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NONLINEAR SCHRÖDINGER EQUATION AND ITS OPTIMAL CONTROL USING LASER INDUCED DYNAMIC POTENTIAL

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Abstract. In this paper we consider a class of nonlinear Schrödinger equations arising in the field of optics. We prove local existence and uniqueness of mild solutions of the nonlinear evolution equation and then formulate its control problems. We consider optimal control problems where the control is given by operator valued functions arising from the electromagnetic field induced by the interaction of laser beam with the target material. We prove existence of optimal controls and present necessary conditions of optimality including a result on convergence of a computational technique based on the necessary conditions developed.

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1. Introduction

Nonlinear Schrödinger equation plays a central role in the field of optical physics and optical engineering. It is well known that the propagation of light in optical fibers and wave guides is governed by nonlinear Schrödinger equation. Also, the subject of quantum computing has inspired renewed interest in this field. Thus the study of nonlinear schrödinger equation has become an interesting area of research because of the enormous potential for commercial prospects arising from the field of fiber-optic communication and future prospects for development of ultra-high speed computers. For its historical development and recent progress see [1].

On the theoretical front, intensive interest has been shown on the question of existence, uniqueness and regularity properties of solutions of nonlinear Schrödinger equations. These questions have been extensively studied in the literature. A partial list of this can be found in the references given in this paper. Because of the presence of nonlinearity, the solutions are often only local, that is, there exists a finite time interval $[0, T)$ such that the solution is well defined on any of its compact subinterval but then blows up at the end point T. In particular, under certain natural assumptions on the nonlinear term, the L_2 solutions corresponding to L_2 initial data remain conserved. On the other hand, under the same assumptions, the $H^1(H^2)$ solutions corresponding to $H^1(H^2)$ initial data, there is only a finite time interval over which the solutions are well defined and then their H^1 norm blows up at the end point of the time interval. This has been a very active area of research for more

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than last three decades and still continue to be so, and historically many authors such as Brezis, Kato, Kazenave, Weissler, Merle, Nakamura, Ozaua, Nawa, Ogawa, Tsutsumi, Rodnianski, Schlag, Strichartz and many others have made significant contributions in this field by either giving a new result or giving a simpler proof of old results or presenting new perspectives, [9, 12–15, 17–20, 22, 24–27].

The objective of this paper is to consider control problems of nonlinear Schrodinger equation where we use some of these results. Several authors have considered the question of controllability of the Schrödinger equation such as Zuazua [28] and Illner, Lange and Teismann [11]. It is well known that Schrödinger equation is not globally controllable. So given any desired state, by using optimal control theory, one can expect to reach a state as close as possible to the desired state subject to control constraints. Several authors have also considered optimal control problems of Quantum mechanical systems such as Atabek and Dion [6], Bandrauk and Lègarè [7], Batista and Brumer [8], Bris et all [10], Lefebvre-Brion [16], Ohtsuki and Rabitz [21], Rabitz and Shi [23]. These papers consider optimal control problems of linear Schrödinger equation with quadratic cost and present necessary conditions of optimality using Lagrange multiplier rule ignoring the question of existence of such multipliers. Also the fundamental question of existence of optimal control is not addressed. In this paper we consider optimal control problems for Nonlinear Schrödinger equation. We prove the existence of optimal controls and also present the necessary conditions of optimality.

The rest of the paper is organized as follows. In section 2, we present the system model with controls which is obtained by transforming the nonlinear Schrödinger equation defined on a complex Hilbert space to a Nonlinear Evolution equation defined on the Cartesian product of two real Hilbert spaces. In section 3, we present briefly some well known results on the questions of existence and uniqueness of solutions of basic nonlinear Schrödinger equation in the L_2 and H^s spaces including the blow up phenomenon. In section 4, we consider the control system introduced in section 2 and prove existence and uniqueness of mild solutions. In section 5, we introduce the class of admissible controls and consider the question of existence of optimal controls. In the last section, we develop necessary conditions of optimality whereby one can determine the optimal (or extremal) control polices. The paper is concluded with a convergence theorem asserting monotone convergence of the cost functional (possibly) to its local optimum.

2. Basic formulation of the system model

The system is governed by the nonlinear Schrödinger equation of the form

$$\begin{aligned} &i\partial_t \psi = (-\Delta + V_0(\xi))\psi + V_c(t,\xi)\psi + \lambda f(|\psi|)\psi, \\ &\psi(0,\xi) = \psi_0, \xi \in \Omega \equiv R^d, t \in I \equiv (0,T], \end{aligned} \tag{2.1}$$

where V_0 is the ground state (natural coulomb) potential independent of time and V_c is the control potential induced by the interaction of laser with the nonlinear medium. We assume that these potentials are real valued measurable functions on Ω. The equation is written in the atomic unit. The symbol ∂_t denotes the partial derivative with respect to time. The operator Δ stands for the standard Laplacian in dimension d (d=1,2,3) with $\Omega \equiv R^d$, and f is a nonnegative function of the amplitude of the complex wave function ψ and λ is a real parameter. The initial state ψ_0 is given and I is a finite time interval. This is the basic dynamic system arising in the study of nonlinear optics and the so called wave function ψ takes values in the field of complex numbers and given by $\psi = \psi_1 + i\psi_2$. We prefer to formulate this as a pair of coupled partial differential equations for ψ_1 and ψ_2.

Let us introduce the multiplication operators on $L_2(\Omega)$ arising from the static and dynamic potentials $V_0 \equiv \{V_0(\xi), \xi \in \Omega\}, V_c(t) \equiv \{V_c(t,\xi), \xi \in \Omega\}$ respectively. The function V_0 denotes the ground state potential and V_c is the dynamic potential induced by the action of a laser source. Define the operator $A_0 \equiv (-\Delta + V_0)$ and introduce the matrix of operators as follows:

$$A \equiv \begin{pmatrix} 0 & A_0 \\ -A_0 & 0 \end{pmatrix}, B(t) \equiv \begin{pmatrix} 0 & V_c(t) \\ -V_c(t) & 0 \end{pmatrix}$$

Using the above notations the Schrödinger equation (2.1) can be written as a system of two coupled partial differential equations in ψ_1 and ψ_2. Defining the vector $x = (x_1, x_2)' \equiv (\psi_1, \psi_2)'$ we can rewrite the system as follows

$$(d/dt)x = Ax + B(t)x + F(x), x(0) = x_0, \tag{2.2}$$

where the nonlinear operator is given by

$$F(x) = \big(\lambda f(|x|)x_2, -\lambda f(|x|)x_1\big)' \tag{2.3}$$

with $|z|$ denoting the Euclidean norm in R^2. The function f is nonnegative given by $f(|z|) = |z|^{p-1}$ for any nonnegative $p \in [1, \infty)$. Thus the nonlinear R^2-valued function F also satisfies the polynomial growth $|F(v)|_{R^2} \leq |\lambda||v|^p_{R^2}$. Later we continue to use the same notation F for the Nemytski operator on various function spaces. The original equation (2.1) is defined on a complex Hilbert space. With the transformation introduced above, we can consider the equivalent system (2.2) defined on a real Hilbert space given by the product space $L_2(\Omega) \times L_2(\Omega) \equiv L_2(\Omega, R^2)$. We denote this space by H and consider this as the state space for the system (2.2). Thus the system (2.2) is considered as an abstract differential equation on the Hilbert space H with F considered as a nonlinear operator on H. Later in section 4, we will see that it is the nonlinearity of F that determines the appropriate choice of the function spaces for its domain and range. In fact the domain space may be continuously embedded in H while H is embedded in the range space.

