}$ satisfies the Perron-Schaefer condition if and only if $A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition. □

In the course of the proof of Corollary 2.4, (b)$\Leftrightarrow$(c), we also establish the following:

Remark 2.5. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. The minimal polynomial of $A|_{W_x}$ over $\mathbb{C}$ (respectively, of $A|_{W_x^{\mathbb{R}}}$ over $\mathbb{R}$) equals $\psi_{A,x}(t)$ (respectively, $\psi^{\mathbb{R}}_{A,x}(t)$), the minimal polynomial of A with respect to x over $\mathbb{C}$ (respectively, over $\mathbb{R}$). Moreover, $\psi^{\mathbb{R}}_{A,x}(t)$ has the same real roots as $\psi_{A,x}(t)$ and with the same multiplicities; the set of non-real complex roots of $\psi^{\mathbb{R}}_{A,x}(t)$ equals the set of non-real complex roots of $\psi_{A,x}(t)$ together with their complex conjugates; and for each non-real complex root λ of $\psi_{A,x}(t)$, the multiplicities of λ and $\bar{\lambda}$ as roots of $\psi^{\mathbb{R}}_{A,x}(t)$ are both equal to the maximum of the multiplicities of λ and $\bar{\lambda}$ as roots of $\psi_{A,x}(t)$. (If $\bar{\lambda}$ is not a root of $\psi_{A,x}(t)$, its multiplicity is taken to be zero.)

Corollary 2.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$, and let K be a closed pointed cone in $\mathbb{C}^n$. If $AK \subseteq K$, then A satisfies the local Perron-Schaefer condition at x for every $x \in K$.*

Proof. For any $0 \neq x \in K$, $W_x^{\mathbb{R}} \cap K$ is an A-invariant proper cone in $W_x^{\mathbb{R}}$ and we have $A|_{W_x^{\mathbb{R}}} \in \pi(W_x^{\mathbb{R}} \cap K)$. So by the "only if" part of Theorem A, $A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition, and by Corollary 2.4 A satisfies the local Perron-Schaefer condition at x. □

Remark 2.7. Let K be a proper cone in $\mathbb{C}^n$ and let $A \in \pi(K)$. Then every distinguished eigenvalue of A for K is a nonnegative real number.

Proof. Let λ be a distinguished eigenvalue of A for K and let $x \in K$ be a corresponding eigenvector. In view of Corollary 2.6, A satisfies the local Perron-Schaefer condition at x; so λ equals $\rho_x(A)$ and is a nonnegative real number. □

Example 2.8. Let $A = \mathrm{diag}(1, i)$ and let $x = (1, 1)^T$. We have $Ax = (1, i)^T, A^2x = (1, -1)^T, A^3x = (1, -i)^T$ and $A^4x = x$. Let $p(t) = t^2 - (1+i)t + i$. Then, $p(A)x = 0$, but no monic polynomial with smaller degree has the same property. So $p(t)$ is the minimal polynomial of $A|_{W_x}$ over $\mathbb{C}$. As $p(t) = (t-1)(t-i)$, it is clear that $A|_{W_x}$ satisfies the Perron-Schaefer condition. Since the representation of x as a sum of generalized eigenvectors of A is given by: $(1, 1)^T = (1, 0)^T + (0, 1)^T$, where $(1, 0)^T, (0, 1)^T$ are respectively eigenvectors of A corresponding to 1 and i, it is clear that A satisfies the local Perron-Schaefer condition at x.

Now note that $p_{\mathbb{R}}(t) := (t-1)(t-i)(t+i) = t^3 - t^2 + t - 1$ is the real monic polynomial of least degree that satisfies $p_{\mathbb{R}}(A)x = 0$. So the minimal polynomial of $A|_{W_x^{\mathbb{R}}}$ is $p_{\mathbb{R}}(t)$ and it is clear that $A|_{W_x^{\mathbb{R}}}$ also satisfies the Perron-Schaefer condition.

In this case, $w_0(A, x)$ is the 3-dimensional pointed polyhedral cone with distinct extreme vectors x, Ax, A^2x, A^3x. The real A-cyclic subspace $W_x^{\mathbb{R}}$ is the 3-dimensional real subspace spanned by x, Ax and A^2x. The complex A-cyclic subspace W_x is $\mathbb{C}^2$. Note also that the complex space W_x is not equal to a complexification of the real space $W_x^{\mathbb{R}}$, as $\dim_{\mathbb{C}} W_x = 2 \neq 3 = \dim_{\mathbb{R}} W_x^{\mathbb{R}}$.

We will need the following elementary lemma (cf. [3, Lemma 1.3.4] and [20, Lemma 4.2]):

Lemma 2.9. *Let Λ be a finite nonempty set of complex numbers such that each $\lambda \in \Lambda$ is off the nonnegative real axis. Then there exists a polynomial $v(t)$ with positive coefficients such that $v(0) = 1$ and $v(\lambda) = 0$ for all $\lambda \in \Lambda$.*

Lemma 2.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that $A|_{W_x}$ has no positive eigenvalue. Then there exists a polynomial $v(t)$ with positive coefficients such that $v(0) = 1$ and $A^{\nu_0}v(A)x = 0$, where $\nu_0 = \nu_0(A|_{W_x})$.*

Proof. If $A|_{W_x}$ is nilpotent, take $p(t)$ to be the constant polynomial 1. So we assume that $A|_{W_x}$ is non-nilpotent. Then $A|_{W_x}$ can be represented by a matrix of the form $B \oplus N$, where B is a nonsingular matrix, each of whose eigenvalues is either complex non-real or negative real, and N is a nilpotent matrix with index of nilpotency equal to ν_0. Note that the summand B must exist, but the summand N may or may not exist, depending on whether or not 0 is an eigenvalue of $A|_{W_x}$. Since B does not have a nonnegative eigenvalue, by Lemma 2.9 there exists a polynomial $u(t) \in \mathbb{R}_+[t]$ of positive degree such that $u(0) = 1$ and $u(\lambda) = 0$ for all eigenvalues λ of B. Then $u(B)$ is a nilpotent matrix and we can find a positive integer l such that $u(B)^l = 0$. Let $v(t)$ be the polynomial $u(t)^l$. Then $v(t)$ is a polynomial with positive coefficients such that $v(0) = 1$ and $v(B) = 0$. Now it should be clear that we have $(A|_{W_x})^{\nu_0}v(A|_{W_x}) = 0$ or, in other words, $A^{\nu_0}v(A)x = 0$. □

3. Characterizations of the local Perron-Schaefer conditions and related conditions

Following [20], for any $A \in \mathcal{M}_n(\mathbb{C})$ and any nonnegative integer k, we denote by $w_k(A)$ the cone $\text{pos}\{A^i : i = k, k+1, \ldots\}$. When x is a vector of $\mathbb{C}^n$, we also denote by $w_k(A, x)$ the cone $\text{pos}\{A^i x : i = k, k+1, \ldots\}$.

Remark 3.1. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Then A satisfies the local Perron-Schaefer condition at x if and only if A satisfies the local Perron-Schaefer condition at $A^k x$ for some (or, for every) positive integer k.

Proof. It suffices to show that A satisfies the local Perron-Schaefer condition at x if and only if A satisfies the local Perron-Schaefer condition at Ax. Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $Ax = Ax_1 + \cdots + Ax_k$ is the representation of Ax as a sum of generalized eigenvectors of A and, moreover, for each $i = 1, \ldots, k$, Ax_i is a generalized eigenvector of A corresponding to λ_i and with the same order as x_i, except that when $\lambda_i = 0$, Ax_i is a generalized eigenvector of order one less than that of x_i (or does not appear if the order of x_i is 1). So from the definition of the local Perron-Schaefer condition, it is clear that A satisfies the local Perron-Schaefer condition at x if and only if A satisfies the local Perron-Schaefer condition at Ax. □

Remark 3.2. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Let k be a nonnegative integer.

(i) $w_k(A, x)$ is a real subspace if and only if $-A^k x \in w_k(A, x)$.

(ii) The cones $w_k(A, x), w_k(A|_{W_x}), w_k(A|_{W_x^{\mathbb{R}}})$ are either all zero (respectively, pointed, closed) or all nonzero (respectively, not pointed, not closed).

(iii) The cones $\text{cl}\, w_k(A, x), \text{cl}\, w_k(A|_{W_x}), \text{cl}\, w_k(A|_{W_x^{\mathbb{R}}})$ are either all zero (respectively, pointed) or are all nonzero (respectively, not pointed, not closed).

(iv) If A is non-nilpotent, then $w_k(A, x)$ is nonzero. If A is nilpotent, then $w_k(A, x)$ is nonzero if and only if $k \leq \nu_0(A|_{W_x}) - 1$.

Proof. (i) It suffices to prove the "if" part. Suppose $-A^kx \in w_k(A,x)$. Then $-A^kx = A^kv(A)x$ for some $v(t) \in \mathbb{R}_+[t]$. So for any $p(t) \in \mathbb{R}_+[t]$, we have $-A^kp(A)x = A^kp(A)v(A)x \in w_k(A,x)$. So $w_k(A,x)$ is a cone and we have $-w_k(A,x) \subseteq w_k(A,x)$; hence $w_k(A,x)$ is a real subspace.

(ii) and (iii): We are going to show that the cones $\operatorname{cl} w_k(A,x), \operatorname{cl} w_k(A|_{W_x})$, $\operatorname{cl} w_k(A|_{W_x^{\mathbb{R}}})$ are either all pointed or all not pointed. The proofs for the remaining parts are similar.

Suppose that the cone cl $w_k(A|_{W_x})$ is not pointed. Then there exist polynomials $p_j(t), q_j(t) \in t^k\mathbb{R}_+[t]$, for $j = 1, 2, \ldots$, such that $\lim_{j\to\infty} p_j(A|_{W_x})$ and $\lim_{j\to\infty} q_j(A|_{W_x})$ both exist, are nonzero, and $\lim_{j\to\infty} p_j(A|_{W_x}) = -\lim_{j\to\infty} q_j(A|_{W_x})$. Then the limits $\lim_{j\to\infty} p_j(A)x$ and $\lim_{j\to\infty} q_j(A)x$ both exist, are nonzero, and are the negative of each other. So the cone $\operatorname{cl} w_k(A,x)$ is not pointed.

Conversely, suppose that the cone $\operatorname{cl} w_k(A,x)$ is not pointed. Then there exist polynomials $p_j(t), q_j(t) \in t^k\mathbb{R}_+[t]$, for $j = 1, 2, \ldots$, such that the limits $\lim_{j\to\infty} p_j(A)x$ and $\lim_{j\to\infty} q_j(A)x$ both exist, are nonzero, and are the negative of each other. Then $\lim_{j\to\infty} p_j(A)v(A)x = -\lim_{j\to\infty} q_j(A)v(A)x$ for every $v(t) \in t^k\mathbb{R}_+[t]$; hence we have $0 \neq \lim_{j\to\infty} p_j(A|_{W_x}) = -\lim_{j\to\infty} q_j(A|_{W_x})$, and so the cone cl $w_k(A|_{W_x})$ is not pointed.

By replacing $A|_{W_x}$ by $A|_{W_x^{\mathbb{R}}}$ in the above argument, we also show that the cones $\operatorname{cl} w_k(A,x), \operatorname{cl} w_k(A|_{W_x^{\mathbb{R}}})$ are either both pointed or both not pointed.

(iv) Obvious. □

Remark 3.3. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Let ν_0 denote the order of the generalized eigenvector corresponding to 0 that appears in the representation of x as a sum of generalized eigenvectors of A. (If there is no such generalized eigenvector, ν_0 is taken to be zero.) Then $w_j(A,x)$ is (linearly) isomorphic to $w_{\nu_0}(A,x)$ for any integer $j > \nu_0$, and for any nonnegative integer $j \leq \nu_0 - 1$, $w_j(A,x)$ is the direct sum of $w_{\nu_0}(A,x)$ and the simplicial cone $\operatorname{pos}\{A^jx, \ldots, A^{\nu_0-1}x\}$.

Proof. Note that $\nu_0 = \nu_0(A|_{W_x^{\mathbb{R}}})$. When $j > \nu_0$, we have, $A^{j-\nu_0}w_{\nu_0}(A,x) = w_j(A,x)$, and since the restriction of A to $\operatorname{span}_{\mathbb{R}}\{A^ix : i = \nu_0, \nu_0+1, \ldots\}$ is an isomorphism, $w_j(A,x)$ is isomorphic to $w_{\nu_0}(A,x)$.

Now suppose $0 \leq j \leq \nu_0 - 1$. Clearly, we have

$$w_j(A,x) = \operatorname{pos}\{A^jx, A^{j+1}x, \ldots, A^{\nu_0-1}x\} + w_{\nu_0}(A,x).$$

To show that the sum is in fact a direct sum, let $a_i, j \leq i \leq \nu_0 - 1$, be real scalars such that $\sum_{i=j}^{\nu_0-1} a_iA^ix = w$, where $w \in \operatorname{span}_{\mathbb{R}} w_{\nu_0}(A,x)$. Write x as $y + z$, where y is a generalized eigenvector of A corresponding to 0 of order ν_0 and z belongs to the direct sum of generalized eigenspaces of A corresponding to nonzero eigenvalues. (If 0 is not an eigenvalue of A then $\nu_0 = 0$, and we are done.) Upon rewriting, the above equality relation becomes $\sum_{i=j}^{\nu_0-1} a_iA^iy = w - \sum_{i=j}^{\nu_0-1} A^iz$. Now the vector on the left side of the equality belongs to the generalized nullspace of A, whereas the vector on the right side belongs to the direct sum of generalized eigenspaces of A corresponding to nonzero eigenvalues because $\sum_{i=j}^{\nu_0-1} A^iz$ is one such vector and so is w (as $w \in \operatorname{span}_{\mathbb{R}} w_{\nu_0}(A,x) = \operatorname{span}_{\mathbb{R}} w_{\nu_0}(A,z)$). So this common vector must be the zero vector. As y is a generalized eigenvector corresponding to 0 of order ν_0, the vectors $y, Ay, \ldots, A^{\nu_0-1}y$ are linearly independent; thus we have $a_i = 0$ for $j \le i \le \nu_0 - 1$, and hence $\sum_{i=j}^{\nu_0-1} a_i A^i x = 0$. This shows that the sum is indeed a direct sum.

Note that the linear independence of $\{y, Ay, \ldots, A^{\nu_0-1}y\}$ also implies that of $\{x, Ax, \ldots, A^{\nu_0-1}x\}$; hence $\text{pos}\{x, Ax, \ldots, A^{\nu_0-1}x\}$ is a simplicial cone. □

It is not difficult to obtain the following analogous result:

Remark 3.4. Let $A \in \mathcal{M}_n(\mathbb{C})$. let W_A be the $\mathcal{L}_A$-cyclic subspace generated by A and denote $\nu_0(A)$ by ν_0. Then:

(i) $W_A = \text{span}\{A, \ldots, A^{\nu_0-1}\} \oplus \text{span}\{A^{\nu_0}, \ldots, A^{m-1}\}$, where m is the degree of the minimal polynomial of A ;

(ii) $w_j(A) = \text{pos}\{A^j, A^{j+1}, \ldots, A^{\nu_0-1}\} \oplus w_{\nu_0}(A)$ for $j = 0, \ldots, \nu_0 - 1$, and the cones $w_j(A)$, for $j = \nu_0, \nu_0 + 1, \ldots$, are pairwise isomorphic.

If A is nilpotent, the second summand in (i) does not appear. If A is nonsingular, the first summands in (i) and (ii) both do not appear.

Using Remark 3.3, we readily obtain the following:

Remark 3.5. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \ne x \in \mathbb{C}^n$.

(i) The following conditions are equivalent:
  (a) For every nonnegative integer k, the cone $w_0(A, k)$ is pointed (respectively, closed, pointed and closed).
  (b) For some nonnegative integer k, the cone $w_0(A, k)$ is pointed (respectively, closed, pointed and closed).
  (c) The cone $w_0(A, x)$ is pointed (respectively, closed, pointed and closed).

(ii) The following conditions are equivalent:
  (a) For every nonnegative integer k, the cone $\text{cl}\, w_0(A, k)$ is pointed.
  (b) For some nonnegative integer k, the cone $\text{cl}\, w_0(A, k)$ is pointed.
  (c) The cone $\text{cl}\, w_0(A, x)$ is pointed.

Theorem 3.6. *For any $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \ne x \in \mathbb{C}^n$. The following conditions are equivalent:*

(a1) *A satisfies the local Perron-Schaefer condition at x.*
(a2) *$A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition.*
(a3) *$A|_{W_x}$ satisfies the Perron-Schaefer condition.*
(b1) *For every nonnegative integer k, the cone $\text{cl}\, w_k(A, x)$ is pointed.*
(b2) *The cone $\text{cl}\, w_0(A, x)$ is pointed.*
(b3) *For some nonnegative integer k, the cone $\text{cl}\, w_k(A, x)$ is pointed.*
(c1) *There is an A-invariant closed, pointed, cone that contains x.*
(c2) *There is an A-invariant proper cone in $W_x^{\mathbb{R}}$ that contains x.*
(c3) *There is an A-invariant proper cone in $W_x^{\mathbb{R}}$.*

Proof. The equivalence of (a1), (a2) and (a3) holds by Corollary 2.4. The equivalence of (b1), (b2) and (b3) follows from Remark 3.5.

(b2) ⇒ (c2): $\text{cl}\, w_0(A, x)$ is the desired proper cone in $W_x^{\mathbb{R}}$.

The implications (c2) ⇒ (c1) and (c2) ⇒ (c3) are both obvious.

(c1) $\Rightarrow$ (b2): Let C be an A-invariant closed pointed cone that contains x. Then we have $\mathrm{cl}\, w_0(A, x) \subseteq C$, and as C is pointed, so is $\mathrm{cl}\, w_0(A, x)$.

(c3) $\Rightarrow$ (a2): Follows from the "only if" part of Theorem A.

We complete the proof by establishing the implication (a1) $\Rightarrow$ (b2) as follows:

Suppose that A satisfies the Perron-Schaefer condition at x, but $\mathrm{cl}\, w_0(A, x)$ is not pointed. Then there exist $p_m(t), q_m(t) \in \mathbb{R}_+[t], m = 1, 2, \ldots,$ such that $\lim_{m\to\infty} p_m(A)x$, $\lim_{m\to\infty} q_m(A)x$ both exist, are nonzero, and are the negative of each other. Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, with $\lambda_1 = \rho_x(A)$. We have $\lim_{m\to\infty}(p_m + q_m)(A)x = 0$ and hence $\lim_{m\to\infty}(p_m + q_m)(A)x_i = 0$ for $i = 1, \ldots, k$; in particular, $\lim_{m\to\infty}(p_m + q_m)(A)x_1 = 0$. By the theory of functions of matrices and, in particular, a formula for $f(J_k(\lambda))$ (see, for instance, [12, Chapter 9]), we have

$$(p_m + q_m)(A)x_1 = \sum_{j=0}^{\nu_{\rho_x}-1} \frac{(p_m^{(j)} + q_m^{(j)})(\rho_x)}{j!}(A - \rho_x I)^j x_1,$$

where $\nu_{\rho_x} = \nu_{\rho_x(A)}(A|_{W_x})$ and a similar equality relation holds for $(p_m + q_m)(A)x_i$ for $i = 2, \ldots, k$. So we have

$$\lim_{m\to\infty}(p_m^{(j)} + q_m^{(j)})(\rho_x) = 0 \text{ for } j = 0, \ldots, \nu_{\rho_x}.$$

Consider any λ_i with modulus ρ_x. By the local Perron-Schaefer condition of A at x, $\nu_{\lambda_i} \leq \nu_{\rho_x}$. For $j = 0, \ldots, \nu_{\lambda_i} - 1$, we have

$$0 \leq |p_m^{(j)}(\lambda_i)| \leq p_m^{(j)}(\rho_x) \leq (p_m^{(j)} + q_m^{(j)})(\rho_x);$$

so $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$. Hence $\lim_{m\to\infty} p_m(A)x_i = 0$ whenever $|\lambda_i| = \rho_x$. Now consider any λ_i with $|\lambda_i| < \rho_x$. Let C denote the circle $|z - \lambda_i| = \rho_x - |\lambda_i|$ in the complex plane. Noting that $\max_{w\in C} |p_m(w)| \leq p_m(\rho_x)$, by Cauchy's inequality ([6, p. 125]) or the Cauchy integral formula for derivatives, we have $0 \leq |p_m^{(j)}(\lambda_i)| \leq \frac{j!}{(\rho_x - |\lambda_i|)^j} p_m(\rho_x)$, which implies $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$ for $j = 0, \ldots, \nu_{\lambda_i}$. So $\lim_{m\to\infty} p_m(A)x = 0$, which is a contradiction. □

Note that in the proof of Theorem 3.6 we are assuming the "only if" part of Theorem A, but not its "if" part. In Section 5, we will establish the "if" part of Theorem A by showing that if A satisfies the Perron-Schaefer condition then there is a proper A-invariant cone which can be expressed as a finite sum of the closures of A-cyclic cones. Our argument will rely on Theorem 3.6, (a2)$\Rightarrow$(b2), which says that if $A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition, then there is a proper cone in $W_x^{\mathbb{R}}$ which is invariant under A (namely, $\mathrm{cl}\, w_0(A, x)$).

Next, we are going to deduce Theorem B, the intrinsic Perron-Frobenius theorem, from Theorem 3.6. We need the following lemma.

Lemma 3.7. *Let $0 \neq A \in \mathcal{M}_n(\mathbb{C})$. Let $\mathcal{L}_A$ be the linear operator on $\mathcal{M}_n(\mathbb{C})$ given by $\mathcal{L}_A(X) = AX$, and let W_A (respectively, $W_A^{\mathbb{R}}$) denote the $\mathcal{L}_A$-cyclic subspace (respectively, real subspace) of $\mathcal{M}_n(\mathbb{C})$ generated by A. Also let m be the degree of the minimal polynomial of A over $\mathbb{C}$ (respectively, over $\mathbb{R}$).*

(i) *If A is nonsingular, then $\{I_n, A, \ldots, A^{m-1}\}$ forms a basis for W_A (respectively, for $W_A^{\mathbb{R}}$), and the minimal polynomial of $\mathcal{L}_A|_{W_A}$ (respectively, of $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) equals the minimal polynomial of A over $\mathbb{C}$ (respectively, over $\mathbb{R}$).*

(ii) *If A is singular, then $\{A,\dots,A^{m-1}\}$ forms a basis for W_A (respectively, for $W_A^{\mathbb{R}}$), and the minimal polynomial of $\mathcal{L}_A|_{W_A}$ (respectively, of $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) is equal to the minimal polynomial of A over $\mathbb{C}$ (respectively, over $\mathbb{R}$) divided by t.*

Proof. Let $q(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ denote the minimal polynomial of A over $\mathbb{C}$ (or over $\mathbb{R}$).

