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WEIGHTED AVERAGES-BASED ALGORITHM FOR A NUMERICAL SOLUTION OF AN INFINITE HORIZON OPTIMAL CONTROL PROBLEM WITH DISCOUTNING IN DISCRETE TIME

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Abstract. In this paper we develop an algorithm for a numerical solution of a deterministic infinite horizon optimal control problem with discoutning in discrete time.

1. Introduction and preliminaries

This paper is devoted to developing an algorithm for a numerical solution of the following infinite horizon optimal control problem with discounting in discrete time:

$$
\begin{aligned}
&\text{Minimize } J(u,y_0)=\sum_{t=0}^{\infty}\alpha^t g(y(t),u(t)),\\
&y(t+1)=f(y(t),u(t)),\ t\in\mathcal{T}:=\{0,1,\dots\},\\
&y(0)=y_0,\\
&y(t)\in Y,\\
&u(t)\in U(y(t)).
\end{aligned}
\tag{1.1}
$$

Here Y is a nonempty compact subset of $I\!R^m$, $U(\cdot):Y\to U_0$ is an upper semicontinuous compact-valued mapping to a compact metric space U_0, $f: I\!R^m\times U_0\to I\!R^m$ is a continuous function, $\alpha\in(0,1)$ is a discount factor.

There are various natural criteria of optimality for infinite horizon optimal control problems, such as strong optimality, overtaking optimality, etc., see, e.g., [3], [4], [5] and references therein. Asymptotic properties of optimal processes in problems with infinite horizon are studied, e.g., in [11], [12] and [13]; in nonsmooth setting discrete problems of optimal control are considered in, e.g. [6] and [7]. In this paper, which continues the line of research started in [8] and [9], we consider optimality with discounting.

In [8] it was established that problem (1.1) is closely related to an infinite dimensional linear programming (LP) problem and its dual. In [9], the results of [8] were used to establish necessary and sufficient optimality conditions for problem (1.1), and the latter conditions were applied for a construction of a near optimal control.

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In this work we present an alternative scheme for finding a near optimal control, which is based on the necessary and sufficient optimality conditions from [9].

The last two constraints of (1.1) can be combined into one by writing

$$u(t) \in A(y(t)),$$

where, for $y \in Y$,

$$A(y) := \{u \in U(y) |\, f(y, u) \in Y\}. \tag{1.2}$$

We say that a process $(y(\cdot), u(\cdot))$ is admissible, if it satisfies the constraints of (1.1). Denote

$$G := \operatorname{graph} A = \{(y, u) |\, y \in Y,\, u \in U(y),\, f(y, u) \in Y\}.$$

It is easy to see that G is a compact subset of $Y \times U_0$. Also denote

$$\mathcal{U}(y_0) := \{u(\cdot) |\, (y(\cdot), u(\cdot)) \text{ is admissible}\}$$

to be the set of admissible controls and let

$$V(y_0) := \min_{u(\cdot)\in\mathcal{U}} J(u, y_0) \tag{1.3}$$

be the value function of problem (1.1).

Throughout the paper we assume that the set $A(y)$ is not empty for any $y \in Y$. This assumption implies that the set $\mathcal{U}(y)$ is not empty for any $y \in Y$. It can be shown that under this assumption, an optimal solution of problem (1.1) exists and the optimal value function $V(\cdot)$ is lower semicontinuous and bounded on Y. Also, $V(\cdot)$ is a solution of the equation

$$V(y) = \min_{u\in A(y)} \{g(y, u) + \alpha V(f(y, u))\} \quad \forall y \in Y, \tag{1.4}$$

which is the dynamic programming principle for problem (1.1) (see, e.g., [1] or [8]).

For a lower semicontinuous function $\psi : Y \to I\!R$ denote

$$H_\psi(y) := \min_{u\in A(y)} \{\alpha(\psi(f(y, u)) - \psi(y)) + g(y, u)\}.$$

Then relation (1.4) can be written as

$$H_V(y) - (1 - \alpha)V(y) = 0,$$

which resembles the Hamilton-Jacobi-Bellman equation for continuous time systems.

Let us outline some notations and results that are used further in the text. For an admissible process $(y(\cdot), u(\cdot))$, a probability measure γ_u is called the *discounted occupational measure* generated by $u(\cdot)$ if, for any Borel set $Q \subset G$,

$$\gamma_u(Q) = (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t 1_Q(y(t), u(t)), \tag{1.5}$$

where $1_Q(\cdot)$ is the indicator function of Q. It can be shown that this definition is equivalent to the validity of the relationship

$$\int_G q(y, u)\gamma_u(dy, du) = (1 - \alpha) \sum_{t=0}^{\infty} \alpha^t q(y(t), u(t)) \tag{1.6}$$

for any Borel measurable function q on G.

To describe convergence properties of occupational measures, we introduce the following metric on $\mathcal{P}(G)$ (the space of probability measures defined on Borel subsets of G):

$$\rho(\gamma', \gamma'') := \sum_{j=1}^{\infty} \frac{1}{2^j} \left| \int_G q_j(y,u)\gamma'(dy,du) - \int_G q_j(y,u)\gamma''(dy,du) \right|$$

for $\gamma', \gamma'' \in \mathcal{P}(G)$, where $q_j(\cdot)$, $j = 1, 2, \ldots$, is a sequence of Lipschitz continuous functions dense in the unit ball of the space of continuous functions $C(G)$ from G to $I\!R$. This metric is consistent with the weak* convergence topology on $\mathcal{P}(G)$, that is, a sequence $\gamma^k \in \mathcal{P}(G)$ converges to $\gamma \in \mathcal{P}(G)$ in this metric if and only if

$$\lim_{k\to\infty} \int_G q(y,u)\gamma^k(dy,du) = \int_G q(y,u)\gamma(dy,du)$$

for any $q \in C(G)$.

The following lower semicontinuity property is valid (see Theorem 2.1 in [2]): if a sequence $\gamma^k \in \mathcal{P}(G)$ converges to $\gamma \in \mathcal{P}(G)$ then for any open set $B \subset G$

$$\liminf_{k\to\infty} \gamma^k(B) \geq \gamma(B).$$

Let $\Gamma(y_0)$ denote the set of all discounted occupational measures generated by the admissible controls, that is,

$$\Gamma(y_0) := \bigcup_{u(\cdot)\in\mathcal{U}(y_0)} \{\gamma_u\}.$$

Notice that $\Gamma(y_0) \neq \emptyset$ since $\mathcal{U}(y_0) \neq \emptyset$. Due to (1.6), problem (1.1) can be rewritten as

$$\min_{\gamma\in\Gamma(y_0)} \int_G g(y,u)\gamma(dy,du). \tag{1.7}$$

Along with problem (1.7) consider the problem

$$\min_{\gamma\in W(y_0)} \int_G g(y,u)\gamma(dy,du) =: g^*(y_0), \tag{1.8}$$

where $W(y_0)$ is a subset of $\mathcal{P}(G)$ defined by

$$\begin{aligned} W(y_0) := \Big\{\gamma \in \mathcal{P}(G) \,|\, \int_G &[\alpha(\varphi(f(y,u)) - \varphi(y)) \\ &+ (1-\alpha)(\varphi(y_0) - \varphi(y))]\gamma(dy,du) = 0 \quad \forall \varphi \in C(Y)\Big\}. \end{aligned} \tag{1.9}$$

Note that (1.8) is an infinite-dimensional problem of linear programming (IDLP) since both the objective functions and the constraints defining $W(y_0)$ are linear in the "decision variable" γ. It can be shown that $W(y_0)$ is equal to the closure of the convex hull of $\Gamma(y_0)$ (see [8], Corollary 2).