3. Well known results on NLS in R^d

Here we present some well known results on the Cauchy problems of Nonlinear Schrödinger equation defined on the whole space $\Omega = R^d$. There are very important interesting results due to many authors such as Kazeneve and Weissler [15], Kato [12-14], Tsutsumi [27], Merle [17] Nakamura and Ozawa [18], Nawa [19], Ogawa and Tsutsumi [20], Rodnianski and Schlag [24], Schlag [25] and many others (as indicated in the reference list). These papers also consider blowup solutions in H^s spaces for $s \geq 1$. We reproduce some of these results here for convenience of the reader. Let $\mathcal{C}$ denote the field of complex numbers and $\mathcal{H} \equiv L_2(R^d, \mathcal{C})$ the complex Hilbert space. Consider the Schrödinger equation

$$i\partial_t \psi = -\Delta \psi + G(\psi), t \geq 0, \psi(0) = \varphi. \tag{3.1}$$

where the function G is given by $G(z) = \lambda |z|^{p-1} z, z \in \mathcal{C}$ for any $\lambda \in R$.

Theorem 3.1 (L_2-Solution)**.** *Consider the nonlinear Schrödinger equation (NLS) (3.1) on the whole space* $\Omega \equiv R^d$ *and suppose* $1 < p < 1 + 4/d$*. Then for every* $\varphi \in \mathcal{H}$*, equation (3.1) has a unique global solution* $\psi \in C(I, \mathcal{H}) \cap L_r(I, L_{p+1}(\Omega, \mathcal{C}))$*, for* $(d/2 - d/(1+p)) = 2/r$*, satisfying the Duhammel's formula*

$$\psi(t) = S(t)\varphi - i \int_0^t S(t-\theta) G(\psi(\theta)) d\theta, t \in I \equiv [0, T], \tag{3.2}$$

where $S(t), t \in R$*, denotes the unitary (Schrödinger) group of operators* $S(t) = e^{it\Delta}$*. Further, the solution satisfies the mass conservation law,* $\parallel \psi(t) \parallel_{\mathcal{H}} = \parallel \varphi \parallel_{\mathcal{H}}$ *for all* $t \in I$*. In fact the interval can be any subset of the real line.*

Proof. For detailed proof see Tsutsumi [27], Ogawa & Tsutsumi [20], Kazenave & Weissler [15], Kato [12-14], Pavlovic & Tzirakis [22]. □

Theorem 3.2 (H^1-Solution)**.** *Consider the NLS (3.1) and suppose* $d \geq 2$ *and* $1 \leq p < 1 + (4/d - 2)$*. Then, for each* $\varphi \in H^1$*, there exists a* $T > 0$ *(depending only on the norm* $|\varphi|_{H^1}$*) and a unique solution* $\psi \in C(I_T, H^1), I_T = [0, T)$*. Further, the solution* ψ *is continuously dependent on the initial data over any compact subinterval* $[0, T_0] \subset I_T$*. If* $[0, T^*)$ *is the maximal interval of existence of* H^1 *solution and* $T^* < \infty$*, the solution blows up leading to* $\lim_{t \to T^*} \parallel \psi(t) \parallel_{H^1} = \infty$*.*

Theorem 3.3 (H^2-Solution)**.** *Consider the (NLS) (3.1) and suppose* $1 \leq p < 1 + (4/(d-4))$ *if* $d \geq 4$*, (for* $d < 4$*, no condition required). Then for each* $\varphi \in H^2$*, there is a* $T > 0$ *depending only on* $|\varphi|_{H^2}$*, and a unique solution* $\psi \in C(I_T, H^2)$ *with* $\psi(0) = \varphi$ *and* $\partial_t \psi \in C(I_T, L_2(R^d, \mathcal{C})) \cap L_r(I, L_p(R^d, \mathcal{C}))$*. If* $[0, T^*)$ *is the maximal interval of existence of* H^2 *solution and* $T^* < \infty$*, the solution blows up leading to* $\lim_{t \to T^*} \parallel \psi(t) \parallel_{H^1} = \infty$*.*

Corresponding to H^1 and H^2 solutions there exists another conservation law known as the conservation of energy. This is given by

$$E(\psi(t)) = (1/2) \int_{R^d} |\nabla \psi(t, x)|^2 dx - \lambda/(p+1) \int_{R^d} |\psi(t, x)|^{1+p} dx = E(\psi(0)).$$

If λ is negative the energy is a positive constant. Therefore blowup can not occur. This is the de-focussing case, and if $\lambda > 0$, the energy can be negative and blowup may occur. For detailed proof of the above results the reader is referred to the papers [9,12-14,15,17,18-20,22,24-25,27]. In fact there is a huge literature on NLS, for example, see the Lecture Notes of Pavlovič and Tzirakis [22] and the extensive references therein.

It is clear from the above results that the L_2 solution exists globally while the H^1 and H^2 solutions exist only locally and may blow up at the end point of the maximal interval of existence. Here in this paper we are mainly interested in L_2 and L_{1+p} (for some $p \geq 1$) solutions which exist globally.

4. Existence and uniqueness of solutions

In this section we study the question of existence and uniqueness of solution of the evolution equation (2.2) arising from the initial value problem (2.1). We assume that the potential V_0 is sufficiently smooth and bounded away from below. Thus by adding a sufficiently large positive number to V_0 and subtracting the same number from the other potentials, we can consider V_0 to be strictly positive bounded away from below by a positive number. So without loss of generality we consider $A_0 \equiv (-\Delta + V_0)$ to be a positive self adjoint operator on the Hilbert space $L_2(\Omega)$. Next we consider the operator A, as defined in section 2, and note that its domain is given by $D(A) = D(A_0) \times D(A_0) \subset H$ and range $R(A) \subset H$, and that $D(A)$ is dense in H. It is not difficult to verify that the operator A is skew adjoint, that is, $(iA)^* = iA$. Thus it follows from Stones theorem [2, Theorem 3.1.4, p71] that A generates a unitary group of bounded linear operators $\{U(t), t \in R\}$ on H with the property that, as $t \to 0$, $U(t)$ converges to the identity operator I_H in the strong operator topology. For simplicity we assume that the potential $V_c \in B_\infty(I \times \Omega) \subset L_\infty(I \times \Omega)$ where $B_\infty(I \times \Omega)$ denotes the Banach space of bounded measurable functions on $I \times \Omega$ with respect to the standard supnorm topology. Under this assumption, the operator $B(t) \in \mathcal{L}(H)$ for all $t \in I$. We may assume that the operator valued function $B = B(\cdot)$ is Borel measurable in the strong operator topology. Using the unitary group $U(t), t \in I$, we can reformulate the system (2.2) as an integral equation on the Hilbert space H as follows:

$$x(t) = U(t)x_0 + \int_0^t U(t-s)B(s)x(s)ds + \int_0^t U(t-s)F(x(s))ds. \tag{4.1}$$

For the time being let us assume that equation (2.2) has an H valued mild solution $x(t), t \in I$, given by the solution of the above integral equation. Then scalar multiplying the equation (2.2) by $x(t)$ in H and integrating we find that

$$\begin{aligned} |x(t)|_H^2 = |x_0|_H^2 &+ 2\int_0^t (Ax(s), x(s))ds + 2\int_0^t (B(s)x(s), x(s))ds \\ &+ 2\int_0^t (F(x(s)), x(s))ds \end{aligned} \tag{4.2}$$