(i) Since $q(t)$ is the minimal polynomial for A and A is nonsingular, we have $a_0 \neq 0, I_n \in W_A$ (respectively, $W_A^{\mathbb{R}}$) and $\{I_n, A, \dots, A^{m-1}\}$ is a spanning set for W_A (respectively, for $W_A^{\mathbb{R}}$). Indeed, the latter set forms an ordered basis for W_A (respectively, for $W_A^{\mathbb{R}}$), else we would obtain a nonzero annihilating polynomial for A of degress less than that of $q(t)$. Straightforward calculation shows that, relative to the said ordered basis for W_A (respectively, for $W_A^{\mathbb{R}}$), $\mathcal{L}_A|_{W_A}$ (respectively, $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) is represented by the companion matrix of $q(t)$, that is, the matrix C_q given by

$$C_q = \begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & -a_{m-2} \\ & & & 1 & -a_{m-1} \end{bmatrix}.$$

As $q(t)$ is the minimal polynomial of C_q (see, for instance, [11, Theorem 3.3.14]), it follows that $q(t)$ is also the common minimal polynomial for A and $\mathcal{L}_A|_{W_A}$ (respectively, for A and $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) .

(ii) When A is singular, by modifying the argument given in the proof for part (i), one can show that the set $\{A, A^2, \dots, A^{m-1}\}$ forms an ordered basis for W_A (respectively, for $W_A^{\mathbb{R}}$) and relative to this ordered basis $\mathcal{L}_A|_{W_A}$ (respectively, $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) is represented by the companion matrix of the polynomial $q(t)/t$. So, in this case, the minimal polynomial of $\mathcal{L}_A|_{W_A}$ (respectively, of $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) is equal to the minimal polynomial of A divided by t. □

The following alternative proof for the complex version of Lemma 3.7 may be of interest:

First, take note of the following: (1) For any $\lambda \in \sigma(A) \setminus \{0\}$, $\lambda E_\lambda^{(0)} + E_\lambda^{(1)}$ is a generalized eigenvector of $\mathcal{L}_A$ corresponding to λ of order $\nu_\lambda(:= \nu_\lambda(C)$; (2) if $0 \in \sigma(A)$, then $E_0^{(1)}$ is a generalized eigenvector of $\mathcal{L}_A$ corresponding to 0 of order $\nu_0 - 1$, provided that $\nu_0 \geq 2$, but $E_0^{(1)} = 0$ when $\nu_0 = 1$. So the spectral resolution of A in terms of its components (see, for instance, [12, p.315, Exercise 1(b)]), i.e.,

$$A = \sum_{\lambda \in \sigma(A)} (\lambda E_\lambda^{(0)} + E_\lambda^{(1)}),$$

where the (nonzero) term $E_0^{(0)} + E_0^{(1)}$ does not appear when $0 \in \sigma(A)$ and $\nu_0 = 1$, is in fact the representation of A as a sum of generalization eigenvectors of $\mathcal{L}_A$. But the minimal polynomial of a matrix C equals $\prod_{i=1}^k (t-\lambda_i)^{\nu_{\lambda_i}}$, where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of C ([11, Theorem 3.3.6]), it follows that the minimal polynomial of $\mathcal{L}_A|_{W_A}$ equals the minimal polynomial of A when A is nonsingular, and equals the minimal polynomial of A divided by t when A is singular.

Corollary 3.8. *Let $0 \neq A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent:*

(a) *A satisfies the Perron-Schaefer condition.*
(b) *$\mathcal{L}_A|_{W_A}$ satisfies the Perron-Schaefer condition.*
(c) *A, treated as a real linear transformation on $\mathbb{C}^n$, satisfies the Perron-Schaefer condition.*
(d) *$\mathcal{L}_A|_{W_A^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition.*

Proof. Let $\phi(t)$ (respectively, $\phi^{\mathbb{R}}(t)$) denote the minimal polynomial of A over $\mathbb{C}$ (respectively, over $\mathbb{R}$). For brevity, denote by T when A is treated as a real linear transformation on $\mathbb{C}^n$.

According to Lemma 2.2, $\phi^{\mathbb{R}}(t)$ is the minimal polynomial of T, $\phi(t)$ and $\phi^{\mathbb{R}}(t)$ share the same real eigenvalues and with the same multiplicities, the non-real complex eigenvalues of $\phi^{\mathbb{R}}(t)$ are precisely the non-real complex eigenvalues of $\phi(t)$ together with their complex conjugates, and for each non-real complex eigenvalue λ, $\nu_\lambda(T) = \max\{\nu_\lambda(A), \nu_{\bar{\lambda}}(A)\}$. Now it should be clear that A satisfies the Perron-Schaefer condition if and only if T satisfies the Perron-Schaefer condition, i.e., conditions (a) and (c) are equivalent.

By Lemma 3.7 the minimal polynomial of $\mathcal{L}_A|_{W_A}$ (respectively, of $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) is equal to $\phi(t)$ (respectively, $\phi^{\mathbb{R}}(t)$) or $\phi(t)$ (respectively, $\phi^{\mathbb{R}}(t)$) divided by t, depending on whether A is nonsingular or singular. So the nonzero part of the spectrum of A (respectively, of T) and that of $\mathcal{L}_A|_{W_A}$ (respectively, of $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$) are the same. Moreover, for each nonzero eigenvalue λ, $\nu_\lambda(A) = \nu_\lambda(\mathcal{L}_A|_{W_A})$ (respectively, $\nu_\lambda(T) = \nu_\lambda(\mathcal{L}_A|_{W_A^{\mathbb{R}}})$). Hence, A satisfies the Perron-Schaefer condition if and only if $\mathcal{L}_A|_{W_A}$ satisfies the Perron-Schaefer condition (respectively, T satisfies the Perron-Schaefer condition if and only if $\mathcal{L}_A|_{W_A^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition), i.e., conditions (a) and (b) (respectively, (c) and (d)) are equivalent. □

The equivalence of conditions (a) and (b) in Corollary 3.8 is implicit in [20, p.265, 2nd paragraph], where it was pointed out that the "only if" part of Theorem A implies the "only if" part of Theorem B. Here we are elaborating and extending the observation.

Proof of Theorem B. First, by Corollary 3.8 A satisfies the Perron-Schaefer condition if and only if $\mathcal{L}_A|_{W_A}$ satisfies the Perron-Schaefer condition. Next, by Theorem 3.6, (a3)⇔(b1)⇔(b3), (with $\mathcal{L}_A$ and A playing the roles of A and x respectively), the condition that $\mathcal{L}_A|_{W_A}$ satisfies the Perron-Schaefer condition is equivalent to that for some (or, for every) nonnegative integer k, the cone $\mathrm{cl}\, w_k(\mathcal{L}_A, A)$ is pointed. As we have $w_k(\mathcal{L}_A, A) = w_{k+1}(A)$ for every nonnegative integer k, the latter condition, in turn, is equivalent to the condition that for some (or, for every) positive integer k, the cone $\mathrm{cl}\, w_k(A)$ is pointed. Finally, by Remark 3.4(ii), for any positive integer k, $\mathrm{cl}\, w_k(A)$ is pointed if and only if $\mathrm{cl}\, w_0(A)$ is pointed. So A satisfies the Perron-Schaefer condition if and only if for some (or, for every) nonnegative integer k, the cone $\mathrm{cl}\, w_k(A)$ is pointed. □

Theorem 3.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{R}^n$. Let ν_0 be the order of the generalized eigenvector corresponding to 0 that appears in the representation of x*

as a sum of generalized eigenvectors of A. (If such vector does not appear in the representation, ν_0 is taken to be 0.) Then the following conditions are equivalent:

(a1) *In the representation of x as a sum of generalized eigenvectors of A, there is a generalized eigenvector corresponding to a positive eigenvalue.*

(a2) *The linear map $A|_{W_x}$ has a positive eigenvalue.*

(a3) *The linear map $A|_{W_x^{\mathbb{R}}}$ has a positive eigenvalue.*

(b1) *For every nonnegative integer k, the cone $w_k(A,x)$ is nonzero, pointed.*

(b2) *For every integer $k \geq \nu_0$, the cone $w_k(A,x)$ is nonzero, pointed.*

(b3) *The cone $w_{\nu_0}(A,x)$ is nonzero pointed.*

(b4) *For some integer $k \geq \nu_0$, the cone $w_k(A,x)$ is nonzero, pointed.*

(c1) *For every integer $k \geq \nu_0$, the cone $w_k(A,x)$ is not a real subspace of $\mathbb{C}^n$.*

(c2) *$w_{\nu_0}(A,x)$ is not a real subspace of $\mathbb{C}^n$.*

(c3) *For some integer $k \geq \nu_0$, the cone $w_k(A,x)$ is not a real subspace of $\mathbb{C}^n$.*

Proof. The equivalence of (a1) and (a2) follows from Lemma 2.3. By Remark 2.5 $A|_{W_x}$ and $A|_{W_x^{\mathbb{R}}}$ share the same real eigenvalues, so the equivalence of (a2) and (a3) follows.

Since an isomorphism preserves the property of being nonzero, pointed (as well as being a real subspace), by Remark 3.3, the equivalence of (b1)–(b4) (and (c1)–(c3)) follows.

(a1) $\Rightarrow$ (b1): Condition (a1) clearly implies that x is not a generalized eigenvector of A corresponding to 0. So, for every nonnegative integer k, $w_k(A,x)$ is a nonzero cone. Since $w_0(A,x) \supseteq w_k(A,x)$ for all positive integers k, it suffices to show that the cone $w_0(A,x)$ is pointed. Assume that the contrary holds. By Remark 3.2(ii), the cone $w_0(A|_{W_x})$ is also not pointed. So there exist nonzero polynomials $p(t), q(t) \in \mathbb{R}_+[t]$ such that $p(A|_{W_x})$ equals $-q(A|_{W_x})$ and is nonzero. Then we have $p(A)y = -q(A)y$ for every vector $y \in W_x$. Let α be a positive eigenvalue of $A|_{W_x}$ and let y be a corresponding eigenvector. We have $p(\alpha)y = -q(\alpha)y$, and hence $p(\alpha) + q(\alpha) = 0$. But $p(t) + q(t)$ is a nonzero polynomial with nonnegative coefficients, so we arrive at a contradiction.

Clearly we have the implication (b2) $\Rightarrow$ (c1). To complete the proof, it remains to establish the implication (c3) $\Rightarrow$ (a2).

(c3) $\Rightarrow$ (a2): Suppose that for some integer $k \geq \nu_0$, $w_k(A,x)$ is not a real subspace. Clearly, $A|_{W_x}$ is non-nilpotent. Assume to the contrary that condition (a2) does not hold. Then by Lemma 2.10, there exists a polynomial $v(t)$ of positive degree r and with positive coefficients such that $v(0) = 1$ and $A^{\nu_0}v(A)x = 0$, where $\nu_0 = \nu_0(A|_{W_x})$. Say, $v(t) = \sum_{i=0}^{r} a_r t^r$. Since $k \geq \nu_0$, we have $A^k v(A)x = 0$, which implies that

$$-A^k x = A^{k+1}(a_r A^{r-1} + a_{r-2}A^{r-2} + \cdots + a_1)x \in w_{k+1}(A,x) \subseteq w_k(A,x).$$

By Remark 3.2(i), $w_k(A,x)$ is a real subspace, which contradicts our assumption on k. □

Using the kind of arguments given in the proof of Theorem 3.9, it is not difficult to show the following:

Remark 3.10. When $x \neq 0$, the condition that $w_0(A,x)$ is a nonzero pointed cone is equivalent to the condition that either $A|_{W_x^{\mathbb{R}}}$ (or, $A|_{W_x}$) is nilpotent or $A|_{W_x^{\mathbb{R}}}$ (or, $A|_{W_x}$) has a positive eigenvalue. Thus, the condition is weaker than the equivalent conditions of Theorem 3.9.

In view of Remark 3.2, to the list of equivalent conditions in Theorem 3.9, one can add further conditions given in terms of the cones $w_k(A|_{W_x})$ or $w_k(A|_{W_x^{\mathbb{R}}})$.

Theorem 3.11. *Let $A \in \mathcal{M}_n(\mathbb{C})$, and let $0 \neq x \in \mathbb{C}^n$. Then $A|_{W_x}$ has a nonnegative eigenvalue if and only if the cone $w_0(A,x)$ is not a real subspace.*

Proof. "Only if" part: Assume to the contrary that $w_0(A,x)$ is a real subspace. Then $-x \in w_0(A,x)$ and so for some $p(t) \in \mathbb{R}_+[t]$, we have $-x = p(A)x$ and hence $-y = p(A)y$ for all $y \in W_x$. Let α be a nonnegative eigenvalue of $A|_{W_x}$ ane let y be a corresponding eigenvector. Then we have $-y = p(A)y = p(\alpha)y$ and so $-1 = p(\alpha)$, which is a contradiction.

"If" part: Suppose that $A|_{W_x}$ does not have a nonnegative eigenvalue. Then $A|_{W_x}$ is nonsingular and by Lemma 2.10 there exists a polynomial $v(t) \in \mathbb{R}_+[t]$ of positive degree m such that $v(0) = 1$ and $v(A)x = 0$. From the latter, we readily obtain $-x \in \text{pos}\{Ax, \ldots, A^m x\} \subseteq w_0(A,x)$ and hence $w_0(A,x)$ is a real subspace, which is a contradiction. ☐

Using Theorem 3.9, we are going to re-derive the following known result ([20, Theorem 1.6 and Theorem 6.4]):

Corollary 3.12. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent:*

(a) *A has a positive eigenvalue.*

(b) *For every (or, for some) integer $k \geq \nu_0(A)$, the cone $w_k(A)$ is nonzero, pointed.*

(c) *For every (or, for some) integer $k \geq \nu_0(A)$, the set $w_k(A)$ is not a real subspace of $\mathcal{M}_n(\mathbb{C})$.*

If, in addition, A is non-nilpotent, the following is also an equivalent condition:

(d) *For every (or, for some) nonnegative integer k, the cone $w_k(A)$ is nonzero, pointed.*

Proof. Denote by $\mathcal{L}_A$ the linear operator on $\mathcal{M}_n(\mathbb{C})$ defined by: $\mathcal{L}_A(X) = AX$. By Lemma 3.7, A and $\mathcal{L}_A|_{W_A}$ share the same nonzero eigenvalues; so condition (a) is equivalent to the condition that $\mathcal{L}_A|_{W_A}$ has a positive eigenvalue. By Theorem 3.9 (with $\mathcal{L}_A$ and A playing the roles of A and x respectively), the latter condition is equivalent to the following:

(b)′ For every (or, for some) integer $k \geq \nu_0(\mathcal{L}_A|_{W_A})$, the cone $w_k(\mathcal{L}_A, A)$ is nonzero pointed.

Note that we always have $w_k(\mathcal{L}_A|_{W_A}, A) = w_{k+1}(A)$. If A is singular, by Lemma 3.7 we have $\nu_0(\mathcal{L}_A|_{W_A}) = \nu_0(A) - 1$; hence condition (b)′ is equivalent to condition (b). If A is nonsingular, then so is $\mathcal{L}_A$ and hence also is $\mathcal{L}_A|_{W_A}$. In this case, $\nu_0(\mathcal{L}_A|_{W_A}) = \nu_0(A)$ and condition (b)′ is equivalent to the condition that $w_{k+1}(A)$ is a nonzero pointed cone for every (or, for some) integer $k \geq \nu_0(A) = 0$. Now by Remark 3.4(ii), the cones $w_0(A), w_1(A), \ldots$ are all isomorphic. So condition (b)′ is also equivalent to condition (b).

In the above, we have established the equivalence of conditions (a) and (b). In a similar way (and also making use of Theorem 3.9), we can show that conditions (a) and (c) are equivalent.

When A is non-nilpotent, $w_{\nu_0}(A,x)$ is nonzero, and again by Remark 3.4(ii), condition (d) is another equivalent condition. ☐

Condition (c) of Corollary 3.12 can also be replaced by the following (which is the condition that appears in [20, Theorem 1.6]):

(c)′ For every (or, for some) integer $k \geq \nu_0(A)$, the set $\mathrm{cl}\, w_k(A)$ is not a real subspace of $\mathcal{M}_n(\mathbb{C})$.

The point is, a convex set C in a finite dimensional (real or complex) vector space is a real subspace if and only if its closure is a real subspace. The latter, in turn, is a consequence of the following basic property for a convex set C in a finite-dimensional vector space: $\mathrm{ri}\, C = \mathrm{ri}\,(\mathrm{cl}\, C)$.

By applying Theorem 3.11 (with A and x replaced by $\mathcal{L}_A$ and A respectively) and Lemma 3.7, one can obtain the following:

Corollary 3.13. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent*:

(a) *Either A has a positive eigenvalue, or 0 is an eigenvalue of A with index ≥ 2.*

(b) *$w_1(A)$ is not a real subspace.*

The following result has appeared in [20, Theorem 6.3]. We give another proof here.

Theorem 3.14. *Let $A \in \mathcal{M}_n(\mathbb{C})$. The following conditions are equivalent*:

(a) *A has a nonnegative eigenvalue.*

(b) *$w_0(A)$ is not a real subspace.*

Proof. (a) ⇒ (b): Suppose that condition (a) holds. If A is nonsingular, then necessarily A has a positive eigenvalue and by Corollary 3.12, (a) ⇒ (c), condition (b) follows. If A is singular then, by part(ii) of Remark 3.4, $w_0(A)$ is the direct sum of the cones $\mathrm{pos}\{I_n, A, \ldots, A^{\nu_0(A)-1}\}$ and $w_{\nu_0(A)}(A)$. But the former cone is pointed, so $w_0(A)$ is not a real subspace.

(b) ⇒ (a): If A is singular, then 0 is an eigenvalue of A and condition (a) clearly holds. If A is nonsingular, then by Corollary 3.12, (c)⇒(a), A has a positive eigenvalue. □

Lemma 3.15. *Let $A \in \mathcal{M}_n(\mathbb{C})$. If x is a generalized eigenvector of A corresponding to a positive eigenvalue λ, then $\mathrm{cl}\, w_0(A, x)$ is a pointed polyhedral cone if and only if $\nu_\lambda(A|_{W_x}) \leq 2$.*

Proof. Clearly, A satisfies the local Perron-Schaefer condition at x. So the cone $\mathrm{cl}\, w_0(A, x)$ is always pointed.

"If" part: If $\nu_\lambda(A|_{W_x}) = 1$, $\mathrm{cl}\, w_0(A, x)$ is simply the ray generated by x. If $\nu_\lambda(A|_{W_x}) = 2$, then $\mathrm{cl}\, w_0(A, x)$ is the pointed polyhedral cone generated by the extreme vectors x and $E_\lambda^{(1)}(A)x$.

"Only if" part: Without loss of generality, we may assume that $A = J_k(\lambda)$ and x is the standard unit vector e_k. Then

$w_0(A, x) = \mathrm{pos}\{\binom{i}{m-1}\lambda^{i-m+1}, \ldots, \binom{i}{2}\lambda^{i-2}, \binom{i}{1}\lambda^{i-1}, \lambda^i)^T : i \geq 0\}$.

Now assume to the contrary that $k(= \nu_\lambda(A|_{W_x}) \geq 3$. Since $\mathrm{cl}\, w_0(A, x)$ is polyhedral, so is the cone $\mathrm{cl}\, C$, where $C = \mathrm{pos}\{\binom{i}{2}\lambda^{i-2}, \binom{i}{1}\lambda^{i-1}, \lambda^i)^T : i \geq 0\}$. Now C is the same as $\mathrm{pos}\{\binom{i}{2}\lambda^{-2}, \binom{i}{1}\lambda^{-1}, 1)^T : i \geq 0\}$ and is linearly isomorphic with the cone $\mathrm{pos}\{i(i-1)/2, i, 1)^T : i \geq 0\}$ under the nonsingular matrix $\mathrm{diag}(\lambda^2, \lambda, 1)^T$. But the latter cone is clearly a non-polyhedral closed pointed cone, so we arrive at a contradiction. □

According to [28, Theorem 7.9], for any $A \in \mathcal{M}_n(\mathbb{R})$, there exists a proper polyhedral cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$ if and only if each peripheral eigenvalue of A equals $\rho(A)$ times a root of unity. For the local version, we have the following partial result:

Theorem 3.16. *Let $A \in \mathcal{M}_n(\mathbb{C})$. If $\operatorname{cl} w_0(A, x)$ is a pointed polyhedral cone, then each of the following conditions is satisfied:*

(a) *A satisfies the local Perron-Schaefer condition at x.*
(b) *Each peripheral eigenvalue of $A|_{W_x}$ equals $\rho_x(A)$ times a root of unity.*
(c) *For each positive eigenvalue λ of $A|_{W_x}$, $\nu_\lambda(A|_{W_x}) \leq 2$.*

Proof. Condition (a) follows from Theorem 3.6, (b2) $\Leftrightarrow$ (a1).

Since $A|_{W_x^{\mathbb{R}}} \in \pi(\operatorname{cl} w_0(A, x))$, by [28, Theorem 7.9] each peripheral eigenvalue of $A|_{W_x^{\mathbb{R}}}$ equals $\rho_x(A)$ times a root of unity. By Remark 2.5 the same can be said for the peripheral eigenvalues of $A|_{W_x}$.

Now we show condition (c). Let $x = x_1 + \cdots + x_k$ be the representation of x as a sum of generalized eigenvectors of A, with x_i corresponding to λ_i for each i. Consider any $y \in \operatorname{cl} w_0(A, x)$. We have $y = \lim_{m\to\infty} p_m(A)x$ and some $p_m(t) \in \mathbb{R}_+[t], m = 1, 2, \ldots$. For each i, clearly $\lim_{m\to\infty} p_m(A)x$ exists, say, equals y_i. Then $y_i \in \operatorname{cl} w_0(A, x_i)$. The assumption that $\operatorname{cl} w_0(A, x)$ is a pointed polyhedral cone implies that if λ_i is a positive eigenvalue then $\operatorname{cl} w_0(A, x)$ is also a pointed polyhedral cone, and so by Lemma 3.15 we have $\nu_\lambda(A|_{W_x}) = \nu_\lambda(A|_{W_{x_i}}) \leq 2$. ☐

We end this section by noting the following observations, which are not difficult to prove:

Remark 3.17. Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Then:

(i) $w_0(A, x)$ is a simplicial cone if and only if the minimal polynomial of $A|_{W_x^{\mathbb{R}}}$ is of the form $t^m - a_1 t^{m-1} - \cdots - a_m$, where $m = \dim W_x^{\mathbb{R}}$ and $a_1, \ldots, a_m$ are all nonnegative real numbers.
(ii) $w_0(A, x)$ is a pointed polyhedral cone if and only if A satisfies the local Perron-Schaefer condition at x and $A|_{W_x^{\mathbb{R}}}$ (or, $A|_{W_x}$) has an annihilating polynomial of the form $t^p - a_1 t^{p-1} - \cdots - a_p$, where p is a positive integer and $a_1, \ldots, a_p$ are nonnegative real numbers.

