Pure and Applied Functional Analysis
Volume 4, Number 1, 2019, 35–43

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

CRITICAL POINTS AND POINT DERIVATIONS OF LIPSCHITZ FUNCTIONS

JERZY GRZYBOWSKI, DIETHARD PALLASCHKE, AND RYSZARD URBAŃSKI

ABSTRACT. For an open subset $U \subset \mathbb{R}^n$ and a point $x_0 \in U$ N. Newns and A. Walker (see [6]) stated in the appendix of their paper a coordinate free characterization of critical points of C^∞-functions, which can also be seen as an algebraic characterization of the algebra of all C^∞-functions, which are defined on U and have $x_0 \in U$ as a critical point. In this paper, we study the characterization of N. Newns and A. Walker for the Banach algebra $\text{Lip}(X, d)$ of Lipschitz functions which are defined on an open subset $X \subset E$ of an real normed vector space $(E, \|\cdot\|)$. This paper is a continuation of our previous work, which was already published partially in [7].

1. Introduction

Let U be an open subset of $\mathbb{R}^n$, $\mathcal{C}^r(U)$ the real algebra of all C^r-functions, $r \geq 2$, defined on U and $x_0 \in U$. In 1956 N. Newns and A. Walker ([6]) gave a purely algebraic characterization of the subalgebra

$$\mathcal{C}^\infty_{x_0}(U) = \{f \in \mathcal{C}^r(U) \mid \nabla f\Big|_{x_0} = 0\} \subset \mathcal{C}^r(U)$$

for of all those functions, which have $x_0 \in U$ as a critical point, i.e. for which the gradient vanishes in x_0. They showed that $\mathcal{C}^\infty_{x_0}(U)$ is the intersection of all maximal subalgebras of $\mathcal{C}^\infty(U)$ which contain the ideal of all functions, which vanish in x_0 of *"second order"*.

We assume throughout our paper that every subalgebra contains the unit element. This paper is a continuation of a previously published paper [7] by the two second authors in Comentationes Mathematica.

We begin our paper with a short description of the Banach algebra $\text{Lip}(X, d)$ of Lipschitz functions on a metric space (X, d). For these algebras D. R. Sherbert [8] gave an algebraic characterization of all point derivations . In particularly Sherbert proved ([8], Lemma 9.4) that all point derivations for Lipschitz functions are given by the evaluation of sequences of special slopes of a Lipschitz function by Banach limits. In the last two sections new results are presented which could be of interest for readers from analysis. The previous sections however contain a discussion of results known in the literature.

2010 *Mathematics Subject Classification.* 49J52, 90B85, 90C46.
Key words and phrases. Lipschitz functions, point derivations, critical points.

2. The Lipschitz algebra

We follow the presentation given in [8]. Let (X, d) be a metric space. Then a function $f : X \longrightarrow \mathbb{R}$ is *Lipschitzian* if there exists a $K \geq 0$ such that for all $x, y \in X$ the inequality $|f(x) - f(y)| \leq Kd(x, y)$ holds. The set of all bounded Lipschitz functions defined on (X, d) is a real algebra and will be denoted by $\mathrm{Lip}(X, d)$. For $f \in \mathrm{Lip}(X, d)$ the following two expressions exist:

$$\|f\|_d := \sup\{\frac{|f(x) - f(y)|}{d(x, y)} \ : \ x, y \in X, \ x \neq y \ \} \text{ and } \|f\|_\infty := \sup\{|f(x)| \ : \ x \in X \ \}.$$

Now

$$\| \cdot \| : \mathrm{Lip}(X, d) \longrightarrow \mathbb{R}_+$$

with

$$\|f\| := \|f\|_d \ + \ \|f\|_\infty$$

is a norm, and in [1] it is shown, that $(\mathrm{Lip}(X, d), \| \cdot \|)$ is always the dual space of some normed linear space and hence complete. Moreover, $(\mathrm{Lip}(X, d), \| \cdot \|)$ is also a Banach algebra, i.e, for all $f, g \in \mathrm{Lip}(X, d)$ the inequality $\|fg\| \leq \|f\| \ \|g\|$ holds. This can be seen as follows: For $f, g \in \mathrm{Lip}(X, d)$ and $x, y \in X, \ x \neq y$ the inequality

