CRITICAL POINTS AND POINT DERIVATIONS OF LIPSCHITZ FUNCTIONS

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Abstract. For an open subset $U \subset \mathbb{R}^n$ and a point $x_0 \in U$, Newns and A. Walker (see [6]) stated in the appendix of their paper a coordinate free characterization of critical points of $C^\infty$-functions, which can also be seen as an algebraic characterization of the algebra of all $C^\infty$-functions, which are defined on $U$ and have $x_0 \in U$ as a critical point. In this paper, we study the characterization of N. Newns and A. Walker for the Banach algebra $\text{Lip}(X,d)$ of Lipschitz functions which are defined on an open subset $X \subset E$ of an real normed vector space $(E,\|\cdot\|)$. This paper is a continuation of our previous work, which was already published partially in [7].

1. Introduction

Let $U$ be an open subset of $\mathbb{R}^n$, $C^r(U)$ the real algebra of all $C^r$-functions, $r \geq 2$, defined on $U$ and $x_0 \in U$. In 1956 N. Newns and A. Walker ([6]) gave a purely algebraic characterization of the subalgebra

$$C^\infty_{x_0}(U) = \{ f \in C^r(U) \mid \nabla f \bigg|_{x_0} = 0 \} \subset C^r(U)$$

for all those functions, which have $x_0 \in U$ as a critical point, i.e. for which the gradient vanishes in $x_0$. They showed that $C^\infty_{x_0}(U)$ is the intersection of all maximal subalgebras of $C^\infty(U)$ which contain the ideal of all functions, which vanish in $x_0$ of "second order".

We assume throughout our paper that every subalgebra contains the unit element. This paper is a continuation of a previously published paper [7] by the two second authors in Comentations Mathematica.

We begin our paper with a short description of the Banach algebra $\text{Lip}(X,d)$ of Lipschitz functions on a metric space $(X,d)$. For these algebras D. R. Sherbert [8] gave an algebraic characterization of all point derivations. In particularly Sherbert proved ([8], Lemma 9.4) that all point derivations for Lipschitz functions are given by the evaluation of sequences of special slopes of a Lipschitz function by Banach limits. In the last two sections new results are presented which could be of interest for readers from analysis. The previous sections however contain a discussion of results known in the literature.

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2. The Lipschitz algebra

We follow the presentation given in [8]. Let \((X, d)\) be a metric space. Then a function \(f : X \to \mathbb{R}\) is Lipschitzian if there exists a \(K \geq 0\) such that for all \(x, y \in X\) the inequality \(|f(x) - f(y)| \leq Kd(x, y)\) holds. The set of all bounded Lipschitz functions defined on \((X, d)\) is a real algebra and will be denoted by Lip\((X, d)\). For \(f \in \text{Lip}(X, d)\) the following two expressions exist:

\[
\|f\|_d := \sup\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\} \quad \text{and} \quad \|f\|_{\infty} := \sup\{|f(x)| : x \in X\}.
\]

Now

\[
\| \cdot \| : \text{Lip}(X, d) \to \mathbb{R}_+
\]

with

\[
\|f\| = \|f\|_d + \|f\|_{\infty}
\]

is a norm, and in [1] it is shown, that \((\text{Lip}(X, d), \| \cdot \|)\) is always the dual space of some normed linear space and hence complete. Moreover, \((\text{Lip}(X, d), \| \cdot \|)\) is also a Banach algebra, i.e, for all \(f, g \in \text{Lip}(X, d)\) the inequality \(|fg| \leq \|f\| \cdot \|g\|\) holds. This can be seen as follows: For \(f, g \in \text{Lip}(X, d)\) and \(x, y \in X, x \neq y\) the inequality

\[
\frac{|(fg)(x) - (fg)(y)|}{d(x, y)} \leq |f(x)| \frac{|g(x) - g(y)|}{d(x, y)} + |g(y)| \frac{|f(x) - f(y)|}{d(x, y)}
\]

implies

\[
\|fg\|_d \leq \|f\|_{\infty} \cdot \|g\|_d + \|g\|_{\infty} \cdot \|f\|_d
\]

and therefore

\[
\|fg\| = \|fg\|_{\infty} + \|fg\|_d \\
\leq \|f\|_{\infty} \cdot \|g\|_{\infty} + \|f\|_{\infty} \cdot \|g\|_d + \|g\|_{\infty} \cdot \|f\|_d \\
= \|f\|_{\infty} \cdot (\|g\|_{\infty} + \|g\|_d) + \|g\|_{\infty} \cdot \|f\|_d \\
\leq \|f\| \cdot \|g\|.
\]

The Banach algebra \((\text{Lip}(X, d), \| \cdot \|)\) will be called the Lipschitz algebra on \((X, d)\). The unit element is the characteristic function on \(X\), which is denoted by \(1\).