4. When the cone $w_0(A, x)$ is closed and pointed

Lemma 4.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Then:*

(i) *For any $0 \neq y \in \operatorname{cl} w_0(A, x)$, we have $\rho_y(A) = \rho_x(A)$ and $\operatorname{sp}_A(y) \preceq \operatorname{sp}_A(x)$.*
(ii) *If $\rho_x(A) > 0$ then $\operatorname{sp}_A(y) = \operatorname{sp}_A(x)$ for every $0 \neq y \in w_0(A, x)$.*

Proof. (i) First, we show that $\rho_y(A) \leq \rho_x(A)$. Since $y \in \operatorname{cl} w_0(A, x)$, there exist $p_m(t) \in \mathbb{R}_+[t]$, $m = 1, 2, \ldots$, such that $y = \lim_{m\to\infty} p_m(A)x$. Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectorsof A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, and with $\lambda_1 = \rho_x(A)$. The existence of $\lim_{m\to\infty} p_m(A)x$ guarantees the existence of $\lim_{m\to\infty} p_m(A)x_i$ for $i = 1, \ldots, k$. Denote $\lim_{m\to\infty} p_m(A)x_i$ by y_i. For each i, since $x_i \in N_{\lambda_i}^{\nu_{\lambda_i}}(A|_{W_x})$, where $\nu_{\lambda_i} =$

$\nu_{\lambda_i}(A|_{W_x})$ and $N_{\lambda_i}^{\nu_{\lambda_i}}(A|_{W_x})$ is A-invariant, $p_m(A)x_i \in N_{\lambda_i}^{\nu_{\lambda_i}}(A|_{W_x})$ for every m. So we have $y = y_1 + \cdots + y_k$ with $y_i \in N_{\lambda_i}^{\nu_{\lambda_i}}(A|_{W_x})$ for each i, and hence

$\rho_y(A) = \max\{|\lambda_i| : 1 \le i \le k, y_i \ne 0\} \le \max_{1\le i\le k} |\lambda_i| \le \rho_x(A)$.

Suppose $\rho_y(A) \ne \rho_x(A)$. Then we must have $\rho_y(A) < \rho_x(A)$ and hence $y_i = 0$ whenever $|\lambda_i| = \rho_x(A)$. In particular, we have, $y_1 = 0$, i.e., $\lim_{m\to\infty} p_m(A)x_1 = 0$. As $p_m(A)x_1 = \sum_{j=0}^{\nu_{\rho_x}-1} \frac{p_m^{(j)}(\rho_x)}{j!}(A - \rho_x I)^j x_1$, where $\nu_{\rho_x} = \nu_{\rho_x}(A|_{W_x})$, it follows that we also have $\lim_{m\to\infty} p_m^{(j)}(\rho_x) = 0$ for $j = 0, \dots, \nu_{\rho_x} - 1$. Then using an argument given in the proof for Theorem 3.6,(a1)⇒(b2), we can show that for each $i = 2, \dots, k$,

$\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$ for $j = 0, \dots, \nu_{\lambda_i} - 1$.

So we have $\lim_{m\to\infty} p_m(A)x = 0$, which is a contradiction. This proves that $\rho_y(A) = \rho_x(A)$. Now, by definition, we have

$\mathrm{ord}_A(y) = \max\{\mathrm{ord}_A(y_i) : |\lambda_i| = \rho_x(A)\} \le \max\{\nu_{\lambda_i} : |\lambda_i| = \rho_x(A)\} = \mathrm{ord}_A(x)$,

where the inequality holds as $y_i \in N_{\lambda_i}^{\nu_{\lambda_i}}(A|_{W_x})$. Therefore, $\mathrm{sp}_A(y) \preceq \mathrm{sp}_A(x)$.

(ii) If $0 \ne y \in w_0(A, x)$, then y can be written as $\sum_{i=0}^{p} a_i A^i x$ for some nonnegative integer p and some nonnegative real numbers $a_0, \dots, a_p$ with at least one positive. By [29, Theorem 4.9(ii)(c)] we have $\mathrm{sp}_A(y) = \max\{\mathrm{sp}_A(A^i x) : a_i > 0\}$, where the maximum is taken in the sense of lexicographic ordering. Since $A|_{W_x^{\mathbb{R}}} \in \pi(\mathrm{cl}\, w_0(A, x))$ and $\rho_x(A) > 0$, by [29, Remark 4.1(ii)] we have $\mathrm{sp}_A(Ax) = \mathrm{sp}_A(x)$ and hence $\mathrm{sp}_A(A^i x) = \mathrm{sp}_A(x)$ for every positive integer x. It follows that $\mathrm{sp}_A(y) = \mathrm{sp}_A(x)$. □

Remark 4.2. In Lemma 4.1, if we drop the assumption that A satisfies the local Perron-Schaefer condition at x, then the result, in its full strength, no longer holds. However, by a slight modification of the original argument, one can show that $\mathrm{sp}_A(y) \preceq \mathrm{sp}_A(x)$ for any $0 \ne y \in \mathrm{cl}\, w_0(A, x)$.

To see this, take $A = \mathrm{diag}(-1, \frac{1}{2})$ and $x = (1, 1)^T$. Then A does not satisfy the local Perron-Schaefer condition at x, as $\rho_x(A) = 1 \notin \sigma(A|_{W_x})$. For $p(t) = 1 + t, p(A)x = (0, \frac{3}{2})^T$; so $\mathrm{sp}_A(p(A)x) = (\frac{1}{2}, 1) \prec (1, 1) = \mathrm{sp}_A(x)$.

Remark 4.3. In part(i) of Lemma 4.1, we cannot replace $\preceq$ by the equality.

To see this, let $A = J_2(1)$ and $x = (0, 1)^T$. Then, as can be readily shown, $w_0(A, x) = \mathbb{R}^2_+ \setminus \{\lambda(1, 0)^T : \lambda > 0\}$, so $\mathrm{cl}\, w_0(A, x) = \mathbb{R}^2_+$. Take $y = (1, 0)^T$. Then $\mathrm{sp}_A(y) = (1, 1) \prec (1, 2) = \mathrm{sp}_A(x)$.

If K is an A-invariant closed pointed cone, then the set $\bigcap_{i=0}^{\infty} A^i K$, denoted by $\mathrm{core}_K(A)$, is called the *core of A relative to K*. By [28, Theorem 2.2] $\mathrm{core}_K(A)$ is always a closed, pointed cone (and usually its dimension is less than that of K) and $A|_{\mathrm{span}_{\mathbb{R}}(\mathrm{core}_K(A))}$ is an automorphism of $\mathrm{core}_K(A)$.

Lemma 4.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \ne x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Let Λ denote the set of peripheral eigenvalues λ of $A|_{W_x}$ for which $\nu_\lambda(A|_{W_x^{\mathbb{R}}}) = \nu_{\rho_x}(A|_{W_x^{\mathbb{R}}})$.*

(i) *If $\rho_x(A) > 0$ then $E_{\rho_x}^{(\nu_{\rho_x}-1)}(A)x$ and $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_{\rho_x}-1)}(A)x$ both belong to $\bigcap_{i=0}^{\infty} \mathrm{cl}\, w_i(A, x)$.*

(ii) *$\mathrm{core}_{\mathrm{cl}\, w_0(A,x)}(A|_{W_x^{\mathbb{R}}}) = \bigcap_{i=0}^{\infty} \mathrm{cl}\, w_i(A, x)$.*

(iii) $\rho_x(A)$ *is the only distinguished eigenvalue of* $A|_{W_x^{\mathbb{R}}}$ *for* $\mathrm{cl}\, w_0(A,x)$ *and* (*up to positive multiples*) $E_{\rho_x(A)}^{(\nu_{\rho_x}-1)}(A)x$ *is the unique corresponding eigenvector in* $\mathrm{cl}\, w_0(A,x)$.

Proof. For brevity, denote $A|_{\mathrm{span}_{\mathbb{R}} w_k(A,x)}$ simply as $A_{k,x}$.

(i) For each nonnegative integer k, since A satisfies the local Perron-Schaefer condition at x, by Theorem 3.6, $\mathrm{cl}\, w_k(A,x)$ is a proper cone in its own real linear span, so $A_{k,x} \in \pi(\mathrm{cl}\, w_k(A,x))$. When $\rho_x(A) > 0$, by [27, the proof of Theorem 3.1], we have

$$\begin{aligned} E_\rho^{(\nu-1)}(A_{k,x}) &= \lim_{j\to\infty}((\nu-1)!)j^{-(\nu-1)}\rho^{-k}(\rho+1)^{-(j-\nu+1)}(A_{k,x}+I)^j A_{k,x}^k \\ &\in \mathrm{cl}\, w_k(A_{k,x}), \end{aligned}$$

where $\rho = \rho_x(A), \nu = \nu_{\rho_x(A)}(A|_{W_x})$, and hence $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}(A)x = E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}(A_{k,x})x \in \mathrm{cl}\, w_k(A,x)$. Since this is true for each nonnegative integer k, we have, $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}x \in \bigcap_{k=0}^\infty \mathrm{cl}\, w_k(A,x)$.

To prove that $(\sum_\lambda E_\lambda^{(\nu_{\rho_x(A)}-1)})x \in \bigcap_{k=0}^\infty \mathrm{cl}\, w_k(A,x)$, we use a similar argument, except that now we apply instead [28, the proof of Theorem 7.1(i)], which says that if $A \in \pi(K)$, then $\sum_\lambda E_\lambda^{(\nu_{\rho(A)}(A)-1)} \in \pi(K)$, where the sum is taken over all eigenvalues λ with $|\lambda| = \rho(A)$ and $\nu_\lambda(A) = \nu_{\rho(A)}(A)$.

(ii) Let $y \in \mathrm{core}_{\mathrm{cl}\, w_0(A,x)}(A|_{W_x^{\mathbb{R}}})$. For each nonnegative integer i, there exist polynomials $p_k(t) \in \mathbb{R}_+[t], k = 1,2,\ldots$ (depending on i), such that $\lim_{k\to\infty} p_k(A)x$ exists and $y = A^i(\lim_{k\to\infty} p_k(A)x)$. As $A^i p_k(A)x \in w_i(A,x)$ for each k, it follows that $y \in \mathrm{cl}\, w_i(A,x)$. This establishes the inclusion $\mathrm{core}_{\mathrm{cl} w_0(A,x)}(A|_{W_x}) \subseteq \bigcap_{i=0}^\infty \mathrm{cl}\, w_i(A,x)$.

To show the reverse inclusion, consider any $y \in \bigcap_{i=0}^\infty \mathrm{cl}\, w_i(A,x)$. For each positive integer i, since $y \in \mathrm{cl}\, w_i(A,x)$, we can find a vector $y_i \in w_i(A,x)(= A^i w_0(A,x))$ such that $\|y_i - y\| \le \frac{1}{i}$, where $\|\cdot\|$ denotes any norm in $\mathbb{C}^n$. Then the sequence $(y_i)_{i\in\mathbb{Z}_+}$ converges to y and as $y_i \in A^i \mathrm{cl}\, w_0(A,x)$ for each i, by [28, Remark 3.10] it follows that $y \in \mathrm{core}_{\mathrm{cl} w_0(A,x)}(A|_{W_x})$, as desired.

(iii) Since $A|_{W_x^{\mathbb{R}}} \in \pi(\mathrm{cl}\, w_0(A,x))$, by [25, Theorem 2.4], every distinguished eigenvalue of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\, w_0(A,x)$ equals $\rho_y(A)$ for some nonzero $y \in \mathrm{cl}\, w_0(A,x)$. By Lemma 4.1(i), for any $0 \ne y \in \mathrm{cl}\, w_0(A,x), \rho_y(A) = \rho_x(A)$. Hence $\rho_x(A)$ is the only distinguished eigenvalue of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\, w_0(A,x)$. Since $A|_{W_x^{\mathbb{R}}}$ is a cyclic linear transformation, each of its eigenvalues has geometric multiplicity 1, and as $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}x \in \mathrm{cl}\, w_0(A,x)$, by part (i) our assertion follows. □

We would like to add that the fact if $A \in \mathcal{M}_n(\mathbb{C})$ satisfies the Perron-Schaefer condition then $E_{\rho(A)}^{(\nu_{\rho(A)}-1)}(A) \in \bigcap_{i=0}^\infty \mathrm{cl}\, \omega_i(A)$ is known (see Schneider [20, Theorem 5.2(iii)]; a short direct proof can also be found in [27, the proof of Theorem 3.1]). In part(i) of Lemma 4.4 we give a local version of the latter result.

Remark 4.5. If $\rho_x(A) = 0$, then Lemma 4.4(i) no longer holds. Instead, we have $E_{\rho_x(A)}^{(\nu_{\rho_x}-1)}(A)x = \sum_{\lambda\in\Lambda} E_\lambda^{(\nu_{\rho_x}-1)}(A)x \in w_0(A,x)$.

This is because, when $\rho_x(A) = 0$, $w_i(A, x) = 0$ for $i \geq \nu_0(A|_{W_x})$, and $E^{(\nu_{\rho_x}-1)}_{\rho_x(A)}(A)$ and $\sum_{\lambda\in\Lambda} E^{(\nu_{\rho_x}-1)}_{\lambda}(A)$ are both equal to $A^{\nu_{\rho_x}-1}$.

Lemma 4.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Then $E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}(A)x \in w_0(A, x)$ if and only if either $\rho_x(A) = 0$ or $\rho_x(A) > 0, \mathrm{ord}_A(x) = 1$ and $A|_{W_x}$ (or $A|_{W_x^{\mathbb{R}}}$) has no positive eigenvalue other than $\rho_x(A)$.*

Proof. Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to the distinct eigenvalue $\lambda_1, \ldots, \lambda_k$ respectively and with $\lambda_1 = \rho_x(A)$.

"Only if" part: Assume that $\rho_x(A) > 0$. Since $E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}(A)x \in w_0(A, x)$, by Lemma 4.1(ii), $\mathrm{sp}_A(x) = \mathrm{sp}_A(E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}(A)x) = (\rho_x(A), 1)$, and so $\mathrm{ord}_A(x) = 1$. On the other hand, we also have, $E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}(A)x = p(A)x$ for some $p(t) \in \mathbb{R}_+[t]$. As $E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}(A)x = E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}(A)x_1$, $p(A)x_1$ is an eigenvector of A corresponding to $\rho_x(A)$ and for $i = 2, \ldots, k$, we have $p(A)x_i = 0$ or, equivalently, $p(A)|_{W_{x_i}}$ is the zero operator. Now for $i = 1, \ldots, k$, the matrix of $A|_{W_{x_i}}$ relative to the ordered basis $\{(A - \lambda_i I)^{n_i-1}x_i, (A - \lambda_i I)^{n_i-2}x_i, \ldots, (A - \lambda_i I)x_i, x_i)\}$ is the elementary Jordan block $J_{n_i}(\lambda_i)$, where n_i is the order of the generalized eigenvector x_i. So the matrix of $p(A)|_{W_{x_i}}$ relative to the said ordered basis is an upper triangular matrix with diagonal entries all equal to $p(\lambda_i)$ (see [12, p.311, Theorem 4]). Thus we have $p(\lambda_i) = 0$ for $i = 2, \ldots, k$. Since the coefficients of the nonzero polynomisl $p(t)$ are all nonnegative, none of the numbers $\lambda_2, \ldots, \lambda_k$ can be a positive real number. This shows that $A|_{W_x}$ (and hence also $A|_{W_x^{\mathbb{R}}}$) has no positive eigenvalue other than $\rho_x(A)$.

"If" part: There is no problem if $\rho_x(A) = 0$. Hereafter, we assume that $\rho_x(A) > 0$. Since $\mathrm{ord}_A(x) = 1$, x_1 is an eigenvector of A corresponding to $\rho_x(A)$. As $A|_{W_x}$ has no positive eigenvalue other than $\rho_x(A)$, by applying Lemma 2.10 (with $x = x_2 + \cdots + x_k$), we can find a nonzero $p(t) \in \mathbb{R}_+[t]$ such that $A^{\nu_0}p(A)(x_2 + \cdots + x_k) = 0$, where $\nu_0 = \nu_0(A|_{W_{x_2+\cdots+x_k}})$. Then we have $A^{\nu_0}p(A)x = A^{\nu_0}p(A)x_1 = \rho_x(A)^{\nu_0}p(\rho_x(A))x_1$. It follows that x_1 belongs to $w_0(A, x)$. Now, sine $\nu_{\rho_x(A)} = \mathrm{ord}_A(x) = 1$, we have, $E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}x = E^{(0)}_{\rho_x(A)}x = x_1$. So $E^{(\nu_{\rho_x(A)}-1)}_{\rho_x(A)}x \in w_0(A, x)$. □

Lemma 4.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Then $\sum_{\lambda\in\Lambda} E^{(\nu_\lambda(A|_{W_x})-1)}_{\lambda}x \in w_0(A, x)$, where Λ has the same meaning as given in Lemma 4.4 if and only if either $\rho_x(A) = 0$ or $\rho_x(A) > 0, \mathrm{ord}_A(x) = 1$ and the following condition is satisfied: if the eigenvalues of $A|_{W_x}$ are not all of the same modulus then every peripheral eigenvalue of $A|_{W_x}$ equals $\rho_x(A)$ times a root of unity and for each non-peripheral eigenvalue μ of $A|_{W_x}$, μ^m is not a positive real number, where m denotes the least positive integer such that $(\frac{\lambda}{|\lambda|})^m = 1$ for every peripheral eigenvalue λ of A.*

Proof. Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to the distinct eigenvalue $\lambda_1, \ldots, \lambda_k$ respectively. Without loss of generality, assume that $\Lambda = \{\lambda_1, \ldots, \lambda_l\}$ and $\lambda_1 = \rho_x(A)$.

"Only if" part: Suppose $\rho_x(A) > 0$. Since $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda(A|_{W_x})-1)} x \in w_0(A,x)$, by Lemma 4.1(ii), $\mathrm{sp}_A(x) = \mathrm{sp}_A(\sum_{i=1}^{l} E_{\lambda_i}^{(\nu_{\lambda_i}-1)} x) = (\rho_x(A),1)$, and so $\mathrm{ord}_A(x) = 1$. On the other hand, we have $\sum_{i=1}^{l} E_{\lambda_i}^{(\nu_{\lambda_i}-1)} x = p(A)x$ for some nonzero $p(t) \in \mathbb{R}_+[t]$. The latter equality relation implies that for $i = 1,\dots,l$, $p(A)x_i$ equals $E_{\lambda_i}^{(\nu_{\lambda_i}-1)} x_i$ and so is an eigenvector of A corresponding to λ_i and, for $l+1 \le i \le k$, $p(A)x_i = 0$. As A satisfies the local Perron-Schaefer condition at x, for $i = 1,\dots,l$, x_i is an eigenvector of λ_i and $p(A)x_i = E_{\lambda_i}^{(\nu_{\lambda_i}-1)} x_i = x_i$.

Now suppose that $l < k$, i.e., $A|_{W_x}$ has an eigenvalue with modulus less than $\rho_x(A)$. In this case the polynomial $p(t)$, considered above, must be of positive degree. Normalizing A, we may assume that $\rho_x(A) = 1$. Assume to the contrary that there exists $r, 2 \le r \le l$, such that $\lambda_r = e^{\sqrt{-1}\,\theta}$, where θ is a real number that is not a rational multiple of π. Say, $p(t) = \sum_{j=0}^{s} a_j t^j$. Note that the condition that $p(A)x_1 = x_1$ and x_1 is an eigenvector of A corresponding to 1 implies that $\sum_{i=0}^{s} a_i = 1$. So the corresponding condition for x_r implies that 1 is a convex combination of the extreme points $e^{\sqrt{-1}\,j\theta}, j = 0,\dots,s$, of the unit circle in the complex plane. As θ is not a rational multiple of π, none of the numbers $e^{\sqrt{-1}\,j\theta}, j = 1,\dots,s$, equals 1. It follows that we must have $a_0 = 1$ and $a_j = 0$ for $j = 1,\dots,s$; hence $p(t)$ is a constant polynomial, which is a contradiction. This proves that every unimodular eigenvalue of A is a root of unity.

A slight modification of the above argument also shows that the condition $p(A)x_i = x_i$, for $i = 1,\dots,l$, implies that for each $j, 0 \le j \le s, a_j$ is nonzero only if j is a multiple of m. So the polynomial $p(t)$ is of the form $q(t^m)$ for some $q(t) \in \mathbb{R}_+[t]$ with positive degree. Then the conditions $p(A)x_i = 0$ for $i = l+1,\dots,k$ become $q(A^m)x_i = 0$ or, equivalently, $q(A)|_{W_{x_i}} = 0$ for $i = l+1,\dots,k$. For each such i, since $q(\lambda_i^m)$ is the only eigenvalue of $q(A^m)|_{W_{x_i}}$, we must have $q(\lambda_i^m) = 0$. As $q(t)$ belongs to $\mathbb{R}_+[t]$ and has positive degree, this implies that λ_i^m is not a positive real number for $i = l+1,\dots,k$.

"If" part: If $\rho_x(A) = 0$ then we have $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)} x = A^{\nu_0(A|_{W_x})-1} x \in w_0(A,x)$. Hereafter we assume that $\rho_x(A) > 0$.

First, consider the case when the eigenvalues of $A|_{W_x}$ are all of the same modulus. Then $k = l$ and since $\mathrm{ord}_A(x) = 1$, we have, $E_{\lambda_i}^{(\nu_{\lambda_i}-1)} x = E_{\lambda_i}^{(0)} x = x_i$ for each i; hence $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_{\lambda_i}-1)} x = \sum_{i=1}^{k} x_i = x \in w_0(A,x)$.

Now consider the case when $A|_{W_x}$ has an eigenvalue with modulus less than $\rho_x(A)$. Since none of the numbers $\lambda_{l+1}^m,\dots,\lambda_k^m$ is a positive real number, by Lemma 2.10 there exists a nonzero $q(t) \in \mathbb{R}_+[t]$ with positive degree such that $q(1) = 1$ and $A^c q(A^m) \sum_{i=1}^{l} x_i = 0$, where c is any common positive integral multiple of m and $\nu_0(A|_{W_{x_{l+1}+\cdots+x_k}})$. Let $p(t) = t^c q(t^m)$. Then $p(t) \in \mathbb{R}_+[t]$. For $i = 1,\dots,l$, since x_i is an eigenvector, $p(A)x_i = A^c q(A^m)x_i = \lambda_i^c q(\lambda_i^m)x_i = q(1)x_i = x_i$. So we have

$$p(A)x = \sum_{i=1}^{l} p(A)x_i + p(A)(\sum_{i=l+1}^{k} x_i) = \sum_{i=1}^{l} x_i = \sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)} x.$$

This shows that $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)} x \in w_0(A,x)$. □

By Lemma 4.6 and Lemma 4.7 we readily obtain the following:

Corollary 4.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. If $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)}x \in w_0(A,x)$, where Λ has the same meaning as given in Lemma 4.4, then $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}(A)x \in w_0(A,x)$.*

When A satisfies the local Perron-Schaefer condition at x, by Lemma 4.4(i), we have $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_{\rho_x}-1)}x \in \mathrm{cl}\, w_0(A,x)$. So $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)}(A)x \in w_0(A,x)$ is a necessary condition for $w_0(A,x)$ to be closed. However, it is not a sufficient condition, as illustrated by the following example.