Since the scalar products $(Ax, x) = 0, (B(t)x, x) = 0, (F(x), x) = 0$, it follows from the identity (4.2) that $|x(t)|_H^2 = |x_0|_H^2$ for all $t \geq 0$. This is easily justified rigorously by replacing the operator A_0 by its Yosida approximation $A_{0,n} = nA_0R(n, A_0), n \in \rho(A_0)$, the resolvent set of the operator A_0. Then the operator A is replaced by A_n where

$$A_n \equiv \begin{pmatrix} 0 & A_{0,n} \\ -A_{0,n} & 0 \end{pmatrix}$$

and the unitary group $U(t), t \geq 0$, by the corresponding unitary group $U_n(t), t \geq 0$. It follows from Trotter-Kato approximations [2, Theorem 4.5.4, Remark 4.5.5] that $U_n(t) \longrightarrow U(t)$ in the strong operator topology on $\mathcal{L}(H)$ uniformly on compact intervals of R. The initial state x_0 is replaced by $x_{0,n} = (R_n x_{0,1}, R_n x_{0,2})'$, where $R_n \equiv nR(n, A_0)$, for $n \in \rho(A_0)$, is the Yosida approximation of the identity operator in the Hilbert space $L_2(\Omega)$. Note that $x_{0,n} \in D(A)$ and that it converges to x_0

strongly in H. In this case the operator A of system (2.2) is replaced by its Yosida approximation A_n and the initial state by $x_{0,n}$ with the corresponding solution denoted by x_n. Again scalar multiplying (in H) on both sides of the following equation

$$\dot{x}_n = A_n x_n + B(t)x_n + F(x_n), x_n(0) = x_{0,n}$$

with x_n and integrating over the interval $[0,t]$ we obtain the following identity,

$$|x_n(t)|_H^2 = |x_{0,n}|_H^2, \text{for all } t \in I.$$

So the approximated system is also conservative. Now as $n \to \infty$, $x_{0,n} \stackrel{s}{\longrightarrow} x_0$ in H and so for each $t \in I$, $x_n(t) \stackrel{s}{\longrightarrow} x(t)$ in H, where x is the mild solution of equation (2.2) or simply the solution of the integral equation (4.1). From these facts it is easy to verify that $|x(t)|_H^2 = |x_0|_H^2$ for all $t \in I$ (the interval of existence of solution). Therefore we conclude that if equation (2.2) has a mild solution with values in H, the H norm of $x(t)$ remains invariant with respect to $t \in I$, that is, $|x(t)|_H^2 = |x(0)|_H^2 = |x_0|_H^2$. In quantum physics this is known as the conservation of mass. As seen above, there is another conservation law called the law of conservation of energy. This involves the H^1 solutions of the Schrödinger equation (2.1). There is an extensive literature on the blowup solutions in H^s spaces for $s \geq 1$. Here we are not interested in this. For details the reader is referred to [2,3,4,5,6].

We are interested in the control of the system (2.2) equivalent to the nonlinear Schrödinger equation (2.1). For nonlinear analysis we introduce the following Banach spaces. First, we have the Hilbert space $H \equiv L_2(\Omega, R^2)$ as already introduced before. Next, for $p \geq 1$, let $X \equiv L_{1+p}(\Omega, R^2)$ with dual $X^* \equiv L_{1+1/p}(\Omega, R^2)$. Using Hölder inequality it is easy to verify that the injection $X \cap X^* \hookrightarrow H$ is continuous. Further, if the space $X \cap X^*$ is given the norm topology $\| z \|_{X\cap X^*} \equiv max\{\| z \|_X, \| z \|_{X^*}\}$, its topological dual is given by $(X \cap X^*)^* = X^* \oplus X$.

In the following theorem we present existence, uniqueness and regularity properties of solutions of the evolution equation (2.2). First let us recall that the free Schrödinger group $S_0(t), t \in R$, satisfies the following estimate

$$\| S_0(t)h \|_{L\infty(\Omega)} \leq (\alpha/t^{d/2}) \| h \|_{L_1(\Omega)}, t \in R \setminus \{0\}$$

for some $\alpha > 0$ finite. The infinitesimal generator of this semigroup (actually unitary group) is perturbed by a bounded linear operator, represented by the multiplication operator $B_0 h \equiv V_0 h$ giving the perturbed operator $(-\Delta + V_0)$. Strichartz, in his paper [26, p.712], states that similar decay property may also hold for the semigroup corresponding to the perturbed generator $(-\Delta + V_0)$. In fact, under certain decay properties of the potential V_0, Rodnianski and Schlag [24, Theorem 1.1, Theorem 1.2, Theorem 2.6] gave a proof of this estimate for the perturbed group for dimension $d = 3$. Schlag also presents the same $L_1 \longrightarrow L_\infty$ estimate in his paper [25, Theorem 2.3]. In other words, the perturbed group $S(t)$ satisfies the following estimate

$$\| S(t)h \|_{L\infty(\Omega)} \leq (c_0/t^{d/2}) \| h \|_{L_1(\Omega)}, t \in R \setminus \{0\}$$

for some constant $c_0 > 0$ possibly depending on α and d. This estimate can be obtained easily from the general theory of perturbation of C_0-semigroups [2, Theorem 4.2.1, p111].

Theorem 4.1. *Consider the evolution equation (2) and suppose the assumptions of Theorem 3.1 hold and that the operator valued function* $B \in B_\infty(I, \mathcal{L}(X, X^*))$ *is measurable in the strong operator topology and uniformly bounded (in operator norm) on* I *and the nonlinear operator* F *is given by (2.3) satisfying the polynomial growth as described above. Then, for any* $x_0 \in X \cap X^*$*, the integral equation (4.1) has a unique solution* $x \in C(I, X)$.

Proof. The proof is based on Banach fixed point theorem. For any $\tau \in I$, define the interval $[0, \tau) \subset I$ and introduce the operator Φ as follows

$$\Phi(x)(t) \equiv U(t)x_0 + \int_0^t U(t-s)B(s)x(s)ds + \int_0^t U(t-s)F(x(s))ds, t \in I_\tau. \tag{4.3}$$

Recall that for each $t \in I_\tau \equiv [0, \tau)$, $\tau \in [0, \infty)$, the operator $U(t)$ is a unitary group on H and hence $\| U(t)h \|_H = \| h \|_H$ for every $h \in H$ and $\| U(t) \|_{\mathcal{L}(H)} = 1$. In fact the operator $U(t)$ is much more smooth and it follows from the basic properties of the Schrödinger group as discussed above that there exists a positive constant c such that

$$\| U(t)h \|_{L_\infty(\Omega, R^2)} \leq c/t^{d/2} \| h \|_{L_1(\Omega, R^2)}, t \neq 0.$$

Based on the unitary property of the group of operators $U(t), t \in R$, and the above inequality, it follows from Riesz-Thorin interpolation theorem [22] that for every $t \neq 0$,

$$\| U(t)f \|_{L_{q'}(\Omega, R^2)} \leq (c/t^{d(1/2-1/q)}) \| f \|_{L_q(\Omega, R^2)}, \ \forall \ f \in L_q(\Omega, R^2), \tag{4.4}$$

for any pair $1 \leq q', q \leq \infty$ satisfying $1/q' + 1/q = 1$. It follows from the definition of the nonlinear operator F (see (2.3)) that

$$|F(u(t, \xi))|_{R^2} = |\lambda| \ |u(t, \xi)|^p_{R^2}, t \in R, \xi \in \Omega.$$

