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INFINITE-DIMENSIONAL INFINITE-HORIZON MULTIOBJECTIVE OPTIMAL CONTROL IN DISCRETE TIME

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Abstract. This paper studies multiobjective optimal control problems in the discrete time framework and in the infinite horizon case when the space of states and the space of controls are infinite-dimensional. The paper generalizes to the multiobjective case existing results for single-objective optimal control problems in that framework. The dynamics are governed by difference equations. Necessary conditions of Pareto optimality are presented namely Pontryagin maximum principles in the weak form.

1. Introduction

This paper studies multiobjective optimal control problems in the discrete time framework and in the infinite horizon case when the space of states and the space of controls are infinite-dimensional. It extends to the multiobjective case results obtained for single-objective optimal control problems in the discrete-time framework and in the infinite-horizon case when the space of states and the space of controls are infinite-dimensional. And it extends to the case of infinite-dimensional spaces of states and controls, results obtained for infinite-horizon multiobjective optimal control problems in the discrete-time framework when the space of states and the space of controls are finite-dimensional. Smooth problems are considered.

The first works on infinite-horizon single-objective optimal control problems are due to Pontryagin and his school [16]. Others followed as Carlson et al [9], Blot and Hayek [5], [6], Blot [3], [4], [7], Zaslavski [17], [18] and [19].

Bachir and Blot [1], [2] recently extended infinite-horizon single-objective optimal control problems in the discrete-time framework, to the case of infinite-dimensional spaces of states and controls. They applied a method of reduction to the finite horizon applied in the setting of the infinite dimension. This method of reduction to the finite horizon was used in the setting of the finite dimension in [4] and [5] for example where an essential difficulty was to extract subsequences of multipliers that do not converge to zero. But when the spaces of states and controls are infinite-dimensional, more difficulties arise as Bachir and Blot show in [1] and [2]. These difficulties are due to the closure of the ranges of linear operators, and to the fact

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that the weak-star closure of the unit sphere is the unit ball and hence contains the origin. Their works [1] and [2] provide answers to these problems.

Results on infinite-horizon multiobjective optimal control problems in the discrete-time framework can be found in Hayek [11] and [12], in Ngo-Hayek [15] where they are obtained by a reduction to a finite-horizon framework and by techniques of infinite horizon.
In Hayek [13] these problems are studied in the special case of the bounded processes and techniques of Banach spaces are used.

In this paper we rely on the results of Bachir and Blot in [1] and [2] to obtain necessary conditions of Pareto optimality under the form of Pontryagin Principles for infinite-horizon multiobjective optimal control problems in an infinite-dimensional setting.

The plan of this paper is as follows. In section 2 the problem is presented: a multiobjective optimal control problem governed by a difference equation when the space of states and the space of controls are infinite-dimensional, in the discrete-time framework and in the infinite-horizon case. The notions of Pareto optimality and weak Pareto optimality are defined. Other notions of optimality are presented. In section 3 the theorems on necessary conditions of Pareto optimality are stated namely Pontryagin maximum principles in the weak form. The proofs are provided in section 4.

2. Problems and Notation

Let X and U be Banach spaces. For all $t \in \mathbb{N}$, let X_t be a nonempty subset of X, U_t be a nonempty subset of U, and $f_t : X_t \times U_t \to X_{t+1}$ be a mapping. Consider the following controlled dynamical system:

(De) $\quad x_{t+1} = f_t(x_t, u_t), \ \ t \in \mathbb{N}.$

Set $\underline{x} := (x_t)_{t\in\mathbb{N}} \in \prod_{t\in\mathbb{N}} X_t$, $\underline{u} := (u_t)_{t\in\mathbb{N}} \in \prod_{t\in\mathbb{N}} U_t$.

Multiobjective optimal control problems governed by (De) will be studied. For $\eta \in X$ let $\mathrm{Adm}(\eta)$ denote the set of all processes $(\underline{x}, \underline{u}) \in \prod_{t\in\mathbb{N}} X_t \times \prod_{t\in\mathbb{N}} U_t$ which satisfy (De) for all $t \in \mathbb{N}$ and such that $x_0 = \eta$. These processes are called admissible. For all $t \in \mathbb{N}$, for all $j \in \{1, .., \ell\}$, let $\phi_t^j : X_t \times U_t \to \mathbb{R}$ be a function. For each $j \in \{1, .., \ell\}$, set $J_j(\underline{x}, \underline{u}) := \sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t)$ and let $\mathrm{Dom}(J_j)$ denote the set of all $(\underline{x}, \underline{u}) \in \mathrm{Adm}(\eta)$ such that the series $\sum_{t=0}^{+\infty} \phi_t^j(x_t, u_t)$ is convergent in $\mathbb{R}$. The optimality criterion considered here is defined by using the vector-function $J := (J_1, .., J_\ell)$. The order for this criterion is the natural order in $\mathbb{R}^\ell$. Now, the domain for the multiobjective optimal control problems with criterion J, is denoted by $\mathrm{DOM}(J) := \left(\bigcap_{j=1}^{\ell} \mathrm{Dom} J_j\right)$. Consider the following multiobjective optimal control problem

(PM^1) Maximize $J(\underline{x}, \underline{u})$ when $(\underline{x}, \underline{u}) \in \mathrm{DOM}(J)$.