Example 4.9. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \oplus [1]$, where θ is not a rational multiple of π, and take $x = (1,0,1)^T$. Then $W_x^{\mathbb{R}} = \mathbb{R}^3$, A is diagonalizable (over $\mathbb{C}$), $\sigma(A) = \{1, e^{i\theta}, e^{-i\theta}\} = \Lambda$, $\rho_x(A) = \rho(A) = 1$, A satisfies the local Perron-Schaefer conditon at x, and $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda-1)} = I$. Thus $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu-1)}x = x \in w_0(A,x)$. In this case, $\mathrm{cl}\, w_0(A,x)$ is the ice-cream cone $K_3 := \{(\xi_1,\xi_2,\xi_3)^T : \xi_3 \geq (\xi_1^2+\xi_2^2)^{1/2}\}$. Also, for any $0 \neq y \in K_3, \mathrm{sp}_A(y) = (1,1) = \mathrm{sp}_A(x)$. However, $w_0(A,x)$ is not closed, as all extreme vectors of K_3, except those of the form $\lambda(\cos k\theta, \sin k\theta, 1)^T$ with $k = 0,1,2,\ldots, \lambda > 0$, do not belong to $w_0(A,x)$.

Lemma 4.10. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that the eigenvalues of $A|_{W_x}$ are all of the same modulus and $\rho_x(A)$ is an eigenvalue of $A|_{W_x}$. Then the cone $w_0(A,x)$ is closed if and only if either $\rho_x(A) = 0$ or $\rho_x(A) > 0$, $\mathrm{ord}_A(x) = 1$, and every eigenvalue of $A|_{W_x}$ equals $\rho_x(A)$ times a root of unity.*

Proof. Let $x = x_1 + \cdots + x_k$, where $x_1,\ldots,x_k$ are generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1,\ldots,\lambda_k$ respectively.

"Only if" part: Suppose that $\rho_x(A) > 0$. Since $w_0(A,x)$ is closed, by Lemma 4.4(i) $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda(A|_{W_x})-1)}x \in w_0(A,x)$, where Λ has the same meaning as before. Then by the "only if" part of Lemma 4.7, $\mathrm{ord}_A(x) = 1$. Assume that $A|_{W_x}$ has an eigenvalue which is not equal to $\rho_x(A)$ times a root of unity. Without loss of generality, assume that $\lambda_1 = \rho_x(A) = 1$ and $\lambda_2 = e^{\sqrt{-1}\theta}$, where θ is not a rational multiple of π. Choose an increasing sequence of positive integers $(m_j)_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} e^{\sqrt{-1}m_j}$ exists and equals $e^{\sqrt{-1}\phi}$, where ϕ is a real number such that $e^{\sqrt{-1}\phi}$ is not a nonnegative integral power of $e^{\sqrt{-1}\theta}$ and $\lim_{j\to\infty} \lambda_i^{m_j}$ exists for $i = 1,\ldots,k$. Then $\lim_{j\to\infty} A^{m_j}x$ exists. Since $w_0(A,x)$ is a closed cone, $\lim_{j\to\infty} A^{m_j}x = p(A)x$ for some $p(t) \in \mathbb{R}_+[t]$. Now we have $p(1)x_1 = p(A)x_1 = \lim_{j\to\infty} A^{m_j}x_1 = x_1$, so $p(1) = 1$. Also $p(e^{i\theta})x_2 = p(A)x_2 = \lim_{j\to\infty} A^{m_j}x_2 = e^{\sqrt{-1}\phi}x_2$. So we have $e^{\sqrt{-1}\phi} = p(e^{\sqrt{-1}\theta})$, i.e., $e^{\sqrt{-1}\phi}$ is a finite convex combination of nonnegative integral powers of $e^{\sqrt{-1}\theta}$. But $e^{\sqrt{-1}\phi}$ is an extreme point of the unit circle in the complex plane and is not a nonnegative integral power of $e^{\sqrt{-1}\theta}$, so we arrive at a contradiction.

"If" part: In this case, we can find a positive integer m such that $\lambda_i^m = \rho_x(A)^m$ for $i = 1,\ldots,k$. Then $A^m x = \rho_x(A)^m x$; so $w_0(A,x)$ is a pointed polyhedral cone generated by $x, Ax, \ldots, A^{m-1}x$. As such it must be a closed cone. □

Corollary 4.11. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Then $A|_{W_x^{\mathbb{R}}}$ is irreducible with respect to*

$\mathrm{cl}\, w_0(A,x)$ *if and only if either* $Ax = 0$ *or* $A|_{W_x^{\mathbb{R}}}$ *is nonsingular,* $\mathrm{ord}_A(x) = 1$ *and* $A|_{W_x^{\mathbb{R}}}$ *has no positive eigenvalue other than* $\rho_x(A)$.

Proof. Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, and with $\lambda_1 = \rho_x(A)$.

"Only if" part: Since $A|_{W_x^{\mathbb{R}}}$ is irreducible with respect to $\mathrm{cl}\, w_0(A,x)$, $E^{(\nu_{\rho_x}-1)}x$, the unique distinguished eigenvector of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\, w_0(A,x)$, belongs to $\mathrm{ri}(\mathrm{cl}\, w_0(A,x)) = \mathrm{ri}\, w_0(A,x)$. By Lemma 4.6, either $\rho_x(A) = 0$ or $\rho_x(A) > 0$, $\mathrm{ord}_A(x) = 1$ and $A|_{W_x^{\mathbb{R}}}$ has no positive eigenvalue other than $\rho_x(A)$. If $\rho_x(A) = 0$ then $\mathrm{cl}\, w_0(A,x)$ equals the simplicial cone with distinct extreme vectors $x, \ldots, A^{\nu_{\rho_x}-1}x$. But $E_{\rho_x}^{(\nu_{\rho_x}-1)}x$ equals $A^{\nu_{\rho_x}-1}x$ and in order that it belongs to $\mathrm{ri}\,(\mathrm{cl}\, w_0(A,x))$, we must have $\nu_{\rho_x} = 1$ and $\mathrm{cl}\, w_0(A,x) = \mathrm{pos}\{x\}$, i.e., $Ax = 0$.

Now consider the case when $\rho_x(A) > 0$. Since $A|_{W_x^{\mathbb{R}}}$ has no positive eigenvalue other than $\rho_x(A)$, by Lemma 2.10 there exists a polynomial $v(t)$ with positive coefficients such that $A^{\nu_0}v(A)(x_2 + \cdots + x_k) = 0$, where $\nu_0 = \nu_0(A|_{W_x^{\mathbb{R}}})$. Then $A^{\nu_0}v(A)x = A^{\nu_0}v(A)x_1 = \rho_x(A)^{\nu_0}v(\rho_x(A))x_1$, where the second equality holds as x_1 is an eigenvector of A corresponding to $\rho_x(A)$. Since $\rho_x(A)^{\nu_0}v(\rho_x(A)) > 0$, $A^{v_0}v(A)x$ is a positive multiple of the eigenvector x_1. Suppose that $A|_{W_x^{\mathbb{R}}}$ is singular. Then $\nu_0 \geq 1$. Note that since $0 \neq v(t) \in \mathbb{R}_+[t]$, $A^{\nu_0}v(A)x \in w_{\nu_0}(A,x)$. According to Remark 3.3, $\mathrm{cl}\, w_0(A,x)$ is the direct sum of the simplicial cone $\mathrm{pos}\,\{x, \ldots, A^{\nu_0-1}x\}$ and $\mathrm{cl}\, w_{\nu_0}(A,x)$. So the eigenvector of $A|_{W_x^{\mathbb{R}}}$ in $\mathrm{cl}\, w_0(A,x)$ belongs to the relative boundary of $\mathrm{cl}\, w_0(A,x)$. This contradicts the irreducibility of $A|_{W_x^{\mathbb{R}}}$.

"If" part: If $Ax = 0$, then $\mathrm{cl}\, w_0(A,x)$ equals the ray generated by x. As $x \in \mathrm{ri}\,\mathrm{pos}\{\lambda : \lambda \geq 0\}$, $A|_{W_x^{\mathbb{R}}}$ is irreducible with respect to $\mathrm{cl}\, w_0(A,x)$.

Consider the case when $A|_{W_x^{\mathbb{R}}}$ is nonsingular. By Lemma 2.10 there exists a polynomial $v(t)$ with positive coefficients such that $v(A)(x_2 + \cdots + x_k) = 0$. Note that the degree of $v(t)$ can be chosen as large as we please. So we may assume that $\deg v(t) \geq \dim W_x^{\mathbb{R}}$. This guarantees that $v(A)x \in \mathrm{ri}\,(\mathrm{cl}\, w_0(A,x))$. According to the argument given in the proof for the "only if" part for the case $\rho_x(A) > 0$, $v(A)x$ is a positive multiple of the eigenvector x_1 which, in turn, equals $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}x$ as $\nu_{\rho_x(A)} = \mathrm{ord}_A(x) = 1$. By Lemma 4.4(iii), (up to positive multiples) $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}x$ is the unique distinguished eigenvector of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\, w_0(A,x)$. So $A|_{W_x^{\mathbb{R}}}$ has no eigenvector in the relative boundary of $\mathrm{cl}\, w_0(A,x)$; hence it is irreducible with respect to $\mathrm{cl}\, w_0(A,x)$. □

Corollary 4.12. *Let* $A \in \mathcal{M}_n(\mathbb{C})$ *and let* $0 \neq x \in \mathbb{C}^n$. *Suppose that* A *satisfies the local Perron-Schaefer condition at* x. *Then* $A|_{W_x^{\mathbb{R}}}$ *is primitive with respect to* $\mathrm{cl}\, w_0(A,x)$ *if and only if* $A|_{W_x^{\mathbb{R}}}$ *is nonsingular,* $\mathrm{ord}_A(x) = 1$, $\rho_x(A)$ *is the only peripheral eigenvalue of* $A|_{W_x^{\mathbb{R}}}$ *, and* $A|_{W_x^{\mathbb{R}}}$ *has no positive eigenvalue other than* $\rho_x(A)$.

Proof. "Only if" part: The condition that $A|_{W_x^{\mathbb{R}}}$ is primitive with respect to $\mathrm{cl}\, w_0(A,x)$ implies that $A|_{W_x^{\mathbb{R}}}$ is irreducible with respect to $\mathrm{cl}\, w_0(A,x)$. So by Corollary 4.11, either $Ax = 0$ or $A|_{W_x^{\mathbb{R}}}$ is nonsingular, $\mathrm{ord}_A(x) = 1$ and $A|_{W_x^{\mathbb{R}}}$ has no positive eigenvalue other than $\rho_x(A)$. Clearly, we can rule out the possibility that $Ax = 0$. As

$A|_{W_x^{\mathbb{R}}}$ is primitive with respect to $\mathrm{cl}\, w_0(A, x)$, by [3, Theorem 1.4.10] $\rho_x(A)$ is the only peripheral eigenvalue of $A|_{W_x^{\mathbb{R}}}$.

"If" part: Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively, and with $\lambda_1 = \rho_x(A)$. By the proof for the "if" part of Corollary 4.11, there exists a polynomial $v(t)$ with positive coefficients such that $v(A)x$ is a positive multiple of $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)} x (= x_1)$, the unique eigenvector of $A|_{W_x^{\mathbb{R}}}$ in $\mathrm{cl}\, w_0(A, x)$ and, moreover, $\deg v(t) \geq \dim W_x^{\mathbb{R}}$. so that $v(A)x \in \mathrm{ri}\, \mathrm{cl}\, w_0(A, x)$. Replacing A by $A/\rho_x(A)$, if necessary, we may assume that $\rho_x(A) = 1$. Since 1 is the only peripheral eigenvalue of $A|_{W_x^{\mathbb{R}}}$, we have, $\lim_{m\to\infty} A^m(x_2 + \cdots + x_k) = 0$ and hence $\lim_{m\to\infty} A^m x = x_1$. As $x_1 \in \mathrm{ri}\,(\mathrm{cl}\, w_0(A, x))$, it follows that there exists a positive integer m such that $A^m x \in \mathrm{ri}\, w_0(A, x)$ and hence $(A|_{W_x^{\mathbb{R}}})^m$ is strictly positive with respect to $\mathrm{cl}\, w_0(A, x)$. Thus $A|_{W_x^{\mathbb{R}}}$ is primitive with respect to $\mathrm{cl}\, w_0(A, x)$. □

Corollary 4.13. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Then $A|_{W_x^{\mathbb{R}}}$ is strictly positive with respect to $\mathrm{cl}\, w_0(A, x)$ if and only if x is an eigenvector of A corresponding to a positive eigenvalue.*

Proof. "If" part: Obvious.

"Only if" part: Since $A|_{W_x^{\mathbb{R}}}$ is strictly positive with respect to $\mathrm{cl}\, w_0(A, x)$, $A|_{W_x^{\mathbb{R}}}$ is also primitive with respect to $\mathrm{cl}\, w_0(A, x)$. By the proof for the "only if" part of Corollary 4.12, $\mathrm{cl}\, w_0(A, x)$ is a pointed polyhedral cone generated by $x, Ax, \ldots, A^l x$ for some nonnegative integer l. As A is strictly positive with respect to $\mathrm{cl}\, w_0(A, x), A^i x \in \mathrm{ri}\,(\mathrm{cl}\, w_0(A, x))$ for $i = 1, 2, \ldots$. So we must have $l = 0$ and $\mathrm{cl}\, w_0(A, x) = \mathrm{pos}\{x\}$, i.e, x is an eigenvector of A corresponding to a positive eigenvalue. □

5. Construction of closed, pointed invariant cones

For brevity, we relax our usage of the terms distinguished eigenvalues and distinguished eigenvectors. For $A \in \mathcal{M}_n(\mathbb{C})$ and any closed, pointed A-invariant cone C, if x is an eigenvector of A in C corresponding to the eigenvalue λ, we say x (respectively, λ) is a distinguished eigenvector (respectively, distinguished eigenvale) of A (instead of $A|_{\mathrm{span}_{\mathbb{R}} C}$) for C.

It is straightfoward to prove the following:

Lemma 5.1. *Let C_1, C_2 be pointed cones in $\mathbb{C}^n$. Then $C_1 + C_2$ is a pointed cone if and only if $C_1 \cap (-C_2) = \{0\}$.*

Lemma 5.2. *Let C_1, C_2 be closed, pointed cones in $\mathbb{C}^n$. The following conditions are equivalent:*

(a) *$C_1 \cap (-C_2) = \{0\}$.*

(b) *$C_1 + C_2$ is pointed.*

(c) *$C_1 + C_2$ is closed and pointed.*

Suppose, in addition, that C_1, C_2 are invariant under a matrix $A \in \mathcal{M}_n(\mathbb{C})$. For $i = 1, 2$, let D_i denote the cone generated by the distinguished eigenvectors of A for C_i. Then the following conditions are also equivalent to conditions (a)—(c).

(d) $C_1 \cap (-C_2)$ *does not contain an eigenvector of* A.

(e) *The cone* $D_1 + D_2$ *is pointed.*

Proof. The equivalence of (a) and (b) follows from Lemma 5.1.

(a) and (b) implies (c): Since C_1, C_2 are closed cones and $C_1 \cap (-C_2) = \{0\}$, the closedness of $C_1 + C_2$ is guaranteed by a standard result in the theory of convex sets (see, for instance, [17, Corollary 9.1.2]).

So conditions (a), (b) and (c) are equivalent.

Last Part: The implication (a)$\Rightarrow$(d) is obvious.

(d)$\Rightarrow$(a): If $C_1 \cap (-C_2) \neq \{0\}$, then $C_1 \cap (-C_2)$ is a nonzero closed pointed cone invariant under A, and by the Perron-Frobenius theorem for cone-preserving maps, A has an eigenvector in $C_1 \cap (-C_2)$.

Note that for $i = 1, 2$, D_i is a closed pointed cone, as C_i is a closed pointed cone and D_i equals $\oplus_\lambda[\mathcal{N}(\lambda I - A) \cap C_i]$, where the direct sum is taken over all distinguished eigenvalues λ of A for C_i. In view of Lemma 5.1, condition (e) is equivalent to the following:

(e)$'$ $D_1 \cap -(D_2) = \{0\}$.

As the implications (a)$\Rightarrow$(e)$'$ and $\sim$(d)$\Rightarrow\sim$(e)$'$ are both obvious, (e) is clearly also an additional equivalent condition. $\square$

Lemma 5.3. *Let* $A \in \mathcal{M}_n(\mathbb{C})$*, and let* K_1, K_2 *be closed, pointed cones in* $\mathbb{C}^n$ *invariant under* A*. Suppose that* $K_1 + K_2$ *is pointed. Then for every distinguished eigenvalue* λ *of* A *for* $K_1 + K_2$*, we have*

$(K_1 + K_2) \cap \mathcal{N}(A - \lambda I) = (K_1 \cap \mathcal{N}(A - \lambda I)) + (K_2 \cap \mathcal{N}(A - \lambda I))$.

Thus the distinguished eigenvalues of A *for* $K_1 + K_2$ *are precisely the distinguished eigenvalues of* A *for* K_1 *or for* K_2.

Proof. It is clear that we have the inclusion

$$(K_1 \cap \mathcal{N}(A - \lambda I)) + (K_1 \cap \mathcal{N}(A - \lambda I)) \subseteq (K_1 + K_2) \cap \mathcal{N}(A - \lambda I).$$

To prove the reverse inclusion, let $x = x_1 + x_2$, with $x_1 \in K_1, x_2 \in K_2$, be an eigenvector of A corresponding to λ. Write each x_i $(i = 1, 2)$ as $u_i + v_i$, where $u_i \in N_\lambda^\nu$ and $v_i \in \oplus_{\mu \neq \lambda} N_\mu^\nu$. Clearly $v_1 + v_2 = 0$. It suffices to show that u_1, u_2 are each either the zero vector or an eigenvector corresponding to λ. Suppose not. Then necessarily u_1, u_2 are both nonzero generalized eigenvectors of A corresponding to λ and with a common order $m \geq 2$. Without loss of generality, assume that $\rho_{x_1}(A) \geq \rho_{x_2}(A)$. In view of the representation $x_2 = u_2 + v_2$ and an equivalent definition of the local spectral radius, we have $\rho_{x_2}(A) \geq \rho_{u_2}(A) = \lambda$. We contend that $\rho_{x_1}(A) = \lambda$. Once this is proved, we will have $\rho_{x_1}(A) = \rho_{x_2}(A) = \lambda$. Assume to the contrary that $\rho_{x_1}(A) > \lambda$. Then in the representation of v_1 as a sum of generalized eigenvectors of A, there must be a generalized eigenvector, say y, that corresponds to $\rho_{x_1}(A)$, and as $v_1 + v_2 = 0$, $-y$ also appears in the representation of v_2 as a sum of generalized eigenvectors of A; hence $\rho_{x_2}(A) = \rho_{v_2}(A) \geq \rho_{x_1}(A)$. As we are assuming $\rho_{x_1}(A) \geq \rho_{x_2}(A)$, we obtain $\rho_{x_1}(A) = \rho_{x_2}(A)$. Clearly, y (respectively, $-y$) is also the generalized eigenvector that appears in the representation of x_1 (respectively, of x_2) as a sum of generalized eigenvectors of A that corresponds to the eigenvalue $\rho_{x_1}(A)$. Let p denote the order of y. Since $x_i \in K_i, A|_{\mathrm{span}_\mathbb{R} K_i} \in \pi(K)$ and $\mathrm{ord}_A(x_i) = p$ $(i = 1, 2)$, by Lemma 4.4(i) we have $E_{\rho_{x_i}}^{(p-1)} x_i \in K_i$ for $i = 1, 2$.

But $E^{(0)}_{\rho_{x_1}} x_1 = y$, $E^{(0)}_{\rho_{x_2}} x_2 = -y$ and $E^{(p-1)}_{\rho_{x_i}} x_i = (A - \rho_{x_1} I)^{p-1} E^{(0)}_{\rho_{x_i}} x_i$, it follows that $0 \neq E_{\rho_{x_1}^{(p-1)}} x_1 \in K_1 \cap (-K_2)$. In view of Lemma 5.1, this contradicts the assumption that $K_1 + K_2$ is pointed.

By what we have done above, $\rho_{x_1}(A) = \rho_{x_2}(A) = \lambda$ and it is also clear that we have $\text{ord}_A(x_1) = \text{ord}_A(x_2) = m$, where $m\,(\geq 2)$ is the common order of u_1 and u_2. Now for $i = 1, 2$, we have, $0 \neq E^{(m-1)}_{\rho_{x_i}} x_i = (A - \lambda I)^{m-1} u_i$ and also $E^{(m-1)}_{\lambda} x_1 = -E^{(m-1)}_{\lambda} x_2$ as

$(A - \lambda I)^{m-1}(u_1 + u_2) = (A - \lambda I)^{m-2}[(A - \lambda I)(u_1 + u_2)] = (A - \lambda I)^{m-2}(A - \lambda I)x = 0$.

So we obtain $0 \neq E^{(m-1)}_{\rho_{x_1}} x_1 \in K_1 \cap (-K_2)$, which again contradicts the pointedness assumption on $K_1 + K_2$. □

Corollary 5.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $x_1, \dots x_k, k \geq 1$, be vectors of $\mathbb{C}^n$ and suppose that A satisfies the local Perron-Schaefer conditions at $x_1, \dots, x_k$ respectively. Then the following conditions are equivalent*:

(a) *The cone* $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_k)$ *is closed and pointed.*
(b) *There exists a closed, pointed A-invarnat cone in $\mathbb{C}^n$ that contains $x_1, \dots, x_k$.*
(c) *The cone* $\text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, k\}$ *is pointed.*

When the equivalent conditions are satisfied, the set of distinguished eigenvalues of A for $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_k)$ *is* $\{\rho_{x_i}(A) : i = 1, \dots, k\}$ *and the cone generated by the distinguished eigenvectors of A for* $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_k)$ *is* $\text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, k\}$.

Proof. (a)⇒(b): When (a) is satisfied, $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_k)$ is the desired closed, pointed A-invariant cone.

(b)⇒(c): Suppose that there exists a closed, pointed A-invariant cone C that contains $x_1, \dots, x_k$. For each $i = 1, \dots, k$, by Lemma 4.4 (i) we have $E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i \in \text{cl}\, w_0(A, x_i) \subseteq C$, and hence $\text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, k\} \subseteq C$. As C is pointed, so is $\text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, k\}$.