$$\frac{|(fg)(x) - (fg)(y)|}{d(x, y)} \leq |f(x)| \frac{|g(x) - g(y)|}{d(x, y)} \ + \ |g(y)| \frac{|f(x) - f(y)|}{d(x, y)}$$

implies

$$\|fg\|_d \leq \|f\|_\infty \cdot \|g\|_d \ + \ \|g\|_\infty \cdot \|f\|_d$$

and therefore

$$\begin{aligned} \|fg\| &= \|fg\|_\infty \ + \ \|fg\|_d \\ &\leq \|f\|_\infty \cdot \|g\|_\infty \ + \ \|f\|_\infty \cdot \|g\|_d \ + \ \|g\|_\infty \cdot \|f\|_d \\ &= \|f\|_\infty \cdot (\|g\|_\infty \ + \ \|g\|_d) \ + \ \|g\|_\infty \cdot \|f\|_d \\ &\leq \|f\| \cdot \|g\|. \end{aligned}$$

The Banach algebra $(\mathrm{Lip}(X, d), \| \cdot \|)$ will be called the *Lipschitz algebra* on (X, d). The unit element is the characteristic function on X, which is denoted by $\mathbf{1}$.

3. Point derivations for Lipschitz functions

A continuous linear functional which satisfies the Leibniz rule is called a point derivation.

Definition 3.1. Let (X, d) be a metric space and $x_0 \in X$. A continuous linear functional $l \in \mathrm{Lip}(X, d)^{'}$ is said to be a point derivation at $x_0 \in X$, if and only if for all $f, g \in \mathrm{Lip}(X, d)$ the Leibniz rule, i.e.,

$$l(fg) \ = \ f(x_0) \cdot l(g) \ + \ g(x_0) \cdot l(f)$$

is satisfied at $x_0 \in X$.

With $\mathbf{1} \in \mathrm{Lip}(X,d)$ we denote the constant function with value one and denote furthermore with ideal $\mathbf{m(x_0)} = \{f \in \mathrm{Lip}(X,d) \mid f(x_0) = 0\}$ of functions, vanishing in $x_0 \in X$ and its algebraic square by

$$\mathbf{m}^2\mathbf{(x_0)} = \{f \in \mathrm{Lip}(X,d) \mid f = \sum_{i=1}^{k} g_i \cdot h_i \ , g_i, h_i \in \mathbf{m(x_0)},\ i = 1,...,k\ ,\ k \geq 1\}.$$

I. Singer and J. Wermer have shown in [9] that point derivations can indeed be characterized by their values for the unit function and their values on the latter ideal.

Proposition 3.2. *Let (X,d) be a metric space and $x_0 \in X$.*
Then for a continuous linear functional $l \in \mathrm{Lip}(X,d)^{'}$ holds

i) $l(\mathbf{1}) \ = \ 0,$
ii) $l\Big|\mathbf{m}^2\mathbf{(x_0)} \ = \ 0.$

if and only if for all $f, g \in \mathrm{Lip}(X,d)$ Leibniz's rule

$$l(fg) \ = \ f(x_0) \cdot l(g) \ + \ g(x_0) \cdot l(f)$$

is satisfied at the point $x_0 \in X$.

Proof. " $\Longleftarrow$ " Let us assume, that the functional $l \in \mathrm{Lip}(X,d)^{'}$ satisfies Leibniz's rule.

Since $\mathbf{1}^2 = \mathbf{1}$, Leibniz's rule implies for $f = g = \mathbf{1}$ that $l(\mathbf{1}) = 2l(\mathbf{1})$, which means that $l(\mathbf{1}) = 0$.
Now assume, that $f, g \in \mathbf{m(x_0)}$. Then $l(fg) \ = \ f(x_0) \cdot l(g) \ + \ g(x_0) \cdot l(f) = 0$, and since the functional l is continuous, it follows that $l|\mathbf{m}^2\mathbf{(x_0)} \ = \ 0$.