From this identity, it is easy to verify that F maps $L_{1+p}(\Omega, R^2)$ to $L_{((1+p)/p)}(\Omega, R^2)$ and it satisfies the estimate

$$\| F(u) \|_{L_{(1+p)/p}(\Omega, R^2)} \leq |\lambda| \| u \|^p_{L_{1+p}(\Omega, R^2)} .$$

In terms of the dual pair of Banach spaces $\{X, X^*\}$, this is equivalent to saying that the nonlinear operator $F : X \longrightarrow X^*$ and that

$$\| F(u) \|_{X^*} \leq |\lambda| \| u \|^p_X \tag{4.5}$$

for all $u \in X$. Since the function f, arising in the definition of the operator F, is purely a homogeneous polynomial of degree $(p-1)$, it is easy to verify that F is locally Lipschitz satisfying

$$\| F(u) - F(v) \|_{X^*} \leq |\lambda| \kappa(\| u \|^{p-1}_X, \| v \|^{p-1}_X) \| u - v \|_X, \tag{4.6}$$

where κ is a nonnegative, continuous and monotone increasing function of it's arguments and bounded on bounded sets. For any arbitrary $r > 0$ (to be chosen later), let

$$C(r) \equiv \sup\{\kappa(\| u \|^{p-1}_X, \| v \|^{p-1}_X), u, v \in B_r(X)\}$$

where $B_r(X)$ is the closed ball in X of radius r centered at the origin. Consider the nonlinear part of the operator Φ given by

$$z(t) \equiv (LF)(x)(t) \equiv \int_0^t U(t-s)F(x(s))ds, t \in I_\tau. \tag{4.7}$$

Using Hölder inequality and the estimate (4.4) for $q = (1+p)/p$ and its conjugate $q' = 1+p$ one can easily verify that

$$\| U(t) \|_{\mathcal{L}(X^*,X)} \leq (c/t^{d(1-p)/2(1+p)}), \text{ for all } t \in R, \tag{4.8}$$

and not only for $t \in I_\tau$. Since by assumption $p \geq 1$, it is clear from the above estimate that $U(t) \in \mathcal{L}(X^*, X)$ for all $t \in R$ and also it is uniformly bounded in operator norm on any bounded subset of R. Further, $t \longrightarrow U(t)$ is continuous on R with respect to the strong operator topology on $\mathcal{L}(X^*, X)$. We have already seen that $F : X \longrightarrow X^*$ satisfying the estimate (4.5). Thus the composition map LF is a continuous nonlinear operator from X to itself and bounded on bounded sets. So the function z defined by the expression (4.7) is an element of $L_\infty(I_\tau, X)$. Since, by assumption, $B \in B_\infty(I_\tau, \mathcal{L}(X, X^*))$ there exists a positive constant b such that

$$\sup\{\| B(t) \|_{\mathcal{L}(X,X^*)}, t \in I_\tau\} \leq b. \tag{4.9}$$

Now taking the X-norm of $\Phi(x)(t)$ it follows from the expression (4.3) and the above estimates that there exists a positive constant C_0 such that

$$\begin{aligned} \| \Phi(x)(t) \|_X \leq C_0 \| x_0 \|_X + C_0 b \int_0^t \| x(s) \|_X \, ds \\ + C_0 |\lambda| \int_0^t \| x(s) \|_X^p \, ds, t \in I_\tau. \end{aligned} \tag{4.10}$$

Consider the restriction of the operator Φ on the closed ball

$$\mathcal{B}_r \equiv \left\{ x \in L_\infty(I_\tau, X) : ess - sup\{\| x(t) \|_X, t \in I_\tau\} \leq r \right\}.$$

Then it follows from (4.10) that, for any $x \in \mathcal{B}_r$, we have

$$\| \Phi(x) \|_{L_\infty(I_\tau,X)} \leq C_0 \| x_0 \|_X + \tau(C_0 br + C_0|\lambda|r^p). \tag{4.11}$$

We choose $r > 0$ large enough so that $r = 2C_0 \| x_0 \|_X$ and then we choose τ sufficiently small so that $\tau(C_0 br + C_0|\lambda|r^p) \leq r/2$. This leads to the inequality $\| \Phi(x) \|_{L_\infty(I_\tau,X)} \leq r$ for all $x \in \mathcal{B}_r$ and hence we conclude that Φ maps $\mathcal{B}_r$ into itself. Now we show that Φ is a contraction on $\mathcal{B}_r$. For any pair $\{x, y\} \in \mathcal{B}_r$, subtracting $\Phi(y)$ from $\Phi(x)$ and taking X-norm, one can easily arrive at the following inequality

$$\| \Phi(x) - \Phi(y) \|_{L_\infty(I_\tau,X)} \leq (C_0 b + C_0|\lambda|C(r))\tau \| x - y \|_{L_\infty(I_\tau,X)} . \tag{4.12}$$

Again, for τ sufficiently small, Φ is a contraction on $\mathcal{B}_r \subset L_\infty(I_\tau, X)$. Hence by Banach fixed point theorem, Φ has a unique fixed point in $\mathcal{B}_r$ and therefore we conclude that the integral equation (4.1) has a unique solution in $L_\infty(I_\tau, X)$ for τ sufficiently small. In other words the evolution equation (2.2) has a unique mild solution $x \in L_\infty(I_\tau, X)$. Since the operator valued function $t \longrightarrow U(t) \in \mathcal{L}(X^*, X)$ is continuous in the strong operator topology, as seen above, the solution actually belongs to $C(I_\tau, X)$, the Banach space (with respect to the usual supnorm topology) of continuous functions on I_τ with values in X. Hence the solution can be extended step by step, $[0, \tau]$, $[\tau, 2\tau]$,$[2\tau, 3\tau]$, and so on to any finite interval $I = [0, T] \subset R$ whereby we can conclude that the solution $x \in C(I, X)$. This completes the proof. □

Remark 4.2. Since $U(t), t \in R$, is a unitary group on H, the H-norm of $x(t)$ remains conserved, that is, $\| x(t) \|_H = \| x_0 \|_H$ for all $t \in R$. Thus the mass conservation law holds for the control system (2.2) but the energy conservation law does not.

5. Optimal control

Recall that our primary objective is to control the evolution equation (2.2) or equivalently the integral equation (4.1). The control here is the operator valued function B constructed from the dynamic (electro-magnetic) potential $V_c(t, \xi), t \in I, \xi \in \Omega$, induced by the action of the laser beam on target materials. In other words, the controls are operator valued functions taking values $B(t) \in \mathcal{L}(X, X^*)$. Using Hölder inequality it is easy to verify that if, $V_c(t) \in L_{((p+1)/p)}(\Omega), 1 \leq p < \infty$, for each $t \in R$, then $B(t) \in \mathcal{L}(X, X^*)$ for each $t \in R$. The space $\mathcal{L}(X, X^*)$ is equipped with the strong operator topology. We denote this topological space by $(\mathcal{L}(X, X^*), \tau_{so}) \equiv \mathcal{L}_{so}(X, X^*)$. Consider the set

$$\Gamma \equiv \big\{L \in \mathcal{L}(X, X^*) : L_{i,j} \in \mathcal{L}(L_{1+p}(\Omega), L_{1+1/p}(\Omega)), \\ \& \ L_{i,j} = 0 \text{ for } i = j, \ \& \ L_{i,j} = -L_{j,i}\big\},$$

and suppose that it is equipped with the relative strong operator topology. Later in the sequel, for proof of existence of optimal control policies, we use compact subsets of the topological space $\mathcal{L}_{so}(X, X^*)$. In the following proposition we present necessary and sufficient conditions characterizing compact subsets of $\mathcal{L}_{so}(X, X^*)$.

