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SECOND ORDER SUFFICIENT CONDITIONS FOR STABLE WELL-POSEDNESS OF φ-PROX-REGULAR FUNCTIONS

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Abstract. With respect to an admissible function φ, we consider the class of all φ-prox-regular functions. Using the Mordukhovich second order sudiferentially, we consider sufficient conditions for the metric φ-regularity of the subdifferential mapping of a φ-prox-regular function. As an application, we provide second order sufficient conditions for the stable well-posedness of a φ-prox-regular function.

1. Introduction

For many smooth optimization problems, one need to construct a sequence $\{x_k\}$ interactively and, for obtaining x_{k+1}, often one uses the gradient $\nabla f(x_k)$ of the objective function f at the k-th interactive point x_k. However, sometimes we cannot obtain the exact gradient $\nabla f(x_k)$ and have to use an approximation $\nabla f(x_k) - u^*$ of $\nabla f(x_k)$, where u^*, an error term, is a continuous linear functional. Correspondingly, linear perturbation f_{u^*} (of f perturbed by u^*) is defined by

$$f_{u^*}(x) := f(x) - \langle u^*, x\rangle \text{ for all } x. \tag{1.1}$$

Then $\nabla f(x_k) - u^*$, as an approximation of the gradient $\nabla f(x_k)$, is just the gradient of f_{u^*} at x_k. So it is natural and useful to consider stability analysis when f undergoes small linear perturbations. In 1998, Poliquin and Rockafellar [19] considered such stability analysis and introduced the tilt-stable minimum: *a proper lower semicontinuous extended-real function f on a Banach space X is said to give a tilt-stable minimum at $\bar{x} \in \mathrm{dom}(f)$ (or say that $\bar{x}$ is a tilt-stable minimizer of f) if there exist $r, \delta, L \in (0, \infty)$ and $M : B_{X^*}(0,\delta) \to B_X(\bar{x}, r)$ with $M(0) = \bar{x}$ such that*

$$f_{u^*}(M(u^*)) = \min_{x\in B_X(\bar{x},r)} f_{u^*}(x) \quad \forall u^* \in B_{X^*}(0,\delta) \tag{1.2}$$

and

$$\|M(x^*) - M(u^*)\| \leq L\|x^* - u^*\| \quad \forall x^*, u^* \in B_{X^*}(0,\delta), \tag{1.3}$$

where $B_X(\bar{x}, r)$ and $B_{X^*}(0,\delta)$ are open balls in X and in its dual X^*, respectively. In the finite dimensional setting (with $X = \mathbb{R}^n$) and under the assumption that

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$f : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is subdifferentially continuous and prox-regular at $(\bar{x}, 0) \in \mathrm{dom}(\partial f)$, they showed that f *gives a tilt-stable minimum at* $\bar{x}$ *if and only if the second order subdifferential* $\partial^2 f(\bar{x}, 0)$ *is positive definite.* In 2000, when f undergoes small linear perturbations, under the name of "uniform second-order growth condition", Bonnans and Shapiro [3] introduced the following notion: $\bar{x}$ *is said to be a stable second order minimizer of* f *if there exist* $r', \delta', L \in (0, +\infty)$ *and a mapping* $\Theta : B_{X^*}(0, \delta') \to B_X(\bar{x}, r')$ *such that* $\Theta(0) = \bar{x}$ *and*

$$L\|x - \Theta(u^*)\|^2 \le f_{u^*}(x) - f_{u^*}(\Theta(u^*)) \quad \forall (x, u^*) \in B_X(\bar{x}, r') \times B_{X^*}(0, \delta'). \tag{1.4}$$

In 2008, Aragón Artacho and Geoffroy [1] established the following characterization for the stable second order minimizer:

Theorem 1.1. *Let X be a Hilbert space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $(\bar{x}, 0) \in \mathrm{gph}(\partial f)$. Then $\bar{x}$ is a stable second order minimizer of f if and only if ∂f is strongly metrically regular at $(\bar{x}, 0)$.*

Under the finite dimension assumption, Drusvyatskiy and Lewis [8] extended Theorem 1.1 in replacing the convexity assumption by the weaker assumption that f is prox-regular and subdifferentially continuous at $(\bar{x}, 0)$; moreover they added another characterization: *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function. Then f gives a tilt-stable minimum at $\bar{x} \in \mathrm{dom}(f)$ if and only if $\bar{x}$ is a stable second order minimizer of f.* Recently, the study on the tilt-stable minima and stable second order minimizers has been pushed further through the works of Mordukhovich and his collaborators (cf. [9, 13, 15, 16] and the references therein). In particular the above mentioned results were extended to the infinite dimensional setting (cf. [9, 13]).

Given two positive numbers p and q, with $\|x^* - u^*\|^p$ and $\|x - \Theta(u^*)\|^q$ replacing $\|x^* - u^*\|$ and $\|x - \Theta(u^*)\|^2$ in (1.3) and (1.4) respectively, Zheng and Ng [22, 23] introduced and studied notions of tilt-stable p-order minima and stable q-order minimizers. Well-posedness is a fundamental notion in optimization and well studied (cf. [7, 10, 11, 20, 21] and the references therein). Let f be a proper lower semicontinuous function on a Banach space X and recall that f is well-posed at $\bar{x} \in \mathrm{dom}(f)$ (in the Tykhonov sense) if every minimizing sequence $\{x_n\}$ of f converges to $\bar{x}$. Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an admissible function if it is a nondecreasing function such that $\varphi(0) = 0$ and $[\varphi(t) \to 0 \Rightarrow t \to 0]$. It is known (cf.[7, P6, Theorem 12]) that f is well-posed at $\bar{x}$ if and only if there exists an admissible function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(\|x - \bar{x}\|) \le f(x) - f(\bar{x}) \quad \forall x \in X. \tag{1.5}$$

Some earlier results mentioned above were further extended to the so-called stable well-posedness in [24]. *Given two admissible functions $\varphi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ and a proper lower semicontinuous function f on a Banach space X, we say that*
(i) *f has stable local well-posedness at $\bar{x} \in \mathrm{dom}(f)$ with respect to φ (in brief, φ-SLWP) if there exist $r, \delta, \tau, \kappa \in (0, +\infty)$ and a mapping $\Theta : B_{X^*}(0, \delta) \to B_X(\bar{x}, r)$ such that $\Theta(0) = \bar{x}$ and*

$$\varphi(\kappa\|x - \Theta(u^*)\|) \le \tau(f_{u^*}(x) - f_{u^*}(\Theta(u^*))) \quad \forall (x, u^*) \in B_X(\bar{x}, r) \times B_{X^*}(0, \delta); \tag{1.6}$$