Also consider the max-min problem

$$
\begin{aligned}
&\max_{\psi\in LS}\inf_{y\in Y}\{H_\psi(y)+(1-\alpha)(\psi(y_0)-\psi(y))\}\\
&\quad=\max_{\psi\in LS}\inf_{(y,u)\in G}\{g(y,u)+\alpha(\psi(f(y,u))-\psi(y))\\
&\qquad+(1-\alpha)(\psi(y_0)-\psi(y))\}=:\mu^*(y_0),
\end{aligned}\tag{1.10}
$$

where maximum is taken over the class of bounded lower semicontinuous functions from Y to $I\!R$ (denoted as LS). It has been established in [8], Theorem 4.1, that the maximum in (1.10) is reached at $\psi = V$, the optimal values in problems (1.8) and (1.10) coincide and are equal to the optimal value of (1.1) multiplied by $(1-\alpha)$, that is,

$$
\mu^*(y_0)=g^*(y_0)=(1-\alpha)V(y_0).\tag{1.11}
$$

Is it clear that a constant shift of a maximizer ψ in (1.10) is also a maximizer, but, in fact, the set of maximizers in (1.10) can be much broader than the function V or its constant shifts (see an example in Section 2 of [9]). In the next theorem necessary and sufficient optimality conditions for problem (1.1) in terms of any such maximizer are established.

Theorem 1.1 ([9, Theorem 2.1])**.** *Let ψ be a solution of* (1.10)*. Optimality of an admissible process $(y(\cdot),u(\cdot))$ is equivalent to the relation*

$$
(y(t),u(t))=\operatorname{argmin}_{(y,u)\in G}\{g(y,u)+\alpha\psi(f(y,u))-\psi(y)\},\tag{1.12}
$$

or, equivalently,

$$
\begin{aligned}
u(t)&=\operatorname{argmin}_{u\in A(y)}\{g(y(t),u)+\alpha\psi(f(y(t),u))\},\\
y(t)&=\operatorname{argmin}_{y\in Y}\{H_\psi(y)-(1-\alpha)\psi(y)\}.
\end{aligned}\tag{1.13}
$$

If a solution ψ of (1.10) is known, optimal control satisfies the first of the formulas (1.13). However, finding exact solution of (1.10) is, in general, difficult. In [9] a procedure of finding an *approximate* solution to this problem was developed. We outline this procedure below.

Let $\{\phi_i\}_{i=1}^\infty$ be a sequence of functions in $C(Y)$ with the following properties: (i) any finite collection of functions from this sequence is linearly independent on any open set, (ii) for any $\psi\in C(Y)$ and any $\delta>0$ there exist N and scalars λ_i^N, $i=1,\dots,N$ such that $\sup_{y\in Y}|\psi(y)-\sum_{i=1}^N\lambda_i^N\phi_i(y)|\le\delta$. (An example of such sequence is the sequence of monomials $y_1^{i_1}\dots y_m^{i_m}$, $i_1,\dots,i_m=0,1,\dots$, where y_j stands for the jth component of y.)

Define the finite dimensional space $D_N\subset C(Y)$ by

$$
D_N:=\{\psi\in C(Y)|\,\psi(y)=\sum_{i=1}^N\lambda_i\phi_i(y),\ \lambda_i\in I\!R,\ i=1,\dots,N\}
$$

and consider the *N-approximating problem* to (1.10)

$$
\sup_{\psi\in D_N}\min_{(y,u)\in G}\{g(y,u)+\alpha(\psi(f(y,u))-\psi(y))+(1-\alpha)(\psi(y_0)-\psi(y))\}=:\mu_N^*(y_0).\tag{1.14}
$$

It is easy to show that

$$\lim_{N\to\infty} \mu_N^*(y_0) = (1-\alpha)V(y_0).$$

Let $\mathcal{R}_{y_0}$ be the reachable set for system (1.1) in finite time. It can be shown (see [9], Proposition 5) that the maximizing function $\psi \in D_N$ in (1.14) exists under a simple controllability-type assumption

$$\operatorname{int}(\operatorname{cl}\mathcal{R}_{y_0}) \neq \emptyset. \tag{1.15}$$

Let (1.15) hold and ψ^N be a solution of the N-approximating problem. Motivated by formula (1.13), define control u^N by

$$u^N(y) = \operatorname{argmin}_{u\in A(y)}\{g(y,u) + \alpha\psi^N(f(y,u))\} \tag{1.16}$$

and let the corresponding trajectory y^N be given by

$$y^N(t+1) = f(y^N(t), u^N(y^N(t))) \tag{1.17}$$

with $y^N(0) = y_0$.

The theorem below asserts the convergence of $u^N(\cdot)$ and $y^N(\cdot)$ to the optimal control and the optimal trajectory $\bar{u}(\cdot)$ and $\bar{y}(\cdot)$, respectively, as $N \to \infty$.

Theorem 1.2 ([9, Theorem 4.1])**.** *In addition to* (1.15) *assume that the functions f and g are Lipschitz continuous and that the optimal solution γ^* of problem* (1.8) *is unique. Assume also that there exists an optimal admissible process $(\bar{y}(\cdot), \bar{u}(\cdot))$ such that:*

(a) *For any $t \in \mathcal{T}$ there exists an open ball Q_t centered at $\bar{y}(t)$ such that the minimizer $u^N(y)$ in the right hand side of* (1.16) *is uniquely defined for $y \in Q_t$;*

(b) *$u^N(\cdot)$ is Lipschitz continuous on Q_t with Lipschitz constant independent of N and t;*

(c) *$y^N(t) \in Q_t$ $\forall t \in \mathcal{T}$ for sufficiently large N.*

Then

$$\begin{aligned} &\lim_{N\to\infty} u^N(y^N(t)) = \bar{u}(t) \quad \forall\ t \in \mathcal{T},\\ &\lim_{N\to\infty} y^N(t) = \bar{y}(t) \quad \forall\ t \in \mathcal{T},\\ &\lim_{N\to\infty} V^N(y_0) = V(y_0), \end{aligned} \tag{1.18}$$

where $V^N(y_0) = \sum_{t=0}^{\infty} \alpha^t g(y^N(t), u^N(y^N(t)))$.

Consider the semi-infinite dimensional problem of linear programming

$$\min_{\gamma\in W_N(y_0)} \int_G g(y,u)\gamma(dy,du) =: g_N^*(y_0), \tag{1.19}$$

where

$$\begin{aligned} W_N(y_0) :=\Big\{\gamma \in \mathcal{P}(G) |\int_G \big[&\alpha(\phi_i(f(y,u)) - \phi_i(y))\\ &+ (1-\alpha)(\phi_i(y_0) - \phi_i(y))\big]\gamma(dy,du) = 0, \quad i = 1,\dots,N\Big\}. \end{aligned} \tag{1.20}$$

Since $W_N(y_0) \supset W(y_0)$ for all N, the set $W_N(y_0)$ is not empty. It is clear that it is compact in weak* topology, therefore the minimum in problem (1.19) is reached.

It can be shown that the optimal value of problem (1.19) is equal to the optimal value of the N-approximating problem (1.14), that is, $g_N^*(y_0) = \mu_N^*(y_0)$ (see [9], Lemma 4.1).