(c)⇒(a) and the last part: We proceed by induction on k. When $k = 1$, conditions (a)—(c) are always satisfied and the last part of our result holds by Lemma 4.4 (iii). Consider $k = p \geq 2$, and assume that our assertion holds for $k = p - 1$. By our induction assumption, the set of distinguished eigenvalues of A for $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_{p-1})$ is $\{\rho_{x_i}(A) : i = 1, \dots, p - 1\}$, and as $\rho_{x_p}(A)$ is the only distinguished eigenvalue of A for $\text{cl}\, w_p(A, x_p)$, by Lemma 5.3 the set of distinguished eigenvalues of A for $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_p)$ is $\{\rho_{x_i}(A) : i = 1, \dots, p\}$. Our induction assumption also guarantees that the cone $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_{p-1})$ is closed and pointed, and the cone generated by the distinguished eigenvectors of A for the latter cone is $\text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, p - 1\}$. Since $\text{pos}\{E^{(\nu_{\rho_{x_p}} - 1)}_{\rho_{x_p}} x_p\} + \text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, p - 1\}$ equals $\text{pos}\{E^{(\nu_{\rho_{x_i}} - 1)}_{\rho_{x_i}} x_i : i = 1, \dots, p\}$ and is pointed, by Lemma 5.2, (e)⇒(c), the cone $\text{cl}\, w_0(A, x_1) + \cdots + \text{cl}\, w_0(A, x_p)$ is also closed and pointed. It remains to show that D, the cone generated by the distinguished eigenvectors of A for $\operatorname{cl} w_0(A, x_1) + \cdots + \operatorname{cl} w_0(A, x_p)$, is precisely $\operatorname{pos}\{E_{\rho_{x_i}}^{(\nu_{\rho_{x_i}}-1)} x_i : i = 1, \ldots, p\}$. It is clear that we have the inclusion $\operatorname{pos}\{E_{\rho_{x_i}}^{(\nu_{\rho_{x_i}}-1)} x_i : i = 1, \ldots, p\} \subseteq D$. To prove the reverse inclusion, let $y \in D$ and suppose that y corresponding to the distinguished eigenvalue λ. By Lemma 5.3 we can write y as $y_1 + y_2$, where y_1 lies in $[\operatorname{cl} w_0(A, x_1) + \cdots + \operatorname{cl} w_0(A, x_{p-1})] \cap \mathcal{N}(A - \lambda I)$ and hence, by the induction assumption, belongs to $\operatorname{pos}\{E_{\rho_{x_i}}^{(\nu_{\rho_{x_i}}-1)} x_i : i = 1, \ldots, p-1\}$, and y_2 belongs to $\operatorname{cl} w_0(A, x_p) \cap \mathcal{N}(A - \lambda I)$, which is included in $\operatorname{pos}\{E_{\rho_{x_p}}^{(\nu_{\rho_{x_p}}-1)} x_p\}$, in view of Lemma 4.4(iii); hence $y \in \operatorname{pos}\{E_{\rho_{x_i}}^{(\nu_{\rho_{x_i}}-1)} x_i : i = 1, \ldots, p+1\}$, as desired. □

Note that the real version of Lemma 5.4 (i.e., the version where $\mathcal{M}_n(\mathbb{C}), \mathbb{C}^n$ are replaced respectively by $\mathcal{M}_n(\mathbb{R}), \mathbb{R}^n$) also holds. With slight modification, the same proof applies.

The following is clearly a necessary condition for the existence of an A-invariant closed pointed cone that contains a given pair of vectors x, y:

For every pair of nonnegative real numbers α, β, A satisfies the local Perron-Schaefer conditions at $\alpha x + \beta y$ and, moreover, we have, $\operatorname{sp}_A(\alpha x + \beta y) = \max\{\operatorname{sp}_A(x), \operatorname{sp}_A(y)\}$*, provided that α, β are both positive.*

However, the condition is not sufficient as can be illustrated by the following example:

Example 5.5. Let $A = J_3(1), x = (0, 0, 1)^T$ and $y = (0, -1, 0)^T$. Clearly, A satisfies the local Perron-Schaefer conditions at x and y respectively, and we have $\operatorname{sp}_A(x) = (1, 3) \succ (1, 2) = \operatorname{sp}_A(y)$. It is readily shown that the above-mentioned necessary condition is satisfied. Now $\operatorname{pos}\{E_{\rho_x}^{(\nu_{\rho_x}-1)} x, E_{\rho_y}^{(\nu_{\rho_y}-1)} y\} = \operatorname{pos}\{(1, 0, 0)^T, (-1, 0, 0)^T\} = \operatorname{span}\{(1, 0, 0)^T\}$ and is not pointed. By Corollary 5.4, there does not exist a closed pointed A-invariant cone that contains both x and y.

Next, for a matrix A that satisfies the Perron-Schaefer condition, we consider the problem of constructing a proper A-invariant cone which is the sum of the closures of finitely many A-cyclic cones.

Lemma 5.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a nonzero nilpotent matrix. For any A-invariant subspaces $W_1, \ldots, W_k$ of $\mathbb{C}^n$, we have,*
$\operatorname{nullity}(A|_{W_1 + \cdots + W_k}) \leq \operatorname{nullity}(A|_{W_1}) + \cdots + \operatorname{nullity}(A|_{W_k})$.

Proof. It is clear that we need only prove our assertion for the case $k = 2$, and since every nilpotent operator on a finite-dimensional space can be written as a direct sum of nilpotent operators, each with nullity one, we may also assume that $\operatorname{nullity}(A|_{W_1}) = 1$.

There is no problem if we have $\mathcal{N}(A|_{W_1+W_2}) \subseteq \mathcal{N}(A|_{W_1}) + \mathcal{N}(A|_{W_2})$. So we assume that there exists $w \in (W_1 + W_2) \cap \mathcal{N}(A)$ such that $w \notin \mathcal{N}(A|_{W_1}) + \mathcal{N}(A|_{W_2})$. (Note that $w \notin W_1 \cup W_2$.) We can express w as $w_1 + w_2$ with $w_i \in W_i, i = 1, 2$. Since W_1, W_2 are both A-invariant and $w \in \mathcal{N}(A)$, we have $A^i w_1 = -A^i w_2 \in W_1 \cap W_2$ for all positive integers i and, moreover, w_1, w_2 are generalized null vectors of A with a common order $p > 1$, and $A^{p-1} w_1 = -A^{p-1} w_2$ is a (nonzero) null vector of A, so

$\mathcal{N}(A|_{W_1}) \subseteq \mathcal{N}(A|_{W_2})$. We contend that $(W_1+W_2)\cap\mathcal{N}(A) = \mathrm{span}\{w\}+(\mathcal{N}(A)\cap W_2)$. Once this is proved, our assertion for the case $k = 2$ and $\mathrm{nullity}(A|_{W_1}) = 1$ will follow, and we are done. It is clear that we have $\mathrm{span}\{w\} + (\mathcal{N}(A) \cap W_2) \subseteq (W_1 + W_2) \cap \mathcal{N}(A)$. To prove the reverse inclusion, let $w' \in [(W_1 + W_2) \cap \mathcal{N}(A)]$. As there is no problem if $w' \in W_2$, hereafter we assume that $w' \notin W_2$. By what we have done for w, we can write w' as $w'_1 + w'_2$, where $w'_i \in W_i, i = 1, 2$, and w'_1, w'_2 are generalized null vectors of A with a common order $p' \geq 2$. Let ν_0 denote $\nu_0(A|_{W_1})$. Recall that w_1 belongs to W_1 and is a generalized null vector of A of order p. Since $A|_{W_1}$ is a nilpotent operator with nullity 1, every vector in W_1 of order less than ν_0 has a pre-image under A in W_1. So we can find a vector $x \in W_1$ such that $A^{\nu_0-p}x = w_1$ and $\{x, Ax, \ldots, A^{\nu_0-1}x\}$ is a basis for W_1. As $w'_1 \in W_1$, we can write w'_1 as $\sum_{i=0}^{\nu_0-1} a_iA^ix$; say, $r = \min\{i : a_i \neq 0\}$. We are going to show that we must have $r = \nu_0 - p$.

First, suppose $r < \nu_0-p$. We have $A^{\nu_0-p-r}w'_1 = A^{\nu_0-p-r}(w'-w'_2) = -A^{\nu_0-p-r}w'_2 \in W_2$. On the other hand, by applying A^{ν_0-p-r} to both sides of the equality relation $w'_1 = \sum_{i=r}^{\nu_0-1} a_iA^ix$ and using the fact that $A^{\nu_0-p}x = w_1$, we also obtain $A^{v_0-p-r}w'_1 = a_rw_1 + a_{r+1}Aw_1 + \cdots + a_{r+p-1}A^{p-1}w_1$. But $a_r \neq 0$ and we have $a_{r+1}Aw_1 + \cdots + a_{r+p-1}A^{p-1}w_1 \in W_2$ as $A^iw \in W_2$ for all positive integers i, it follows that w_1, and hence also w, belongs to W_2, which is a contradiction.

Now suppose $r > \nu_0 - p$. By applying $A^{r-\nu_0+p}$ to both sides of the equality relation $A^{\nu_0-p}x = w_1$ and using the fact that in this case we have $w'_1 = a_rA^rx + a_{r+1}A^{r-\nu_0+p+1}w_1 + \cdots + a_{\nu_0-1}A^{p-1}w_1$, we obtain

$$a_rA^{r-\nu_0+p}w_1 = a_rA^rx = w'_1 - a_{r+1}A^{r-\nu_0+p+1}w_1 - \cdots - a_{\nu_0-1}A^{p-1}w_1.$$

But $A^iw_1 \in W_2$ for all positive integer i, it follows that w'_1, and hence also w', belongs to W_2, which is a contradiction.

So we must have $r = \nu_0 - p$. In view of $w_2 = w - w_1$ and $w'_2 = w' - w'_1 = w' - \sum_{i=0}^{p-1} a_{r+i}A^iw_1$, a little calculation yields

$w' = a_rw + (w'_2 - a_rw_2 + \sum_{i=1}^{p-1} a_{r+i}A^iw_1) \in \mathrm{span}\{w\} + (\mathcal{N}(A) \cap W_2)$,
as desired. □

Theorem 5.7. *Let $A \in \mathcal{M}_n(\mathbb{R})$ (respectively, $\mathcal{M}_n(\mathbb{C})$) satisfy the Perron-Schaefer condition. Let $m = \max\{\mathrm{nullity}(A - \lambda I) : \lambda \in \sigma(A)\}$ (respectively, $m' = \max\{2\,\mathrm{nullity}(A - \alpha I), \mathrm{nullity}(A - \lambda I) + \mathrm{nullity}(A - \bar{\lambda} I) : \alpha \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \mathbb{R}\}$). Then there exists a proper A-invariant cone K in $\mathbb{R}^n$ (respectively, in $\mathbb{C}^n$), which is the sum of the closures of m (respectively, m') A-cyclic cones, but there is no A-invariant proper cone in $\mathbb{R}^n$ (respectively, in $\mathbb{C}^n$) that is the sum of the closures of less than m (respectively, m') A-cyclic cones.*

Proof. We will treat the real case first and then deduce the complex case as a consequence.

Let $A \in \mathcal{M}_n(\mathbb{R})$, and let $\lambda_1 = \rho(A), \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of A. For each $i = 1, \ldots, k$, let m_i denote $\mathrm{nullity}(\lambda_iI - A)$ or, equivalently, the number of Jordan blocks in the Jordan form of A corresponding to λ_i, and let the sizes of the Jordan blocks associated with λ_i, arranged in nonincreasing order, be $l_1^{(i)}, \ldots, l_{m_i}^{(i)}$. (Clearly $l_1^{(1)} = \nu_{\rho(A)}$.) Choose a Jordan basis β for $\mathbb{C}^n$ associated with A such that the generalized eigenvectors in β corresponding to real eigenvalues are real and the generalized eigenvectors corresponding to conjugate complex eigenvalues occur in conjugate pairs. Suppose the Jordan chain corresponding to the jth ($1 \le j \le m_i$) Jordan block for λ_i ($i = 1, \ldots, k$) is $x_{i1}^{(j)}, x_{i2}^{(j)}, \ldots, x_{il_j^{(i)}}^{(j)}$ ($x_{i1}^{(j)}$ being a generalized eigenvector of order $l_j^{(i)}$ and $x_{il_j^{(i)}}^{(j)}$ being an eigenvector).

For $i = 1, \ldots, k$, choose the "top vector" of the first Jordan chain for λ_i, and denote the sum of the "top vectors" by y_1, i.e., $y_1 = x_{11}^{(1)} + x_{21}^{(1)} + \cdots + x_{k1}^{(1)}$. Let $K_1 = \mathrm{cl}\, w_0(A, y_1)$. Note that by our choice of β, if x_{r1} ($1 \le r \le k$) is a non-real complex vector then $\bar{x}_{r1} = x_{s1}$ for some $s, 1 \le s \le k$; so $y_1 \in \mathbb{R}^n$ and $K_1 \subseteq \mathbb{R}^n$. For $k = 2, \ldots, m$, let $K_j = \mathrm{cl}\, w_0(A, y_j)$, where $y_j = x_{11}^{(1)} + x_{11}^{(j)} + x_{21}^{(j)} + x_{31}^{(j)} + \cdots + x_{k1}^{(j)}$. Here we adopt the convention that $x_{i1}^{(j)}$ is taken to be the zero vector if $j > m_i$. Note that we also have $K_j \subseteq \mathbb{R}^n$ for $2 \le j \le k$, and for $j = 1, \ldots, m$, $\rho_{y_j}(A) = \rho(A)$, $\mathrm{ord}_A(y_j) = \nu_{\rho(A)}$ and A satisfies the local Perron-Schaefer condition at y_j. Moreover, $E_{\rho_{y_j}}^{(\nu_{\rho_{y_j}}-1)} y_j$ equals $x_{1\nu_{\rho(A)}}^{(1)} + x_{1\nu_{\rho(A)}}^{(j)}$ if $l_j^{(1)} = \nu_{\rho(A)}$ and $2 \le j \le m_1$, and equals $x_{1\nu_{\rho(A)}}^{(1)}$, otherwise. So the cone $\mathrm{pos}\{E_{\rho_{y_j}}^{(\nu_{\rho_{y_j}}-1)} y_j : j = 1, \ldots, m\}$ is included in the simplicial cone $\mathrm{pos}\{x_{1\nu_{\rho(A)}}^{(j)} : l_j^{(1)} = \nu_{\rho(A)}\}$ and hence is pointed. By the real version of Corollary 5.4 it follows that $K_1 + \cdots + K_m$ is a closed, pointed A-invariant cone in $\mathbb{R}^n$.

It remains to show that the cone $K_1 + \cdots + K_m$ is full in $\mathbb{R}^n$. For $j = 1, \ldots, m$, since $x_{i1}^{(j)}$ ($i = 2, \ldots, k$) appears in the representation of y_j as a sum of generalized eigenvectors of A, all vectors in the Jordan chain $x_{i1}^{(j)}, x_{i2}^{(j)}, \ldots, x_{il_j^{(i)}}^{(j)}$ lies in the A-invariant subspace $\mathrm{span}_{\mathbb{C}} w_0(A, y_j)(= \mathrm{span}_{\mathbb{C}} K_j)$ of $\mathbb{C}^n$. (If $j > m_i$, ignore the argument.) Similarly, all vectors in the Jordan chain $x_{11}^{(1)}, x_{12}^{(1)}, \ldots, x_{1\nu_{\rho(A)}}^{(1)}$ also lie in $\mathrm{span}_{\mathbb{C}} K_1$. For $j = 2, \ldots, m_1$, since $x_{11}^{(1)} + x_{11}^{(j)} \in \mathrm{span}_{\mathbb{C}} K_j$ and $x_{11}^{(1)} \in \mathrm{span}_{\mathbb{C}} K_1$, $x_{11}^{(j)} \in \mathrm{span}_{\mathbb{C}}(K_1 + K_j)$; hence, all vectors in the Jordan chain $x_{11}^{(j)}, x_{12}^{(j)}, \ldots, x_{1l_j^{(1)}}^{(j)}$ lie in $\mathrm{span}_{\mathbb{C}}(K_1 + K_j)$. Thus, all vectors in β belong to $\mathrm{span}_{\mathbb{C}}(K_1 + \cdots + K_m)$; or, in other words, $\mathrm{span}_{\mathbb{C}}(K_1 + \cdots + K_m) = \mathbb{C}^n$. But the cone $K_1 + \cdots + K_m$ is included in $\mathbb{R}^n$, so it is a full cone in $\mathbb{R}^n$.

Last Part. Let K be an A-invariant proper cone in $\mathbb{R}^n$ which is the sum of the closures of p A-cyclic cones; say, $K = \mathrm{cl} w_0(A, u_1) + \cdots + \mathrm{cl} w_0(A, u_p)$. Then we have $\mathbb{R}^n = \mathrm{span}_{\mathbb{R}} K = \mathrm{span}_{\mathbb{R}} w_0(A, u_1) + \cdots + \mathrm{span}_{\mathbb{R}} w_0(A, u_p)$, which, in turn, implies that $\mathbb{C}^n = \mathrm{span}_{\mathbb{C}} w_0(A, u_1) + \cdots + \mathrm{span}_{\mathbb{C}} w_0(A, u_p)$. Now let μ be an eigenvalue of A for which $\mathrm{nullity}(A - \mu I) = m$. Then

$$N_\mu^{\nu_\mu}(A) = (\mathrm{span}_{\mathbb{C}} w_0(A, u_1) \cap N_\mu^{\nu_\mu}(A)) + \cdots + (\mathrm{span}_{\mathbb{C}} w_0(A, u_p) \cap N_\mu^{\nu_\mu}(A)).$$

For each $i = 1, \ldots, p$, if the subspace $\mathrm{span}_{\mathbb{C}} w_0(A, u_p) \cap N_\mu^{\nu_\mu}(A)$ is nonzero then the restriction of $A - \mu I$ to this subspace is a nilpotent operator with nullity 1. By Lemma 5.6 it follows that $m = \mathrm{nullity}(A - \mu I) \le p$, as desired.

Now suppose $A \in \mathcal{M}_n(\mathbb{C})$. We treat A as a linear transformation acting on $\mathbb{C}^n$ as a real vector space. As such, by [28, Lemma 8.1], A is similar to the matrix $\mathrm{diag}(A, \bar{A})$. It is readily shown that the maximum geometric multiplicity of the eigenvalues of $\mathrm{diag}(A, \bar{A})$ equals m'. So our assertion follows from what we have done for the real case. $\square$

Note that Theorem 5.7 does not say that if A satisfies the Perron-Schaefer condition, then every proper A-invariant cone can be written as the sum of the closures of finitely many A-cyclic cones. Indeed, the latter is far from being true. For instance, consider the case when A equals the identity matrix I_n. Clearly, every proper cone in $\mathbb{R}^n$ is I_n-invariant. However, every non-polyhedral proper cone cannot be written as the sum of the closures of finitely many I-cyclic cones, as every I-cyclic cone, and hence also its closure, is a single ray.

For a closed pointed cone K in $\mathbb{R}^n$ (respectively, in $\mathbb{C}^n$), by the *dual cone* of K, denoted by K^*, we mean the (closed) cone $\{z \in \mathbb{R}^n : z^T x \geq 0\}$ (respectively, $\{z \in \mathbb{C}^n : \mathrm{Re}(z^* x) \geq 0\}$).

Corollary 5.8. *Let $A \in \mathcal{M}_n(\mathbb{R})$ (respectively, $\mathcal{M}_n(\mathbb{C})$) satisfy the Perron-Schaefer condition. For any closed pointed A-invariant cone C in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$), there exists a proper A-invariant cone K in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) which includes C.*

Proof. We give the proof only for the real case, as the complex case will follow as a consequence.

Let C be a closed pointed A-invariant cone in $\mathbb{R}^n$. Choose any nonzero vector z in the interior of the dual cone of C in $\operatorname{span} C$. Then $C \setminus \{0\}$ is included in the open half-space $\{x \in \mathbb{R}^n : z^T x > 0\}$. Choose a basis $\{x_{1\nu}^{(1)}, \ldots, x_{1\nu}^{(r)}\}$ for $(A - \rho(A)I)^{\nu-1} E_{\rho(A)}^{\nu}$, where $\nu = \nu_{\rho(A)}(A)$ and $r = \dim(A - \rho(A)I)^{\nu-1} E_{\rho(A)}^{\nu}$, in such a way that $\{x_{1\nu}^{(1)}, \ldots, x_{1\nu}^{(r)}\}$ is included in the closed half-space $\{x \in \mathbb{R}^n : z^T x \geq 0\}$ and extend it to a Jordan basis β for $\mathbb{C}^n$ associated with A in the standard way so that the generalized eigenvectors corresponding to real eigenvalues are real and the generalized eigenvectors corresponding to conjugate complex eigenvalues occur in conjugate pairs. Now construct a proper A-invariant cone C' in $\mathbb{R}^n$ (as the sum of the closures of m A-cyclic cones, where m is the maximum geometric multiplicity of the eigenvalues of A) in the way as described in the proof of Theorem 5.7. Note that by construction $\rho(A)$ is the only distinguished eigenvalue of A for C' and the cone generated by the distinguished eigenvectors of A for C' is included in $\operatorname{pos}\{x_{1\nu}^{(1)}, \ldots, x_{1\nu}^{(r)}\}$ and hence in the closed half-space $\{x \in \mathbb{R}^n : z^T x \geq 0\}$. But $C \setminus \{0\}$ is included in the open half-space $\{x \in \mathbb{R}^n : z^T x > 0\}$. So $C \cap (-C')$ does not contain an eigenvector of A and by Lemma 5.2 $K = C + C'$ is the desired proper A-invariant cone. □

Corollary 5.9. *When $A \in \mathbf{M}_n(\mathbb{C})$ satisfies the Perron-Schaefer condition, to the list of equivalent conditions in Corollary 5.4, we can add the following condition:*

(d) *There exists a proper A-invariant cone in $\mathbb{C}^n$ that contains $x_1, \ldots, x_k$.*

6. Automorphisms

Theorem 6.1. *Let $A \in \mathbf{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Consider the following conditions:*

(a) *$A|_{W_x^{\mathbb{R}}}$ is nonsingular and the cone $\operatorname{cl}\operatorname{pos}\{(A|_{W_x^{\mathbb{R}}})^i x : i = 0, \pm 1, \pm 2, \ldots\}$ is pointed (and $A|_{W_x^{\mathbb{R}}}$ is an automorphism of the cone).*

(b) *There exists a proper cone C in $W_x^{\mathbb{R}}$ containing x such that $A|_{W_x^{\mathbb{R}}} \in \operatorname{Aut}(C)$.*

(c) *There exists a closed, pointed cone C in $\mathbb{C}^n$ containing x such that $A|_{\mathrm{span}_{\mathbb{R}} C} \in \mathrm{Aut}(C)$.*

(d) *$A|_{W_x^{\mathbb{R}}}$ is nonsingular, $A|_{W_x^{\mathbb{R}}}$ and $(A|_{W_x^{\mathbb{R}}})^{-1}$ both satisfy the Perron-Schaefer condition.*

(e) *Let $x = x_1 + \ldots + x_k$ be the representation of A as a sum of generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively. Then $\lambda_1, \ldots, \lambda_k$ are all nonzero, and there exist i, j such that $\lambda_i = \rho_x(A), \mathrm{ord}_A(x_i) = ord_A(x)$ and $\lambda_j = \min\{|\lambda_i| : 1 \le i \le k\}, \mathrm{ord}_A(x_j) = \max\{\mathrm{ord}_A(x_l) : |\lambda_l| = \lambda_j\}$.*

Then (a), (b) *and* (c) *are equivalent, and so are* (d) *and* (e). *Furthermore, we have* (a) $\Rightarrow$ (d).