(ii) f *gives a* ψ*-tilt-stable local minimum at* $\bar{x}$ *(in brief,* ψ*-TSLM) if there exist* $r, \delta, \tau, \kappa \in (0, +\infty)$ *and* $M : B_{X^*}(0, \delta) \to B_X(\bar{x}, r)$ *with* $M(0) = \bar{x}$ *such that* (1.2) *holds and*

$$\kappa\|M(u_1^*) - M(u_2^*)\| \leq \psi(\tau\|u_1^* - u_2^*\|) \quad \forall u_1^*, u_2^* \in B_{X^*}(0, \delta). \tag{1.7}$$

Clearly, in the case when $\varphi(t) = t^2$ and $\psi(t) = t$ (resp. $\varphi(t) = t^q$ and $\psi(t) = t^p$), φ-SLWP and ψ-TSLM reduce to the stable second order minimizer and tilt-stable minimum (resp. stable q-order minimizer and tilt-stable p-order minimum). Under the assumption that the admissible function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and strictly convex admissible function with $\varphi'(0) = 0$, Zheng and Zhu [24] proved that a proper lower semicontinuous function f on a Banach space has φ-*SLWP* at $\bar{x} \in \mathrm{dom}(f)$ if and only if f has $(\varphi')^{-1}$-*TSLM* at $\bar{x}$. Moreover, related to the result by Poliquin and Rockafellar, the following result was established in [24].

Theorem 1.2. *Let* ψ *be a convex admissible function and* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper lower semicontinuous convex function. Let* $\bar{x} \in \mathrm{dom}(f)$ *and* $0 \in \partial f(\bar{x})$*. Suppose that there exist* $\kappa, r \in (0, +\infty)$ *such that*

$$\kappa\|h\|^2\psi'_+(d(x, (\partial f)^{-1}(v - h))) \leq \langle z, h\rangle$$

for all $(x, v, h) \in (\mathrm{gph}(\partial f) \times \mathbb{R}^n) \cap B_{\mathbb{R}^n}(\bar{x}, r) \times B_{\mathbb{R}^n}(0, r) \times B_{\mathbb{R}^n}(0, r)))$ *and* $z \in \partial^2 f(x, v)(h)$*, where* $\partial^2 f(x, v)$ *denotes the second order subdifferential (see Section 2 for its definition). Then,* f *has* φ*-SLWP at* $\bar{x}$ *with* $\varphi(t) := \int_0^t \psi(t)dt$*.*

In this paper, given a convex admissible function φ, we first prove that ∂f is locally monotone at $(\bar{x}, 0)$ (i.e. $\mathrm{gph}(\partial f) \cap V$ is monotone for some neighborhood V of $(\bar{x}, 0)$) whenever f is subdifferentially continuous and prox-regular at $(\bar{x}, 0)$ and f has the φ-SLWP at $\bar{x}$. As an extension of the local monotonicity, we adopt the notion of ξ-D-hypomonotonicity of ∂f. Given a proper lower semicontinuous function f, we prove that if ∂f is metrically φ'_+-regular and ξ-D-hypomonotone at $(\bar{x}, 0)$ then f has the φ-SLWP; in particular, we extend and improve (assuming the metric regularity rather than the strong one) the sufficiency part of Theorem 1.1. Using the second order subdifferential $\partial^2 f$ of f, we provide a sufficient condition for the first order subdifferential ∂f to be metrically ψ-regular at some $(\bar{x}, 0)$ in $\mathrm{gph}(\partial f)$ in the case when f is subdifferentially continuous and ψ-prox-regular at $(\bar{x}, 0)$. As an application, we establish second order sufficient conditions for φ-*SLWP* in terms of a kind of positive definiteness of the Mordukhovich second order subdifferential $\partial^2 f$. In particular, we extend Theorem 1.2 to the φ-prox-regularity case.

2. Preliminaries.

In this section, we give some known notions and results of variational analysis (see [4, 12] for more details).

Let X and Y be Banach spaces. The topological dual of X is denoted by X^*. We denote by B_X the closed unit ball of X. For $\bar{x} \in X$ and $\delta > 0$, let $B_X(\bar{x}, \delta)$ denote the open ball centered at $\bar{x}$ with radius δ in X. For a proper lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, we denote by $\mathrm{dom}(f)$ and $\mathrm{epi}(f)$ the domain and the epigraph of f respectively, that is,

$$\mathrm{dom}(f) := \{x \in X : f(x) < +\infty\} \quad \text{and} \quad \mathrm{epi}(f) := \{(x, \alpha) \in X \times \mathbb{R} : f(x) \le \alpha\}.$$

For $x \in \mathrm{dom}(f)$ and $h \in X$, let $f^{\uparrow}(x, h)$ denote the generalized directional derivative introduced by Rockafellar [4]; that is,

$$f^{\uparrow}(x, h) := \lim_{\varepsilon \downarrow 0} \limsup_{u \xrightarrow{f} x, t \downarrow 0} \inf_{w \in h + \varepsilon B_X} \frac{f(u + tw) - f(u)}{t} \quad \forall h \in X,$$

where $u \xrightarrow{f} x$ means that $u \to x$ and $f(u) \to f(x)$. Let $\partial f(x)$ denote the Clarke-Rockafellar subdifferential of f at x; that is,

$$\partial f(x) := \{x^* \in X^* : \langle x^*, h\rangle \le f^{\uparrow}(x, h) \quad \forall h \in X\}.$$

In the case when X is an Asplund space, the Fréchet subdifferential and Mordukhovich limiting subdifferential are more suitable than Clarke's one (for the details see [12]). Indeed, under the Asplund space framework, some results (eg. Proposition 3.4) in this paper hold still with the Mordukhovich limiting subdifferential replacing the Clarke subdifferential.

When f is convex, the Clarke-Rockafellar subdifferential reduces to the one in the sense of convex analysis; that is, for all $x \in \mathrm{dom}(f)$, one has

$$\begin{aligned} \partial f(x) &= \{x^* \in X^* : \langle x^*, y - x\rangle \le f(y) - f(x) \quad \forall y \in X\} \\ &= \{x^* \in X^* : \langle x^*, h\rangle \le \lim_{t \to 0^+} \frac{f(x + th) - f(x)}{t} \quad \forall h \in X\}. \end{aligned} \tag{2.1}$$

For $(x, x^*) \in \mathrm{gph}(\partial f)$, the Mordukhovich second-order subdifferential $\partial^2 f(x, x^*)$ of f at (x, x^*) is defined as follows:

$$\partial^2 f(x, x^*)(h^{**}) = \{z^* \in X^* : (z^*, -h^{**}) \in N(\mathrm{gph}(\partial f), (x, x^*))\} \quad \forall h^{**} \in X^{**}$$

(see [12] for more details).