Proposition 1.3 ([9, Proposition 7]). *Among the optimal solutions of problem* (1.19)*, there exists one (denoted below as γ^N) that is presented as a convex combination of at most $N+1$ Dirac measures with concentration points in G. More precisely,*

(1.21)
$$\gamma^N = \sum_{j=1}^{K_N} \beta_j^N \delta_{(y_j^N, u_j^N)}, \quad \text{where} \quad \beta_j^N > 0,\ j = 1, \ldots, K_N \leq N+1, \quad \sum_{j=1}^{K_N} \beta_j^N = 1$$

and where $\delta_{(y_j^N, u_j^N)}$ are the Dirac measures concentrated at $(y_j^N, u_j^N) \in G$. Moreover, the concentration points (y_j^N, u_j^N), $j = 1, \ldots, K_N$ satisfy the following relationships:

$$\begin{aligned} u_j^N &= \operatorname{argmin}_{u \in A(y_j^N)} \{ g(y_j^N, u) + \alpha \psi^N(f(y_j^N, u)) \}, \\ y_j^N &= \operatorname{argmin}_{y \in Y} \{ H_{\psi^N}(y) - (1-\alpha)\psi^N(y) \}, \end{aligned} \tag{1.22}$$

where ψ^N is a solution of the N-approximating problem (1.14).

In the next proposition it is established that any point along the optimal process is a limiting point of the set of concentration points $\{(y_{j_N}^N, u_{j_N}^N)\}$ as $N \to \infty$.

Proposition 1.4 ([9, Proposition 8]). *Let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be an optimal process in* (1.1) *such that the conditions (a),(b) and (c) of Theorem 1.2 are satisfied and let γ^N be an optimal solution of* (1.19) *that is represented in the form* (1.21)*. Then, for any t, there exist points $(u_{j_N}^N, y_{j_N}^N) \in \{(y_j^N, u_j^N),\ j = 1, \ldots, K_N\}$ such that*

$$(\bar{y}(t), \bar{u}(t)) = \lim_{N \to \infty} (y_{j_N}^N, u_{j_N}^N). \tag{1.23}$$

An optimal solution (1.21) of the semi-infinite LP problem (1.19), its optimal value g_N^* and an optimal solution ψ^N of the N-approximating problem (1.14) can be found numerically. When ψ^N is found, a control $u^N(y)$ can constructed as a minimizer in (1.16), and this $u^N(y)$ is near optimal in (1.1) for sufficiently large N due to Theorem 1.2. This numerical approach is carried out in an example in Section 5 of [9]. However, finding a minimizer in (1.16) may be difficult, since the problem on the right-hand-side of (1.16) is generally not of the convex programming class. A simple heuristic algorithm that circumvents this difficulty was suggested in Section 6 of [9]. The latter algorithm is appealing for its simplicity, but it does not take into account the weights $\{\beta_j^N\}$, $j = 1, ..., K_N$, and convergence of this algorithm cannot be, in general, asserted. In Section 3 of this paper we propose a mathematically rigorous algorithm that accounts for the aforementioned weights and converges to the optimal process under appropriate assumptions. To justify the algorithm, we establish a few theoretical results that have independent significance in Section 2. A numerical example is illustrated in Section 5.

2. Asymptotic properties of measures

Throughout the rest of the paper we assume the following.

(A1) The optimal solution γ^* of the IDLP problem (1.8) is unique.

(A2) For any optimal trajectory in (1.1), either all points are distinct, or it is periodic, in which case no point is visited more than once within one period.

Proposition 2.1. *Under Assumptions (A1)-(A2) the optimal process in* (1.1) *is unique.*

Proof. Let $(y(\cdot), u(\cdot))$ be an admissible process in (1.1) that generates occupational measure γ_u. From the definition of occupational measure (1.5), it follows that when all points of the trajectory $y(\cdot)$ are distinct, we have for all t

$$\gamma_u(\{(y(t), u(t))\}) = (1-\alpha)\alpha^t, \tag{2.1}$$

and, for a periodic trajectory with period T, we have

$$\gamma_u(\{(y(t), u(t))\}) = (1-\alpha)\sum_{k=0}^{\infty} \alpha^{\tau + kT} = \frac{(1-\alpha)\alpha^\tau}{1-\alpha^T}, \tag{2.2}$$

where $\tau \in \{0, \dots, T-1\}$ is the first time when the trajectory enters the state $y(t)$.

Assume that there exist two different optimal processes in (1.1): $(y(\cdot), u(\cdot))$ and $(\bar{y}(\cdot), \bar{u}(\cdot))$. It is clear that both processes must generate γ^*, and the sets of points $\{(y(t), u(t))\}_{t=0}^{\infty}$ and $\{(\bar{y}(t), \bar{u}(t))\}_{t=0}^{\infty}$ coincide (otherwise, these processes can't generate the same occupational measure). Hence, if one of the optimal trajectories is periodic with period T, the other also has to be periodic with the same period. Since the processes are different, there exist τ_1 and τ_2, $\tau_1 \neq \tau_2$, such that

$$(y(\tau_1), u(\tau_1)) = (\bar{y}(\tau_2), \bar{u}(\tau_2)). \tag{2.3}$$

If the points along both the trajectories are distinct, the latter equality implies via (2.1) that

$$(1-\alpha)\alpha^{\tau_1} = (1-\alpha)\alpha^{\tau_2},$$

which is not possible. If both trajectories are periodic with the same period T, then equality (2.3) is not possible either due to (2.2). $\square$

Denote the unique optimal process in (1.1) by $(\bar{y}(\cdot), \bar{u}(\cdot))$.

Let γ^N be an optimal solution of (1.19) that is represented in the form (1.21). Denote

$$_{\mathtt{N}} := \{(\mathtt{y}^{\mathtt{N}}_{\mathtt{j}}, \mathtt{u}^{\mathtt{N}}_{\mathtt{j}}),\ \mathtt{j} = \mathtt{1}, \dots, \mathtt{K}_{\mathtt{N}}\}, \tag{2.4}$$

that is, $_{\mathtt{N}}$ is the set of points from representation (1.21). Also denote

$$B_r(\bar{y}, \bar{u}) := \{(y, u) |\, |y - \bar{y}| + |u - \bar{u}| < r\}.$$

Proposition 1.4 asserts that each point along the optimal process is a limiting point of a sequence from $_{\mathtt{N}}$ as $N \to \infty$. Propositions 2.2 and 2.3 below are related to a converse statement; they imply that the total γ^N-measure of points in $_{\mathtt{N}}$ outside any neighborhood of the optimal process tends to zero as $N \to \infty$.