3. Point derivations for Lipschitz functions

A continuous linear functional which satisfies the Leibniz rule is called a point derivation.

**Definition 3.1.** Let \((X, d)\) be a metric space and \(x_0 \in X\). A continuous linear functional \(l \in \text{Lip}(X, d)^{\prime}\) is said to be a point derivation at \(x_0 \in X\), if and only if for all \(f, g \in \text{Lip}(X, d)\) the Leibniz rule, i.e.,

\[
l(fg) = f(x_0) \cdot l(g) + g(x_0) \cdot l(f)
\]

is satisfied at \(x_0 \in X\).
Let $l \in \text{Lip}(X, d)$ we denote the constant function with value one and denote furthermore with ideal $\mathfrak{m}(x_0) = \{ f \in \text{Lip}(X, d) \mid f(x_0) = 0 \}$ of functions, vanishing in $x_0 \in X$ and its algebraic square by

$$
\mathfrak{m}^2(x_0) = \{ f \in \text{Lip}(X, d) \mid f = \sum_{i=1}^{k} g_i \cdot h_i, g_i, h_i \in \mathfrak{m}(x_0), \; i = 1, \ldots, k, \; k \geq 1 \}.
$$

I. Singer and J. Wermer have shown in [9] that point derivations can indeed be characterized by their values for the unit function and their values on the latter ideal.

**Proposition 3.2.** Let $(X, d)$ be a metric space and $x_0 \in X$.

Then for a continuous linear functional $l \in \text{Lip}(X, d)'$ holds

i) $l(1) = 0$,

ii) $l(\mathfrak{m}^2(x_0)) = 0$.

if and only if for all $f, g \in \text{Lip}(X, d)$ Leibniz’s rule

$$
l(fg) = f(x_0) \cdot l(g) + g(x_0) \cdot l(f)
$$

is satisfied at the point $x_0 \in X$.

**Proof.** “$\Leftarrow$” Let us assume, that the functional $l \in \text{Lip}(X, d)'$ satisfies Leibniz’s rule.

Since $1^2 = 1$, Leibniz’s rule implies for $f = g = 1$ that $l(1) = 2l(1)$, which means that $l(1) = 0$.

Now assume, that $f, g \in \mathfrak{m}(x_0)$. Then $l(fg) = f(x_0) \cdot l(g) + g(x_0) \cdot l(f) = 0$, and since the functional $l$ is continuous, it follows that $l(\mathfrak{m}^2(x_0)) = 0$.

“$\Rightarrow$” Let us assume, that the continuous linear functional $l$ satisfies conditions i) and ii). Then for every $f, g \in \text{Lip}(X, d)$ holds

$$
l(fg) = l(fg - f(x_0)g(x_0)1)
$$

$$
= l\left((f - f(x_0)1) \cdot (g - g(x_0)1) + f(x_0)(g - g(x_0)1) + g(x_0)(f - f(x_0)1)\right)
$$

$$
= l((f - f(x_0)1) \cdot (g - g(x_0)1)) + f(x_0) \cdot l(g - g(x_0)1) + g(x_0) \cdot l(f - f(x_0)1)
$$

$$
= f(x_0) \cdot l(g - g(x_0)1) + g(x_0) \cdot l(f - f(x_0)1)
$$

$$
= f(x_0) \cdot l(g) + g(x_0) \cdot l(f),
$$

since $(f - f(x_0)1) \cdot (g - g(x_0)1) \in \mathfrak{m}^2(x_0)$. \qed

The linear space of all point derivations at $x_0 \in X$ will be denoted by $\text{Der}_{x_0}(\text{Lip}(X, d))$ and is a weak-*-closed subspace of $\text{Lip}(X, d)'$. D. R. Sherbert determines in [8] all point derivations in $\text{Lip}(X, d)$. We will now repeat his construction:

Let us consider the real Banach space

$$
1^\infty := \{ x := (x_n)_{n \in \mathbb{N}} \mid (x_n)_{n \in \mathbb{N}} \text{ bounded sequence } \}
$$

endowed with the supremum norm $\|x\|_\infty := \sup_{n \in \mathbb{N}} |x_n|$. Let $c \subset 1^\infty$ denote the closed subset of all convergent sequences, and $\lim : c \longrightarrow \mathbb{R}$ the continuous linear
functional which assigns to every convergent sequence its limit. We consider a norm-preserving Hahn-Banach extension “LIM” of the functional “lim” to $l^\infty$ as indicated:

\[
\begin{array}{ccc}
\mathcal{C} & \subset & l^\infty \\
\text{lim} & \uparrow & \text{LIM} \\
& & \mathbb{R}
\end{array}
\]

with the following additional properties:

i) $\text{LIM}_{n \to \infty} x_n = \text{LIM}_{n \to \infty} x_{n+1}$,

ii) $\liminf_{n \to \infty} x_n \leq \text{LIM}_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n$,

where we used the notation $\text{LIM}_{n \to \infty} x_n = \text{LIM}(x)$ for $x := (x_n)_{n \in \mathbb{N}} \in l^\infty$. This functionals “LIM” are called translation invariant Banach limits. For its construction, we refer to [4], Chapter II.4, Exercise 22.

We will denote the linear space of all translation invariant Banach limits by $\text{Lim}(l^1)'$ and for a single Banach limit $\text{LIM}$:

Let $x_0 \in X$ be a nonisolated point of a metric space $(X,d)$ and $w := (x_n,y_n)_{n \in \mathbb{N}} \subset \{(s,t) \in X \times X \mid s \neq t\}$ which converges to the point $(x_0,x_0)$. Then for the sequence of slopes of $f$ given by $(\frac{f(y_n) - f(x_n)}{d(y_n,x_n)})_{n \in \mathbb{N}}$, which is bounded, the mapping

\[
T_w : \text{Lip}(X,d) \to l^\infty \text{ with } T_w(f) := \left( \frac{f(y_n) - f(x_n)}{d(y_n,x_n)} \right)_{n \in \mathbb{N}}
\]

is a continuous linear operator, since $\|T_w(f)\|_{\infty} \leq \|f\|_d \leq \|f\|$.

Now we repeat the proof of D. Sherbert, that for a translation invariant Banach limit the continuous linear functional

\[
D_w : \text{Lip}(X,d) \to \mathbb{R} \text{ with } D_w(f) = \text{LIM}(T_w(f))
\]

is a point derivation at $x_0 \in X$ (see [8] Lemma 9.4). For abbreviation let us put $\Delta := \{(s,t) \in X \times X \mid s = t\}$.

**Proposition 3.3.** Let $x_0 \in X$ be a nonisolated point of a metric space $(X,d)$ and $w := (x_n,y_n)_{n \in \mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s,t) \in X \times X \mid s \neq t\}$ a sequence, which converges to the point $(x_0,x_0)$. Then for every translation invariant Banach limit $\text{LIM} : l^\infty \to \mathbb{R}$ the continuous linear functional

\[
D_w : \text{Lip}(X,d) \to \mathbb{R} \text{ with } D_w(f) = \text{LIM}(T_w(f))
\]

is a point derivation for the Lipschitz algebra $(\text{Lip}(X,d), \| \cdot \|)$ at $x_0 \in X$.

**Proof.** First observe, that for every convergent sequence $(a_n)_{n \in \mathbb{N}} \in \mathcal{C}$ and every bounded sequence $(b_n)_{n \in \mathbb{N}} \in l^\infty$ the formula $\text{LIM}_{n \to \infty}(a_n \cdot b_n) = \lim_{n \to \infty} a_n \cdot \text{LIM}_{n \to \infty} b_n$ holds.
Proposition 3.4. Let \(\alpha := \lim_{n \to \infty} a_n\). Then \(\lim_{n \to \infty}(a_n \cdot b_n - \alpha b_n) = 0\), since \((a_n \cdot b_n - \alpha b_n)_{n \in \mathbb{N}}\) is a sequence converging to zero and hence \(\lim_{n \to \infty}(a_n \cdot b_n) = \alpha \lim_{n \to \infty} b_n\).