Proposition 5.1. *Let $\{e_n\} \subset X$ be a normalized Schauder orthogonal basis with $\{e_n^*\} \subset X^*$ being the dual basis. A set $\Gamma \subset \mathcal{L}(X, X^*)$ is conditionally compact in the strong operator topology τ_{so} if, and only if, the following conditions hold:*

(i): *Γ is a (norm) bounded subset of $\mathcal{L}(X, X^*)$,*

(ii): *For each $x \in X$, and any $\varepsilon > 0$, there exists an integer $N_\varepsilon \in N$ such that $\sup_{T\in\Gamma}\{\sum_{i=n}^{\infty} |(Tx, e_i)_{X^*,X}|\} < \varepsilon$ for all $n \geq N_\varepsilon$.*

Proof. (Outline) Since both X and X^* are Banach spaces, the space $\mathcal{L}(X, X^*)$ endowed with the strong operator topology is a locally convex sequentially complete Hausdorff topological vector space. Thus by the uniform boundedness principle the limit of any sequence in the strong operator topology is an element of $\mathcal{L}(X, X^*)$. Reflexivity of the Banach spaces (X, X^*) also imply that any bounded subset of $\mathcal{L}(X, X^*)$ is conditionally compact in the weak operator topology. Using this fact one can show that the conditions (i) and (ii) are necessary and sufficient for conditional compactness in the strong operator topology. This completes the outline of our proof. □

Since the above result has wider and independent interest, a detailed proof will appear in another paper.

Let $\Gamma_b \subset \Gamma \subset \mathcal{L}(X, X^*)$ be a set compact in the strong operator topology with the operator norms being bounded from above by a finite positive number b. For admissible controls, we choose the class of Γ_b-valued functions defined on I and measurable in the strong operator topology. We denote this class by $\mathcal{B}_{ad} \equiv B_\infty(I, \Gamma_b)$. Furnished with the Tychonoff product topology τ_π, the set $\mathcal{B}_{ad}$ is compact.

For proof of existence of optimal controls we shall need results asserting continuous dependence of solutions with respect to controls. This is stated in the following theorem.

Theorem 5.2. *Consider the System (2.2) and suppose the assumptions of theorem 4.1 hold. Then, the (mild) solution of equation (2.2) is continuously dependent on the controls in the sense that whenever a generalized sequence $\{B_n\} \in \mathcal{B}_{ad}$ converges in the Tychonoff product topology to B_o, the sequence of solutions $\{x_n\}$ of equation (2.2), corresponding to the sequence $\{B_n\}$, converges in the sup-norm topology of $C(I, X)$ to the solution x_o of equation (2.2) corresponding to the control B_o.*

Proof. Let $x(B) \in C(I, X)$ denote the solution of the integral equation (4.1) corresponding to the control operator $B \in \mathcal{B}_{ad}$ and let $\mathcal{X} \equiv \{x(B), B \in \mathcal{B}_{ad}\}$ denote the set of solutions corresponding to the admissible set of controls $\mathcal{B}_{ad}$. Since the set Γ_b is contained in the ball $B_b(\mathcal{L}(X, X^*))$, for the given $T > 0$ finite, there exists a sufficiently large positive number r so that the set of solutions $\mathcal{X}$ (corresponding to the set of admissible controls) is contained in the ball

$$B_r(C(I, X)) \equiv \{x \in C(I, X) : \parallel x \parallel_{C(I,X)} \leq r\}.$$

Let $\{B_n\}$ be a generalized sequence from the admissible set $\mathcal{B}_{ad}$ and let $B_o \in \mathcal{B}_{ad}$ to which B_n converges in the Tychonoff's product topology τ_π. Let $\{x_n\}$ and x_o denote respectively the corresponding solutions of the integral equation (4.1). We show that $x_n \stackrel{s}{\longrightarrow} x_o$ in $C(I, X)$ where $\stackrel{s}{\longrightarrow}$ denotes convergence in the usual sup-norm topology. Subtracting x_o from x_n it follows from the integral equation (4.1) that

$$\begin{aligned} x_n(t) - x_o(t) &= \int_0^t U(t-s)[B_n(s)x_n(s) - B_o(s)x_o(s)]ds \\ &+ \int_0^t U(t-s)[F(x_n(s)) - F(x_o(s))]ds, \ t \in I. \end{aligned} \tag{5.1}$$

Define the function e_n by

$$e_n(t) \equiv \int_0^t U(t-s)\big(B_n(s) - B_o(s)\big)x_o(s)ds, t \in I. \tag{5.2}$$

Using the expression (5.2) we can rewrite equation (5.1) as follows:

$$\begin{aligned} x_n(t) - x_o(t) &= e_n(t) + \int_0^t U(t-s)B_n(s)(x_n(s) - x_o(s))ds \\ &+ \int_0^t U(t-s)[F(x_n(s)) - F(x_o(s))]ds, t \in I. \end{aligned} \tag{5.3}$$

Taking the X-norm of either side of the expression (5.3) and using (4.6) and (4.8) and the fact that for all $t \in I$, $x_n(t), x_o(t) \in B_r(X)$ (the closed ball in X of radius

r centered at the origin) for all $n \in N$, we obtain

$$\| x_n(t) - x_o(t) \|_X \leq \| e_n(t) \|_X + bc \int_0^t (t-s)^{d(p-1)/2(p+1)} \| x_n(s) - x_o(s) \|_X \, ds$$
$$+ C(r)|\lambda|c \int_0^t (t-s)^{d(p-1)/2(p+1)} \| x_n(s) - x_o(s) \|_X \, ds. \tag{5.4}$$

Since by our assumption the exponent p, related to the nonlinear operator F (see (2.3)), is equal or greater than 1 ($p \geq 1$) the above inequality reduces to

$$\| x_n(t) - x_o(t) \|_X \leq \| e_n(t) \|_X + \beta \int_0^t \| x_n(s) - x_o(s) \|_X \, ds, t \in I_T \equiv I. \tag{5.5}$$

where $\beta \equiv (b + C(r)|\lambda|)cT^{d(p-1)/2(p+1)} < \infty$. Now by virtue of Gronwall inequality, it follows from (5.5) that

$$\| x_n(t) - x_o(t) \|_X \leq \| e_n(t) \|_X + \beta \int_0^t e^{\beta(t-s)} \| e_n(s) \|_X \, ds, t \in I. \tag{5.6}$$

Returning to the expression (5.2) and using the estimate (4.8) we arrive at the following inequality

$$\| e_n(t) \|_X \leq cT^{d(p-1)/2(p+1)} \int_0^t \| [B_n(s) - B_o(s)]x_o(s) \|_{X^*} \, ds, t \in I. \tag{5.7}$$

Clearly, this implies that

$$\sup\{\| e_n(t) \|_X, t \in I\} \leq cT^{d(p-1)/2(p+1)} \int_0^T \| [B_n(s) - B_o(s)]x_o(s) \|_{X^*} \, ds. \tag{5.8}$$