Proposition 2.2. *Assume that all points of the optimal trajectory $\bar{y}(\cdot)$ are distinct. Let $\delta > 0$ be arbitrary and $S \in \mathcal{T}$ be such that*

$$\sum_{t=S+1}^{\infty} \gamma^*\{(\bar{y}(t), \bar{u}(t))\} = \alpha^{S+1} < \delta/6. \tag{2.5}$$

Then for any $r > 0$ there exists N_0 such that for all $N \geq N_0$

$$\gamma^N(\mathtt{N} \setminus \cup_{\mathtt{t}=0}^{\mathtt{S}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))) < \delta. \tag{2.6}$$

Proof. Assume that the proposition is not true. Then there exist $r_0 > 0$ and a sequence $N_i \to \infty$ such that

$$\gamma^{N_i}(\mathtt{N_i} \setminus \cup_{\mathtt{t}=0}^{\mathtt{S}} \mathtt{B_{r_0}}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))) \geq \delta. \tag{2.7}$$

Taking into account that

$$\gamma^{N_i}(\mathtt{N_i} \setminus \cup_{\mathtt{t}=0}^{\mathtt{S}} \mathtt{B_{r_0}}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))) = 1 - \gamma^{\mathtt{N_i}}(\cup_{\mathtt{t}=0}^{\mathtt{S}} \mathtt{B_{r_0}}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))), \tag{2.8}$$

the convergence $\gamma^{N_i} \to \gamma^*$ (due to (A1)) and semicontinuity property of occupational measures

$$\gamma^{N_i}(\cup_{t=0}^{S} B_{r_0}(\bar{y}(t), \bar{u}(t))) > \gamma^*(\cup_{t=0}^{S} B_{r_0}(\bar{y}(t), \bar{u}(t))) - \delta/2 \quad \text{for sufficiently large } N_i, \tag{2.9}$$

we obtain from (2.7)-(2.9) that

$$\begin{aligned} &\gamma^*(\cup_{t=0}^{S} B_{r_0}(\bar{y}(t), \bar{u}(t))) < \gamma^{N_i}(\cup_{t=0}^{S} B_{r_0}(\bar{y}(t), \bar{u}(t))) + \delta/2 \\ &= (1 - \gamma^{N_i}(\mathtt{N_i} \setminus \cup_{\mathtt{t}=0}^{\mathtt{S}} \mathtt{B_{r_0}}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t})))) + \delta/2 \leq (1 - \delta) + \delta/2 = 1 - \delta/2. \end{aligned} \tag{2.10}$$

Therefore, taking into account (2.5), we obtain

$$\begin{aligned} 1 = \sum_{t=0}^{\infty} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) &= \sum_{t=0}^{S} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) + \sum_{t=S+1}^{\infty} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) \\ &\leq \gamma^*(\cup_{t=0}^{S} B_{r_0}(\bar{y}(t), \bar{u}(t))) + \sum_{t=S+1}^{\infty} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) < (1 - \delta/2) + \delta/6 < 1, \end{aligned}$$

which is a contradiction. □

Proposition 2.3. *Assume that the optimal trajectory $\bar{y}(\cdot)$ is periodic with period T. Then for any $\delta > 0$ and $r > 0$ there exists N_0 such that for all $N \geq N_0$*

$$\gamma^N(\mathtt{N} \setminus \cup_{\mathtt{t}=0}^{\mathtt{T-1}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))) < \delta.$$

Proof. Assume that the proposition is not true. Then there exist $r_0 > 0$ and a sequence $N_i \to \infty$ such that (2.7)-(2.10) hold with S replaced with $T - 1$. Furthermore,

$$1 = \sum_{t=0}^{T-1} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) \leq \gamma^*(\cup_{t=0}^{T-1} B_{r_0}(\bar{y}(t), \bar{u}(t))) < 1 - \delta/2,$$

where the last inequality is due to (2.10). This contradiction completes the proof of the proposition. □

Proposition 2.4. *For any $\delta > 0$ and $t \in \mathcal{T}$ there exists $\rho = \rho(\delta, t) > 0$ with the property: for any $r \in (0, \rho]$ there exists N_0 such that for all $N \geq N_0$*

$$|\gamma^N(B_r(\bar{y}(t), \bar{u}(t))) - \gamma^*(\{(\bar{y}(t), \bar{u}(t))\})| \leq \delta. \tag{2.11}$$

This proposition can be interpreted as follows: each point along the optimal process is surrounded by a cluster of points from $\mathtt{N}$. As $N \to \infty$, the radius of each cluster shrinks to zero and its total γ^N-measure approaches $\gamma^*(\{(\bar{y}(t), \bar{u}(t))\})$ (given by (2.1) or (2.2)).

Proof. Suppose now that the proposition is violated for some $\delta > 0$ at a point $\tau \in \mathcal{T}$. Then for any $\rho > 0$ there exists $r \in (0, \rho]$ and a sequence $N_i \to \infty$ such that

$$|\gamma^{N_i}(B_r(\bar{y}(\tau), \bar{u}(\tau))) - \gamma^*(\{(\bar{y}(\tau), \bar{u}(\tau))\})| > \delta. \tag{2.12}$$

Assume first that all the points of the optimal trajectory are *distinct.* Let $S \in \mathcal{T}$ be such that (2.5) holds. By increasing S, if necessary, we can assume without loss of generality that $S \geq \tau$. Take $\rho > 0$ such that

$$B_\rho(\bar{y}(t'), \bar{u}(t')) \cap B_\rho(\bar{y}(t''), \bar{u}(t'')) = \emptyset \text{ for } 0 \leq t' < t'' \leq S. \tag{2.13}$$

Then, by construction, for any $r \in (0, \rho]$,

$$\begin{aligned} \gamma^*(B_r(\bar{y}(\tau), \bar{u}(\tau))) &\leq \gamma^* \left(\{(\bar{y}(\tau), \bar{u}(\tau))\} \cup (\cup_{t=S+1}^{\infty}\{(\bar{y}(t), \bar{u}(t))\})\right) \\ &\leq \gamma^*(\{(\bar{y}(\tau), \bar{u}(\tau))\}) + \delta/6. \end{aligned} \tag{2.14}$$

From (2.12) and (2.14) we obtain

$$|\gamma^{N_i}(B_r(\bar{y}(\tau), \bar{u}(\tau))) - \gamma^*(B_r(\bar{y}(\tau), \bar{u}(\tau)))| > 5\delta/6. \tag{2.15}$$

From the semicontinuity property of probability measures

$$\liminf_{N\to\infty} \gamma^N(B_r(\bar{y}(t), \bar{u}(t))) \geq \gamma^*(B_r(\bar{y}(t), \bar{u}(t))) \quad \text{for all } t \tag{2.16}$$

we conclude that (2.15) can only hold if

$$\gamma^{N_i}(B_r(\bar{y}(\tau), \bar{u}(\tau))) - \gamma^*(B_r(\bar{y}(\tau), \bar{u}(\tau))) > 5\delta/6. \tag{2.17}$$

It follows from (2.5) that

$$\begin{aligned} \sum_{t=0}^{S} \gamma^*(B_r(\bar{y}(t), \bar{u}(t))) &\geq \sum_{t=0}^{S} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) \\ &= \sum_{t=0}^{\infty} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) - \sum_{t=S+1}^{\infty} \gamma^*(\{(\bar{y}(t), \bar{u}(t))\}) \\ &> 1 - \delta/6. \end{aligned} \tag{2.18}$$

Due to (2.16) we can assume that for all N_i and $0 \leq t \leq S$ we have

$$\gamma^{N_i}(B_r(\bar{y}(t), \bar{u}(t))) \geq \gamma^*(B_r(\bar{y}(t), \bar{u}(t))) - \delta/(6S). \tag{2.19}$$

Taking into account (2.17)-(2.19) and that $0 \le \tau \le S$, we obtain

$$\begin{aligned}\sum_{t=0}^{S} \gamma^{N_i}(B_r(\bar{y}(t), \bar{u}(t))) &= \gamma^{N_i}(B_r(\bar{y}(\tau), \bar{u}(\tau))) \\ &\quad + \sum_{t=0,\, t\neq\tau}^{S} \gamma^{N_i}(B_r(\bar{y}(t), \bar{u}(t))) \\ &> (\gamma^*(B_r(\bar{y}(\tau), \bar{u}(\tau))) + 5\delta/6) \\ &\quad + \sum_{t=0,\, t\neq\tau}^{S} \left(\gamma^*(B_r(\bar{y}(t), \bar{u}(t))) - \frac{\delta}{6S}\right) \\ &= \sum_{t=0}^{S} \gamma^*(B_r(\bar{y}(t), \bar{u}(t))) + 5\delta/6 - \delta/6 \\ &> (1 - \delta/6) + 5\delta/6 - \delta/6 \\ &= 1 + \delta/2,\end{aligned} \tag{2.20}$$

which contradicts the inequality $\sum_{t=0}^{S} \gamma^{N_i}(B_r(\bar{y}(t), \bar{u}(t))) \le 1$ and, thus, proves the validity of (2.11).