Given an admissible function ψ and a closed multifunction F between two Banach spaces X and Y. Recall that F is said to be metrically ψ-regular at $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ if there exist $\kappa, \tau, \delta \in (0, +\infty)$ such that

$$\psi(\kappa d(x, F^{-1}(y))) \le \tau d(y, F(x)) \quad \forall (x, y) \in B(\bar{x}, \delta) \times B(\bar{y}, \delta), \tag{2.2}$$

while F is said to be strongly metrically ψ-regular at $(\bar{x}, \bar{y})$ if F is metrically ψ-regular at $(\bar{x}, \bar{y})$ and there exist $r, \delta \in (0, +\infty)$ such that $F^{-1}(y) \cap B_X(\bar{x}, r)$ is a singleton for all $y \in B_Y(\bar{y}, \delta)$.

We will need the following lemma on the metric ψ-regularity for a subdifferential mapping, which was recently established in [24].

Lemma 2.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a convex admissible function and f be a proper lower semicontinuous function on a Banach space X. Let $\bar{x} \in \mathrm{dom}(f)$ be a local minimizer of f. Suppose that ∂f is strongly metrically φ'_+-regular at $(\bar{x}, 0)$. Then f has φ-SLWP at $\bar{x}$.*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and recall that f is prox-regular at $(\bar{x}, \bar{x}^*) \in \mathrm{gph}(\partial f)$ if there exist $\rho, r \in (0, +\infty)$ such that

$$f(y) \ge f(x) + \langle x^*, y - x\rangle - \rho\|y - x\|^2 \tag{2.3}$$

for all $x, y \in B_X(\bar{x}, r)$ with $|f(x) - f(\bar{x})| < r$ and $x^* \in \partial f(x) \cap B_{X^*}(\bar{x}^*, r)$. Clearly, the convexity of f implies the prox-regularity. The prox-regularity is a useful notion in variational analysis and has been well studied (cf. [2, 5, 6, 18]). Given a nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\psi(\|y-x\|)\|y-x\|$ replacing $\|y-x\|^2$ in (2.3), one can adopt the following notion.

Definition 2.2. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function. We say that f is ψ-prox-regular at $(\bar{x}, \bar{x}^*) \in \operatorname{gph}(\partial f)$ if there exist $\rho, r \in (0, +\infty)$ such that

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - \rho\psi(\|y-x\|)\|y-x\| \quad \forall y \in B_X(\bar{x}, r)$$

whenever $x \in B_X(\bar{x}, r)$, $x^* \in \partial f(x) \cap B_{X^*}(\bar{x}^*, r)$ and $f(x) < f(\bar{x}) + r$.

In the case when $\psi(t) = t$, the ψ-prox-regularity reduces to the usual prox-regularity. Clearly, the convexity of f implies the ψ-prox-regularity.

Recall that f is subdifferentially continuous at $(\bar{x}, \bar{x}^*) \in \operatorname{gph}(\partial f)$ if

$$\lim_{(x,x^*) \overset{\operatorname{gph}(\partial f)}{\longrightarrow} (\bar{x}, \bar{x}^*)} f(x) = f(\bar{x}).$$

Most of the existing results on the stable second order minimizer and tilt-stable minimum require the subdifferential continuity (cf. [8, 9, 13, 15, 16, 19] and the references therein). It is easy to verify that if f is subdifferentially continuous at $(\bar{x}, \bar{x}^*)$ then f is ψ-prox-regular at $(\bar{x}, \bar{x}^*)$ if and only if there exist $\rho, r \in (0, +\infty)$ such that

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - \rho\psi(\|y-x\|)\|y-x\| \quad \forall y \in B_X(\bar{x}, r)$$

whenever $x \in B_X(\bar{x}, r)$ and $x^* \in \partial f(x) \cap B_{X^*}(\bar{x}^*, r)$.

3. Main result

Under the assumption that f is subdifferentially continuous and prox-regular at $(\bar{x}, 0)$, we first provide a necessary condition for f to have the φ-SLWP at $\bar{x}$.

Proposition 3.1. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex admissible function. Let f be a proper lower semicontinuous extended-real function on a Hilbert space X such that f is subdifferentially continuous and prox-regular at $(\bar{x}, 0)$. Suppose that f has the φ-SLWP at $\bar{x}$. Then ∂f is locally monotone at $(\bar{x}, 0)$, namely there exist $r_0, \delta_0 \in (0, +\infty)$ such that*

$$\langle x_2^* - x_1^*, x_2 - x_1\rangle \geq 0 \quad \forall (x_1, x_1^*), (x_2, x_2^*) \in \operatorname{gph}(\partial f) \cap (B_X(\bar{x}, r_0) \times B_{X^*}(0, \delta_0)).$$

Proof. Take $r, \delta, \tau, \kappa \in (0, +\infty)$ and a mapping $\Theta : B_{X^*}(0, \delta) \to B_X(\bar{x}, r)$ such that $\Theta(0) = \bar{x}$ and (1.6) holds. Using (1.6) twice, we have that

$$\begin{aligned} 2\varphi(\kappa\|\Theta(u_2^*) - \Theta(u_1^*)\|) &\leq \tau\langle u_2^* - u_1^*, \Theta(u_2^*) - \Theta(u_1^*)\rangle \\ &\leq \tau\|u_2^* - u_1^*\|\|\Theta(u_2^*) - \Theta(u_1^*)\| \\ &\leq 2\tau r\|u_2^* - u_1^*\| \end{aligned} \tag{3.1}$$

for all $u_1^*, u_2^* \in B_{X^*}(0, \delta)$. Moreover, (1.6) also implies that $\Theta(u^*)$ is a minimizer of $f_{u^*} = f - u^*$ on $B_X(\bar{x}, r)$ and so the conjugate function $(f + \delta_{B_X(\bar{x},r)})^*$ of $f + \delta_{B_X(\bar{x},r)}$ satisfies

$$(f + \delta_{B_X(\bar{x},r)})^*(u^*) = \langle u^*, \Theta(u^*)\rangle - f(\Theta(u^*)) \quad \forall u^* \in B_{X^*}(0, \delta). \tag{3.2}$$

Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$\mathrm{epi}(g) = \overline{\mathrm{co}}(\mathrm{epi}(f + \delta_{B_X(\bar{x},r)})). \tag{3.3}$$

Then g is a lower semicontinuous convex function, $g^* = (f + \delta_{B_X(\bar{x},r)})^*$ and

$$g(x) \le f(x) + \delta_{B_X(\bar{x},r)}(x) = f(x) \quad \forall x \in B_X(\bar{x}, r). \tag{3.4}$$

Hence, by (3.2), one has

(3.5)
$$g^*(u^*) = \langle u^*, \Theta(u^*)\rangle - f(\Theta(u^*)) \text{ and } f(\Theta(u^*)) = g(\Theta(u^*)) \quad \forall u^* \in B_{X^*}(0, \delta).$$