Now let \(f, g \in \text{Lip}(X, d)\) be given. From the above observation follows that
\[
D_w(fg) = \lim_{n \to \infty} \left( \frac{(fg)(y_n) - (fg)(x_n)}{d(y_n, x_n)} \right)
\]
\[
= \lim_{n \to \infty} \left( \frac{g(y_n) - g(x_n)}{d(y_n, x_n)} f(y_n) - \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} g(x_n) \right)
\]
\[
= \left( \lim_{n \to \infty} \frac{g(y_n) - g(x_n)}{d(y_n, x_n)} f(y_n) \right) + \left( \lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} g(x_n) \right)
\]
\[
\begin{align*}
D_w(fg) &= f(x_0) \lim_{n \to \infty} \frac{g(y_n) - g(x_n)}{d(y_n, x_n)} + g(x_0) \lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} \\
&= f(x_0) \lim_{n \to \infty} (T_w(g)) + g(x_0) \lim_{n \to \infty} (T_w(f)) \\
&= f(x_0) D_w(g) + g(x_0) D_w(f)
\end{align*}
\]

Since \(D_w\) is continuous, it is a point derivation at \(x_0 \in X\).

Now the following representation theorem holds (see [8], Theorem 9.5).

**Proposition 3.4.** Let \(x_0 \in X\) be a nonisolated point of a metric space \((X, d)\) and \(W_{x_0} := \{ w := (x_n, y_n)_{n \in \mathbb{N}} \subset X \times X \setminus \Delta \mid \lim x_n = \lim y_n = x_0 \}\). Moreover let \(\lim : \text{Lip} \to \mathbb{R}\) be a fixed translation invariant Banach limit. Then
\[
\text{Der}_{x_0}(\text{Lip}(X, d)) = \text{cl} (\text{span}\{D_w = \lim_{n \to \infty} T_w \mid w \in W_{x_0} \})
\]
where ”\(\text{cl span}\)” denotes the weak-* closure of the linear hull in \(\text{Lip}(X, d)\).

4. Characterization of Newns and Walker

Newns and Walker [6] proved in 1956 a coordinate free characterization of critical points of \(C^\infty\)-functions intrinsically in algebraic terms:

Let \(U \subseteq \mathbb{R}^n\) be an open subset and \(x_0 \in U\) and let \(\delta_{x_0} : C^\infty(U) \to \mathbb{R}\) be given by \(\delta_{x_0}(f) = f(x_0)\). As above consider the ideal \(\mathfrak{m}(x_0)\) of the algebra \(A = C^\infty(U)\) given by \(\mathfrak{m}(x_0) = \{ f \in A \mid \delta_{x_0}(f) = 0 \}\). Let \(\mathfrak{m}^2(x_0)\) be the algebraic square of the ideal \(\mathfrak{m}(x_0)\), i.e.
\[
\mathfrak{m}^2(x_0) = \{ f \in A \mid f = \sum_{i=1}^k g_i \cdot h_i, g_i, h_i \in \mathfrak{m}(x_0), i = 1, \ldots, k, \ k \geq 1 \}.
\]

Consider now the intersection of all maximal subalgebras \(a \subset A\), which contain \(\mathfrak{m}^2(x_0)\). In analogy to Frattini groups, (Frattini (1885)), this set is called a Frattini algebra of \(A\) at \(x_0 \in X\) and denoted by
\[
\mathcal{F}(A)(x_0) = \bigcap \{ a \mid \mathfrak{m}^2(x_0) \subset a \subset A, \ a \ is \ a \ maximal \ closed \ subalgebra \}
\]
In this notation they proved in the appendix of their paper [6]:

**Theorem 4.1.** Let \(U \subseteq \mathbb{R}^n\) be an open subset and let \(x_0 \in U\). A function \(f \in A = C^\infty(U)\) has a critical point in \(x_0 \in U\) if and only if \(f(x) - f(x_0) \in \mathcal{F}(A)(x_0)\).
Let us point out, that the proof of this Theorem is based on directional derivatives which are not in a close analogy to point derivations. This can for instance be seen in the case of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) := x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. This function is differentiable in $x_0 = 0$, but has in $x_0 = 0$ point derivatives with values in the interval $[-1, 1]$ (see also the remark on p. 266 in [8]).