If the optimal trajectory is *periodic with period T*, we take S equal to $T - 1$ and the proof follows the steps above with small adjustments, which we omit. □

3. Numerical algorithm based on weighted averages

In this section we describe the construction of an approximating trajectory $(y_N(t), u_N(t))$, $t \in \mathcal{T}$ starting with the case when all point of the optimal trajectory are *distinct.*

For the initial point of the trajectory we set $y_N(0) = y_0$.

Take $\delta : 0 < \delta < 1 - \alpha$ and let S be such that (2.5) holds. Take $r > 0$ such that for all $y \in Y$ we have

$$|y - y_0| < r \ \text{ implies that } |y - \bar{y}(t)| \ge r, \ \text{ for all } t \in \{1, \ldots, S\}. \tag{3.1}$$

Then

$$B_r(y_0, \bar{u}(0)) \cap B_r(\bar{y}(t), \bar{u}(t)) = \emptyset \ \text{ for all } t \in \{1, \ldots, S\}.$$

By reducing r, if necessary, we can assume that $r \le \rho(0, \delta)$, where $\rho(\cdot, \cdot)$ is from Proposition 2.4. Due to the latter proposition, for sufficiently large N, relation (2.11) holds with $t = 0$. Since $\gamma^*(\{(y_0, \bar{u}(0))\}) = 1 - \alpha$, we conclude from (2.11) that

$$1 - \alpha - \delta \le \sum_j \beta_j^N \le 1 - \alpha + \delta, \tag{3.2}$$

where β_j^N are the weights from representation (1.21) and the summation is taken with respect to the indices j such that $(u_j^N, y_j^N) \in B_r(y_0, \bar{u}(0))$. We can also assume that N is large enough to ensure that (2.6) holds.

Select the point (η_1, ν_1) from $\mathtt{N}$ whose first component is closest to y_0, that is,

$$\eta_1 = \text{argmin}\{|y - y_0|,\ (y, u) \in_{\mathtt{N}}\}.$$

If there is a tie, take a point arbitratily. (Here and below we will not indicate explicitly dependence of η_1, etc., on N.)

Assume without loss of generality that (η_1, ν_1) is the first point in ${}_{\mathtt{N}}$, that is, $(\eta_1, \nu_1) = (y_1^N, u_1^N)$. Notice that we must have $|\eta_1 - y_0| < r$ because the contrary would mean that ${}_{\mathtt{N}} \cap \mathtt{B_r}(\mathtt{y_0}, \bar{\mathtt{u}}(0)) = \emptyset$, contradicting (3.2). Continuing to select points from ${}_{\mathtt{N}}$ closest to y_0, we obtain a sequence

$$\begin{aligned} &\eta_2 = \operatorname{argmin}\{|y - y_0|,\, (y,u) \in_{\mathtt{N}} \setminus \{(\eta_1, \nu_1)\}\}, \\ &\dots \\ &\eta_{k_1} = \operatorname{argmin}\{|y - y_0|,\, (y,u) \in_{\mathtt{N}} \setminus \cup_{\mathtt{j=1}}^{\mathtt{k_1-1}} \{(\eta_{\mathtt{j}}, \nu_{\mathtt{j}})\}\} \end{aligned}$$

and we can assume without loss of generality that $(\eta_j, \nu_j) = (y_j^N, u_j^N)$, $j = 1, \dots, k_1$.

Let us show that there exists $k_1 : 1 \le k_1 \le K_N$ such that

$$1 - \alpha - \delta \le \sum_{j=1}^{k_1} \beta_j^N \le 1 - \alpha + 2\delta, \tag{3.3}$$

that is, the total γ^N-measure of the points selected in this process must fall in the interval $[1 - \alpha - \delta, 1 - \alpha + 2\delta]$ at some stage. Indeed, assume that this is not the case. Then there exists $k : 1 \le k \le K_N$ for which

$$\sum_{j=1}^{k-1} \beta_j^N < 1 - \alpha - \delta \quad \text{and} \quad \sum_{j=1}^{k} \beta_j^N > 1 - \alpha + 2\delta. \tag{3.4}$$

Denote

$$\Omega_N := \{(\eta_j, \nu_j)\}_{j=1}^k.$$

Consider the equality

$$\Omega_N = \big(\Omega_N \cap B_r(y_0, \bar{u}(0))\big) \cup \big(\Omega_N \cap (\cup_{t=1}^S B_r(\bar{y}(t), \bar{u}(t))\big) \cup \big(\Omega_N \setminus (\cup_{t=0}^S B_r(\bar{y}(t), \bar{u}(t))\big),$$

which follows from the fact that the right side is the union of three disjoint sets. From the inclusion $\Omega_N \subset_{\mathtt{N}}$ the latter implies that

$$\Omega_N \subset \big({}_{\mathtt{N}} \cap \mathtt{B_r}(\mathtt{y_0}, \bar{\mathtt{u}}(0))\big) \cup \big(\Omega_{\mathtt{N}} \cap (\cup_{\mathtt{t=1}}^{\mathtt{S}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))\big) \cup \big({}_{\mathtt{N}} \setminus (\cup_{\mathtt{t=0}}^{\mathtt{S}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))\big). \tag{3.5}$$

Let us show that both inequalities $|\eta_k - y_0| < r$ and $|\eta_k - y_0| \ge r$ lead to a contradiction, thus confirming (3.3).

If $|\eta_k - y_0| < r$ then, by construction, $|\eta_j - y_0| < r$ for all $j \le k$, therefore, due to (3.1),

$$(\eta_j, \nu_j) \notin \cup_{t=1}^S B_r(\bar{y}(t), \bar{u}(t)),$$

or, equivalently,

$$\Omega_N \cap (\cup_{t=1}^S B_r(\bar{y}(t), \bar{u}(t))) = \emptyset. \tag{3.6}$$

From (3.5) we obtain

$$\begin{aligned} \sum_{j=1}^{k} \beta_j^N \equiv& \gamma^N(\Omega_N) \le \gamma^N({}_{\mathtt{N}} \cap \mathtt{B_r}(\mathtt{y_0}, \bar{\mathtt{u}}(0))) + \gamma^{\mathtt{N}}\big(\Omega_{\mathtt{N}} \cap (\cup_{\mathtt{t=1}}^{\mathtt{S}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t})))\big) \\ &+ \gamma^N\big({}_{\mathtt{N}} \setminus (\cup_{\mathtt{t=0}}^{\mathtt{S}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))\big). \end{aligned}$$

Taking into account the second inequality in (3.2), along with (3.6) and (2.6) we get

$$\sum_{j=1}^{k} \beta_j^N < (1-\alpha+\delta) + 0 + \delta = 1-\alpha+2\delta,$$

which contradicts the second inequality in (3.4).