This implies that $\Theta(u^*) \in \partial g^*(u^*)$ for all $u^* \in B_{X^*}(0, \delta)$. Noting that Θ is continuous on $B_X(\bar{x}, \delta)$ (thanks to (3.1) and $\varphi(t) \to 0 \Rightarrow t \to 0$), it follows from [17, Proposition 28] that the convex function g^* is Fréchet differentiable on $B_{X^*}(0, \delta)$ and $\nabla g^*(u^*) = \Theta(u^*)$ for all $u^* \in B_{X^*}(0, \delta)$. Hence

$$\mathrm{gph}(\partial g^*) \cap (B_{X^*}(0, \delta) \times X) = \{(u^*, \Theta(u^*)) : \ u^* \in B_{X^*}(0, \delta)\}.$$

Noting that $\mathrm{gph}(\partial g^*) = \{(x^*, x) : \ (x, x^*) \in \mathrm{gph}(\partial g)\}$ (thanks to the convexity of g), one has

$$\mathrm{gph}(\partial g) \cap (X \times B_{X^*}(0, \delta)) = \{(\Theta(u^*), u^*) : \ u^* \in B_{X^*}(0, \delta)\}. \tag{3.6}$$

Since $\partial g(x) = \{x^* \in X : \ \langle x^*, y - x\rangle \le g(y) - g(x) \ \ \forall y \in X\}$, it follows from (3.4) and (3.5) that

$$\mathrm{gph}(\partial g) \cap (B_X(\bar{x}, r) \times B_{X^*}(0, \delta)) \subset \mathrm{gph}(\partial f). \tag{3.7}$$

On the other hand, by the subdifferential continuity and prox-regularity of f at $(\bar{x}, 0)$, there exist $r_1 \in (0, \ r)$, $\delta_1 \in (0, \ \delta)$ and $\rho > 0$ such that

$$f(y) \ge f(x) + \langle x^*, y - x\rangle - \rho\|y - x\|^2$$

for all $x, y \in B_X(\bar{x}, r_1)$ and $x^* \in \partial f(x) \cap B_{X^*}(0, \delta_1)$. This implies that

$$\langle x_2^* - x_1^*, x_2 - x_1\rangle + 2\rho\|x_2 - x_1\|^2 \ge 0$$

for any $(x_1, x_1^*), (x_2, x_2^*) \in \mathrm{gph}(\partial f) \cap (B_X(\bar{x}, r_1) \times B_{X^*}(0, \delta_1))$, and so

$$\mathrm{gph}(\partial f + 2\rho I) \cap (B_X(\bar{x}, r_1) \times B_{X^*}(0, \delta_1)) \text{ is monotone.} \tag{3.8}$$

Moreover, by the continuity of Θ and $\Theta(0) = \bar{x}$, there exists $\delta_2 \in (0, \ \delta_1)$ such that

$$\|\Theta(u^*) + 2\rho u^* - \bar{x}\| = \|\Theta(u^*) - \Theta(0) + 2\rho u^*\| < r_1 \quad \forall u^* \in B_{X^*}(0, \delta_2).$$

Hence, by (3.6) and (3.7), one has

(3.9)
$$\{(\Theta(u^*) + 2\rho u^*, u^*) : \ u^* \in B_{X^*}(0, \delta_2)\} \subset \mathrm{gph}(\partial f + 2\rho I) \cap (B_X(\bar{x}, r_1) \times B_{X^*}(0, \delta_2)).$$

The inclusion in (3.9) can in fact be replaced by the equality as the intersection set on the right-hand side is monotone (by (3.8)) while the set on the left-hand side is a maximal monotone subset of $B_X(\bar{x}, r_1) \times B_{X^*}(0, \delta_2))$ by [14, Lemma 2.1] (noting that $(\Theta + 2\rho I)(B_{X^*}(0, \delta_2)) \subset B_X(\bar{x}, r_1))$ and Θ is monotone and continuous on $B_{X^*}(0, \delta)$ (thanks to (3.1))). By the just established equality version of (3.9) together with the continuity of Θ and $\Theta(0) = \bar{x}$, it follows that there exist $r_0 > 0$ and $\delta_0 \in (0, \ \delta_2)$ such that

$$\mathrm{gph}(\partial f) \cap (B_X(\bar{x}, r_0) \times B_{X^*}(0, \delta_0)) = \{(\Theta(u^*), u^*) : \ u^* \in B_{X^*}(0, \delta_0)\}.$$

Therefore $\mathrm{gph}(\partial f) \cap (B_X(\bar{x}, r_0) \times B_{X^*}(0, \delta_0))$ is monotone. The proof is complete. $\square$

Note that the φ-SLWP of f at $\bar{x}$ implies $0 \in \partial f(\bar{x})$. Based on Proposition 3.1, to consider sufficient conditions for the φ-SLWP of f at $\bar{x}$, it is reasonable to require that ∂f is monotone on a neighborhood of $(\bar{x}, 0)$. This leads us to introduce the following notion which can be regarded as generalizations of the notions of hypomonotonicity and D-hypomonotonicity.

Definition 3.2. Given a function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, a mapping $T : X \rightrightarrows X^*$ is said to be ξ-hypomonotone (resp. ξ-D-hypomonotone) at $(\bar{x}, \bar{x}^*) \in \mathrm{gph}(T)$ if there exist $\rho > 0$ and $\delta > 0$ such that

$$\langle y^* - x^*, y - x\rangle \geq -\rho\xi(\|y - x\|)\|y - x\| \tag{3.10}$$

$$\big(\text{resp. } \langle y^* - x^*, y - x\rangle \geq -\rho\xi(\|y^* - x^*\|)\|y^* - x^*\|\big)$$

for all $(x, x^*), (y, y^*) \in \mathrm{gph}(T) \cap (B_X(\bar{x}, \delta) \times B_{X^*}(\bar{x}^*, \delta))$.

Clearly, T is ξ-D-hypomonotone at $(\bar{x}, \bar{x}^*)$ if and only if T^{-1} is ξ-hypomonotone at $(\bar{x}^*, \bar{x})$. Moreover, monotonicity of T implies trivially both ξ-hypomonotonicity and ξ-D-hypomonotonicity of T. The following example shows that monotonicity may be strictly stronger than both ξ-hypomonotonicity and ξ-D-hypomonotonicity. Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $f(x) := -c_0\|x\|^2 + \langle u, x\rangle + c$ for all $x \in \mathbb{R}^n$ with $(c_0, u, c) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}$. Then f is a nonconvex and smooth function on $\mathbb{R}^n$ with $\triangledown f(x) = -2c_0x + u$ for all $x \in \mathbb{R}^n$. It follows that

$$\langle \triangledown f(x_2) - \triangledown f(x_1), x_2 - x_1\rangle = -2c_0\|x_2 - x_1\|^2 \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

This shows that $\triangledown f$ is not monotone at $(0, 0)$ but it is both ξ-hypomonotone and ξ-D-hypomonotone at $(0, 0)$ with $\xi(t) = t$ for all $t \geq 0$.