" $\Longrightarrow$ " Let us assume, that the continuous linear functional l satisfies conditions i) and ii). Then for every $f, g \in \mathrm{Lip}(X,d)$ holds

$$\begin{aligned} l(fg) \ &= \ l(fg - f(x_0)g(x_0)\mathbf{1}) \\ &= \ l((f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1}) \\ &\quad +f(x_0)(g - g(x_0)\mathbf{1}) + g(x_0)(f - f(x_0)\mathbf{1})) \\ &= \ l((f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1})) + f(x_0) \cdot l(g - g(x_0)\mathbf{1}) \\ &\quad +g(x_0) \cdot l(f - f(x_0)\mathbf{1}) \\ &= \ f(x_0) \cdot l(g - g(x_0)\mathbf{1}) + g(x_0) \cdot l(f - f(x_0)\mathbf{1}) \\ &= \ f(x_0) \cdot l(g) + g(x_0) \cdot l(f), \end{aligned}$$

since $(f - f(x_0)\mathbf{1}) \cdot (g - g(x_0)\mathbf{1}) \in \mathbf{m}^2\mathbf{(x_0)}$. $\square$

The linear space of all point derivations at $x_0 \in X$ will be denoted by $\mathrm{Der}_{x_0}(\mathrm{Lip}(X,d))$ and is a weak-*-closed subspace of $\mathrm{Lip}(X,d)^{'}$. D. R. Sherbert determines in [8] all point derivations in $\mathrm{Lip}(X,d)$. We will now repeat his construction:

Let us consider the real Banach space

$$\mathbf{l}^\infty := \{x := (x_n)_{n\in\mathbb{N}} \ \mid \ (x_n)_{n\in\mathbb{N}} \ \text{bounded sequence}\ \}$$

endowed with the *supremum norm* $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$. Let $\mathbf{c} \subset \mathbf{l}^\infty$ denote the closed subset of all convergent sequences, and $\lim : \mathbf{c} \longrightarrow \mathbb{R}$ the continuous linear functional which assigns to every convergent sequence its limit. We consider a norm-preserving Hahn-Banach extension "LIM" of the functional "lim" to $\mathbf{l}^\infty$ as indicated:

$$\begin{tikzcd} \mathbf{c} \arrow[r, "\subset"] \arrow[dr, "\lim"'] & \mathbf{l}^\infty \arrow[d, "\mathrm{LIM}"] \\ & \mathbb{R} \end{tikzcd}$$

with the following additional properties:

i) $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_{n+1}$,

ii) $\liminf_{n\to\infty} x_n \leq \mathrm{LIM}_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$,

where we used the notation $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}(x)$ for $x := (x_n)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$. This functionals "LIM" are called *translation invariant Banach limits.* For its construction , we refer to [4], Chapter II.4, Exercise 22.

We will denote the linear space of all translation invariant Banach limits by $\mathbf{Lim} \subset (\mathbf{l}^\infty)'$ and for a single Banach limit $\mathrm{LIM} \in \mathbf{Lim}$.

Let $x_0 \in X$ be a nonisolated point and $w := (x_n, y_n)_{n\in\mathbb{N}} \subset \{(s,t) \in X \times X \mid s \neq t\}$ which converges to the point (x_0, x_0). Then for the sequence of *slopes* of f given by $\left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right)_{n\in\mathbb{N}}$, which is bounded, the mapping

$$T_w : \mathrm{Lip}(X,d) \longrightarrow \mathbf{l}^\infty \text{ with } T_w(f) := \left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right)_{n\in\mathbb{N}}$$

is a continuous linear operator, since $\|T_w(f)\|_\infty \leq \|f\|_d \leq \|f\|$.

Now we repeat the proof of D. Sherbert, that for a translation invariant Banach limit the continuous linear functional

$$D_w : \mathrm{Lip}(X,d) \longrightarrow \mathbb{R} \text{ with } D_w(f) = \mathrm{LIM}(T_w(f))$$

is a point derivation at $x_0 \in X$ (see [8] Lemma 9.4). For abbreviation let us put $\Delta := \{(s,t) \in X \times X \mid s = t\}$.