Since B_n converges to B_o in the Tychonoff product topology, $B_n(t) \xrightarrow{\tau_{so}} B_o(t)$ for each $t \in I$, and since $x_o(s) \in X$ for each $s \in I$, it is clear that the integrand in the expression (5.8) converges to zero for each $t \in I$, and further, it is bounded from above by $2br$. Thus, it follows from Lebesgue bounded convergence theorem that the integral converges to zero implying

$$\lim_{n \to \infty} \sup\{\| e_n(t) \|_X, t \in I\} = 0. \tag{5.9}$$

Hence it follows from (5.6) and (5.9) that

$$\lim_{n \to \infty} \sup\{\| x_n(t) - x_o(t) \|_X, t \in I\} = 0.$$

This shows that the (mild) solution $x(B)$ of the evolution equation (2.2) is continuously dependent on the control B . This completes the proof. □

Now we are prepared to consider control problems. The objective functional is given by

$$J(B) \equiv \int_0^T \ell(t, x(t)) \, dt + \Psi(x(T)), \tag{5.10}$$

where $x \in C(I, X)$ ($I = I_T$) is the mild solution of equation (2.2) corresponding to the control policy (operator valued function) $B \in \mathcal{B}_{ad}$. The integral term represents the running cost and Ψ denotes the terminal cost. The objective is to find an operator valued function $B \in \mathcal{B}_{ad}$ that minimizes the functional $J(B)$ subject to the dynamic constraint (2.2). In this section we consider the question of existence of optimal control. In section 6, we develop the necessary conditions of optimality characterizing optimal policies.

Theorem 5.3. *Consider the dynamic system (2.2) and suppose the assumptions of Theorem 5.2 hold and that the set of admissible controls $\mathcal{B}_{ad}$ is compact in the Tychonoff product topology τ_π. Let $\ell : I \times X \longrightarrow R_+ \cup +\infty$ be measurable in the first argument and lower semicontinuous in the second satisfying*

$$|\ell(t,x)| \leq \alpha(t) + k \parallel x \parallel_X^q,$$

with $k \geq 0$ and $\alpha \in L_1^+(I)$ and $\infty > q \geq 1$. The function $\Psi : X \longrightarrow R \cup \{+\infty\}$ is lower semicontinuous and $|\Psi(x)| \leq \gamma + \delta \parallel x \parallel_X^q$, with $\gamma, \delta \geq 0$. Then, there exists an optimal policy $B_o \in \mathcal{B}_{ad}$ at which J attains its minimum.

Proof. Since the solution set $\mathcal{X} \subset C(I,X)$ is bounded and the functions $\{\ell, \Psi\}$ satisfy the inequalities as stated in the theorem and $\alpha \in L_1^+(I)$, it is clear that $\inf\{J(B), B \in \mathcal{B}_{ad}\} > -\infty$. We prove that J is lower semicontinuous on $\mathcal{B}_{ad}$ in the product topology τ_π. Let $\{B_n\}$ be any sequence from $\mathcal{B}_{ad}$ and let $\{x_n\} \in C(I,X)$ be the corresponding sequence of mild solutions of equation (2.2). Since $\mathcal{B}_{ad}$ is compact in the Tychonoff product topology, there exists a subsequence $\{B_{n_k}\}$ of the sequence $\{B_n\}$, relabeled as B_n, and a $B_o \in \mathcal{B}_{ad}$ such that $B_n \xrightarrow{\tau_\pi} B_o$. Then it follows from Theorem 5.1 that the corresponding sequence of solutions $\{x_n\}$, along the subsequence if necessary, converges to x_o in the usual norm topology of $C(I,X)$. As the function ℓ is lower semicontinuous in its second argument, it is clear that

$$\ell(t, x_o(t)) \leq \underline{\lim}\, \ell(t, x_n(t)), a.e\ t \in I.$$

Hence it follows from generalized Fatou's lemma that

$$\int_I \ell(t, x_o(t))dt \leq \underline{\lim} \int_I \ell(t, x_n(t))dt.$$

Since $\{x_n, x_o\} \in C(I,X)$ and, by our assumption Ψ is lower semicontinuous on X, it is obvious that

$$\Psi(x_o(T)) \leq \underline{\lim}\, \Psi(x_n(T)).$$

As the sum of a finite number of lower semicontinuous functions is a lower semicontinuous function, we conclude that $B \longrightarrow J(B)$ is a lower semicontinuous functional on $\mathcal{B}_{ad}$ in the Tychonoff product topology τ_π. Thus J attains its minimum on $\mathcal{B}_{ad}$ proving the existence of an optimal control. □

Remark 5.4. Since there is hard constraint on controls through the set Γ_b, it is not essential to introduce control cost. However, if it is necessary to impose an extra Levy on the usage of controls one may extend the cost functional (5.10) by adding the following control cost

$$J_3(B) \equiv \int_0^T \sum_{i=1}^\infty \gamma_i \ (\langle B(t)e_i, e_i\rangle_{X^*,X})^2 dt,$$

where $\gamma_i \geq 0, \{\gamma_i\} \in \ell_1$, and $\{e_i\}$ is a normalized Schauder basis of the Banach space X. Since the admissible set $\mathcal{B}_{ad}$ is bounded, the infinite series is well defined and bounded above by $\beta^2 \parallel \gamma \parallel_{\ell_1}$.

Remark 5.5. In the field of photonics, laser pulses of the order of femto seconds (10^{-15} seconds) or even shorter (10^{-18} seconds) are used to interact with molecules and atoms. Such controls can be regarded as impulsive controls. In this case the admissible controls can be chosen as operator valued measures which contain impulsive controls as special case. That is, the set $\mathcal{B}_{ad}$ is now a proper subset of the space of operator valued measures $\mathcal{M}_{casbsv}(\Sigma, \mathcal{L}(X, X^*))$ which are countably additive in the strong operator topology having bounded semivariation. For characterization of weakly compact sets in such spaces see Ahmed [3,4,5]. In view of the application mentioned above, it would be interesting to prove existence of optimal controls in this broader class. We leave this as an open problem for future consideration.

6. Necessary conditions of optimality

In this section we develop the necessary conditions of optimality whereby one can construct a computational algorithm to determine the optimal control policy. For this we need additional regularity of the functions ℓ and Ψ and the nonlinear operator F. We denote the Gâteaux differential of F at the point $\xi \in X$ and in the direction $\eta \in X$ by $DF(\xi;\eta) \equiv DF(\xi)\eta$. Clearly, since the operator F originates purely from a polynomial f, its Gâteaux derivatives do exist and they are also continuous.

Theorem 6.1. *Consider the system (2.2) and suppose the assumptions of Theorem 5.3 hold. Further suppose the set Γ_b, determining the admissible controls $\mathcal{B}_{ad}$, is closed and convex, and that Ψ is once continuously Gâteaux differentiable on X with the Gâteax derivative belonging to X^*, and ℓ is also once continuously Gâteaux differentiable on X with respect to its second argument with the Gâteaux derivative belonging to $L_1(I, X^*)$. Then, for a pair $(B^o, x^o) \in \mathcal{B}_{ad} \times \mathcal{X}$ to be optimal, it is necessary that there exists a $\psi \in C(I, X)$ such that the triple $\{B^o, x^o, \psi\}$ satisfies the following inequality and the evolution equations:*

$$\int_0^T \langle (B(s) - B^o(s))x^o(s), \psi(s)\rangle_{X^*,X} ds \geq 0, \ \forall \ B \in \mathcal{B}_{ad}, \tag{6.1}$$

$$\begin{aligned} -(d/dt)\psi &= A^*\psi + (B^o(t))^*\psi + (DF(x^o(t)))^*\psi \\ &\quad + \ell_x(t, x^o(t)), \psi(T) = \Psi(x^o(T)), \end{aligned} \tag{6.2}$$

$$(d/dt)x^o = Ax^o + B^o(t)x^o + F(x^o(t)), x^o(0) = x_0. \tag{6.3}$$

Proof. Let $B^o \in \mathcal{B}_{ad}$ be the optimal policy and $B \in \mathcal{B}_{ad}$ any other element. Since $\mathcal{B}_{ad}$ is closed and convex, it is clear that $B^o+\varepsilon(B-B^o) \in \mathcal{B}_{ad}$ and $J(B^o+\varepsilon(B-B^o) \geq J(B^o)$ for all $\varepsilon \in [0,1]$. Thus

$$(1/\varepsilon)\big(J(B^o + \varepsilon(B - B^o)) - J(B^o)\big) \geq 0 \ \forall \ \varepsilon \in [0,1], B \in \mathcal{B}_{ad}.$$