The following lemma is useful in our later analysis, which is a generalization of [24, Lemma 4.2].

Lemma 3.3. *Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing function such that $\lim_{t\to 0^+} \xi(t) = 0$. Let $T : X \rightrightarrows X^*$ be ξ-hypomonotone at $(\bar{x}, \bar{x}^*) \in \mathrm{gph}(T)$. Suppose further that there exist a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $r, \eta \in (0, +\infty)$ such that $\lim_{t\to 0^+} \psi(t) = \psi(0) = 0$ and*

$$T(x_1) \cap B_{X^*}(\bar{x}^*, r) \subset T(x_2) + \psi(\|x_1 - x_2\|)B_{X^*} \quad \forall x_1, x_2 \in B_X(\bar{x}, \eta). \tag{3.11}$$

Then there exist $\gamma, \delta' \in (0, +\infty)$ such that $T(x) \cap B_{X^*}(\bar{x}^*, \gamma)$ is a singleton for all $x \in B_X(\bar{x}, \delta')$.

Proof. Since $T : X \rightrightarrows X^*$ is ξ-hypomonotone at $(\bar{x}, \bar{x}^*)$, there exist $\rho, \delta \in (0, +\infty)$ such that (3.10) holds for all $(x, x^*), (y, y^*) \in \mathrm{gph}(T) \cap (B_X(\bar{x}, \delta) \times B_{X^*}(\bar{x}^*, \delta))$. By the assumption that $\lim_{t\to 0^+} \psi(t) = \psi(0) = 0$, one has

$$r' := \sup\{t \geq 0 : \ \psi([0, t]) \subset [0, \ \min\{r, \delta\})\} > 0.$$

Let $\delta' := \min\{\delta, \eta, r'\}$. We claim that $T(x) \cap B_{X^*}(\bar{x}^*, \min\{r, \delta\})$ is a singleton for all $x \in B_X(\bar{x}, \delta')$. To do this, let $x \in B_X(\bar{x}, \delta')$. Then, $\|x - \bar{x}\| < r'$, and so $\psi(\|x - \bar{x}\|) < \min\{r, \delta\}$; moreover, by (3.11), one has $\bar{x}^* \in T(x) + \psi(\|x - \bar{x}\|)B_{X^*}$. Hence there exists $v_x^* \in T(x)$ such that

$$\|v_x^* - \bar{x}^*\| \leq \psi(\|x - \bar{x}\|) < \min\{r, \delta\}. \tag{3.12}$$

It suffices to show that $T(x) \cap B_{X^*}(\bar{x}^*, \min\{r, \delta\}) = \{v_x^*\}$. To do this, suppose to the contrary that there exists $z^* \in T(x) \cap B_{X^*}(\bar{x}^*, \min\{r, \delta\})$ such that $v_x^* \neq z^*$. Then, there exists $h \in X$ such that

$$\langle v_x^* - z^*, h\rangle < 0. \tag{3.13}$$

Since $\|x - \bar{x}\| < \delta'$, there exists a sequence $\{\varepsilon_n\} \subset (0, +\infty)$ converging to 0 such that $\{x + \varepsilon_n h\} \subset B_X(\bar{x}, \delta')$. It follows from (3.11) that

$$v_x^* \in T(x) \cap B_{X^*}(\bar{x}^*, r) \subset T(x + \varepsilon_n h) + \psi(\varepsilon_n\|h\|)B_{X^*} \quad \forall n \in \mathbb{N}.$$

Hence, for any $n \in \mathbb{N}$ there exists $x_n^* \in T(x + \varepsilon_n h)$ such that

$$\|x_n^* - v_x^*\| \le \psi(\varepsilon_n\|h\|) \to 0.$$

Thus, by (3.12), we can assume without loss of generality that $x_n^* \in B_{X^*}(\bar{x}^*, \delta)$ for all $n \in \mathbb{N}$. It follows from (3.10) that

$$\begin{aligned}
-\rho\xi(\varepsilon_n\|h\|)\varepsilon_n\|h\| &\le \langle x_n^* - z^*, \varepsilon_n h\rangle \\
&= \varepsilon_n\langle x_n^* - z^*, h\rangle \\
&= \varepsilon_n(\langle x_n^* - v_x^*, h\rangle + \langle v_x^* - z^*, h\rangle) \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Hence

$$\langle v_x^* - z^*, h\rangle \ge \lim_{n\to\infty}(-\rho\xi(\varepsilon_n\|h\|)\|h\| - \langle x_n^* - v_x^*, h\rangle) = 0,$$

contradicting (3.13). The proof is complete. □

In contrast to [24, Theorem 3.3], the following proposition provides a sufficient condition for f to have φ-*SLWP* at $\bar{x}$ in terms of the metrical φ'_+-regularity of ∂f instead of the strongly metrical φ'_+-regularity of ∂f.

Proposition 3.4. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly convex admissible function, $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\lim_{t\to 0^+}\xi(t) = 0$, and let f be a proper lower semicontinuous extended-real function on a Banach space X such that ∂f is ξ-D-hypomonotone at $(\bar{x}, 0) \in \mathrm{gph}(\partial f)$. Suppose that ∂f is metrically φ'_+-regular at $(\bar{x}, 0)$. Then f has the φ-SLWP at $\bar{x}$.*

Proof. By the metrical φ'_+-regularity assumption on ∂f, there exist $\tilde{\kappa}, \tilde{\tau}, \tilde{r} \in (0, +\infty)$ such that

$$\varphi'_+(\tilde{\kappa}d(u, (\partial f)^{-1}(v^*)) \le \tilde{\tau}d(v^*, \partial f(u)) \quad \forall (u, v^*) \in B(\bar{x}, \tilde{r}) \times B(0, \tilde{r}). \tag{3.14}$$

Since the admissible function φ is strictly convex, φ'_+ is an inverse function. Hence

$$d(u, (\partial f)^{-1}(v^*)) \le \frac{1}{\tilde{\kappa}}(\varphi'_+)^{-1}(\tilde{\tau}d(v^*, \partial f(u))) \quad \forall (u, v^*) \in B(\bar{x}, \tilde{r}) \times B(0, \tilde{r}).$$

It follows that

$$(\partial f)^{-1}(u^*) \cap B(\bar{x}, \tilde{r}) \subset (\partial f)^{-1}(v^*) + \frac{1}{\tilde{\kappa}}(\varphi'_+)^{-1}(\tilde{\tau}\|v^* - u^*\|)B_{\mathbb{R}^n} \quad \forall u^*, v^* \in B(0, \tilde{r}).$$