Proposition 3.3. *Let $x_0 \in X$ be a nonisolated point of a metric space (X,d) and $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s,t) \in X \times X \mid s \neq t\}$ a sequence, which converges to the point (x_0, x_0). Then for every translation invariant Banach limit $\mathrm{LIM} : \mathbf{l}^\infty \longrightarrow \mathbb{R}$ the continuous linear functional*

$$D_w : \mathrm{Lip}(X,d) \longrightarrow \mathbb{R} \textit{ with } D_w(f) = \mathrm{LIM}(T_w(f))$$

is a point derivation for the Lipschitz algebra $(\mathrm{Lip}(X,d), \|\cdot\|)$ at $x_0 \in X$.

Proof. First observe, that for every convergent sequence $(a_n)_{n\in\mathbb{N}} \in \mathbf{c}$ and every bounded sequence $(b_n)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$ the formula $\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \mathrm{LIM}_{n\to\infty} b_n$ holds.

Namely put $\alpha := \lim_{n\to\infty} a_n$. Then $\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n - \alpha b_n) = 0$, since $(a_n \cdot b_n - \alpha b_n)_{n\in\mathbb{N}}$ is a sequence converging to zero and hence $\mathrm{LIM}_{n\to\infty}(a_n \cdot b_n) = \alpha \mathrm{LIM}_{n\to\infty} b_n$.

Now let $f, g \in \mathrm{Lip}(X, d)$ be given. From the above observation follows that

$$\begin{aligned}
D_w(fg) &= \mathrm{LIM}(T_w(fg)) \\
&= \mathrm{LIM}_{n\to\infty}\left(\frac{(fg)(y_n) - (fg)(x_n)}{d(y_n, x_n)}\right) \\
&= \mathrm{LIM}_{n\to\infty}\left(f(y_n)\frac{g(y_n) - g(x_n)}{d(y_n, x_n)} \ + \ g(x_n)\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)\mathrm{LIM}_{n\to\infty}\left(\frac{g(y_n) - g(x_n)}{d(y_n, x_n)}\right) \ + \ g(x_0)\mathrm{LIM}_{n\to\infty}\left(\frac{f(y_n) - f(x_n)}{d(y_n, x_n)}\right) \\
&= f(x_0)\mathrm{LIM}(T_w(g)) + g(x_0)\mathrm{LIM}(T_w(f)) \\
&= f(x_0)D_w(g) + g(x_0)D_w(f).
\end{aligned}$$

Since D_w is continuous, it is a point derivation at $x_0 \in X$. □

Now the following representation theorem holds (see [8], Theorem 9.5).

Proposition 3.4. *Let $x_0 \in X$ be a nonisolated point of a metric space (X, d) and $\mathcal{W}_{x_0} := \{w := (x_n, y_n)_{n\in\mathbb{N}} \subset X \times X \setminus \Delta \ \mid \ \lim x_n = \lim y_n = x_0 \ \}$. Moreover let $\mathrm{LIM} : \mathbf{l}^\infty \longrightarrow \mathbb{R}$ be a fixed translation invariant Banach limit . Then*

$$\mathrm{Der}_{x_0}(\mathrm{Lip}(X, d)) = \mathrm{cl}\ (\mathrm{span}\{D_w = \mathrm{LIM}(T_w) \ \mid \ w \in \mathcal{W}_{x_0} \ \}),$$

where "cl span" *denotes the weak-*-closure of the linear hull in* $\mathrm{Lip}(X, d)'$.

4. Characterization of Newns and Walker

Newns and Walker [6] proved in 1956 a coordinate free characterization of critical points of C^∞-functions intrinsically in algebraic terms :

Let $U \subseteq \mathbb{R}^n$ be an open subset and $x_0 \in U$ and let $\delta_{x_0} : C^\infty(U) \to \mathbb{R}$ be given by $\delta_{x_0}(f) = f(x_0)$. As above consider the ideal $\mathbf{m(x_0)}$ of the algebra $\mathcal{A} = C^\infty(U)$ given by $\mathbf{m(x_0)} = \{f \in \mathcal{A} \ \mid \ \delta_{x_0}(f) = 0\}$. Let $\mathbf{m}^2\mathbf{(x_0)}$ be the algebraic square of the ideal $\mathbf{m(x_0)}$, i.e.