5. Critical points of Lipschitz functions

In this section we consider the algebraic characterization of critical points for Lipschitz functions. Therefore we will consider for a metric space $(X, d)$ the algebra $A = \text{Lip}(X, d)$ of all bounded Lipschitz functions defined on $(X, d)$. Moreover we will assume that $x_0 \in X$ is a nonisolated point of a metric space $(X, d)$ and that

$$\delta_{x_0} : \text{Lip}(X, d) \rightarrow \mathbb{R}$$

is defined by $\delta_{x_0}(f) = f(x_0)$.

As previously we consider for $A = \text{Lip}(X, d)$ the ideal $\mathfrak{m}(x_0) = \{ f \in A \mid \delta_{x_0}(f) = 0 \}$ and its algebraic square

$$\mathfrak{m}^2(x_0) = \{ f \in A \mid f = \sum_{i=1}^{k} g_i \cdot h_i, g_i, h_i \in \mathfrak{m}(x_0), i = 1, \ldots, k, k \geq 1 \}.$$

Following the characterization given in Theorem 4.1 we say that an element $f \in \text{Lip}(X, d)$ has an "(algebraic) critical point" at $x_0 \in X$ if and only if $f(x) - f(x_0) \in \mathcal{F}(A)(x_0)$.

First we show:

**Proposition 5.1.** Let $(X, d)$ be a metric space and $x_0 \in X$. Then for every proper subalgebra $a$ with $1 \in a$ and $\mathfrak{m}^2(x_0) \subset a \subset A = \text{Lip}(X, d)$ there exists a nontrivial point derivation $D \in \text{Der}_{x_0}(\text{Lip}(X, d)) \setminus \{ 0 \}$ with

$$a \subset \ker(D) = \{ f \in \text{Lip}(X, d) \mid D(f) = 0 \}.$$

Moreover $\ker(D) = \{ f \in \text{Lip}(X, d) \mid D(f) = 0 \}$ is a maximal subalgebra.

**Proof.** Let $a \subset A = \text{Lip}(X, d)$ be a proper subalgebra. Then $a$ is also a proper linear subspace of $\text{Lip}(X, d)$ and hence there exists a hyperspace $F$, i.e. a linear subspace of codimension 1 with $a \subset F$. Since $1 \in a$ and $\mathfrak{m}^2(x_0) \subset a \subset A = \text{Lip}(X, d)$ we have by Proposition 3.2 that the canonical projection $\pi : \text{Lip}(X, d) \rightarrow \text{Lip}(X, d)/F \simeq \mathbb{R}$ is a point derivation at $x_0 \in X$.

Now change the notation and put $D = \pi$. Then we show that $\ker(D)$ is a subalgebra of $\text{Lip}(X, d)$. Namely if $f, g \in \ker(D)$ then obviously $f + g \in \ker(D)$ and from the Leibniz rule follows $f \cdot g \in \ker(D)$ since $D(fg) = f(x_0)Dg + g(x_0)Df = 0$.

From the construction follows, that $\ker(D)$ is maximal. \hfill $\square$

Now we use the explicit representations of point derivations given in Proposition 3.3 and Proposition 3.4.

Let us denote by $\mathcal{F}(\text{Lip}(X, d))(x_0)^0 = \bigcap_{D_w = \text{LIM}(T_w), w \in W_{x_0}, \text{LIM} \in \text{Lim}} \ker(D_w)$. Then we have:

**Proposition 5.2.** Let $D \in \text{Der}_{x_0}(\text{Lip}(X, d))$ be a point derivation. Then for every $f \in \mathcal{F}(\text{Lip}(X, d))(x_0)^0$ holds $D(f) = 0$. 

Proof. By Proposition 3.4 we have
\[ \text{Der}_{x_0}((\text{Lip}(X,d))(x_0)) = \text{cl} (\text{span}\{ D_w = \text{LIM}(T_w) \mid w \in W_{x_0} \}) \]
where "cl span" denotes the weak-*-closure of the linear hull in \( \text{Lip}(X,d)' \).

By the above statements it follows immediately from the definition of \( \mathcal{F}(\text{Lip}(X,d))(x_0)^0 \).

Now assume that \( D \in \text{Der}_{x_0}((\text{Lip}(X,d))(x_0)) \) is a weak-*-accumulation point of \( \text{span}\{ D_w = \text{LIM}(T_w) \mid w \in W_{x_0} \} \). Then for every \( \varepsilon > 0 \) and arbitrary points \( f_1, \ldots, f_k \in \text{Lip}(X,d) \) there exists a \( D_w = \text{LIM}(T_w) \) with \( w \in W_{x_0} \) and \( \text{LIM} \in \text{Lim} \) such that \( D_w - D \in U_{f_1,\ldots,f_k,\varepsilon} \) where \( U_{f_1,\ldots,f_k,\varepsilon} = \{ l \in \text{Lip}(X,d)' \mid |l(f_i)| < \varepsilon \} \) is a weak-*-neighborhood of \( 0 \in \text{Lip}(X,d)' \).