Definition 2.1.

- A process $(\underline{\hat{x}}, \underline{\hat{u}}) \in \mathrm{DOM}(J)$ is called a Pareto optimal solution of Problem (PM^1), if there does not exist a process $(\underline{x}, \underline{u}) \in \mathrm{DOM}(J)$ such that for all $j \in \{1,..,\ell\}$, $J_j(\underline{x},\underline{u}) \geq J_j(\underline{\hat{x}},\underline{\hat{u}})$ and for some $i \in \{1,..,\ell\}$, $J_i(\underline{x},\underline{u}) > J_i(\underline{\hat{x}},\underline{\hat{u}})$.

- A process $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{DOM}(J)$ is called a weak Pareto optimal solution of Problem (PM^1), if there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{DOM}(J)$ such that for all $j \in \{1,..,\ell\}$, $J_j(\underline{x},\underline{u}) > J_j(\underline{\hat{x}},\underline{\hat{u}})$.

It is clear that a Pareto optimal solution of Problem (PM^1) is a weak Pareto optimal solution of Problem (PM^1).

Consider now the following problems for the cases where the infinite series do not necessarily converge:

(PM^2) Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}(\eta)$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}(\eta)$ satisfying for all $j \in \{1,\dots,\ell\}$, $\limsup_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) \geq 0$ and for some $i \in \{1,\dots,\ell\}$, $\limsup_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^i(x_t,u_t) - \sum_{t=0}^{h}\phi_t^i(\hat{x}_t,\hat{u}_t)) > 0$.

$(PM^{2'})$ Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}(\eta)$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}(\eta)$ satisfying for all $j \in \{1,\dots,\ell\}$, $\limsup_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) > 0$.

(PM^3) Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}(\eta)$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}(\eta)$ satisfying for all $j \in \{1,\dots,\ell\}$, $\liminf_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) \geq 0$ and for some $i \in \{1,\dots,\ell\}$, $\liminf_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^i(x_t,u_t) - \sum_{t=0}^{h}\phi_t^i(\hat{x}_t,\hat{u}_t)) > 0$.

$(PM^{3'})$ Find $(\underline{\hat{x}},\underline{\hat{u}}) \in \mathrm{Adm}(\eta)$ such that, there does not exist a process $(\underline{x},\underline{u}) \in \mathrm{Adm}(\eta)$ satisfying for all $j \in \{1,\dots,\ell\}$, $\liminf_{h\to+\infty}(\sum_{t=0}^{h}\phi_t^j(x_t,u_t) - \sum_{t=0}^{h}\phi_t^j(\hat{x}_t,\hat{u}_t)) > 0$.

Let T be a fixed number in $\mathbb{N}_*$, set $(\mathbf{x^T},\mathbf{u^T}) := ((x_t)_{1\leq t\leq T},(u_t)_{0\leq t\leq T})$ and set $J_j^T(\mathbf{x^T},\mathbf{u^T}) := \sum_{t=0}^{T}\phi_t^j(x_t,u_t)$ and $J^T := (J_1^T,..,J_\ell^T)$. Consider the following reduced problem

$$(FM^T) \qquad \left.\begin{array}{ll} \text{Maximize} & J^T(\mathbf{x^T},\mathbf{u^T}) \\ & x_{t+1} = f_t(x_t,u_t), \forall t \in \{0,\dots,T\} \\ & x_0 = \eta,\ x_{T+1} = \hat{x}_{T+1}. \end{array}\right\}$$

Definition 2.2.

- $(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ is called a Pareto optimal solution of Problem (FM^T), if there does not exist any $(\mathbf{x^T},\mathbf{u^T})$ admissible for Problem (FM^T) such that for all $j \in \{1,..,\ell\}$, $J_j^T(\mathbf{x^T},\mathbf{u^T}) \geq J_j^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ and for some $i \in \{1,..,\ell\}$, $J_i^T(\mathbf{x^T},\mathbf{u^T}) > J_i^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$.
- $(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ is called a weak Pareto optimal solution of Problem (FM^T), if there does not exist any $(\mathbf{x^T},\mathbf{u^T})$ admissible for Problem (FM^T) such that for all $j \in \{1,..,\ell\}$, $J_j^T(\mathbf{x^T},\mathbf{u^T}) > J_j^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$.