$$\mathbf{m}^2\mathbf{(x_0)} = \{f \in \mathcal{A} \ \mid \ f = \sum_{i=1}^{k} g_i \cdot h_i \ , g_i, h_i \in \mathbf{m(x_0)}, \ i = 1, ..., k \ , \ k \geq 1\}.$$

Consider now the intersection of all maximal subalgebras $\mathbf{a} \subset \mathcal{A}$, which contain $\mathbf{m}^2\mathbf{(x_0)}$, In analogy to Frattini groups, (Frattini (1885)), this set is called a *Frattini algebra of* $\mathcal{A}$ at $x_0 \in X$ and denoted by

$$\mathcal{F}(\mathcal{A})(x_0) = \bigcap\{ \ \mathbf{a} \ \mid \ \mathbf{m}^2\mathbf{(x_0)} \ \subset \ \mathbf{a} \subset \mathcal{A}, \ \mathbf{a} \textit{ is a maximal closed subalgebra}\}$$

In this notation they proved in the appendix of their paper [6]:

Theorem 4.1. *Let $U \subseteq \mathbb{R}^n$ be an open subset and let $x_0 \in U$. A function $f \in \mathcal{A} = C^\infty(U)$ has a critical point in $x_0 \in U$ if and only if $f(x) - f(x_0) \in \mathcal{F}(\mathcal{A})(x_0)$.*

Let us point out, that the proof of this Theorem is based on directional derivatives which are not in a close analogy to point derivations. This can for instance be seen in the case of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) := x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. This function is differentiable in $x_0 = 0$, but has in $x_0 = 0$ point derivatives with values in the interval $[-1, 1]$ (see also the remark on p. 266 in [8]).

5. Critical points of Lipschitz functions

In this section we consider the algebraic characterization of critical points for Lipschitz functions. Therefore we will consider for a metric space (X, d) the algebra $\mathcal{A} = \mathrm{Lip}(X, d)$ of all bounded Lipschitz functions defined on (X, d). Moreover we will assume that $x_0 \in X$ is a nonisolated point of a metric space (X, d) and that

$$\delta_{x_0} : \mathrm{Lip}(X, d) \to \mathbb{R} \quad \text{ is defined by } \quad \delta_{x_0}(f) = f(x_0).$$

As previously we consider for $\mathcal{A} = \mathrm{Lip}(X, d)$ the ideal $\mathbf{m}(\mathbf{x_0}) = \{f \in \mathcal{A} \mid \delta_{x_0}(f) = 0\}$ and its algebraic square

$$\mathbf{m}^2(\mathbf{x_0}) = \{f \in \mathcal{A} \mid f = \sum_{i=1}^{k} g_i \cdot h_i \ , g_i, h_i \in \mathbf{m}(\mathbf{x_0}), \ i = 1, ..., k \ , \ k \geq 1\}.$$

Following the characterization given in Theorem 4.1 we say that an element $f \in \mathrm{Lip}(X, d)$ has an *"(algebraic) critical point"* at $x_0 \in X$ if and only if $f(x) - f(x_0) \in \mathcal{F}(\mathcal{A})(x_0)$.

First we show:

Proposition 5.1. *Let (X, d) be a metric space and $x_0 \in X$. Then for every proper subalgebra $\mathbf{a}$ with $\mathbf{1} \in \mathbf{a}$ and $\mathbf{m}^2(\mathbf{x_0}) \subset \mathbf{a} \subset \mathcal{A} = \mathrm{Lip}(X, d)$ there exists a nontrivial point derivation $D \in \mathrm{Der}_{x_0}(\mathrm{Lip}(X, d)) \setminus \{0\}$ with*

$$\mathbf{a} \subset \ker(D) = \{f \in \mathrm{Lip}(X, d) \mid D(f) = 0\}.$$

Moreover $\ker(D) = \{f \in \mathrm{Lip}(X, d) \mid D(f) = 0\}$ *is a maximal subalgebra.*