We claim that

$$\lim_{t\to 0^+}(\varphi'_+)^{-1}(t) = 0. \tag{3.15}$$

Indeed, if this is not the case, there exist $\varepsilon_0 > 0$ and a sequence $\{t_n\}$ in $(0, +\infty)$ such that $t_n \to 0$ and $(\varphi'_+)^{-1}(t_n) > \varepsilon_0$ for all $n \in \mathbb{N}$, that is, $\varphi'_+(\varepsilon_0) < t_n$. Hence $\varphi'_+(\varepsilon_0) =$

0, and so $\varphi'_+(t) = 0$ for all $t \in [0,\ \varepsilon_0]$. Since φ is convex and $\varphi(0) = 0$, $\varphi(t) = 0$ for all $t \in (0,\ \varepsilon_0]$, contradicting the fact that φ is an admissible function. On the other hand, since ∂f is ξ-D-hypomonotone at $(\bar{x}, 0)$, $(\partial f)^{-1}$ is ξ-hypomonotone at $(0, \bar{x})$. Thus, by (3.14), (3.15) and Lemma 3.3, there exist $\gamma, \delta_0 \in (0,\ \infty)$ such that $(\partial f)^{-1}(v^*) \cap B_X(\bar{x}, \gamma)$ is a singleton for all $v^* \in B_{X^*}(0, \delta_0)$. This and (3.14) imply that ∂f is strongly metrically φ'_+-regular at $(\bar{x}, 0)$. Hence f has the φ-SLWP at $\bar{x}$ (thanks to Lemma 2.1). The proof is complete. □

Under the ξ-D-hypomonotonicity assumption on ∂f at $(\bar{x}, 0)$, we can see from the proof of Proposition 3.4 that metric φ'_+-regularity of ∂f at $(\bar{x}, 0)$ is equivalent to strong metric φ'_+-regularity of ∂f at $(\bar{x}, 0)$ (because strong metric φ'_+-regularity of ∂f at $(\bar{x}, 0)$ implies trivially metric φ'_+-regularity of ∂f at $(\bar{x}, 0)$). In the case when $\varphi(t) = t^2$, Drusvyatskiy, Mordukhovich and Nghia [9] considered the relationship between the metric regularity and strong metric regularity of ∂f. In particular, they made the following conjecture: *Consider a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that is both prox-regular and subdifferentially continuous at $\bar{x}$ for $\bar{x}^* = 0$, where $\bar{x}$ is a local minimizer of f. Then ∂f is metrically regular at $(\bar{x}, 0)$ if and only if ∂f is strongly metrically regular at $(\bar{x}, 0)$.*

Next we consider the sufficient conditions for the metric ψ-regularity of ∂f. In the remainder of this section, we mainly deal with the case when $X = \mathbb{R}^n$. For convenience, we denote the open ball of $\mathbb{R}^n$ with center $\bar{x}$ and radius r by $B(\bar{x}, r)$.

Theorem 3.5. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex admissible function and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function such that f is subdifferentially continuous and ψ-prox-regular at $(\bar{x}, 0) \in \mathrm{gph}\partial f$. Suppose that there exist $\kappa, r \in (0, +\infty)$ such that*

$$\kappa\|h\|^2\psi'_+(d(x, (\partial f)^{-1}(v - h))) \le \langle z, h\rangle \tag{3.16}$$

for all $(x, v, h) \in (\mathrm{gph}(\partial f) \times \mathbb{R}^n) \cap (B(\bar{x}, r) \times B(0, r) \times B(0, r))$ and $z \in \partial^2 f(x, v)(h)$. Then, ∂f is metrically ψ-regular at $(\bar{x}, 0)$.

Proof. Since f is subdifferentially continuous and ψ-prox-regular at $(\bar{x}, 0)$, there exist $\rho \in (0, +\infty)$ and $\bar{r} \in (0,\ r)$ such that

$$f(y) \ge f(x) + \langle x^*, y - x\rangle - \rho\psi(\|y - x\|)\|y - x\| \tag{3.17}$$

for all $x, y \in B(\bar{x}, 2\bar{r})$ and $x^* \in \partial f(x) \cap B(0, 2\bar{r})$. We claim that

$$\Omega := \mathrm{gph}(\partial f) \cap ((\bar{x} + \bar{r}B_{\mathbb{R}^n}) \times \bar{r}B_{\mathbb{R}^n}) \tag{3.18}$$

is closed. Let $(u, u^*) \in \mathrm{cl}(\Omega)$. Then there exists a sequence $\{(x_n, x_n^*)\}$ in Ω such that $\|x_n - u\| + \|x_n^* - u^*\| \to 0$. Hence

$$(u, u^*) \in (\bar{x} + \bar{r}B_{\mathbb{R}^n}) \times \bar{r}B_{\mathbb{R}^n}, \tag{3.19}$$

and

$$f(y) \ge f(x_n) + \langle x_n^*, y - x_n\rangle - \rho\psi(\|y - x_n\|)\|y - x_n\| \quad \forall (y, n) \in B(\bar{x}, 2\bar{r}) \times \mathbb{N}$$

(thanks to (3.17)). Noting that $\liminf_{x\to u} f(x) \ge f(u)$, it follows that

$$f(y) \ge f(u) + \langle u^*, y - u\rangle - \rho\psi(\|y - u\|)\|y - u\| \quad \forall y \in B(\bar{x}, 2\bar{r}).$$

Noting that $\lim_{t\to 0^+}\psi(t)=\psi(0)=0$, for any $\varepsilon>0$ there exists $\delta\in(0,\ \bar r)$ such that $\psi(\|y-u\|)\leq\frac{\varepsilon}{\rho}$ for all $y\in B(u,\delta)\subset B(\bar x,2\bar r)$. Hence

$$f(y)\geq f(u)+\langle u^*,y-u\rangle-\varepsilon\|y-u\|\quad\forall y\in B(u,\delta).$$

This implies that $u^*\in\hat\partial f(u)\subset\partial f(u)$, and so $(u,u^*)\in\mathrm{gph}(\partial f)$. It follows from (3.19) and the definition of Ω that $(u,u^*)\in\Omega$. This shows that Ω is closed.