Proof. (a)$\Rightarrow$(b): The cone $\mathrm{cl\,pos}\{(A|_{W_x})^i x : i = 0, \pm1, \pm2, \ldots\}$, which is mapped onto itself by A, is the desired proper cone in $W_x^{\mathbb{R}}$.

(b)$\Rightarrow$(c): Obvious.

(c)$\Rightarrow$(a): Since A maps $\mathrm{span}_{\mathbb{R}} C$ onto $\mathrm{span}_{\mathbb{R}} C$ and $W_x^{\mathbb{R}}$ is an A-invariant subspace of $\mathrm{span}_{\mathbb{R}} C$, A maps $W_x^{\mathbb{R}}$ onto itself, i.e., $A|_{W_x^{\mathbb{R}}}$ is nonsingular. As $A|_{\mathrm{span}_{\mathbb{R}} C} \in \mathrm{Aut}(C)$ and $x \in C$, we have, $(A|_{\mathrm{span}_{\mathbb{R}} C})^i x \in C$ for $i = 0, \pm1, \pm2, \ldots$. Note that we have $(A|_{\mathrm{span}_{\mathbb{R}} C})^{-1} x = (A|_{W_x^{\mathbb{R}}})^{-1} x$ as the pre-image of x under $A|_{\mathrm{span}_{\mathbb{R}} C}$ must lie in $W_x^{\mathbb{R}}$. Thus we have $(A|_{\mathrm{span}_{\mathbb{R}} C})^i x = (A|_{W_x^{\mathbb{R}}})^i x$ for all integers i. So $\mathrm{cl\,pos}\{(A|_{W_x^{\mathbb{R}}})^i x : i = 0, \pm1, \pm2, \ldots\}$, as a subset of C, is necessarily pointed.

By Theorem 3.6, (a1)$\Leftrightarrow$(a2), condition (d) is equivalent to the following:

(d)$'$ $A|_{W_x^{\mathbb{R}}}$ is nonsingular, and $A|_{W_x^{\mathbb{R}}}$ and $(A|_{W_x^{\mathbb{R}}})^{-1}$ both satisfy the local Perron-Schaefer condition at x.

Condition (d)$'$, in turn, can be rewritten as condition (e).

(a)$\Rightarrow$(d): Note that condition (a) implies that the cones $\mathrm{cl\,pos}\{A^i x : i \ge 0\}$ and $\mathrm{cl\,pos}\{((A|_{W_x^{\mathbb{R}}})^{-1})^i x : i \ge 0\}$ are both pointed. The former cone is the same as $\mathrm{cl}\, w_0(A, x)$, and by Theorem 3.6, that $\mathrm{cl}\, w_0(A, x)$ is pointed is equivalent to the condition that $A|_{W_x^{\mathbb{R}}}$ satisfies the Perron-Schaefer condition. By applying (a more general version of) Theoem 3.6 to $(A|_{W_x^{\mathbb{R}}})^{-1}$, we also conclude that $(A|_{W_x^{\mathbb{R}}})^{-1}$ satisfies the Perron-Schaefer condition. □

For the conditions (a)–(e) of Theorem 6.1, we do not have (e) $\Rightarrow$ (a). For instance, let $A = J_n(1)$ with n even, and take x to be the nth standard unit vector e_n. Clearly, condition (e) is satisfied. In this case, we have $W_x^{\mathbb{R}} = \mathbb{R}^n$ and so $A|_{W_x^{\mathbb{R}}} = A$. If the cone $\mathrm{cl\,pos}\{J_n(1)^i e_n : i = 0, \pm1, \pm2, \ldots\}$ is pointed, then $J_n(1)$ is its automorphism. On the other hand, according to [28, Theorem 7.13], when n is even, there does not exist a proper cone K in $\mathbb{R}^n$ such that $J_n(1) \in \mathrm{Aut}(K)$. So the cone $\mathrm{cl\,pos}\,\{J_n(1)^i e_n : i = 0, \pm1, \pm2, \ldots\}$ is not pointed, i.e., condition (a) is not fulfilled.

Condition (a) of Theorem 6.1 suggests the following observation:

Remark 6.2. If $\lambda_1, \ldots, \lambda_n$ are distinct positive real numbers, then there exists a proper cone K in $\mathbb{R}^n$ such that $\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathrm{Aut}(K)$.

To see this, denote $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ by A and let $K = \mathrm{cl\,pos}\,\{A^i x : i = 0, \pm1, \pm2, \ldots\}$, where $x = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. Then $W_x^{\mathbb{R}} = \mathbb{R}^n$. As $K \subseteq \mathbb{R}_+^n$, K is pointed; so K is a proper cone in $\mathbb{R}^n$. It is clear that we have $A \in \mathrm{Aut}(K)$.

Theorem 6.3. *Let* $A \in \mathbf{M}_n(\mathbb{C})$ *and let* $0 \neq x \in \mathbb{C}^n$. *Then* $\mathrm{cl}\, w_0(A,x)$ *is pointed and* $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(\mathrm{cl}\, w_0(A,x))$ *if and only if* $A|_{W_x^{\mathbb{R}}}$ *is nonzero, diagonalizable* (*over* $\mathbb{C}$), *all eigenvalues of* $A|_{W_x^{\mathbb{R}}}$ *are of the same modulus and* $\rho_x(A)$ *is an eigenvalue of* $A|_{W_x^{\mathbb{R}}}$.

Proof. "Only if" part: Since $\mathrm{cl}\, w_0(A,x)$ is pointed, A must satisfy the local Perron-Schaefer condition at x. So $\rho_x(A)$ is an eigenvalue of $A|_{W_x^{\mathbb{R}}}$.

As $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(\mathrm{cl}\, w_0(A,x))$, $(A|_{W_x^{\mathbb{R}}})^{-1} \in \pi(\mathrm{cl}\, w_0(A,x))$. So $A|_{W_x^{\mathbb{R}}}$ is nonzero, and $\rho((A|_{W_x^{\mathbb{R}}})^{-1})$ is a distinguished eigenvalue of $(A|_{W_x^{\mathbb{R}}})^{-1}$ for $\mathrm{cl}\, w_0(A,x)$. The latter, in turn, implies that $(\rho((A|_{W_x^{\mathbb{R}}})^{-1}))^{-1}$ is a distinguished eigenvalue of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\, w_0(A,x)$. Now according to Lemma 4.4(iii), $\rho_x(A)(= \rho(A|_{W_x^{\mathbb{R}}})$ is the only distinguished eigenvalue of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\, w_0(A,x)$. So we have $(\rho((A|_{W_x^{\mathbb{R}}})^{-1}))^{-1} = \rho_x(A)$. But $(\rho((A|_{W_x^{\mathbb{R}}})^{-1}))^{-1}$ is the least modulus of the eigenvalues of $A|_{W_x^{\mathbb{R}}}$, it follows that all eigenvalues of $A|_{W_x^{\mathbb{R}}}$ have the same modulus. It remains to show that $\nu_{\rho_x(A)}(A|_{W_x^{\mathbb{R}}} = 1$.

Since $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(\mathrm{cl}\, w_0(A,x))$, there exists $y \in \mathrm{cl}\, w_0(A,x)$ such that $Ay = x$. Then $Ay_1 = x_1$, where x_1 (respectively, y_1) is the generalized eigenvector of A corresponding to $\rho_x(A)$ that appears in the representation of x (respectively, of y) as a sum of generalized eigenvectors of A. So x_1, y_1 both belong to $W_{x_1}^{\mathbb{R}}$; indeed, y_1 equals $\frac{(-1)^{\nu-1}}{\rho^{\nu}}(A-\rho I)^{\nu-1}x_1 + \frac{(-1)^{\nu-2}}{\rho^{\nu-1}}(A-\rho I)^{\nu-2}x_1 + \cdots + \frac{(-1)}{\rho^2}(A-\rho I)x_1 + \frac{1}{\rho}x_1$, where for brevity we denote $\rho_x(A)$ and $\nu_{\rho_x(A)}(A|_{W_x^{\mathbb{R}}})$ respectively by ρ and ν, because the latter vector is the unique pre-image of x_1 under A in $W_{x_1}^{\mathbb{R}}$. Note that if $\nu > 1$ then in the preceding representation of y_1 as a linear combination of the generalized eigenvectors $x_1, (A-\rho I)x_1, \ldots, (A-\rho I)^{\nu-1}x_1$ of A, some of the coefficients are negative. On the other hand, since $y \in \mathrm{cl}\, w_0(A,x)$, for some $p_m(t) \in \mathbb{R}_+[t], m = 0, 1, 2, \ldots$, we have $y = \lim_{m\to\infty} p_m(A)x$, and hence $y_1 = \lim_{m\to\infty} p_m(A)x_1$. Then a little calculation shows that each $p_m(A)x_1$, and hence also y_1, is a nonnegative linear combination of the generalized eigenvectors $x_1, (A-\rho I)x_1, \ldots, (A-\rho I)^{\nu-1}x_1$. So we arrive at a contradiction. Therefore, we must have $\nu = 1$, as desired.

"If" part: Under the given assumptions, clearly A satisfies the local Perron-Schaefer condition at x; so $A|_{W_x^{\mathbb{R}}} \in \pi(\mathrm{cl}\, w_0(A,x))$ and as $A|_{W_x^{\mathbb{R}}}$ is nonzero, diagonalizable, and with all eigenvalues of the same modulus and $\rho_x(A) \in \sigma(A|_{W_x^{\mathbb{R}}})$, by [28, Theorem 5.9,(a)⇔(c)] we have $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(\mathrm{cl}\, w_0(A,x))$. □

Theorem 6.4. *Let* $A \in \mathbf{M}_n(\mathbb{C})$. *Consider the following conditions*:

(a) *A is nonzero, diagonalizable* (*over* $\mathbb{C}$), *and all eigenvalues of A are of the same modulus.*
(b) *There is a subsequence of* $((\frac{1}{\rho(A)}A)^k)_{k\in\mathbb{N}}$ *that converges to* I.
(c) *A is nonsingular and there is a subsequence of* $((\frac{1}{\rho(A)}A)^k)_{k\in\mathbb{N}}$ *that converges to* $\rho(A)A^{-1}$.
(d) *A is nonsingular and* $A^{-1} \in \mathrm{cl}\, w_0(A)$.
(e) $I \in \mathrm{cl}\, w_1(A)$.
(f) $\mathcal{L}_A$ *maps* $\mathrm{cl}\, w_0(A)$ *onto itself.*

We always have (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)*,* (d)$\Leftrightarrow$(e)$\Leftrightarrow$(f)*, and* (a)$\Rightarrow$(d)*. When A satisfies the Perron-Schaefer condition, conditions* (a)—(f) *are all equivalent.*

Proof. (a)$\Rightarrow$(b): Modify the proof of [28, Theorem 3,9(i)] or prove it directly.

(b)$\Rightarrow$(c): Suppose that $((A/\rho(A))^{k_i})_{i\in\mathbb{N}}$ converges to I. As I is nonsingular, $(A/\rho(A))^{k_i}$ is nonsingular for i sufficiently large; hence A is nonsingular, and $\left(\frac{A^{k_i-1}}{\rho(A)^{k_i-1}}\right)_{i\in\mathbb{N}}$ converges to $\rho(A)A^{-1}$.

Retracing the above argument backward, we readily obtain (c)$\Rightarrow$(b).

(b)$\Rightarrow$(a): Clearly, condition (b) guaranteeds that $\rho(A) > 0$. Suppose that $((A/\rho(A))^{k_i})_{i\in\mathbb{N}}$ converges to I. Then the said sequence is bounded, and so for any peripheral eigenvalue λ of A, necessarily, $\nu_\lambda(A) = 1$. On the other hand, for any eigenvalue λ of A, we also have, $\lim_{i\to\infty}(\frac{\lambda}{\rho(A)})^{k_i} = 1$, which implies that $|\lambda| = \rho(A)$. Thus, the eigenvalues of A are all of the same modulus, and condition (a) follows.

(c)$\Rightarrow$(d): Obvious.

(d)$\Rightarrow$(e): Since $A^{-1} \in \mathrm{cl}\, w_0(A)$, there exist $p_m(t) \in \mathbb{R}_+[t], m = 1, 2, \ldots$, such that $\lim_{m\to\infty} p_m(A) = A^{-1}$. Then we have $\lim_{m\to\infty} Ap_m(A) = I$ and hence $I \in \mathrm{cl}\, w_1(A)$.

(e)$\Rightarrow$(d): Since $I \in \mathrm{cl}\, w_1(A)$, there exist $p_m(t) \in \mathbb{R}_+[t], m = 1, 2, \ldots$, such that $\lim_{m\to\infty} Ap_m(A) = I$. Then A is necessarily nonsingular and we have $A^{-1} = \lim_{m\to\infty} p_m(A)$. So $A^{-1} \in \mathrm{cl}\, w_0(A)$.

(d) and (e) $\Rightarrow$ (f): Since A is nonsingular, so is $\mathcal{L}_A$. Clearly, we have $w_0(A) = \mathrm{pos}\,\{I\} + w_1(A) = w_0(A)$. On the other hand, we also have $I \in \mathrm{cl}\, w_1(A)$, so $w_0(A) \subseteq \mathrm{cl}\, w_1(A)$ and hence $\mathrm{cl}\, w_0(A) = \mathrm{cl}\, w_1(A)$. Now since $\mathcal{L}_A w_0(A) = w_1(A)$ and $\mathcal{L}_A$ is nonsingular, we have $\mathcal{L}_A \mathrm{cl}\, w_0(A) = \mathrm{cl}\, w_1(A) = \mathrm{cl}\, w_0(A)$.

(f)$\Rightarrow$(d): Since $I \in w_0(A)$, by condition (f) there exists $B \in \mathrm{cl}\, w_0(A)$ such that $AB = I$. So A^{-1} exists, equals B, and belongs to $\mathrm{cl}\, w_0(A)$.

(d) and (e)$\Rightarrow$(a) (assuming that A satisfies the Perron-Schaefer condition): Since A is nonsingular, $\rho(A) > 0$. Replacing A by $\frac{1}{\rho(A)}A$, hereafter, we assume that $\rho(A) = 1$. As $I \in \mathrm{cl}\, w_1(A)$, there exist $p_m(t) \in \mathbb{R}_+[t], t = 1, 2, \ldots$, such that $I = \lim_{m\to\infty} Ap_m(A)$. Then $\lim_{m\to\infty} \lambda p_m(\lambda) = 1$ for every eigenvalue λ of A. In particular, since $1 \in \sigma(A)$, we have $\lim_{m\to\infty} p_m(1) = 1$. Assume to the contrary that A has an eigenvalue λ with modulus less than 1. For each $m \in \mathbb{Z}_+$, since $p_m(t) \in \mathbb{R}_+[t]$ and $|\lambda| < 1$, we have, $|\lambda p_m(\lambda)| \leq |\lambda| p_m(|\lambda|) \leq |\lambda| p_m(1)$. Letting $m \to \infty$, we obtain $\lim_{m\to\infty} |\lambda p_m(\lambda)| \leq |\lambda| < 1$, which is a contradiction.

Next, we show that $\nu_{\rho(A)}(A) = 1$. Assume to the contrary that the Jordan form of A contains a Jordan block $J_r(1)$ with $r \geq 2$. Since $\lim_{m\to\infty} Ap_m(A) = I$, we have $\lim_{m\to\infty} J_r(1)p_m(J_r(1)) = I$. By considering the $(1,1)$ and $(1,2)$ entries of both sides, we obtain, $\lim_{m\to\infty} p_m(1) = 1$ and $\lim_{m\to\infty}(p'_m(1) + p_m(1)) = 0$; hence $\lim_{m\to\infty} p'_m(1) = -1$, and so $p'_m(1) < 0$ for m sufficiently large, which violates the assumption that $p_m(t) \in \mathbb{R}_+[t]$. So $\nu_{\rho(A)}(A) = 1$, and since A satisfies the Perron-Schafer condition, we also have $\nu_\lambda(A) = 1$ for all peripheral eigenvalues, and hence for all eigenvalues, λ of A. This establishes condition (a). ☐

Remark 6.5. The following conditions are also equivalent to the equivalent conditions (d)—(f) of Theorem 6.4:

(e1) $I \in \mathrm{cl}\, w_k(A)$ for every positive integer k.

(e2) $I \in \mathrm{cl}\, w_k(A)$ for some positive integer k.

Proof. Clearly we have the implications (e1)⇒(e) and (e)⇒(e2).

(e2)⇒(d): Suppose that $I \in \mathrm{cl}\, w_k(A)$ for some positive integer k. Then $I = \lim_{m\to\infty} A^k p_m(A)$, where $p_m(t) \in \mathbb{R}_+[t], m = 1, 2, \ldots$. The latter condition implies that A is nonsingular and we have $A^{-1} = \lim_{m\to\infty} A^{k-1} p_m(A) \in \mathrm{cl}\, w_{k-1}(A) \subseteq \mathrm{cl}\, w_0(A)$, so condition (d) holds.

(d)⇒(e1): Suppose $A^{-1} = \lim_{m\to\infty} p_m(A)$, where $p_m(t) \in \mathbb{R}_+[t], m = 1, 2, \ldots$ For any positive integer k, we have $(A^{-1})^k = \lim_{m\to\infty} p_m(A)^k \in \mathrm{cl}\, w_0(A)$. So $I = \lim_{m\to\infty} A^k p_m(A)^k \in \mathrm{cl}\, w_k(A)$. □

Remark 6.6. According to [28, the complex version of Theorem 5.9], the condition $\rho(A) \in \sigma(A)$, together with condition (a) of Theorem 6.4, is equivalent to the condition that there exists a proper cone K in $\mathbb{C}^n$ such that $A \in \mathrm{Aut}(K)$ and A has an eigenvector in $\mathrm{int}\, K$. Four other equivalent conditions can also be found in the same theorem.

Before we conclude this section, we provide the proofs for two results which were announced but not proved in the review paper [26, Theorem 6.10 and Theorem 6.11]. We prove a lemma first.

For a closed pointed cone K, we denote by $\mathrm{Ext}K$ the set of nonzero extreme vectors of K.

Lemma 6.7. *Let K_1, K_2 be closed pointed cones in $\mathbb{R}^n$, both of dimension at least 2. Suppose that $\mathrm{span}K_1 \cap \mathrm{span}K_2 = \mathrm{span}\{u\}$, where $u \in \mathrm{ri}\,(K_1) \cap \mathrm{ri}\,(K_2)$. If, at least one of the cones K_1, K_2 is indecomposable or K_1 and K_2 are both 2-dimensional, then $K_1 + K_2$ is an indecomposable closed, pointed cone and $u \in \mathrm{ri}\,(K_1 + K_2)$.*

Proof. The assumption $\mathrm{span}K_1 \cap \mathrm{span}K_2 = \mathrm{span}\{u\}$ clearly implies that $K_1 \cap (-K_2) = \{0\}$, so $K_1 + K_2$ is a closed pointed cone. As $u \in \mathrm{ri}(K_1) \cap \mathrm{ri}(K_2)$, using the known fact that for a convex cone K, $y \in \mathrm{ri}K$ if and only if for any $0 \neq x \in \mathrm{span}K$ there exists $\varepsilon > 0$ such that $y \pm \varepsilon x \in K$, one readily shows that $u \in \mathrm{ri}\,(K_1 + K_2)$. It remains to prove that $K_1 + K_2$ is indecomposable.

If K_1, K_2 are both 2-dimensional, then clearly $K_1 + K_2$ is a 3-dimensional indecomposable polyhedral cone with 4 extreme rays. So assume that one of the cones K_1, K_2 is indecomposable, say, K_1. Assume to the contrary that $K_1 + K_2$ is decomposable. Then there exist nonzero closed pointed cones C_1, C_2 such that $K_1 + K_2 = C_1 \oplus C_2$. Clearly, $\mathrm{Ext}(K_1 + K_2) \subseteq \mathrm{Ext}K_1 \cup \mathrm{Ext}K_2$. To prove the reverse inclusion, consider any $x_1 \in \mathrm{Ext}K_1$, and suppose we have $x_1 = (y_1 + y_2) + (z_1 + z_2)$, where $y_1, z_1 \in K_1$ and $y_2, z_2 \in K_2$. After rewriting, we obtain $x_1 - y_1 - z_1 = y_2 + z_2 = \alpha u$ for some $\alpha \geq 0$. Hence $x_1 = y_1 + z_1 + \alpha u$, and as $x_1 \in \mathrm{Ext}(K_1)$, $u \in \mathrm{ri}K_1$ and $\dim K_1 \neq 1$ it follows that y_1, z_1 are both nonnegative multiples of x_1. Then from the relation $y_2 + z_2 = 0$ and the pointedness assumption of K_2 we also obtain $y_2 = z_2 = 0$. This shows that x_1 is an extreme vector of $K_1 + K_2$; so we have $\mathrm{Ext}K_1 \subseteq \mathrm{Ext}(K_1 + K_2)$. Similarly, we also have $\mathrm{Ext}K_2 \subseteq \mathrm{Ext}(K_1 + K_2)$. This establishes the equality relation $\mathrm{Ext}(K_1 + K_2) = \mathrm{Ext}K_1 \cup \mathrm{Ext}K_2$. Now since $\mathrm{Ext}(C_1 \oplus C_2) = \mathrm{Ext}C_1 \cup \mathrm{Ext}C_2$ and K_1 is indecomposable, we have either $\mathrm{Ext}K_1 \subseteq \mathrm{Ext}C_1$ or $\mathrm{Ext}K_1 \subseteq \mathrm{Ext}C_2$; say, the former holds. Then $K_1 \subseteq C_1$, and as $u \in K_1$ and C_1 is included in the relative boundary of $C_1 \oplus C_2 (= K_1 + K_2)$, $u \notin \mathrm{ri}\,(K_1 + K_2)$. This contradicts what we have obtained at the beginning of the proof. □

Theorem 6.8. *For an $n \times n$ real matrix A, with $n \geq 3$, there exists an indecomposable proper polyhedral cone K such that $A \in \mathrm{Aut}(K)$ if and only if A is nonzero, diagonalizable, $\rho(A)$ is an eigenvalue of A, and every eigenvalue of A equals $\rho(A)$ times a root of unity.*

Proof. "Only if" part: Here we adapt an argument, due to Pullman [16], given in some detail in [28, Section 3]. Since $A \in \mathrm{Aut}(K)$, A permutes the extreme rays of K. Let τ_A denote the induced permutation. As a permutation τ_A can be written as a composition of unique (up to the ordering) disjoint cycles. By abuse of language, τ_A distributes the extreme rays of K into various (disjoint) cycles. Suppose that σ is one such cycle and is of length d. Choose a nonzero vector, say x, from one of the extreme rays in the cycle. Then there exists a positive real number λ such that $A^d x = \lambda^d x$. Let $v_\sigma = \sum_{i=0}^{d-1} \lambda^{-i} A^i x$. Then a little calculcation reveals that v_σ is, in fact, a distinguished eigenvector of A for K corresponding to the distinguished eigenvalue λ. Since $\mathrm{core}_K(A)(= K)$ is polyhedral and indecomposable, by [28, Corollary 3.3], $\rho(A)$ is the only distinguished eigenvalue of A for K. So, necessarily, we have $\lambda = \rho(A)$. Let m be the order of the permutation τ_A or, in other words, the least common multiple of the length of the cycles associated with τ_A. Then we have $A^m = \rho(A)^m$. So A is nonzero, diagonalizable and every eigenvalue of A equals $\rho(A)$ times a root of unity, and certainly $\rho(A)$ is an eigenvalue of A.