Let $\{x^o, x^\varepsilon\} \in C(I, X)$ denote the (mild) solutions of the evolution equation (2.2) corresponding to the control operators $\{B^o, B^\varepsilon\}$ respectively where $B^\varepsilon \equiv B^o+\varepsilon(B-B^0)$. Let $DF(x^o(t))$ denote the Gâteaux differential of the nonlinear operator F and $\ell_x(t, x^o(t))$ and $\Psi_x(x^o(T))$ the Gâteaux differentials of ℓ and Ψ respectively. Since the entries of the operator F are homogeneous polynomials of degree $p > 1$, it is clear that it has well defined Gâteaux differentials. Since, by our assumption, both ℓ

and Ψ are continuously Gâteaux differentiable, it follows from the above inequality that the Gâteaux differential of J at B^o in the direction $(B - B^o)$ satisfies the following inequality

$$dJ(B^o, B - B^o) = \int_0^T \langle \ell_x(t, x^o(t)), y(t)\rangle_{X^*,X} dt + \langle \Psi_x(x^o(T)), y(T)\rangle_{X^*,X} \geq 0, \tag{6.4}$$

for all $B \in \mathcal{B}_{ad}$ where $y \in C(I, X)$ is the (mild) solution of the following variational equation

$$(d/dt)y = Ay + B^o(t)y + DF(x^o(t))y + (B - B^o)x^o, y(0) = 0, \tag{6.5}$$

driven by the process $(B - B^o)x^o$ with $DF(x^o(t))$ satisfying the following inequality

$$\| DF(x^o(t)) \|_{\mathcal{L}(X,X^*)} \leq C \| x^o(t) \|_X^{p-1} .$$

Since equation (6.5) is a linear evolution equation, in particular a special case of the original nonlinear equation (2.2), it follows from Theorem 4.1, with minor variation in the steps, that this variational equation has a unique solution $y \in C(I, X)$ and that it is given by the $\lim_{n\to\infty}(1/\varepsilon)(x^\varepsilon - x^o)$ uniformly on I. Clearly, if $B = B^o$, the solution y of equation (6.5) is identically zero, $y(t) \equiv 0$ for $t \in I$. It follows from the properties of the admissible set $\mathcal{B}_{ad}$ that $(B - B^o)x^o \in L_\infty(I, X^*) \subset L_1(I, X^*)$. Thus the map

$$(B - B^o)x^o \longrightarrow y$$

is a continuous linear (bounded) map from $L_1(I, X^*)$ to $C(I, X)$. Define the functional L by

$$L(y) \equiv \int_0^T \langle \ell_x(t, x^o(t)), y(t)\rangle_{X^*,X} dt + \langle \Psi_x(x^o(T)), y(T)\rangle_{X^*,X}. \tag{6.6}$$

By our assumption, ℓ and Ψ are continuously Gâteaux differentiable on X and $\ell_x(\cdot, x^o(\cdot)) \in L_1(I, X^*)$ and $\Psi_x(x^o(T))) \in X^*$. Thus the map $y \longrightarrow L(y)$ is a continuous linear functional on $C(I, X) \subset L_\infty(I, X)$. Combining the above results, we conclude that the composition map

$$(B - B^o)x^o \longrightarrow y \longrightarrow L(y) \equiv \tilde{L}((B - B^o)x^o) \tag{6.7}$$

is a continuous linear functional on $L_1(I, X^*)$. Hence by Riesz representation theorem and the fact that X is a reflexive Banach space, there exists a $\psi \in L_\infty(I, X) \equiv L_\infty(I, X^{**})$ so that

$$\tilde{L}((B - B^o)x^o) = \int_0^T \langle B - B^o)x^o, \psi\rangle_{X^*,X} dt. \tag{6.8}$$

Now it follows from (6.4), (6.6),(6.7) and (6.8) that

$$dJ(B^o, B - B^o) = \int_0^T \langle B - B^o)x^o, \psi\rangle_{X^*,X} dt \geq 0, \ \forall\ B \in \mathcal{B}_{ad}. \tag{6.9}$$

This proves the necessary condition (6.1). To prove the necessary condition (6.2), formally we substitute the variational equation (6.5) into the righthand side of the expression (6.9) to obtain

$$RHS(36) = \int_0^T \langle (d/dt)y - Ay - B^o y - DF(x^o)y, \psi(t)\rangle_{X^*,X} dt. \tag{6.10}$$

Integrating by parts, it follows from (6.10) that

$$\begin{aligned} RHS(36) &= \langle y(T), \psi(T)\rangle_{X,X^*} \\ &+ \int_0^T < y, -\dot{\psi} - A^*\psi - (B^o)^*\psi - (DF(x^o))^*\psi >_{X,X^*} dt. \end{aligned} \tag{6.11}$$

This is justified by use of the well known Yosida approximation A_n of A and then letting n go to infinity. Now setting

$$-\dot{\psi} = A^*\psi + (B^o)^*\psi + (DF(x^o))^*\psi + \ell_x(t, x^o(t)), \psi(T) = \Psi_x(x^o(T)), \tag{6.12}$$

it follows from (6.9) and (6.11) that

$$\begin{aligned} dJ(B^o, B - B^o) = &\int_0^T \langle y(t), \ell_x(t, x^o(t))\rangle_{X,X^*} dt \\ &+ \langle y(T), \Psi_x(x^o(T))\rangle_{X,X^*}, \end{aligned} \tag{6.13}$$

where the expression on the right hand side of (6.13) coincides with $L(y)$ as required. This proves that, for $\{B^o, x^o\}$ to be an optimal control-state pair, it is necessary that ψ satisfies the adjoint equation (6.12) which is the same as (6.2). The necessary condition (6.3) is the system equation (2.2) corresponding to the control operator B^o and so is natural. This completes the proof. □

Remark 6.2. The solution of the adjoint equation (6.12) (or (6.2)) is given by the solution of the following (backward) integral equation

$$\begin{aligned} \psi(t) = &U^*(T - t)\Psi_x(x^o(T)) + \int_t^T U^*(\theta - t)(B^o(\theta))^*\psi(\theta)d\theta \\ &+ \int_t^T U^*(\theta - t)(DF(x^o(\theta)))^*\psi(\theta)d\theta + \int_t^T U^*(\theta - t)\ell_x(\theta, x^o(\theta))d\theta. \end{aligned} \tag{6.14}$$

Since $U(t)$ is a unitary group, $U^*(t) = U(-t)$. The proof of existence of solution of this equation again follows from Banach fixed point theorem as in Theorem 4.1.