Proof. Let $\mathbf{a} \subset \mathcal{A} = \mathrm{Lip}(X, d)$ be a proper subalgebra. Then $\mathbf{a}$ is also a proper linear subspace of $\mathrm{Lip}(X, d)$ and hence there exists a hyperspace F, i.e. a linear subspace of codimension 1 with $\mathbf{a} \subset F$. Since $\mathbf{1} \in \mathbf{a}$ and $\mathbf{m}^2(\mathbf{x_0}) \subset \mathbf{a} \subset \mathcal{A} = \mathrm{Lip}(X, d)$ we have by Proposition 3.2 that the canonical projection $\pi : \mathrm{Lip}(X, d) \longrightarrow \mathrm{Lip}(X, d)/_F \simeq \mathbb{R}$ is a point derivation at $x_0 \in X$.

Now change the notation and put $D = \pi$. Then we show that $\ker(D)$ is a subalgebra of $\mathrm{Lip}(X, d)$. Namely if $f, g \in \ker(D)$ then obviously $f + g \in \ker(D)$ and from the Leibniz rule follows $fg \in \ker(D)$ since $D(fg) = f(x_0)Dg + g(x_0)Df = 0$.

From the construction follows, that $\ker(D)$ is maximal. □

Now we use the explicit representations of point derivations given in Proposition 3.3 and Proposition 3.4.

Let us denote by $\mathcal{F}(\mathrm{Lip}(X, d))(x_0)^0 = \bigcap_{D_w = \mathrm{LIM}(T_w),\, w \in \mathcal{W}_{x_0},\, \mathrm{LIM} \in \mathbf{Lim}} \ker(D_w)$. Then we have:

Proposition 5.2. *Let $D \in \mathrm{Der}_{x_0}(\mathrm{Lip}(X, d))$ be a point derivation. Then for every $f \in \mathcal{F}(\mathrm{Lip}(X, d))(x_0)^0$ holds $D(f) = 0$.*

Proof. By Proposition 3.4 we have

$$\mathrm{Der}_{x_0}(\mathrm{Lip}(X,d)) = \mathrm{cl}\ (\mathrm{span}\{D_w = \mathrm{LIM}(T_w) \ \mid \ w \in \mathcal{W}_{x_0}\ \}),$$

where "cl span" denotes the weak-*-closure of the linear hull in $\mathrm{Lip}(X,d)'$.

If $D \in \mathrm{span}\{D_w = \mathrm{LIM}(T_w) \ \mid \ w \in \mathcal{W}_{x_0}\ \}$ this follows immediately from the definition of $\mathcal{F}(\mathrm{Lip}(X,d))(x_0)^0$.

Now assume that $D \in \mathrm{Der}_{x_0}(\mathrm{Lip}(X,d))$ is a proper weak-*-accumulation point of $\mathrm{span}\{D_w = \mathrm{LIM}(T_w) \ \mid \ w \in \mathcal{W}_{x_0}\ \}$. Then for every $\varepsilon > 0$ and arbitrary points $f_1, ..., f_k \in \mathrm{Lip}(X,d)$ there exists a $D_w = \mathrm{LIM}(T_w)$ with $w \in \mathcal{W}_{x_0}$ and $\mathrm{LIM} \in \mathbf{Lim}$ such that $D_w - D \in U_{f_1,....,f_k,\varepsilon}$ where $U_{f_1,....,f_k,\varepsilon} = \left\{ l \in \mathrm{Lip}(X,d)' \ \mid \ |l(f_i)| < \varepsilon \right\}$ is a weak-*-neighborhood of $0 \in \mathrm{Lip}(X,d)'$.