Next we show that there exist $\tilde\kappa,\tilde\tau,\tilde r\in(0,\ +\infty)$ such that

(3.20)
$$\psi(\tilde\kappa d(u,(\partial f)^{-1}(v^*)))\leq\tilde\tau d(v^*,\partial f(u))\quad\forall(u,v^*)\in B(\bar x,\tilde r)\times(\partial f(B(\bar x,\tilde r))\cap B(0,\tilde r)).$$

To do this, suppose to the contrary that there exists a sequence $\{(u_i,x_i,v_i^*)\}\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ converging to $(\bar x,\bar x,0)$ such that

$$v_i^*\in\partial f(u_i)\ \text{ and }\ \psi\big(\frac{1}{i}d(x_i,(\partial f)^{-1}(v_i^*))\big)>id(v_i^*,\partial f(x_i))\quad\forall i\in\mathbb{N}.$$

Then

$$0<d(x_i,(\partial f)^{-1}(v_i^*))\leq\|x_i-u_i\|\to 0,\tag{3.21}$$

and there exists $y_i^*\in\partial f(x_i)$ such that

$$\|v_i^*-y_i^*\|<\frac{1}{i}\psi\big(\frac{1}{i}d(x_i,(\partial f)^{-1}(v_i^*))\big)\leq\frac{1}{i}\psi\big(\frac{1}{i}\|x_i-u_i\|\big)\to 0.\tag{3.22}$$

Define

$$g_i(u,v^*):=\|v^*-v_i^*\|+\delta_\Omega(u,v^*)\quad\forall(u,v^*)\in\mathbb{R}^n\times\mathbb{R}^n.$$

Then, g_i is lower semicontinuous, and

$$g_i(x_i,y_i^*)<\inf_{(u,v^*)\in\mathbb{R}^n\times\mathbb{R}^n}g_i(u,v^*)+\frac{1}{i}\psi\big(\frac{1}{i}d(x_i,(\partial f)^{-1}(v_i^*))\big).$$

For any $j\in\mathbb{N}$, let

$$\|(u,v^*)\|_j:=\|u\|+\frac{1}{j}\|v^*\|\quad\forall(u,v^*)\in\mathbb{R}^n\times\mathbb{R}^n.$$

By the Ekeland variational principle, there exists $(x_{ij},y_{ij}^*)\in\Omega$ such that

$$\|(x_{ij},y_{ij}^*)-(x_i,y_i^*)\|_j<\frac{1}{i}d(x_i,(\partial f)^{-1}(v_i^*)),\tag{3.23}$$

$$\|y_{ij}^*-v_i^*\|=g_i(x_{ij},y_{ij}^*)\leq g_i(x_i,y_i^*)=\|y_i^*-v_i^*\|\tag{3.24}$$

and

$$g_i(x_{ij},y_{ij}^*)\leq g_i(u,v^*)+\frac{\psi\big(\frac{1}{i}d(x_i,(\partial f)^{-1}(v_i^*))\big)}{d(x_i,(\partial f)^{-1}(v_i^*))}\|(u,v^*)-(x_{ij},y_{ij}^*)\|_j\tag{3.25}$$

for all $(u,v^*)\in\mathbb{R}^n\times\mathbb{R}^n$. Since Ω is a bounded closed subset of $\mathbb{R}^n\times\mathbb{R}^n$, we can assume without loss of generality that $\lim_{j\to\infty}(x_{ij},y_{ij}^*)=(\bar x_i,\bar v_i^*)\in\Omega$ (passing to a subsequence if necessary). It follows from (3.23)—(3.25) that

$$\|\bar x_i-x_i\|\leq\frac{1}{i}d(x_i,(\partial f)^{-1}(v_i^*)),\ \ \|\bar v_i^*-v_i^*\|\leq\|y_i^*-v_i^*\|$$

and

$$\|\bar{v}_i^* - v_i^*\| \leq \|v^* - v_i^*\| + \delta_\Omega(u, v^*) + \frac{\psi\big(\frac{1}{i}d(x_i, (\partial f)^{-1}(v_i^*))\big)}{d(x_i, (\partial f)^{-1}(v_i^*))}\|u - \bar{x}_i\| \tag{3.26}$$

for all $(u, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$. Hence, by (3.21), (3.22) and $(x_i, v_i^*) \to (\bar{x}, 0)$, one has

$$0 < d(x_i, (\partial f)^{-1}(v_i^*))) \leq \frac{i}{i-1} d(\bar{x}_i, (\partial f)^{-1}(v_i^*)) \tag{3.27}$$

and

$$\bar{v}_i^* \neq v_i^* \text{ and } (\bar{x}_i, \bar{v}_i^*) \to (\bar{x}, 0).$$

It follows from (3.27) and the convexity of ψ that

$$0 < \frac{\psi\big(\frac{1}{i}d(x_i, (\partial f)^{-1}(v_i^*))\big)}{\frac{1}{i}d(x_i, (\partial f)^{-1}(v_i^*))} \leq \frac{\psi\big(\frac{1}{i-1}d(\bar{x}_i, (\partial f)^{-1}(v_i^*))\big)}{\frac{1}{i-1}d(\bar{x}_i, (\partial f)^{-1}(v_i^*))} \leq \psi'_+(d(\bar{x}_i, (\partial f)^{-1}(v_i^*)))$$

for all $i > 1$. This and (3.26) imply that

$$\|\bar{v}_i^* - v_i^*\| \leq \|v^* - v_i^*\| + \delta_\Omega(u, v^*) + \frac{1}{i}\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(v_i^*))\big)\|u - \bar{x}_i\|$$

for all $(u, v^*) \in \mathbb{R}^n \times \mathbb{R}^n$. Hence,

$$\begin{aligned}(0,0) &\in \{0\} \times \partial\| \cdot - v_i^*\|(\bar{v}_i^*) + \partial\delta_\Omega(\bar{x}_i, \bar{v}_i^*) + \frac{1}{i}\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(v_i^*))\big)B_{\mathbb{R}^n} \times \{0\} \\ &= \{0\} \times \Big\{\frac{\bar{v}_i^* - v_i^*}{\|\bar{v}_i^* - v_i^*\|}\Big\} + N(\Omega, (\bar{x}_i, \bar{v}_i^*)) + \frac{1}{i}\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(v_i^*))\big)B_{\mathbb{R}^n} \times \{0\},\end{aligned}$$

and so there exists $x_i^* \in B_{\mathbb{R}^n}$ such that

$$\left(\frac{1}{i}\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(v_i))\big)x_i^*, -\frac{\bar{v}_i^* - v_i^*}{\|\bar{v}_i^* - v_i^*\|}\right) \in N(\Omega, (\bar{x}_i, \bar{v}_i^*)). \tag{3.28}$$