Let us now assume that our statement is not true. Then there exists an element \( f^* \in \mathcal{F}(\text{Lip}(X,d))(x_0)^0 \) with \( |D(f^*)| = c > 0 \). Now let \( \varepsilon = \frac{c}{2} \) and choose arbitrary elements \( f_2, \ldots, f_k \in \text{Lip}(X,d) \). Then for the weak-*-neighborhood of \( 0 \in \text{Lip}(X,d)' \) given by \( U_{f^*,f_2,\ldots,f_k,\varepsilon} \) there exists a \( D_w = \text{LIM}(T_w) \) such that \( D_w - D \in U_{f^*,f_2,\ldots,f_k,\varepsilon} \). For the element \( f^* \in \mathcal{F}(\text{Lip}(X,d))(x_0)^0 \) this implies: \( c = |D(f^*)||D(f^*) - D_w(f^*)| < \varepsilon = \frac{c}{2} \) which is a contradiction.

The above two propositions imply that \( \mathcal{F}(\text{Lip}(X,d))(x_0) = \mathcal{F}(\text{Lip}(X,d))(x_0)^0 \).

Finally we prove our main result:

**Theorem 5.3.** Let \( x_0 \in X \) be a nonisolated point of a metric space \( (X,d) \). Then \( f \in \text{Lip}(X,d) \) has an algebraic critical point in \( x_0 \in X \) if and only if for every point sequence \( w := (x_n, y_n)_{n \in \mathbb{N}} \subset (X \times X) \setminus \Delta = \{ (s, t) \in X \times X \mid s \neq t \} \) which converges to the point \( (x_0, x_0) \), the corresponding sequence of slopes of \( f \) given by \( T_w(f) := \left( \frac{f(y_n) - f(x_n)}{d(y_n,x_n)} \right)_{n \in \mathbb{N}} \in \mathbb{R}^\infty \) converges to zero.

**Proof.** By the above statements \( f \in \text{Lip}(X,d) \) has an algebraic critical point in \( x_0 \in X \) if and only if \( D(f) = 0 \) for all point derivations \( D \in \text{Der}_{x_0}((\text{Lip}(X,d))(x_0)) \setminus \{ 0 \} \).

Now by the characterization of G.G. Lorentz [5] on almost convergent sequences, that are bounded real sequence \( (a_n)_{n \in \mathbb{N}} \) for which all translation invariant Banach limits have the same value \( L \in \mathbb{R} \), we have that for every \( \varepsilon > 0 \) there exists a \( p_0 \in \mathbb{N} \) so that for all \( p > p_0 \) and for all \( n \in \mathbb{N} \) the condition
\[
|a_n + \ldots + a_{n+p-1}|_p - L < \varepsilon
\]
holds. In our case \( L = 0 \) which means that for every \( \varepsilon > 0 \) there exists a \( p_0 \in \mathbb{N} \) so that for all \( p > p_0 \) and for all \( n \in \mathbb{N} \) the condition
\[
\left| \frac{f(y_n) - f(x_n)}{d(y_n,x_n)} \right| + \ldots + \left| \frac{f(y_{n+p-1}) - f(x_{n+p-1})}{d(y_{n+p-1},x_{n+p-1})} \right| < \varepsilon
\]
holds. Since condition \( (*) \) has to hold for every point sequence \( w := (x_n, y_n)_{n \in \mathbb{N}} \subset (X \times X) \setminus \Delta = \{ (s, t) \in X \times X \mid s \neq t \} \) which converges to the point \( (x_0, x_0) \), this implies, that the corresponding sequence of slopes \( \left( \frac{f(y_n) - f(x_n)}{d(y_n,x_n)} \right)_{n \in \mathbb{N}} \in \mathbb{R}^\infty \) converges to zero.
The converse direction is obvious, because if for every point sequence \( w := (x_n, y_n)_{n \in \mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s, t) \in X \times X \mid s \neq t\} \) which converges to the point \((x_0, x_0)\), the corresponding sequence of slopes of \( f \) given by \( T_w(f) := \left( \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} \right)_{n \in \mathbb{N}} \in \mathbb{R}^\infty \) converges to zero, then condition (*) is satisfied. □