Let us now assume that our statement is not true. Then there exists an element $f^* \in \mathcal{F}(\mathrm{Lip}(X,d))(x_0)^0$ with $|D(f^*)| = c > 0$. Now let $\varepsilon = \frac{c}{2}$ and choose arbitrary elements $f_2, ..., f_k \in \mathrm{Lip}(X,d)$. Then for the weak-*-neighborhood of $0 \in \mathrm{Lip}(X,d)'$ given by $U_{f^*,f_2,....,f_k,\varepsilon}$ there exists a $D_w = \mathrm{LIM}(T_w)$ such that $D_w - D \in U_{f^*,f_2,....,f_k,\varepsilon}$. For the element $f^* \in \mathcal{F}(\mathrm{Lip}(X,d))(x_0)^0$ this implies: $c = |D(f^*)||D(f^*) - D_w(f^*)| < \varepsilon = \frac{c}{2}$ which is a contradiction. □

The above two propositions imply that $\mathcal{F}(\mathrm{Lip}(X,d))(x_0) = \mathcal{F}(\mathrm{Lip}(X,d))(x_0)^0$. Finally we prove our main result:

Theorem 5.3. *Let $x_0 \in X$ be a nonisolated point of a metric space (X,d). Then $f \in \mathrm{Lip}(X,d)$ has an algebraic critical point in $x_0 \in X$ if and only if for every point sequence $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s,t) \in X \times X \ \mid \ s \neq t\ \}$ which converges to the point (x_0, x_0), the corresponding sequence of slopes of f given by $T_w(f) := \left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$ converges to zero.*

Proof. By the above statements $f \in \mathrm{Lip}(X,d)$ has an algebraic critical point in $x_0 \in X$ if and only if $D(f) = 0$ for all point derivations $D \in \mathrm{Der}_{x_0}(\mathrm{Lip}(X,d)) \setminus \{0\}$.

Now by the characterization of G.G. Lorentz [5] on almost convergent sequences, that are bounded real sequence $(a_n)_{n\in\mathbb{N}}$ for which all translation invariant Banach limits have the same value $L \in \mathbb{R}$, we have that for every $\varepsilon > 0$ there exists a $p_0 \in \mathbb{N}$ so that for all $p > p_0$ and for all $n \in \mathbb{N}$ the condition

$$\left| \frac{a_n + \ldots + x_{a+p-1}}{p} - L \right| < \varepsilon$$

holds. In our case $L = 0$ which means that for every $\varepsilon > 0$ there exists a $p_0 \in \mathbb{N}$ so that for all $p > p_0$ and for all $n \in \mathbb{N}$ the condition

$$\left| \frac{\left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right) + \ldots + \left(\frac{f(y_{n+p-1})-f(x_{n+p-1})}{d(y_{n+p-1},x_{n+p-1})}\right)}{p} \right| < \varepsilon \qquad (*)$$

holds. Since condition $(*)$ has to hold for every point sequence $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s,t) \in X \times X \ \mid \ s \neq t\ \}$ which converges to the point (x_0, x_0), this implies, that the corresponding sequence of slopes $\left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$ converges to zero.

The converse direction is obvious, because if for every point sequence $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s,t) \in X \times X \mid s \neq t\}$ which converges to the point (x_0, x_0), the corresponding sequence of slopes of f given by $T_w(f) := \left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$ converges to zero, then condition $(*)$ is satisfied. □

For Lipschitz functions on open sets in normed linear spaces, this implies that the gradient is zero, as we shall prove now:

Proposition 5.4. *Let $(E, \|\cdot\|_E)$ be a real normed vector space, $U \subset E$ an open subset, $x_0 \in U$, and $f \in \mathrm{Lip}(U,d)$, where the metric d on U is induced by the norm. If $D(f) = 0$ holds for every $D \in \mathrm{Der}_{x_0}(\mathrm{Lip}(U,d))$ then $f \in \mathrm{Lip}(U,d)$ is Fréchet differentiable at $x_0 \in X$ and its differential vanishes, i.e., $df\Big|_{x_0} = 0$.*

Proof. Let us assume that the function $f \in \mathrm{Lip}(U,d)$ is not differentiable in $x_0 \in U$. Then for every $l \in E'$ there exists an $\varepsilon_l > 0$ such that for all $\delta > 0$ there exists a vector $h_\delta \in E$ with $\|h_\delta\|_E \leq \delta$ and

$$|f(x_0 + h_\delta) - f(x_0) - l(h_\delta)| > \varepsilon_l \|h_\delta\|_E.$$

For the special case $l = 0 \in E'$ and $\delta = \frac{1}{n}$ we get with $h_n := h_\delta$ that there exists an $\varepsilon_0 > 0$ such that for $n \in \mathbb{N}$

$$\frac{|f(x_0 + h_n) - f(x_0)|}{\|h_n\|_E} > \varepsilon_0.$$

Now choose a subsequence $(n_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$ the nominator $f(x_0 + h_{n_k}) - f(x_0)$ has a constant sign, for instance $f(x_0 + h_{n_k}) - f(x_0) > 0$.