Here admissibility means that all the constraints, including the dynamical system, the initial and final conditions, are satisfied. Then we have the following result :

Lemma 2.3.

- *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a Pareto optimal solution of Problem (PM^1) (respectively, solution of (PM^2), (PM^3)) and let $T \in \mathbb{N}^*$. Then the restriction $(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})$ is a Pareto optimal solution of the finite-horizon problem (FM^T).*
- *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem (PM^1) (respectively, solution of $(PM^{2'})$, $(PM^{3'})$) and let $T \in \mathbb{N}^*$. Then the restriction $(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})$ is a weak Pareto optimal solution of the finite-horizon problem (FM^T).*

The proof of this lemma is analogous to the proof given in [11] for the finite-dimensional case.

3. The main theorems

Theorem 3.1. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem (PM^1) (respectively, a solution of $(PM^{2'})$, $(PM^{3'})$). We assume that the following assumptions are fulfilled*

(i) *For all $t \in \mathbb{N}$, X_t is a nonempty open convex subset of X, U_t is a nonempty convex subset of U.*
(ii) *X is separable.*
(iii) *For all $t \in \mathbb{N}$, for all $j \in \{1, \ldots, \ell\}$, ϕ_t^j are Fréchet-differentiable at $(\hat{x}_t, \hat{u}_t)$ and f_t is continuously Fréchet-differentiable at $(\hat{x}_t, \hat{u}_t)$.*
(iv) *For all $t \in \mathbb{N}$, $ImD_2 f_t(\hat{x}_t, \hat{u}_t)$ is closed and its dimension is finite*
or
(v) *For all $t \in \mathbb{N}$, $ImD_2 f_t(\hat{x}_t, \hat{u}_t)$ is closed and its codimension is finite.*

Then, for all $T \in \mathbb{N}$, $T \geq 2$, there exist $(\theta_1^T, .., \theta_\ell^T) \in \mathbb{R}^\ell$, $(p_t^T)_{1 \leq t \leq T+1} \in (X^*)^{T+1}$, satisfying the following conditions.

(a) $(\theta_1^T, .., \theta_\ell^T, (p_t^T)_{1 \leq t \leq T+1}) \neq (0, .., 0)$
(b) $\theta_j^T \geq 0$ *for all* $j \in \{1, \ldots, \ell\}$.
(c) $p_t^T = p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \sum_{j=1}^{\ell} \theta_j^T . D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)$, *for all* $t \in \{1, \ldots, T\}$.
(d) $< \sum_{j=1}^{\ell} \theta_j^T . D_2 \phi_t^j(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_2 f_t(\hat{x}_t, \hat{u}_t), u_t - \hat{u}_t > \leq 0$, *for all* $t \in \{0, \ldots, T\}$, *for all* $u_t \in U_t$.

Theorem 3.2. *Let $(\underline{\hat{x}}, \underline{\hat{u}})$ be a weak Pareto optimal solution of Problem (PM^1) (respectively, a solution of $(PM^{2'})$, $(PM^{3'})$). We assume that all assumptions of Theorem 3.1 are satisfied together with the following assumptions :*

(vi) *For all $t \in \mathbb{N}$, the partial differential $D_1 f_t(\hat{x}_t, \hat{u}_t)$ is invertible.*
(vii) *The tangent cone of U_1 at the point $\hat{u}_1$, denoted by $T_{U_1}(\hat{u}_1)$, is a vector space.*
(viii) $ImD_1 f_1(\hat{x}_1, \hat{u}_1) \subset D_2 f_1(\hat{x}_1, \hat{u}_1)(T_{U_1}(\hat{u}_1))$.

Then, there exist $\theta_1, .., \theta_\ell \in \mathbb{R}$, $(p_t)_{t \in \mathbb{N}_*} \in (X^*)^{\mathbb{N}_*}$, satisfying the following conditions.

(a) $(\theta_1, .., \theta_\ell, p_1) \neq (0, \ldots, 0, 0)$

(b) $\theta_j \geq 0$ *for all* $j \in \{1, \ldots, \ell\}$.

(c) $p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t, \hat{u}_t) + \sum_{j=1}^{\ell} \theta_j . D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)$, *for all* $t \in \mathbb{N}_*$.

(d) $< \sum_{j=1}^{\ell} \theta_j . D_2 \phi_t^j(\hat{x}_t, \hat{u}_t) + p_{t+1} \circ D_2 f_t(\hat{x}_t, \hat{u}_t), u_t - \hat{u}_t > \leq 0$, *for all* $t \in \mathbb{N}$, *for all* $u_t \in U_t$.