On the other hand, if $|\eta_k - y_0| \geq r$, then, by construction, there there are no points in $\mathtt{N} \cap \mathtt{B_r}(\mathtt{y_0}, \mathtt{\bar{u}(0)})$ other than those in the set $\{(\eta_j, \nu_j)\}_{j=1}^{k-1}$, therefore, due to the first inequality in (3.4)

$$\gamma^N(B_r(y_0, \bar{u}(0))) \equiv \gamma^N(\mathtt{N} \cap \mathtt{B_r}(\mathtt{y_0}, \mathtt{\bar{u}(0)})) \leq \sum_{\mathtt{j=1}}^{\mathtt{k-1}} \beta_{\mathtt{j}}^{\mathtt{N}} < 1 - \alpha - \delta, \tag{3.7}$$

which contradicts the first inequality in (3.2). Existence of k_1 satisfying (3.3) is proved.

Let k_1 be the *minimum* integer such that (3.3) holds. This ensures that $|\eta_j - y_0| < r$ for all $j : 1 \leq j \leq k_1$ (otherwise, we arrive at a contradiction to the first inequality in (3.2)) and, consequently,

$$(\eta_j, \nu_j) \notin \cup_{t=1}^{S} B_r(\bar{y}(t), \bar{u}(t))), \ j = 1, \ldots, k_1. \tag{3.8}$$

Set

$$u_N(0) := \frac{\sum_{j=1}^{k_1} \beta_j^N \nu_j}{\sum_{j=1}^{k_1} \beta_j^N}, \tag{3.9}$$

that is, $u_N(0)$ is the weighted average of $\{\nu_j\}_{j=1}^{k_1}$. Note that if the mapping $U(y)$ does not depend on y (i.e, $U(y) \equiv U_0$) and is convex, then $u_N(0) \in U_0$.

Let us estimate the distance $|u_N(0) - \bar{u}(0)|$. In the set $\{(\eta_j, \nu_j)\}_{j=1}^{k_1}$ there must exist at least one point (η_j, ν_j) that lies in $B_r(y_0, \bar{u}(0))$ (otherwise (3.2) cannot hold), hence, for this point $|\nu_j - \bar{u}(0)| < r$. Assume that for the first k^* points from the set $\{(\eta_j, \nu_j)\}_{j=1}^{k_1}$ we have $|\nu_j - \bar{u}(0)| < r$, that is,

$$|\nu_j - \bar{u}(0)| < r, \ j = 1, \ldots, k^* \tag{3.10}$$

and, for the remaining points,

$$r \leq |\nu_j - \bar{u}(0)| \leq d_{U_0}, \ j = k^* + 1, \ldots, k_1, \tag{3.11}$$

where $d_{U_0} := \max_{u', u'' \in U_0} |u' - u''|$ is the diameter of U_0. From (3.9) we have

$$\begin{aligned} u_N(0) - \bar{u}(0) &= \frac{\sum_{j=1}^{k_1} \beta_j^N (\nu_j - \bar{u}(0))}{\sum_{j=1}^{k_1} \beta_j^N} \\ &= \frac{\sum_{j=1}^{k^*} \beta_j^N (\nu_j - \bar{u}(0))}{\sum_{j=1}^{k_1} \beta_j^N} + \frac{\sum_{j=k^*+1}^{k_1} \beta_j^N (\nu_j - \bar{u}(0))}{\sum_{j=1}^{k_1} \beta_j^N}. \end{aligned} \tag{3.12}$$

For the first sum, due to (3.10), we have the estimate

$$\frac{\sum_{j=1}^{k^*} \beta_j^N |\nu_j - \bar{u}(0)|}{\sum_{j=1}^{k_1} \beta_j^N} < r. \tag{3.13}$$

Due to (3.8) and (3.11) we have

$$(\eta_j, \nu_j) \in_{\mathtt{N}} \setminus \left(\cup_{\mathtt{t}=0}^{\mathtt{S}} \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{t}), \bar{\mathtt{u}}(\mathtt{t}))\right), \ \mathtt{j} = \mathtt{k}^* + 1, \ldots, \mathtt{k_1},$$

hence, $\sum_{j=k^*+1}^{k_1} \beta_j^N < \delta$ due to (2.6). Therefore, for the second sum in (3.12) we have the estimate

$$\frac{\sum_{j=k^*+1}^{k_1} \beta_j^N |\nu_j - \bar{u}(0)|}{\sum_{j=1}^{k_1} \beta_j^N} < \frac{d_{U_0}}{\sum_{j=1}^{k_1} \beta_j^N} \delta.$$

Thus, from (3.12), (3.13), and (3.3), we get

$$|u_N(0) - \bar{u}(0)| < r + \frac{d_{U_0}}{\sum_{j=1}^{k_1} \beta_j^N} \delta \le r + \frac{d_{U_0}}{1 - \alpha - \delta} \delta. \tag{3.14}$$

(Notice that this estimate deteriorates when $\alpha + \delta$ close to 1.) Set

$$y_N(1) := f(y_N(0), u_N(0)). \tag{3.15}$$

If f is Lipschitz with constant l, we have

$$|y_N(1) - \bar{y}(1)| \le l(|y_N(0) - \bar{y}(0)| + |u_N(0) - \bar{u}(0)|) \le l(r + \frac{d_{U_0}}{1 - \alpha - \delta} \delta). \tag{3.16}$$

Taking into account that r and δ can be taken arbitrarily small, we completed the proof of the following proposition.

Proposition 3.1. *Assume that f is Lipschitz. For any $\varepsilon > 0$ there exists N_0 such that for all $N \ge N_0$ for the approximation $u_N(0)$ of the optimal control given by formula* (3.9) *and the corresponding approximation of the optimal trajectory $y_N(1) := f(y_0, u_N(0))$ we have*

$$\begin{aligned} &|u_N(0) - \bar{u}(0)| \le \varepsilon, \\ &|y_N(1) - \bar{y}(1)| \le \varepsilon. \end{aligned} \tag{3.17}$$

Moreover, if $U(y) \equiv U_0$ and U_0 is convex, then $u_N(0) \in U_0$.

For the construction of $u_N(1)$ (an approximation of the optimal control at $t = 1$) we use an approach similar to the one used in the construction of $u_N(0)$. However, there is an additional factor to take into account: since $y_N(0) = y_0$, the point y_0 is in the center, with respect to the y variable, of the set $_{\mathtt{N}} \cap \mathtt{B_r}(\mathtt{y_0}, \bar{\mathtt{u}}(\mathtt{0}))$. At the same time, since $y_N(1) \ne \bar{y}(1)$, the point $y_N(1)$ is not the center of $_{\mathtt{N}} \cap \mathtt{B_r}(\bar{\mathtt{y}}(\mathtt{1}), \bar{\mathtt{u}}(\mathtt{1}))$ with respect to the y variable (the error is estimated by (3.16)). Next we outline the procedure of construction of $u_N(1)$ and $y_N(2)$.