Since $(\bar{x}_i, \bar{v}_i^*) \to (\bar{x}, 0)$, (3.18) implies that

$$N(\Omega, (\bar{x}_i, \bar{v}_i^*)) = N(\mathrm{gph}(\partial f), (\bar{x}_i, \bar{v}_i^*))$$

for all sufficiently large i. Hence, by (3.28),

$$\frac{1}{i}\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(v_i^*))\big)x_i^* \in \partial^2 f(\bar{x}_i, \bar{v}_i^*)\left(\frac{\bar{v}_i^* - v_i^*}{\|\bar{v}_i^* - v_i^*\|}\right)$$

for all sufficiently large i. Let $h_i^* := \bar{v}_i^* - v_i^*$. Then, $v_i^* = \bar{v}_i^* - h_i^*$,

$$z_i^* := \frac{1}{i}\|h_i^*\|\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(\bar{v}_i^* - h_i^*))\big)x_i^* \in \partial^2 f(\bar{x}_i, \bar{v}_i^*)(h_i^*)$$

and so

$$\begin{aligned}\langle z_i^*, h_i^*\rangle &= \frac{1}{i}\|h_i^*\|\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(\bar{v}_i^* - h_i^*))\big)\langle x_i^*, h_i^*\rangle \\ &\leq \frac{1}{i}\|h_i^*\|^2\psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(\bar{v}_i^* - h_i^*))\big).\end{aligned}$$

Noting that $0 < \psi'_+\big(d(\bar{x}_i, (\partial f)^{-1}(\bar{v}_i^* - h_i^*)\big)$, it follows from (3.16) that $\kappa \leq \frac{1}{i}$ for all sufficiently large i, a contradiction. Therefore, there exist $\tilde{\kappa}, \tilde{\tau}, \tilde{r} \in (0, +\infty)$ such that (3.20) holds for all $(u, v^*) \in B(\bar{x}, \tilde{r}) \times (\partial f(B(\bar{x}, \tilde{r})) \cap B(0, \tilde{r}))$.

Let $r' \in (0,\ \tilde{r})$. We claim that there exists $\tilde{\delta} \in (0,\ r')$ such that $B(0,\tilde{\delta}) \subset \partial f(B[\bar{x},r'])$. Granting this, one has

$$B(\bar{x},\tilde{\delta}) \times B(0,\tilde{\delta}) \subset B(\bar{x},\tilde{r}) \times (\partial f(B(\bar{x},\tilde{r}) \cap B(0,\tilde{r})).$$

This and (3.20) imply that ∂f is metrically ψ-regular at $(\bar{x},0)$. It remains to show that there exists $\tilde{\delta} \in (0,\ r')$ such that $B(0,\tilde{\delta}) \subset \partial f(B[\bar{x},r'])$. Indeed, if this is not the case, there exists a sequence $\{y_k^*\}$ converging to 0 such that each $y_k^* \notin \partial f(B[\bar{x},r'])$. Noting that $\partial f(B[\bar{x},r'])$ is closed (thanks to the compactness of $B[\bar{x},r']$ and the closedness of Ω), there exists $w_k^* \in \partial f(B[\bar{x},r'])$ such that

$$0 < \|y_k^* - w_k^*\| = d(y_k^*, \partial f(B[\bar{x},r'])) \le \|y_k^*\| \to 0, \tag{3.29}$$

and so $w_k^* \to 0$. It follow from (3.20) that

$$\psi(\tilde{\kappa} d(\bar{x},(\partial f)^{-1}(w_k^*))) \le \tilde{\tau} d(w_k^*, \partial f(\bar{x})) \le \tilde{\tau}\|w_k^*\| \to 0$$

Hence, $\tilde{\kappa} d(\bar{x},(\partial f)^{-1}(w_k^*)) \to 0$ and so there exists $a_k \in (\partial f)^{-1}(w_k^*)$ such that $a_k \to \bar{x}$. On the other hand, by the equality of (3.29), one has

$$\langle y_k^* - w_k^*, y^* - w_k^*\rangle \le d(y_k^*, \partial f(B[\bar{x},r']))\|y^* - w_k^*\| \le \|y^* - w_k^*\|^2 \quad \forall y^* \in \partial f(B[\bar{x},r']).$$

Hence

$$\langle (0, y_k^* - w_k^*), (x,y^*) - (a_k, w_k^*)\rangle \le \|(x,y^*) - (a_k,w_k^*)\|^2$$

for all $(x,y^*) \in \mathrm{gph}(\partial f) \cap (B[\bar{x},r'] \times \mathbb{R}^n)$. This implies that

$$(0, y_k^* - w_k^*) \in N(\mathrm{gph}(\partial f) \cap B[\bar{x},r'] \times \mathbb{R}^n), (a_k, w_k^*)).$$

Since (a_k, w_k^*) is an interior point of $B[\bar{x},r'] \times \mathbb{R}^n$ for all k large enough, $(0, y_k^* - w_k^*) \in N(\mathrm{gph}(\partial f), (a_k, w_k^*))$, namely $0 \in \partial^2 f(a_k, w_k^*)(w_k^* - y_k^*)$. It follows from (3.16) that

$$\kappa\|y_k^* - w_k^*\|^2 \psi'_+(d(a_k, (\partial f)^{-1}(y_k^*))) \le \langle 0, y_k^* - w_k^*\rangle = 0.$$

By the first inequality of (3.29), one has $\psi'_+(d(a_k,(\partial f)^{-1}(y_k^*))) = 0$. Since ψ is a convex admissible function, $d(a_k,(\partial f)^{-1}(y_k^*)) = 0$, and so $y_k^* \in \partial f(a_k)$. This contradicts that $a_k \to \bar{x}$ and $y_k^* \notin \partial f(B[\bar{x},r'])$. The proof is complete. $\square$

The proof of Theorem 3.5 follows the line of the one of [24, Proposition 6.1] which requires that the graph $\mathrm{gph}(\partial f)$ is a closed set.

The following corollary is immediate from Proposition 3.4 and Theorem 3.5 and can be regarded as an extension of Theorem 1.2 (because the convexity of f implies trivially its ψ-prox-regularity and ξ-D-hypomonotonicity at $(\bar{x},0)$)).

Corollary 3.6. *Let ψ be a convex admissible function and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lower semicontinuous function such that f is ψ-prox-regular at $(\bar{x},0) \in \mathrm{gph}(\partial f)$. Suppose that ∂f is ξ-D-hypomonotone at $(\bar{x},0)$, where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $\lim_{t\to 0^+} \xi(t) = 0$. Further suppose that that there exist $\kappa, r \in (0,+\infty)$ such that (3.16) holds for all $(x,v,h) \in (\mathrm{gph}(\partial f) \times \mathbb{R}^n) \cap B(\bar{x},r) \times B(0,r) \times B(0,r)))$ and $z \in \partial^2 f(x,v)(h)$. Then, f has φ-SLWP at $\bar{x}$ with $\varphi(t) := \int_0^t \psi(t)dt$.*

In the case when $\psi(t) = t$ for all $t \ge 0$, some authors (cf. [9, 13, 16, 19]) considered second-order subdifferential characterizations for f to have φ-*SLWP* at $\bar{x}$ with $\varphi(t) := \int_0^t \psi(t)dt$ (under the name of tilt stable minimizer). It is worth mentioning that (3.16) is equivalent to the positive definiteness of $\partial^2 f(x,v)$ when $\psi(t) = t$ for all $t \ge 0$.

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