For Lipschitz functions on open sets in normed linear spaces, this implies that the gradient is zero, as we shall prove now:

**Proposition 5.4.** Let \((E, \| \cdot \|_E)\) be a real normed vector space, \(U \subset E\) an open subset, \(x_0 \in U\), and \(f \in \text{Lip}(U, d)\), where the metric \(d\) on \(U\) is induced by the norm. If \(D(f) = 0\) holds for every \(D \in \text{Der}_{x_0}(\text{Lip}(U, d))\) then \(f \in \text{Lip}(U, d)\) is Fréchet differentiable at \(x_0 \in X\) and its differential vanishes, i.e., \(df|_{x_0} = 0\).

**Proof.** Let us assume that the function \(f \in \text{Lip}(U, d)\) is not differentiable in \(x_0 \in U\). Then for every \(l \in E'\) there exists an \(\varepsilon_l > 0\) such that for all \(\delta > 0\) there exists a vector \(h_\delta \in E\) with \(\|h_\delta\|_E \leq \delta\) and

\[
|f(x_0 + h_\delta) - f(x_0) - l(h_\delta)| > \varepsilon_l \|h_\delta\|_E.
\]

For the special case \(l = 0\in E'\) and \(\delta = \frac{1}{n}\) we get with \(h_n := h_\delta\) that there exists an \(\varepsilon_0 > 0\) such that for \(n \in \mathbb{N}\)

\[
\frac{|f(x_0 + h_n) - f(x_0)|}{\|h_n\|_E} > \varepsilon_0.
\]

Now choose a subsequence \((n_k)_{k \in \mathbb{N}}\) such that for all \(k \in \mathbb{N}\) the nominator \(f(x_0 + h_{n_k}) - f(x_0)\) has a constant sign, for instance \(f(x_0 + h_{n_k}) - f(x_0) > 0\).

For the sequence

\[
\hat{w} := (x_0, x_0 + h_{n_k})_{k \in \mathbb{N}} \in W_{x_0}
\]

holds then

\[
D_{\hat{w}}(f) = \lim(T_{\hat{w}}(f)) > \varepsilon_0,
\]

since every translation invariant Banach limit of a bounded sequence is greater or equal then the lower limit of this sequence. Hence there exists a point derivation at \(x_0 \in U\) with \(D_{\hat{w}}(f) \neq 0\), which is a contradiction. □

In the case of Lipschitz functions, the algebraic characterization of a critical is stronger than the critical point concept of Bonnisseau and Cornet [2] and of Clarke [3]. For instance the function \(f : \mathbb{R} \rightarrow \mathbb{R}\) defined by \(f(x) := x^2 \sin \frac{1}{x}\) for \(x \neq 0\) and \(f(0) = 0\) is differentiable in \(x_0 = 0\) with \(f'(x_0) = 0\), but \(x_0 = 0\) is not a critical point in the algebraic sense.

6. Conclusion

From the coordinate free characterization of Newns and Walker [6] of critical points of \(C^\infty\)-functions, we derive for Lipschitz functions on metric space a characterization of critical points, which is stronger than the concept of Clarke [3]. According to this characterization one has for a nonisolated point \(x_0 \in X\) of a metric space \((X, d)\) the following condition:

\[
\text{there exists a } \varepsilon > 0 \text{ such that for all } \delta > 0 \text{ there exists a vector } h_\delta \in E \text{ with } \|h_\delta\|_E \leq \delta \text{ and }
\]

\[
|f(x_0 + h_\delta) - f(x_0) - l(h_\delta)| > \varepsilon \|h_\delta\|_E.
\]
Namely, a Lipschitz function \( f \in \text{Lip}(X, d) \) has an algebraic critical point in \( x_0 \in X \) if and only if for every point sequence \( w := (x_n, y_n)_{n \in \mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s, t) \in X \times X \mid s \neq t\} \) which converges to the point \((x_0, x_0)\), the corresponding sequence of slopes of \( f \) given by \( T_w(f) := \left( \frac{f(y_n) - f(x_n)}{d(y_n, x_n)} \right)_{n \in \mathbb{N}} \) converges to zero.

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