Take $\delta : 0 < \delta < (1 - \alpha)\alpha$ and let S be such that (2.5) holds. Take ε and $r > 0$ such that for all $y \in Y$ we have

$$|y - \bar{y}(1)| < r + \varepsilon \ \text{ implies that } |y - \bar{y}(t)| \ge r, \ \text{ for all } t \in \{0, 2, 3 \ldots, S\}.$$

By reducing r, if necessary, we can assume that $r \le \rho(1, \delta)$, where $\rho(\cdot, \cdot)$ is from Proposition 2.4. Then, for sufficiently large N, relation (2.11) holds for $t = 1$ and (3.17) is true. Since $\gamma^*(\{(\bar{y}(1), \bar{u}(1))\}) = (1 - \alpha)\alpha$, we conclude from (2.11) that

$$(1 - \alpha)\alpha - \delta \le \sum_j \beta_j^N \le (1 - \alpha)\alpha + \delta, \tag{3.18}$$

where β_j^N are the weights from representation (1.21) and the summation is taken with respect to the indices j such that $(u_j^N, y_j^N) \in B_r(\bar{y}(1), \bar{u}(1))$.

Select the point $(\eta_{k_1+1}, \nu_{k_1+1})$ from the set $\mathtt{N}$ (minus the points $\{(\eta_j, \nu_j)\}_{j=1}^{k_1}$), whose y-component is closest to $y_N(1)$, that is, let

$$\eta_{k_1+1} = \operatorname{argmin}\{|y - y_N(1)|, (y,u) \in_{\mathtt{N}} \setminus \cup_{\mathtt{j=1}}^{\mathtt{k_1}} \{(\eta_{\mathtt{j}}, \nu_{\mathtt{j}})\},$$

where k_1 satisfies (3.3). Notice that we must have $|\eta_{k_1+1} - y_N(1)| < r + \varepsilon$ because the contrary would mean that $|\eta_{k_1+1} - \bar{y}(1)| \geq r$, that is, $\mathtt{N} \cap \mathtt{B_r}(\bar{\mathtt{y}}(1), \bar{\mathtt{u}}(1)) = \emptyset$, which contradicts (3.18). Continuing to select points in $\mathtt{N}$ closest to $y_N(1)$, we obtain a sequence

$$\eta_{k_1+j} = \operatorname{argmin}\{|y - y_N(1)|, (y,u) \in_{\mathtt{N}} \setminus \cup_{\mathtt{j=k_1}}^{\mathtt{k_1+j-1}} (\eta_{\mathtt{j}}, \nu_{\mathtt{j}})\}\}, \ \mathtt{j} = 2, \ldots, \mathtt{k_2}.$$

Assume without loss of generality that $(\eta_{k_1+j}, \nu_{k_1+j}) = (y_{k_1+j}^N, u_{k_1+j}^N)$, $j = 1, \ldots, k_2 - k_1$. It can be shown similarly to the proof of (3.3) that the number $k_2 : k_1 + 1 \leq k_2 \leq K_N$ can be chosen as the minimum integer with the property

$$(1-\alpha)\alpha - \delta \leq \sum_{j=k_1+1}^{k_2} \beta_j^N \leq (1-\alpha)\alpha + 2\delta.$$

Set

$$u_N(1) := \frac{\sum_{j=k_1+1}^{k_2} \beta_j^N \nu_j}{\sum_{j=k_1+1}^{k_2} \beta_j^N} \text{ and } y_N(2) := f(y_N(1), u_N(1)).$$

Similarly to the estimates (3.14) and (3.16), we obtain

$$|u_N(1) - \bar{u}(1)| < r + \frac{d_{U_0}}{(1-\alpha)\alpha - \delta}\delta$$

and

$$|y_N(2) - \bar{y}(2)| \leq l(|y_N(1) - \bar{y}(1)| + |u_N(1) - \bar{u}(1)|) \leq l(\varepsilon + r + \frac{d_{U_0}}{(1-\alpha)\alpha - \delta}\delta),.$$

where l is a Lipschitz constant of f. By reducing r and δ and increasing N, if necessary, we obtain the estimates

$$|u_N(1) - \bar{u}(1)| \leq \varepsilon,$$
$$|y_N(2) - \bar{y}(2)| \leq 2l\varepsilon.$$

Continuing in a similar manner, we obtain a sequence $(y_N(t), u_N(t))$, $t \geq 1$ such that

$$|u_N(t) - \bar{u}(t)| \leq \varepsilon,$$
$$|y_N(t) - \bar{y}(t)| \leq (t+1)l\varepsilon,$$

for which the error remains bounded on any bounded time interval. We summarize the described procedure in a form convenient for numerical implementation below.

Summary of the algorithm of constructing an approximating optimal process in the case if all points of the optimal trajectory are distinct.

1. Set $y_N(0) = y_0$. Select the interval $0 \leq t \leq S$ on which the optimal process will be approximated and take $\delta \in (0, (1-\alpha)\alpha^S)$.

2. Keep selecing points $\{(\eta_j, \nu_j)\} \in_{\mathtt{N}}$ with the corresponding weights β_j^N whose first component η_j is closest to y_0. Let k_1 be the minimum integer such that

$$1 - \alpha - \delta \leq \sum_{j=1}^{k_1} \beta_j^N \leq 1 - \alpha + 2\delta.$$

(It is shown above that for sufficiently large N such k_1 is guaranteed to exists.) Set

$$u_N(0) := \frac{\sum_{j=1}^{k_1} \beta_j^N \nu_j}{\sum_{j=1}^{k_1} \beta_j^N},$$

that is, $u_N(0)$ is the weighted average of the second components of the selected points, and set

$$y_N(1) := f(y_0, u_N(0)).$$

3. Continue a similar process for all natural $p : 1 \leq p \leq S$. That is, find points $\{(\eta_j, \nu_j)\}_{j=k_p+1}^{k_{p+1}}$ from $\mathtt{N} \setminus \left(\cup_{\mathtt{j=1}}^{\mathtt{k_p}} \{(\nu_{\mathtt{j}}, \eta_{\mathtt{j}})\} \right)$ with the corresponding weights β_j^N, $j = k_p+1, \ldots, k_{p+1}$ whose first component is closest to $y_N(p)$. Here k_{p+1} is the minimum integer for which

$$(1 - \alpha)\alpha^p - \delta \leq \sum_{j=k_p+1}^{k_{p+1}} \beta_j^N \leq (1 - \alpha)\alpha^p + 2\delta. \tag{3.19}$$

Set

$$u_N(p) := \frac{\sum_{j=k_p+1}^{k_{p+1}} \beta_j^N \nu_j}{\sum_{j=k_p+1}^{k_{p+1}} \beta_j^N},$$

and

$$y_N(p + 1) := f(y_N(p), u_N(p)).$$

If f is Lipschitz, then for any $\varepsilon > 0$ there exists N_0 such that for $N \geq N_0$ the estimate

$$|u_N(t) - \bar{u}(t)| + |y_N(t) - \bar{y}(t)| \leq \varepsilon \tag{3.20}$$

holds for all $0 \leq t \leq S$.

□

Assume now that the optimal trajectory is *periodic with period* T, in which case formula (2.2) rather than (2.1) is valid. Therefore, the algorithm above can be adjusted by setting $S = T - 1$ and replacing condition (3.19) with

$$\frac{(1 - \alpha)\alpha^p}{1 - \alpha^T} - \delta \leq \sum_{j=k_p+1}^{k_{p+1}} \beta_j^N \leq \frac{(1 - \alpha)\alpha^p}{1 - \alpha^T} + \delta,$$

in which case the estimate (3.20) still holds.

Of course, it is not known *a priori* if the optimal trajectory is periodic. However, it is possible to determine this, as well as to find the period of a periodic trajectory, by using the fact that if the optimal trajectory is periodic with period T, for any sufficiently small $r > 0$ the γ^N-measure of the set $\mathtt{N} \cap \mathtt{B_r}(\mathtt{y_0}, \bar{\mathtt{u}}(0))$ can be made arbitrarily close to $\frac{1-\alpha}{1-\alpha^T}$ by increasing N.