For the sequence

$$\hat{w} := (x_0, x_0 + h_{n_k})_{k\in\mathbb{N}} \in \mathcal{W}_{x_0}$$

holds then

$$D_{\hat{w}}(f) = \mathrm{LIM}(T_{\hat{w}}(f)) > \varepsilon_0,$$

since every translation invariant Banach limit of a bounded sequence is greater or equal then the lower limit of this sequence. Hence there exists a point derivation at $x_0 \in U$ with $D_{\hat{w}}(f) \neq 0$, which is a contradiction. □

In the case of Lipschitz functions, the algebraic characterization of a critical is stronger than the critical point concept of Bonnisseau and Cornet [2] and of Clarke [3]. For instance the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) := x^2 \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$ is differentiable in $x_0 = 0$ with $f'(x_0) = 0$, but $x_0 = 0$ is not a critical point in the algebraic sense.

6. Conclusion

From the coordinate free characterization of Newns and Walker [6] of critical points of C^∞-functions, we derive for Lipschitz functions on metric space a characterization of critical points, which is stronger than the concept of Clarke [3]. According to this charaterization one has for a nonisolated point $x_0 \in X$ of a metric space (X, d) the following condition:

Namely, a Lipschitz function $f \in \mathrm{Lip}(X,d)$ has an algebraic critical point in $x_0 \in X$ if and only if for every point sequence $w := (x_n, y_n)_{n\in\mathbb{N}} \subset (X \times X) \setminus \Delta = \{(s,t) \in X \times X \ \mid \ s \neq t\ \}$ which converges to the point (x_0, x_0), the corresponding sequence of slopes of f given by $T_w(f) := \left(\frac{f(y_n)-f(x_n)}{d(y_n,x_n)}\right)_{n\in\mathbb{N}} \in \mathbf{l}^\infty$ converges to zero.

References

[1] R. Arens and J. Eells, Jr. *On embedding uniform and topological spaces*, Pacific Journ. Math. **6** (1956), 397–403.

[2] J.-M. Bonnisseau and B. Cornet, *Fixed-point theorems and Morse's lemma for Lipschitzian functions,* Journ. Math. Anal. Appl. **146** (1990), 318–332.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis,* CRM, Université de Montréal, Quebec, Canada, 1989.

[4] N. Dunford and J. T. Schwartz, *Linear Operators: Part I,* Interscience Publishers, Inc., New York, 1957.

[5] G. G. Lorentz, *A contribution to the Theory of divergent sequences,* Acta Math. **80** (1948), 167–190

[6] N. Newns and A. Walker, *Tangent planes to differentiable manifolds,* Journal London Math. Soc. **31** (1956), 400–407.

[7] D. Pallaschke and R. Urbański, *Critical points and point derivations for Lipschitz functions,* Comment. Math. **26** (1998),165 – 172.

[8] D. R. Sherbert, *The structure of ideals and point derivations in Banach Algebras of Lipschitz functions,* Trans AMS **111** (1964), 240–272.

[9] I. Singer and J. Wermer, *Derivations on commutative normed algebras,* Math. Ann. **129** (1955), 260–264.

Manuscript received August 27 2017
revised November 3 2017

J. Grzybowski
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, PL-61-614 Poznań, Poland
E-mail address: jgrz@amu.edu.pl

D. Pallaschke
Institute of Operations Research, University of Karlsruhe, Karlsruhe Institute of Technology (KIT), D-76131 Karlsruhe, Germany
E-mail address: diethard.pallaschke@kit.edu

R. Urbański
Faculty of Mathematics and Computer Science, Adam Mickiewicz University , PL-61-614 Poznań, Poland
E-mail address: rich@amu.edu.pl