Computing Optimal Policy: Using the above result one can develop an algorithm for computing the optimal policy. First we need some preliminaries. An element $C \in \mathcal{L}(X^*, X)$ is said to be nuclear if there exists a sequence $\{x_n, y_n\} \subset X$ such that for every $x^* \in X^*$

$$C(x^*) = \sum_{n=1}^{\infty} x^*(x_n)y_n.$$

Using the notation for tensor products, a nuclear operator can be written as $C \equiv \sum x_n \otimes y_n$. Since the Banach space X is reflexive $x^*(x_n) = \hat{x}_n(x^*)$ defines a continuous linear functional on X^* where $\hat{x}_n$ denotes the canonical embedding of $x_n \in X$

to $\hat{x}_n \in X^{**}$. Let $\mathcal{L}_1(X^*, X)$ denote the class of nuclear operators from X^* to X. The nuclear norm of C is given by

$$\| C \|_{\mathcal{L}_1(X^*,X)} = \inf\{\sum_{n=1}^{\infty} \| x_n \|_X \| y_n \|_X\}$$

where the infimum is taken over all pairs $\{x_n, y_n\} \subset X$ such that $C(x^*) = \sum_{n=1} x^*(x_n) y_n$ for all $x^* \in X^*$. With respect to this norm topology, $\mathcal{L}_1(X^*, X)$ is a Banach space and hence it follows from Hahn-Banach theorem that it has a nonempty dual and its topological dual is given by the space of bounded linear operators $\mathcal{L}(X, X^*)$. In symbol $(\mathcal{L}_1(X^*, X))^* = \mathcal{L}(X, X^*)$. For any $L \in \mathcal{L}(X, X^*)$ and $C \in \mathcal{L}_1(X^*, X)$, the duality pairing of L and C is given by

$$\langle L, C \rangle = \sum_{n=1}^{\infty} (L x_n, y_n)_{X^*,X}.$$

It is clear that $|\langle L, C\rangle_{\mathcal{L}(X,X^*),\mathcal{L}_1(X^*,X)}| \leq \| L \|_{\mathcal{L}(X,X^*)} \| C \|_{\mathcal{L}_1(X^*,X)}$. Now we can state the following corollary asserting convergence of any computational algorithm based on the necessary conditions of optimality.

Corollary 6.3 (A convergence theorem). *Suppose the assumptions of Theorem 6.1 hold. Then there exists a sequence of operators $\{B_n\} \in \mathcal{B}_{ad}$ such that the cost functional J is monotone decreasing along the sequence $\{B_n\}$ and convergent to a (possibly local) minimum.*

Proof. Let $B_1 \in \mathcal{B}_{ad}$ be any element. Let $x_1 \in C(I, X)$ be the (mild) solution of the state equation (2.2) corresponding to the (control) operator B_1. Considering the adjoint equation (6.2) and using the pair $\{B_1, x_1\}$ in place of the pair $\{B^o, x^o\}$, we solve this equation and denote the solution by $\psi_1 \in C(I, X)$. Using the tensor product notation we define an operator valued function C_1 by $C_1(t) \equiv x_1(t) \otimes \psi_1(t), t \in I$. Since the solutions of the state equation and the adjoint equations are unique, the operator is uniquely defined. Clearly, this operator takes values in the space of nuclear operators $\mathcal{L}_1(X^*, X)$ and $C_1(t)x^* = x^*(x_1(t))\psi_1(t) \in X$ for all $t \in I$. For each $t \in I$, the nuclear norm of this operator valued function is given by $\| C_1(t) \|_{\mathcal{L}_1(X^*,X)} = (\| x_1(t) \|_X)(\| \psi_1(t) \|_X)$. Using the triple $\{B_1, x_1, \psi_1\}$, we can compute the Gâteaux differential of J at B_1 in any direction $(B - B_1)$ for any $B \in \mathcal{B}_{ad}$ giving

$$dJ(B_1, B - B_1) = \int_0^T \langle B(t) - B_1(t), C_1(t)\rangle_{\mathcal{L}(X,X^*),\mathcal{L}_1(X^*,X)} dt. \tag{6.15}$$

Here we need the duality map Γ mapping $\mathcal{L}_1(X^*, X)$ to its dual $\mathcal{L}(X, X^*)$ given by

$$\Gamma(C) \equiv \{L \in \mathcal{L}(X, X^*) : \langle L, C\rangle = \| L \|^2_{\mathcal{L}(X,X^*)} = \| C \|^2_{\mathcal{L}_1(X^*,X)}\}. \tag{6.16}$$

In general, Γ is a multivalued map and demicontinuous in the sense that, whenever C_n converges strongly to C_0, the multivalued map $\Gamma(C_n)$ converges to $\Gamma(C_0)$ in the weak star topology. For any $\varepsilon > 0$, let us define the operator B_2 as

$$B_2 = B_1 - \varepsilon L_1 \text{ for any } L_1 \in \Gamma(C_1).$$

Choosing $\varepsilon > 0$ sufficiently small so that $B_2 \in \mathcal{B}_{ad}$ and substituting this in equation (6.15) and using the property of the duality map Γ we obtain

$$\begin{aligned} dJ(B_1, B_2 - B_1) &= \int_0^T \langle B_2(t) - B_1(t), C_1(t)\rangle_{\mathcal{L}(X,X^*),\mathcal{L}_1(X^*,X)} dt \\ &= -\varepsilon \int_0^T \| L_1 \|^2_{\mathcal{L}(X,X^*)} dt = -\varepsilon \int_0^T \| C_1 \|^2_{\mathcal{L}_1(X^*,X)} dt. \end{aligned} \tag{6.17}$$

Using classical Lagrange formula and evaluating the cost functional J at B_2 we arrive at the following expression,

$$\begin{aligned} J(B_2) &= J(B_1) + dJ(B_1, B_2 - B_1) + o(\| B_2 - B_1 \|) \\ &= J(B_1) - \varepsilon \int_0^T \| C_1 \|^2_{\mathcal{L}_1(X^*,X)} dt + o(\varepsilon). \end{aligned} \tag{6.18}$$

Clearly, for sufficiently small $\varepsilon > 0$, we have $J(B_2) < J(B_1)$. Starting with B_2 and repeating the above procedure we obtain $B_3 \in \mathcal{B}_{ad}$ satisfying the inequality $J(B_3) < J(B_2)$. Thus repeating this process add infinitum we construct a sequence $\{B_n\} \in \mathcal{B}_{ad}$ satisfying

$$J(B_{n+1}) < J(B_n) < \cdots J(B_2) < J(B_1).$$

This shows that the sequence of cost functionals $\{J(B_n)\}_n$ is monotone decreasing and since $J(B)$ is bounded away from $-\infty$, it is clear that $J(B_n)$ converges to (possibly) a local minimum. This completes the proof. $\square$

Remark 6.4 A question of significant interest is whether or not it is possible to find a global minimum. Since the problem is nonlinear and the functional $B \longrightarrow J(B)$ is not necessarily convex, the answer to the above question is generally negative. However, using any heuristic technique such as Random recursive search technique, Simulated annealing, Threshold accepting, Genetic algorithm, etc, one can cover a large region of $\mathcal{B}_{ad}$ which is satisfactory in many scientific applications. One can easily find these techniques well described in Wikipedia.

An Open Problem Extension of the results presented here to stochastic Nonlinear Schrödinger equation is of significant theoretical interest. This will require stochastic integration on Banach spaces with respect to cylindrical Brownian motion on Hilbert spaces.

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