4. Numerical Example

Consider the optimal control problem (1.1) with

$$y = (y_1, y_2), \quad u = (u_1, u_2), \quad g(y, u) = -y_1(t)u_2(t) + y_2(t)u_1(t)$$

and with $f(y, u) = (f_1(y, u), f_2(y, u))$, where

$$f_i(y, u) = \frac{1}{2}y_i - \frac{1}{2}u_i, \qquad i = 1, 2. \tag{4.1}$$

The sets U and Y are given by $U(y) = U = [-1, 1] \times [-1, 1]$ and $Y := [-1, 1] \times [-1, 1]$. It is easy to see that for all $u \in U$ and $y \in Y$ we have $f(y, u) \in Y$, therefore, $A(y) = U$ (see (1.2)).

The semi-infinite LP problem (1.19) was formulated for this problem with the monomials $\phi_{i_1,i_2}(y) = y_1^{i_1} y_2^{i_2}$, $i_1, i_2 = 0, 1, ..., 7$, as the functions $\phi_i(\cdot)$ used in defining $W_N(y_0)$ in (1.20). This problem and the corresponding N-approximating problem (1.14) were solved numerically using the algorithm similar to the one described in [10] with the number of constraints $N = (7 + 1)^2 = 64$. The discount factor α was taken to be equal to 0.7, and the initial conditions were taken to be

$$y_1(0) = -0.5, \qquad y_2(0) = 0.$$

In particular, the optimal value of the semi-infinite LP problem was evaluated to be approximately equal to $g_N^*(y_0) = -0.89$.

The construction of the optimal trajectory is illustrated in Figure 1, in which the points from the set $\mathtt{N}$ (see formula (2.4)) and the calculated trajectory are shown.

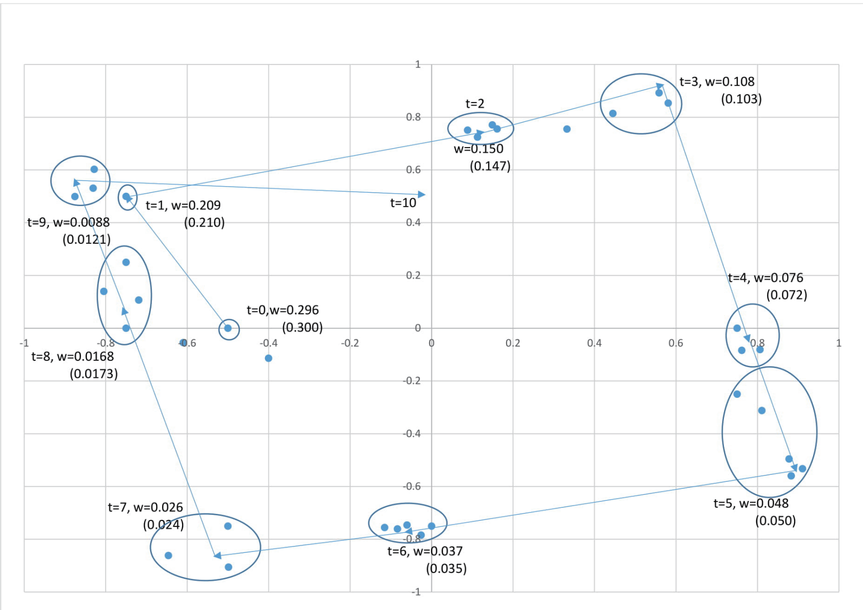

Figure 1. The state trajectory between $t = 0$ and $t = 10$.

At $t = 0$, one point from the set $\mathbb{N}$ happens to coincide with $(y_1^N(0), y_2^N(0)) = (-0.5, 0)$. The weight of this point is 0.296, which is very close to the value of $1 - \alpha = 0.3$ (see formula (3.2)). The values of the u-components at this point are $(u_1^N(0), u_2^N(0)) = (1, -1)$ and $(y_1^N(1), y_2^N(1))$ are evaluated as

$$y_1^N(1) = \frac{1}{2}(-0.5 - 1) = -0.75, \quad y_2^N(1) = \frac{1}{2}(0 + 1) = 0.5.$$

This point, again, happens to coincide with a point from $\mathbb{N}$ with the weight 0.204, which is close to $(1 - \alpha)\alpha = 0.21$ (see formula (3.18)). The u-components at this point are $(u_1^N(1), u_2^N(1)) = (-1, -1)$, therefore,

$$y_1^N(2) = \frac{1}{2}(-0.75 + 1) = 0.125, \quad y_2^N(2) = \frac{1}{2}(0.5 + 1) = 0.75.$$

In the vicinity of the latter point there are four points in $\mathbb{N}$ with the total weight 0.150, while $(1 - \alpha)\alpha^2 = 0.147$. (The values of $(1 - \alpha)\alpha^t$ are given in Figure 1 in parentheses.) The weighted average of the u-components of these four points is $(u_1^N(2), u_2^N(2)) = (-1, -0.984)$ and we can evaluate $(y_1^N(3), y_2^N(3)) = (0.56, 0.87)$.

Continuing in a similar manner we obtain a sequence of point on the trajectory until $t = 10$, when the total weight of the remaining points in $\mathbb{N}$ not used in the preceding steps becomes very small, making further calculations not possible. Parameter N would have to be increased to calculate the trajectory for $t > 10$. However, it can be conjectured that the optimal trajectory approaches a square-like limiting cycle, similarly to the example in Section 5 of [9], where the same dynamical system with different initial conditions was treated using a different numerical approach.

It can be noticed that for $t \leq 8$ the relative error of the weight of the clusters $|w - (1 - \alpha)\alpha^t|/(1 - \alpha)\alpha^t$ is small, but at $t = 9$ it becomes significant: $(0.0121 - 0.0088)/0.0121 = 0.27$. Therefore, it is not surprising that the calculated position of the trajectory at $t = 10$ is noticeably off its expected location. The value of the cost functional until time $t = 9$ is $v_9^N := \sum_{t=0}^{9} \alpha^t g(y^N(t), u^N(t)) = -2.87$. Since

$$g_N^*(y_0) = \mu_N^*(y_0) \leq \mu^*(y_0) = (1 - \alpha)V(y_0) \leq (1 - \alpha)V^N(y_0),$$

we have

$$V^N(y_0) - V(y_0) \leq V^N(y_0) - g_N(y_0)(1 - \alpha)^{-1},$$

hence,

$$v_9^N - V(y_0) \leq v_9^N - g_N(y_0)(1 - \alpha)^{-1}.$$

Taking into account that $g_N(y_0) = -0.89$, we have

$$v_9^N - V(y_0) \leq -2.87 + 0.89/0.3 = 0.10.$$

Since $g(y(t), u(t)) \leq 2$ for any admissible trajectory, we have

$$|V^N(y_0) - v_9^N| \leq 2 \sum_{t=10}^{\infty} \alpha^t = 0.19,$$

therefore,

$$v_9^N - V(y_0) = (v_9^N - V^N(y_0)) + (V^N(y_0) - V(y_0)) \geq -0.19.$$

Combining this with the estimate above, we obtain

$$-0.19 \leq v_9^N - V(y_0) \leq 0.10,$$

which shows that the value of v_9^N is close to the optimal value of the cost function.

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