"If" part: By [28, Theorem 7.9 and Theorem 5.9] we can find a proper polyhedral cone K such that $A \in \mathrm{Aut}(K)$. The difficult part of the proof is to show that there is one such K which is indecomposable. This is achieved by taking the sum of some closed, pointed cones, on each of which (a restriction of) A is an automorphism, and applying Lemma 6.7 repeatedly (cf. the argument given in the proof of [28, Lemma 7.6 and 7.8]).

Normalizing A, we may assume that $\rho(A) = 1$. Then $\mathbb{R}^n$ is a direct sum of the following A-invariant subspaces: $\mathcal{N}(A - I)$, $\mathcal{N}(A + I)$ (provided that $-1 \in \sigma(A)$) and certain 2-dimensional A-invariant subspaces with basis $\{x, y\}$ for which $Ax = \cos\theta\, x + \sin\theta\, y$ and $Ay = -\sin\theta\, x + \cos\theta\, y$, where θ is a pth root of unity for some positive integer p (provided that A has non-real complex eigenvalues). Choose any nonzero vector $u \in \mathcal{N}(A - I)$. If $\dim \mathcal{N}(A - I) = r > 1$, choose a basis $\{u, u_1, \ldots, u_{r-1}\}$ for $\mathcal{N}(A - I)$ and let K_1 denote the polyhedral cone generated by the (extreme) vectors $u_1, \ldots, u_{d-1}, 2u - u_1, \ldots, 2u - u_{r-1}$. Then K_1 is a proper polyhedral cone in $\mathcal{N}(A - I)$ such that $u \in \mathrm{ri}K_1$ and $A|_{\mathrm{span}K_1} \in \mathrm{Aut}(K_1)$. If $\dim \mathcal{N}(A + I) = s > 0$, choose a basis $\{v_1, \ldots, v_s\}$ for $\mathcal{N}(A + I)$ and let K_2 be the polyhedral cone generated by the (extreme) vectors $v_1, \ldots, v_s, 2u - v_1, \ldots, 2u - v_s$. Then K_2 is a pointed polyhedral cone such that $u \in \mathrm{ri}K_2$ and $A|_{\mathrm{span}K_2} \in \mathrm{Aut}(K_2)$. Corresponding to a 2-dimensional A-invariant subspace $\mathrm{span}\{x, y\}$ described above, we can also construct a 3-dimensional indecomposable pointed polyhedral cone K_3 with extreme vectors $\cos\frac{2k\pi}{p}\, x + \sin\frac{2k\pi}{p}\, y + u$, $k = 0, \ldots, p-1$ (and in case $p = 3$, we may have to replace it by $\mathrm{pos}\{\cos\frac{2k\pi}{6}\, x + \sin\frac{2k\pi}{6}\, y + u : 0 \leq k \leq 5\}$ in order to guarantee indecomposability).

The desired indecomposable proper polyhedral cone can be constructed by taking the sum of suitable pointed polyhedral cones on which A is an automorphsim. We illustrate the argument by considering the case $\dim \mathcal{N}(A - I) > 1$. If 1 is the only eigenvalue of A, then since $n \geq 3$, it is readily shown that K_1 is indecomposable (see, for instance,[23, proof of Theorem 2]), and so it is the desired proper polyhedral cone. If K_2 exists, then by Lemma 6.7 $K_1 + K_2$ is an indecomposable pointed polyhedral cone such that $u \in \text{ri}(K_1 + K_2)$. As A permutes the extreme rays of $K_i, i = 1, 2$, A also permutes the extreme rays of $K_1 + K_2$; hence $A|_{\text{span}(K_1+K_2)} \in \text{Aut}(K_1+K_2)$. If K_1+K_2 is a proper cone in $\mathbb{R}^n$, we are done. Otherwise, there exists a K_3, and by Lemma 6.7 $K_1 + K_2 + K_3$ is an indecomposable pointed polyhedral cone such that $u \in \text{ri}(K_1 + K_2 + K_3)$ and $A|_{\text{span}(K_1+K_2+K_3)} \in \text{Aut}(K_1 + K_2 + K_3)$. If $K_1 + K_2 + K_3$ is a proper cone in $\mathbb{R}^n$, then we are done. Otherwise, we can add a different K_3 and continue the process until we obtain the desired proper cone. Similarly, we can also treat the case when only K_1 and some K_3s exist. □

For completeness and for possible future use, we would like to add that with a little more work Lemma 6.7 can be strengthened as follows:

Remark 6.9. Let K_1, K_2 be closed pointed cones in $\mathbb{R}^n$, both of dimension at least 2. Suppose that $\text{span}K_1 \cap \text{span}K_2 = \text{span}\{u\}$, where $u \in \text{ri}\,(K_1) \cap \text{ri}\,(K_2)$. Then $K_1 + K_2$ is an indecomposable closed, pointed cone and $u \in \text{ri}\,(K_1 + K_2)$.

Theorem 6.10. *For an $n \times n$ real matrix A, there exists a proper polyhedral cone K in $\mathbb{R}^n$ such that $A \in \text{Aut}(K)$ if and only if A is nonsingular, and for any eigenvalue λ of A, λ equals $|\lambda|$ times a root of unity and $|\lambda|$ is also an eigenvalue of A.*

Proof. "Only if" part: Since $A \in \text{Aut}(K)$, A is clearly nonsingular. Let τ_A denote the permmutation induced by A on the set of extreme rays of K, and let m be the order of τ_A. By an argument given in the proof for the "only if" part of Theorem 6.8, we find that each nonzero extreme vector of K is a distinguished eigenvector of A^m (for K) or, more precisely, we have $\text{Ext}K \subseteq \bigoplus_{\lambda \in \sigma_d(A)} [\mathcal{N}(A^m - \lambda^m I) \cap \text{Ext}K]$, where $\sigma_d(A)$ is the set all nonzero distinguished eigenvalues λ of A. From the latter inclusion relation we obtain $K = \bigoplus_{\lambda \in \sigma_d(A)} K_\lambda$, where K_λ denotes the closed pointed cone $\mathcal{N}(A^m - \lambda^m I) \cap K$; hence $\sigma(A) = \cup_{\lambda \in \sigma_d(A)} \sigma(A|_{\text{span}K_\lambda})$. Now for each $\lambda \in \sigma_d(A)$, since $t^m - \lambda^m$ is an annihilating polynomial for $A|_{\text{span}K_\lambda}$, the spectrum of $A|_{\text{span}K_\lambda}$ consists of λ together with λ times some mth roots of unity. So our assertion follows.

"If" part: In this case $\mathbb{R}^n$ admits a direct decomposition $\mathbb{R}^n = W_1 \oplus \cdots W_k$, where each W_i is A-invariant and $\sigma(A|_{W_i})$ consists of $\rho(A|_{W_i})$ (different from zero), together with $\rho(A|_{W_i})$ times some roots of unity. By the "if" part of Theorem 6.8, for each i, there exists an (indecomposable) proper polyhedral cone in W_i such that $A|_{W_i} \in \text{Aut}(K)$. Let $K = K_1 \oplus \cdots \oplus K_k$. Clearly, K is a proper cone in $\mathbb{R}^n$ and we have $A \in \text{Aut}(K)$. □

7. Local Perron-Frobenius theory for cross-positive matrices

Let K be a proper cone in $\mathbb{C}^n$ and let $A \in \mathcal{M}_n(\mathbb{C})$. A is said to be *cross-positive on* K if for all $x \in K, z \in K^*$, $\text{Re}(z^*x) = 0$ implies $\text{Re}(z^*Ax) \geq 0$; A is said to be *exponentially K-nonnegative* if $e^{tA} \in \pi(K)$ for all nonnegative integers t. It is known that A is cross-positive on K if and only if A is exponentially K-nonnegative. (For other equivalent definitions and the Perron-Frobenius type theorems for the class of cross-positive matrices, see [9], [21], [22] and [2].)

By the *spectral abscissa* of a matrix A, denoted by $\xi(A)$, we mean the maximum of the real part of the eigenvalues of A. By the *local spectral abscissa* of A at x, denoted by $\xi_x(A)$, we mean the quantity $\xi(A|_{W_x})$.

The following result is stated, without proof, by Elsner ([8, Satz 4.1]) :

Theorem C. *For an $n \times n$ real matrix A, the following conditions are equivalent*:

(a) $\xi(A) \in \sigma(A)$ *and* $\nu_{\xi(A)}(A) \geq \nu_\lambda(A)$ *for all* $\lambda \in \sigma(A)$ *with* $\mathrm{Re}\lambda = \xi(A)$.

(b) *There exists a proper cone K in $\mathbb{R}^n$ such that A is exponentially K-nonnegative.*

Whereas the implication (b)$\Rightarrow$(a) follows readily from the fact that a K-nonnegative matrix satisfies the Perron-Schaefer condition, the reverse implication seems not obvious. One purpose of this section is to supply a proof for the reverse implication.

For $A \in \mathcal{M}_n(\mathbb{C})$, we refer to condition (a) of the preceding theorem as the Elsner-Schneider-Vidyasagar condition or, in short, the ESV condition.

We also say $A \in \mathcal{M}_n(\mathbb{C})$ satisfies the *local ESV condition* at x if in the representation of x as a sum of generalized eigenvectors of A there is a generalized eigenvectors y corresponding to $\xi_x(A)$, and moreover the order of y is not less than that of any other generalized eigenvector in the representation that corresponds to an eigenvalue with real part equal to $\xi_x(A)$.

Lemma 7.1. *Let K be a proper cone in $\mathbb{C}^n$. If A is cross-positive on K then A satisfies the local ESV condition at x for every $x \in K$.*

Proof. Consider any $0 \neq x \in K$. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of $A|_{W_x}$. According to Lemma 2.3, the Jordan canonical form of $A|_{W_x}$ is $J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$, where, for each i, $n_i = \nu_{\lambda_i}(A|_{W_x})$. Consider any $t_0 > 0$. As the derivative of the analytic function $f(z) = e^{t_0 z}$ has no zeros in the complex plane, by a known result concerning the elementary divisors of functions of matrices (see, for instance, [12, p.313, Theorem 7(a)]), the Jordan canonical form of $e^{t_0 A|_{W_x}}$ is $J_{n_1}(e^{t_0\lambda_1}) \oplus \cdots \oplus J_{n_k}(e^{t_0\lambda_k})$. Note that $\rho(e^{t_0 A|_{W_x}}) = e^{t_0 \xi_x(A)}$. Since $e^{t_0 A}$ is K-nonnegative (as A is cross-positive), $e^{t_0 A}|_{W_x}$, which is the same as $e^{t_0 A|_{W_x}}$ satisfies the Perron-Schaefer condition. So $\rho(e^{t_0 A|_{W_x}})$ is an eigenvalue of $e^{t_0 A|_{W_x}}$ with index not less than that of any other eigenvalue of $e^{t_0 A|_{W_x}}$ with the same modulus. To be specific, say, $e^{t_0\lambda_1} = \rho(e^{t_0 A|_{W_x}})$. Then, we have, $n_1 \geq n_j$ whenever $|e^{t_0\lambda_j}| = |e^{t_0\lambda_1}|$ or, equivalently, whenever $\mathrm{Re}\lambda_j = \mathrm{Re}\lambda_1$. Now choose a positive real number t_0 such that $t_0 \notin \{t : t > 0, t = \frac{2\pi p}{\mathrm{Im}\lambda_j}, p \text{ an integer}, \mathrm{Re}\lambda_j = \xi_x(A)\}$. With such choice of t_0, the condition $e^{t_0\lambda_1} = e^{t_0\xi_x(A)}$ guarantees that $\lambda_1 = \xi_x(A)$. This proves that A satisfies the local ESV condition at x. □

Lemma 7.2. *Let K be a proper cone in $\mathbb{C}^n$ (or $\mathbb{R}^n$). If A is cross-positive on K then A satisfies the ESV condition.*

Proof. We modify the argument given in the proof of Lemma 7.1. Now let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of A, and choose a positive real number t_0 such that $t_0 \notin \{t : t > 0, t = \frac{2\pi p}{\mathrm{Im}\lambda_j}, p \text{ an integer}, \mathrm{Re}\lambda_j = \xi(A)\}$. Since A is cross-positive on K, $e^{t_0 A}$ is K-nonnegative and hence satisfies the Perron-Schaefer condition. With our choice of t_0, the latter condition implies that A satisfies the ESV condition. □

For convenience, hereafter, we extend the usage of the terms distinguished eigenvalues and distinguished eigenvectors to the class of cross-positive matrices. That is, when A is cross-positive on K, if $0 \neq x \in K$ and $\lambda \in \mathbb{C}$ satisfy $Ax = \lambda x$, we say x (respectively, λ) is a distinguished eigenvector (respectively, distinguished eigenvalue) of A for K.

By Lemma 7.1 we readily obtain the following analogous result of Remark 2.7.

Remark 7.3. Let K be a proper cone in $\mathbb{C}^n$. If A is cross-positive on K, then every distinguished eigenvalue of A for K is a real number.

Theorem 7.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. The following conditions are equivalent:*

(a) *A satisfies the local ESV condition at x.*

(b) *$A|_{W_x^{\mathbb{R}}}$ (or $A|_{W_x}$) satisfies the ESV condition.*

(c) *The cone $\mathrm{cl}(\mathrm{pos}\{e^{tA}x : t \geq 0\})$ is pointed.*

(d) *There exists a proper cone K in $W_x^{\mathbb{R}}$ containing x such that $A|_{W_x^{\mathbb{R}}}$ is cross-positive on K.*

(e) *There exists a closed pointed cone K that contains x such that $A|_{\mathrm{span}_{\mathbb{R}}K}$ is cross-positive on K.*

(f) *There exists a proper cone K in $W_x^{\mathbb{R}}$ such that $A|_{W_x^{\mathbb{R}}}$ is cross-positive on K.*

Proof. In view of Remark 2.5 and an equivalent formulation of the ESV condition given in terms of the minimal polynomial (cf. Remark 2.1), it is clear that $A|_{W_x^{\mathbb{R}}}$ satisfies the ESV condition if and only if $A|_{W_x}$ satisfies the ESV condition.

The equivalence of (a) and (b) follows from Lemma 2.3. The implications (d) $\Rightarrow$ (e) and (d) $\Rightarrow$ (f) are both obvious.

(f) $\Rightarrow$ (b): Apply Lemma 7.2 to $A|_{W_x^{\mathbb{R}}}$.

(e) $\Rightarrow$ (c): Since A is cross-positive on K and $x \in K$, we have $e^{tA}x \in K$ for all $t \geq 0$; thus $\mathrm{cl}\,\mathrm{pos}\{e^{tA}x : t \geq 0\} \subseteq K$. But K is pointed, hence so is $\mathrm{cl}\,\mathrm{pos}\{e^{tA}x : t \geq 0\}$.

(c) $\Rightarrow$ (d): Note that for any $t \geq 0, e^{tA} \in \mathrm{cl}\, w_0(A)$. So we have

$\mathrm{span}_{\mathbb{R}}\{e^{tA}x : t \geq 0\} \subseteq \mathrm{span}_{\mathbb{R}}\mathrm{cl}\, w_0(A,x) = \mathrm{span}_{\mathbb{R}} w_0(A,x) = W_x^{\mathbb{R}}$.

Since $A = \lim_{t \to 0^+} \frac{e^{tA}-I}{t}$ for all nonnegative integers k, we also have,

$A^k x = \lim_{t \to 0^+} \left(\frac{e^{tA}-I}{t}\right)^k x \in \mathrm{span}_{\mathbb{R}}\{e^{tA}x : t \geq 0\}$;

hence $\mathrm{span}_{\mathbb{R}} w_0(A,x) \subseteq \mathrm{span}_{\mathbb{R}}\{e^{tA}x : t \geq 0\}$. This shows that $\mathrm{span}_{\mathbb{R}}\{e^{tA}x : t \geq 0\}$ equals the A-invariant subspace $W_x^{\mathbb{R}}$. If the cone $\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$ is pointed, then it is a proper cone in $W_x^{\mathbb{R}}$ that contains x. Clearly the cone is invariant under e^{sA} for all nonnegative integers s. So it is the desired proper cone in $W_x^{\mathbb{R}}$.

(a) $\Rightarrow$ (c): Let $x = x_1 + \cdots + x_k$, where $x_1, \ldots, x_k$ are generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively and with $\lambda_1 = \max_{1 \leq i \leq k} \mathrm{Re}\,\lambda_i$. If the cone $\mathrm{cl}\,(\mathrm{pos}\,\{e^{tA}x : t \geq 0\})$ is not pointed, we can find convergent sequences $(y_m)_{m \in \mathbb{Z}_+}, (z_m)_{m \in \mathbb{Z}_+}$ in $\mathrm{pos}\,\{e^{tA}x, t \geq 0\}$ such that $\lim_{m \to \infty} y_m = -\lim_{m \to \infty} z_m \neq 0$. For each m, we have, $y_m = \sum_{l=1}^{k_m} a_l^{(m)} e^{s_l^{(m)}A} x$ for some $k_m \in \mathbb{Z}_+$ and some nonnegative real numbers $a_l^{(m)}, s_l^{(m)}, m = 1, \ldots, k_m$, and $z_m = \sum_{l=1}^{i_m} b_l^{(m)} e^{r_l^{(m)} A} x$ for some $i_m \in \mathbb{Z}_+$ and some nonnegative real numbers $b_l^{(m)}, r_l^{(m)}, l = 1, \ldots, i_m$. For each m, let $p_m(t) = \sum_{l=1}^{k_m} a_l^{(m)} e^{s_l^{(m)} t}$ and $q_m(t) = \sum_{l=1}^{i_m} b_l^{(m)} e^{r_l^{(m)} t}$. Then $y_m = p_m(A)x$ and $z_m = q_m(A)x$. A little calculation yields

$$p_m^{(j)}(t) = \sum_{l=1}^{k_m} a_l^{(m)} (s_l^{(m)})^j e^{s_l^{(m)} t} \text{ for all } j \in \mathbb{Z}_+,$$

and a similar expression for $q_m^{(j)}(t)$. From the equality relation $\lim_{m\to\infty} y_m = -\lim_{m\to\infty} z_m$ we obtain $\lim_{m\to\infty}(p_m + q_m)(A)x_i = 0$ for $i = 1, \ldots, k$. Now we have

$$(p_m + q_m)(A)x_1 = \sum_{j=0}^{\nu_{\lambda_1}-1} \frac{(p_m^{(j)}+q_m^{(j)})(\lambda_1)}{j!}(A - \lambda_1 I)^j x_1$$

and hence

$$\lim_{m\to\infty}(p_m^{(j)} + q_m^{(j)})(\lambda_1) = 0 \text{ for } j = 0, \ldots, \nu_{\lambda_1} - 1.$$

Similarly, for $i = 2, \ldots, k$, we also have

$$\lim_{m\to\infty}(p_m^{(j)} + q_m^{(j)})(\lambda_i) = 0 \text{ for } j = 0, \nu_{\lambda_i} - 1.$$

For $j = 0, \ldots, \mu_{\lambda_1} - 1$, we have

$$0 \le |p_m^{(j)}(\lambda_1)| = \sum_{l=1}^{k_m} a_l^{(m)} (s_l^{(m)})^j e^{s_l^{(m)} \lambda_1} \le (p_m^{(j)} + q_m^{(j)})(\lambda_1),$$

which implies $\lim_{m\to\infty} p_m^{(j)}(\lambda_1) = 0$.

Consider $\lambda_i, 2 \le i \le k$ with $\operatorname{Re}\lambda_i = \lambda_1$. By the local ESV condition at x, $\nu_{\lambda_i} \le \nu_{\lambda_1}$. For $j = 0, \ldots, \nu_{\lambda_i} - 1$, we have

$$0 \le |p_m^{(j)}(\lambda_i)| \le \sum_{l=1}^{k_m} a_l^{(m)} (s_l^{(m)})^j |e^{s_l^{(m)} \lambda_i}| = \sum_{l=1}^{k_m} a_l^{(m)} (s_l^{(m)})^j e^{s_l^{(m)} \lambda_1} = p_m^{(j)}(\lambda_1);$$

thus, $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$.

Now consider λ_i with $\operatorname{Re}\lambda_i < \lambda_1$. Let C denote the circle $|z - \lambda_i| = \lambda_1 - \operatorname{Re}\lambda_i$ in the complex plane. Noting that $\max_{w\in C} |p_m(w)| \le p_m(\lambda_1)$, by Cauchy's inequality, we have $0 \le |p_m^{(j)}(\lambda_i)| \le \frac{j!}{(\lambda_1 - \operatorname{Re}\lambda_i)^{j+1}} p_m(\lambda_1)$; thus, $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$. We have shown that $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, \nu_{\lambda_i} - 1$. Since $p_m(A) = \sum_{i=1}^{k} \sum_{j=0}^{\nu_{\lambda_i}-1} \frac{p_m^{(j)}(\lambda_i)}{j!}(A - \lambda_i I)^j x_i$ and $y_m = p_m(A)$, we obtain $\lim_{m\to\infty} y_m = 0$, which is a contradiction. □

Before proceeding further, we need to introduce the concept of real spectral pair of a vector relative to a matrix.

Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \ne x \in \mathbb{C}^n$. Let $x = x_1 + \cdots + x_k$ be the representation of x as a sum of generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively. By the *real order of x relative to A*, denoted by $\operatorname{ord}_A^{\mathbb{R}}(x)$, we mean $\max\{\operatorname{ord}_A(x_i) : \operatorname{Re}(\lambda_i) = \xi_x(A)\}$. We denote the ordered pair $(\xi_x(A), \operatorname{ord}_A^{\mathbb{R}}(x))$ by $\operatorname{sp}_A^{\mathbb{R}}(x)$ and refer to it as the *real spectral pair of x relative to A*. We also adopt the convention $\operatorname{sp}_A^{\mathbb{R}}(0) = (0, 0)$.

Using the definition of real spectral pair of a vector and the fact that if x_i is a generalized eigenvector of A corresponding to λ_i then for any $t \ge 0$, $e^{tA}x_i$ is also a generalized eigenvector of A corresponding to λ_i and with the same order as x_i, then one can readily establish the following (cf. the corresponding properties for spectral pairs as given in [29, Remark 4.1]):

Remark 7.5. For any $A \in \mathcal{M}_n(\mathbb{C})$, $x, y \in \mathbb{C}^n$, $0 \ne \lambda \in \mathbb{C}$, we have

(i) $\operatorname{sp}_A^{\mathbb{R}}(\lambda x) = \operatorname{sp}_A^{\mathbb{R}}(x)$.
(ii) $\operatorname{sp}_A^{\mathbb{R}}(e^{tA}x) = \operatorname{sp}_A^{\mathbb{R}}(x)$ for any nonnegative integer t.

(iii) $\mathrm{sp}_A^{\mathbb{R}}(x+y) \preceq \max\{\mathrm{sp}_A^{\mathbb{R}}(x), \mathrm{sp}_A^{\mathbb{R}}(x)\}$, where the maximum is taken in the sense of lexicographic ordering.

Borrowing the argument given in the proofs for Lemma 4.1 and Theorem 7.4, (a) $\Rightarrow$ (c), one can obtain the following:

Lemma 7.6. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local ESV condition at x. Then:*

(i) *For any $0 \neq y \in \mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$, we have, $\xi_y(A) = \xi_x(A)$ and $\mathrm{sp}_A^{\mathbb{R}}(y) \preceq \mathrm{sp}_A^{\mathbb{R}}(x)$.*

(ii) *For any $0 \neq y \in \mathrm{pos}\,\{e^{tA}x : t \geq 0\}$, we have, $\mathrm{sp}_A^{\mathbb{R}}(y) = \mathrm{sp}_A^{\mathbb{R}}(x)$.*

We will make use of the following known result ([27, Theorem 3.2]): If A is cross-positive on K, then $E_{\xi(A)}^{(\nu_\xi - 1)}(A) \in \pi(K)$, where $\nu_\xi = \nu_{\xi(A)}(A)$.

Lemma 7.7. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Suppose that A satisfies the local ESV condition at x. Then $\xi_x(A)$ is the only distinguished eigenvalue of A for $\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$ and (up to multiples) $E_{\xi_x(A)}^{(\nu_{\xi_x(A)}-1)}(A)x$ is the unique distinguished eigenvector of A for $\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$.*

Proof. Since A satisfies the local ESV condition at x, by Theorem 7.4 (and its proof), $\mathrm{span}_{\mathbb{R}}\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\} = W_x^{\mathbb{R}}$ and $A|_{W_x^{\mathbb{R}}}$ is cross-positive on $\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$. By applying [27, Theorem 3.2] to $A|_{W_x^{\mathbb{R}}}$, we find that the eigenvector $E_{\xi_x(A)}^{(\nu_{\xi_x(A)}-1)}(A)x$ of A belongs to $\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$. Since $A|_{W_x^{\mathbb{R}}}$ is a cyclic operator, each of its eigenvalues and, in particular, the eigenvalue $\xi_x(A)$, has geometric multiplicity 1. Now let α be a distinguished eigenvalue of $A|_{W_x^{\mathbb{R}}}$ for $\mathrm{cl}\,\mathrm{pos}\,\{e^{tA}x : t \geq 0\}$ and let w be a corresponding distinguished eigenvector. By Lemma 7.6 we have $\alpha = \xi_w(A) = \xi_{y_j}(A)$. So we can draw the desired conclusions. □

Lemma 7.8. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Let K_1, K_2 be closed, pointed cones in $\mathbb{C}^n$ and assume that $A|_{\mathrm{span}_{\mathbb{R}} K_i}$ is cross-positive on K_i for $i = 1, 2$. Suppose that $K_1 + K_2$ is pointed. Then for every distinguished eigenvalue λ of A for $K_1 + K_2$, we have*

$$(K_1 + K_2) \cap \mathcal{N}(A - \lambda I) = (K_1 \cap \mathcal{N}(A - \lambda I)) + (K_1 \cap \mathcal{N}(A - \lambda I)).$$

Thus the distinguished eigenvalues of A for $K_1 + K_2$ are precisely the distinguished eigenvalues of A for K_1 or for K_2.

Proof. Let $\lambda_j = \sigma_j + \sqrt{-1}\omega_j$ ($\sigma_j, \omega_j \in \mathbb{R}$), $j = 1, \ldots, n$, be the eigenvalues of A. Choose a positive real number t_0 such that

$$t_0 \notin \{t : t > 0, t = \tfrac{2\pi p}{\omega_j - \omega_k}, p \text{ an integer}, \sigma_j = \sigma_k, \omega_j \neq \omega_k\}.$$

According to the proof of [21, Lemma 8], $e^{t_0\lambda_j} \neq e^{t_0\lambda_k}$ whenever $\lambda_j \neq \lambda_k$, and for each j, we have $\mathcal{N}(A - \lambda_j I) = \mathcal{N}(e^{t_0 A} - e^{t_0\lambda_j} I)$. So λ is a distinguished eigenvalue of A for K_1 (respectively, for $K_2, K_1 + K_2$) if and only if $e^{t_0\lambda}$ is a distinguished eigenvalue of $e^{t_0 A}$ for K_1 (respectively, for $K_2, K_1 + K_2$), and a similar remark can be said for the distinguished eigenvectors. By applying Lemma 5.3 to $e^{t_0 A}$, we can draw the desired conclusions. □

Lemma 7.9. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Let $x_1, \ldots x_k, k \geq 2$, be vectors of $\mathbb{C}^n$ and suppose that A satisfies the local ESV condition at $x_1, \ldots, x_k$ respectively. Let K denote* *the cone* $\mathrm{cl\,pos}\{e^{tA}x_1 : t \geq 0\} + \cdots + \mathrm{cl\,pos}\{e^{tA}x_k : t \geq 0\}$. *Then the following conditions are equivalent*:

(a) *The cone* K *is closed and pointed.*

(b) *There exists a closed, pointed cone* C *in* $\mathbb{C}^n$ *such that* C *contains* $x_1, \ldots, x_k$ *and* $A|_{\mathrm{span}_{\mathbb{R}} C}$ *is cross-positive on* C.

(c) *The cone* $\mathrm{pos}\{E_{\xi_{x_i}(A)}^{(\nu_{\xi_{x_i}(A)}-1)}(A)x_i : i = 1, \ldots, k\}$ *is pointed.*

When the equivalent conditions are satisfied, the set of distinguished eigenvalues of A *for* K *is* $\{\xi_{x_i}(A) : i = 1, \ldots, k\}$ *and the cone generated by the distinguished eigenvectors of* A *for* K *is* $\mathrm{pos}\{E_{\xi_{x_i}(A)}^{(\nu_{\xi_{x_i}(A)}-1)}(A)x_i : i = 1, \ldots, k\}$.

Proof. (a)⇒(b): For $i = 1, \ldots, k$, since A satisfies the local ESV condition at x_i, $A|_{W_{x_i}^{\mathbb{R}}}$ is cross-positive, and hence exponentially nonnegative, on $\mathrm{cl\,pos}\{e^{tA}x_i : t \geq 0\}$. Thus, $A|_{W_x^{\mathbb{R}}}$ is exponentially nonnegative, and hence cross-positive, on K.

(b)⇒(c): The argument is similar to that given in the proof for Corollary 5.4, (b)⇒(c), except that now we apply Lemma 7.7 instead of Lemma 4.4.

(c)⇒(a) and the last part: The argument is again similar to that given in the proof for the corresponding part of Corollary 5.4. Here instead of invoking Lemma 4.4(iii) and Lemma 5.3, we invoke Lemma 7.7 and Lemma 7.8 (and its proof) respectively. Also, instead of applying Lemma 5.2, (a)⇒(c) to A, we apply it to e^{t_0A}, where t_0 is a positive real number, as chosen in the proof of Lemma 7.8. □

Theorem 7.10. *Let* $A \in \mathcal{M}_n(\mathbb{R})$ (*respectively,* $\mathcal{M}_n(\mathbb{C})$) *satisfy the ESV condition. Let* $m = \max\{\mathrm{nullity}(A - \lambda I) : \lambda \in \sigma(A)\}$ (*respectively,* $m' = \max\{2\,\mathrm{nullity}(A - \alpha I), \mathrm{nullity}(A - \lambda I) + \mathrm{nullity}(A - \bar{\lambda} I) : \alpha \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \mathbb{R}\}$). *Then there exists a proper cone* K *in* $\mathbb{R}^n$ (*respectively, in* $\mathbb{C}^n$), *which is the sum of* m (*respectively,* m') *cones of the form* $\mathrm{cl\,pos}\,\{e^{tA}x : x \geq 0\}$, *such that* A *is cross-positive on* K, *but there is no proper cone in* $\mathbb{R}^n$ (*respectively, in* $\mathbb{C}^n$), *which is the sum of less than* m (*respectively, less than* m') *cones of the form* $\mathrm{cl\,pos}\,\{e^{tA}x : x \geq 0\}$, *on which* A *is cross-positive.*

Proof. We modify the argument given in the proof of Theorem 5.7. As before, we deal with the real case of our result first.

Let $\lambda_1 = \xi(A), \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of A. For $i = 1, \ldots, k$, let $m_i = \mathrm{nullity}(\lambda_i I - A)$, and let $l_1^{(i)}, \ldots, l_{m_i}^{(i)}$ denote the sizes of the Jordan blocks in the Jordan form of A associated with λ_i, arranged in nonincreasing order. As before, choose a Jordan basis β for $\mathbb{C}^n$ associated with A such that the generalized eigenvectors corresponding to real eigenvalues are real and the generalized eigenvectors corresponding to non-real eigenvalues occur in conjugate pairs. For $i = 1, \ldots, k$, let the Jordan chain in β corresponding to the jth Jordan block for λ_i be $x_{i1}^{(j)}, x_{i2}^{(j)}, \ldots, x_{il_j^{(i)}}^{(j)}$. Define the vectors $y_1, \ldots, y_m$ in the same way as before, and set $K_j = \mathrm{cl\,pos}\,\{e^{tA}y_j : t \geq 0\}$ for $j = 1, \ldots, m$. Note that A satisfies the local ESV condition at each y_j. So by Theorem 7.4 (and its proof), for each j, $\mathrm{span}_{\mathbb{R}} K_j = W_{y_j}^{\mathbb{R}}(A)$, K_j is a closed pointed cone and $A|_{\mathrm{span}_{\mathbb{R}} K_j}$ is cross-positive on K_j. By Lemma 7.7, $\xi_{y_j}(A)$ is the only distinguished eigenvalue of A for K_j and (up to multiples) $E_{\xi_{y_j}(A)}^{(\nu_{\xi_{y_j}(A)}-1)}(A)y_j$ is the unique distinguished eigenvector of A for K_j. Using the argument given in the proof of Theorem 5.7, we can show that the cone $\mathrm{pos}\{E_{\xi_{y_j}(A)}^{(\nu_{\xi_{y_j}(A)}-1)}(A)y_j : j = 1, \ldots, m\}$ is included in the simplicial cone $\mathrm{pos}\{x_{1\nu_{\xi(A)}}^{(j)} : l_j^{(1)} = \nu_{\xi(A)}\}$ and hence is pointed. Then by Lemma 7.9 we conclude that the cone $K := K_1 + \cdots + K_m$ is closed and pointed. With slight modification, the argument given in the proof of Theorem 5.7 can also be used to show that K is full in $\mathbb{R}^n$; so K is a proper cone in $\mathbb{R}^n$. Since A is exponentially nonnegative on each K_j, A is exponentially nonnegative on their sum; hence A is cross-positive on K. Likewise, by the argument given in the proof of Theorem 5.7 we also show that there is no proper cone in $\mathbb{R}^n$, which is the sum of less than m cones of the form $\mathrm{cl\,pos}\,\{e^{tA}x : x \geq 0\}$, on which A is cross-positive. Finally, we can also deduce the complex case of our result from its real case. □

By modifying the proof of Corollary 5.8, we obtain the following:

Corollary 7.11. *Let $A \in \mathcal{M}_n(\mathbb{R})$ (respectively, $\mathcal{M}_n(\mathbb{C})$) satisfy the ESV condition. If C is a closed pointed cone in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) such that $A|_{\mathrm{span}_{\mathbb{R}}\mathrm{C}}$ is cross-positive on C, then there exists a proper cone K in $\mathbb{R}^n$ (respectively, $\mathbb{C}^n$) which includes C such that A is cross-positive on C.*

Similarly, we also have the following result:

Corollary 7.12. *When A satisfies the ESV condition, to the list of equivalent conditions in Corollary 7.9, we can add the following condition*:

(d) *There exists a proper cone in $\mathbb{C}^n$ that contains $x_1, \ldots, x_k$ such that A is cross-positive on C.*

According to Corollary 3.8, A satisfies the Perron-Schaefer condition if and only if $\mathcal{L}_A|_{W_A}$ satisfies the Perron-Schaefer condition. However, the corresponding result for the ESV condition does not hold.

Remark 7.13. When A satisfies the ESV condition, $\mathcal{L}_A|_{W_A}$ need not satisfy the ESV condition.

As a counter-example, consider $A = \mathrm{diag}(1, i)$. Clearly A satisfies the ESV condition. Since the minimal polynomial of A is $t(t-i)$, by Lemma 3.7 the minimal polynomial of $\mathcal{L}_A|_{W_A}$ is $t - i$. So $\mathcal{L}_A|_{W_A}$ does not satisfy the ESV condition.

Nevertheless, we have the following counterpart for Theorem B.

Theorem 7.14. *Let $A \in \mathcal{M}_n(\mathbb{C})$. Then A is cross-positive on K for some proper cone K in $\mathbb{C}^n$ if and only if the cone* $\mathrm{cl\,pos}\,\{e^{tA} : t \geq 0\}$ *is pointed.*

Proof. "Only if" part: If A is cross-positive on a proper cone K, then A is exponentially K-nonnegative; hence $\mathrm{cl\,pos}\,\{e^{tA} : t \geq 0\}$ is included in the proper cone $\pi(K)$ and so it is pointed.

"If" part: Suppose that the cone $\mathrm{cl\,pos}\,\{e^{tA} : t \geq 0\}$ is not pointed. Then there exist convergent sequences $(Y_m)_{m \in \mathbb{Z}_+}, (Z_m)_{m \in \mathbb{Z}_+}$ in $\mathrm{cl\,pos}\,\{e^{tA} : t \geq 0\}$ such that $\lim_{m \to \infty} Y_m = -\lim_{m \to \infty} Z_m \neq 0$. For each m, we have, $Y_m = \sum_{l=1}^{k_m} a_l^{(m)} e^{s_l^{(m)} A}$

for some $k_m \in \mathbb{Z}_+$ and some nonnegative real numbers $a_l^{(m)}, s_l^{(m)}, m = 1, \ldots, k_m$, and $Z_m = \sum_{l=1}^{i_m} b_l^{(m)} e^{r_l^{(m)} A}$ for some $i_m \in \mathbb{Z}_+$ and some nonnegative real numbers $b_l^{(m)}, r_l^{(m)}, l = 1, \ldots, i_m$. For each m, let $p_m(t) = \sum_{l=1}^{k_m} a_l^{(m)} e^{s_l^{(m)} t}$ and $q_m(t) = \sum_{l=1}^{i_m} b_l^{(m)} e^{r_l^{(m)} t}$. Then $Y_m = p_m(A)$ and $Z_m = q_m(A)$. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. As A satisfies the ESV condition, we may assume that $\lambda_1 = \xi(A)$. By a standard result in the theory of functions of matrices, for each positive integer m, we have $p_m(A) = \sum_{i=1}^{k} \sum_{j=0}^{\nu_{\lambda_i}-1} \frac{p_m^{(j)}(\lambda_i)}{j!} E_{\lambda_i}^{(j)}(A)$, and similar expressions for $q_m(A)$ and $(p_m + q_m)(A)$. The fact that $\lim_{m\to\infty}(Y_m + Z_m) = 0$ implies that $\lim_{m\to\infty}(p_m^{(j)} + q_m^{(j)})(\lambda_i) = 0$ for $i = 1, \ldots, k, j = 0, \ldots, \nu_{\lambda_i} - 1$. Then using an argument similar to the one given in the proof of Theorem 7.4, (a)⇒(c), one can show that $\lim_{m\to\infty} p_m^{(j)}(\lambda_i) = 0$ for $i = 1, \ldots, k, j = 0, \ldots, \nu_{\lambda_i} - 1$; hence $\lim_{m\to\infty} Y_m = \lim_{m\to\infty} p_m(A) = 0$, which is a contradiction. □

8. Open problems

Inspite of this work, many natural questions remain unanswered. In below we collect some of them.

In Remark 3.17(ii) we have provided an equivalent condition for $w_0(A, x)$ to be a pointed polyhedral cone. The result is unsatisfactory in that the condition is not spectral nor is it readily checkable.

Question 8.1. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and let $0 \neq x \in \mathbb{C}^n$. Find a spectral or a readily checkable equivalent condition for $w_0(A, x)$ to be a pointed polyhedral cone.*

Question 8.2. *Find an equivalent condition for $\operatorname{cl} w_0(A, x)$ to be a pointed polyhedral cone.*

We do not know whether the set of necessary conditions given in Theorem 3.16 for $\operatorname{cl} w_0(A, x)$ to pointed, polyhedral is also sufficient.

Question 8.3. *Find an equivalent condition for $w_0(A, x)$ (or $\operatorname{cl} w_0(A, x)$) to be an indecomposable closed pointed cone.*

For $x \neq 0$, in view of Remark 3.3, a necessary condition for $\operatorname{cl} w_0(A, x)$ to be indecomposable is that either $A|_{W_x}$ is nonsingular or x is a null vector of A. However, the condition is not sufficient. For a counter-example, consider $\operatorname{cl} w_0(A, x)$, where $A = J_2(1)$ and $x = (0, 1)^T$.

Question 8.4. *Find an equivalent condition for $w_0(A, x)$ to be a closed, pointed cone.*

Clearly, a necessary condition for $w_0(A, x)$ to be closed, pointed is that A satisfies the local Perron-Schaefer condition at x and $\sum_{\lambda\in\Lambda} E_\lambda^{(\nu_\lambda(A|_{W_x})-1)} x \in w_0(A, x)$ and, as a consequence, the spectral conditions given in Lemma 4.7 are fulfilled. For an equivalent condition for $w_0(A, x)$ closed, pointed in the special case when the eigenvalues of $A|_{W_x}$ are all of the same modulus and $\rho_x(A)$ is one of the eigenvalues, see Lemma 4.10.

Question 8.5. *Given $A \in \mathcal{M}_n(\mathbb{C})$ and $0 \neq x \in \mathbb{C}^n$, when does there exist a proper cone C in $W_x^{\mathbb{R}}$ containing x such that $A|_{W_x^{\mathbb{R}}} \in \mathrm{Aut}(C)$?*

Question 8.6. *Find an equivalent condition on a given $n \times n$ real matrix A so that $A \in \mathrm{Aut}(K)$ for some proper cone K.*

Question 8.7. *Is it true that when $\rho(A) \in \sigma(A)$, conditions* (a)—(f) *of Theorem 6.4 are all equivalent ?*

According to the proof of Theorem 6.4, the problem is reduced to proving that we have (d) and (e) $\Rightarrow$ (a) when $\rho(A) \in \sigma(A)$. By modifying the argument given in the last part of the proof of Theorem 6.4, we can show that the answer to the preceding question is *in the negative* if one can find a nonzero complex number λ for which there exist $p_m(t) \in \mathbb{R}_+[t], m = 1, 2, \dots$, such that $\lim_{m\to\infty} p_m(\lambda) = \frac{1}{\lambda}$ and $\lim_{m\to\infty} p_m'(\lambda) = -\frac{1}{\lambda^2}$.

In Section 5, for an $n \times n$ complex matrix A, we consider the question of when there exists a proper A-invariant cone in $\mathbb{C}^n$ that contains certain given vectors $x_1, \dots, x_k$. Recently, motivated by applications in Glass networks and joint spectral radius [5, Theorem 1 and Theorem 3], the dual question of the existence and construction of common proper invariant cones for families of real matrices has also been treated by several authors ([7], [18]) and complete solutions have been offered for some special cases. We take this opportunity to point out that it is not difficult to prove the following modest result:

Lemma 8.8. *Let $A, B \in \mathcal{M}_n(\mathbb{R})$. A necessary condition for the existence of a proper cone K in $\mathbb{R}^n$ such that $A, B \in \pi(K)$ is that the cone* $\mathrm{cl}\,\mathrm{pos}\{p(A,B) : p(r,s)$ *is a monomial in the noncommuting indeterminates* $r, s\}$ *is pointed.*

It is clear that this necessary condition implies the condition that $\mathrm{cl}\, w_0(A) + \mathrm{cl}\, w_0(B)$ is pointed or, equivalently, $\mathrm{cl}\, w_0(A) \cap (-\mathrm{cl}\, w_0(B)) = \{0\}$. However, we do not know whether this necessary condition is a sufficient condition.

Example 8.9. Let $A_1 = \mathrm{diag}(1, -1, -1), A_2 = \mathrm{diag}(-1, -1, 1)$. Then $A_1^2 = A_2^2 = I_3$ and $A_1 A_2 = \mathrm{diag}(-1, 1, -1)$. In this case,

$$\mathrm{cl}\,\mathrm{pos}\,\{p(A_1, A_2) : p(r, s)\ \text{a monomial in the noncommuting indeterminates}\ r, s\}$$

equals $\mathrm{pos}\{A_1, A_2, A_1A_2, I_3\}$ and is not pointed, as $A_1 + A_2 + I_3 + A_1A_2 = 0$. By Theorem 8.8 there is no $\{A_1, A_2\}$-invariant proper cone.

The above pair of matrices A_1, A_2 has been considered in [18, Example 5], where the nonexistence of an $\{A_1, A_2\}$-invariant proper cone is obtained by applying Theorem 12 of the paper.

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