Following Bachir and Blot in [2] for the single-objective case, one can weaken some assumptions of Theorem 3.2 as follows:

Theorem 3.3. *Let* $(\underline{\hat{x}}, \underline{\hat{u}})$ *be a weak Pareto optimal solution of Problem* (PM^1) *(respectively, a solution of* $(PM^{2'})$, $(PM^{3'})$*). We assume that all assumptions of Theorem 3.1 are satisfied together with the following assumptions :*

(ix) *for all* $t \in \mathbb{N}$, $0 \in int[Df_t(\hat{x}_t, \hat{u}_t)((X \times T_{U_t}(\hat{u}_t)) \cap B_{X \times U})]$ *where* $B_{X \times U}$ *denotes the closed unit ball of* $X \times U$.

(x) *there exists* $s \in \mathbb{N}$ *such that* $A_s = D_2 f_s(\hat{x}_s, \hat{u}_s)(T_{U_s}(\hat{u}_s))$ *contains a closed convex subset* K *with* $ri(K) \neq \emptyset$ *and such that* $\overline{Aff(K)}$ *is of finite codimension in* X.

Then, there exist $\theta_1, .., \theta_\ell \in \mathbb{R}$, $(p_t)_{t \in \mathbb{N}_*} \in (X^*)^{\mathbb{N}_*}$, *such that*

(a) $(\theta_1, .., \theta_\ell, p_t) \neq (0, \ldots, 0, 0)$, *for all* $t \geq s$

and conclusions (b), (c) and (d) of Theorem 3.2 are satisfied.

Remark. Notice that the invertibility of the partial differential $D_1 f_t(\hat{x}_t, \hat{u}_t)$ (condition (vi) of Theorem 3.2) is avoided in this theorem and replaced by condition (ix) of Theorem 3.3 which is weaker. And notice that conditions (vii) and $(viii)$ of Theorem 3.2 are replaced by condition (x) of Theorem 3.3 which is weaker. Condition (x) is satisfied and is included in condition (v), whenever there exists an $s \in \mathbb{N}$ such that $T_{U_s}(\hat{u}_s) = X$, in particular, if $\hat{u}_s$ belongs to the interior of U_s.

4. Proofs of the main theorems

4.1. Proof of Theorem 3.1.

The following lemmas will be useful in the proof.

Lemma 4.1. *Under assumption (iii) of theorem 3.1 ,* J^T *is Fréchet-differentiable at* $(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})$ *and*

$$DJ^T(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})(\delta\mathbf{x^T}, \delta\mathbf{u^T}) = (DJ_1^T(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})(\delta\mathbf{x^T}, \delta\mathbf{u^T}), \ldots, DJ_\ell^T(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})(\delta\mathbf{x^T}, \delta\mathbf{u^T}))$$

where

$$DJ_j^T(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})(\delta\mathbf{x^T}, \delta\mathbf{u^T}) = \sum_{t=0}^{T} D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)\delta x_t + \sum_{t=0}^{T} D_2 \phi_t^j(\hat{x}_t, \hat{u}_t)\delta u_t$$

Proof. For all $j = 1, \ldots, \ell$, J_j^T is Fréchet-differentiable at $(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})$ as a sum of Fréchet-differentiable mappings that are compositions of Fréchet-differentiable mappings. Hence J^T is Fréchet-differentiable at $(\mathbf{\hat{x}^T}, \mathbf{\hat{u}^T})$ □

Define $H_t^T : \prod_{t=1}^{T} X_t \times \prod_{t=0}^{T} U_t \to X_{t+1}$ by setting:

$$H_t^T(\mathbf{x^T},\mathbf{u^T}) := \begin{cases} -x_1 + f_0(\eta, u_0) & if\ t=0 \\ -x_{t+1} + f_t(x_t, u_t) & if\ 0<t<T \\ -\hat{x}_{T+1} + f_T(x_T, u_T) & if\ t=T \end{cases}$$

Define $H^T : \prod_{t=1}^T X_t \times \prod_{t=0}^T U_t \to \prod_{t=0}^T X_{t+1}$ by setting

$$H^T(\mathbf{x^T},\mathbf{u^T}) := (H_0^T(\mathbf{x^T},\mathbf{u^T}), \ldots, H_T^T(\mathbf{x^T},\mathbf{u^T}))$$

Lemma 4.2. *Under assumption (iii) of theorem 3.1 , H^T is of class C^1 at $(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ and*

$$DH^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})(\delta\mathbf{x^T},\delta\mathbf{u^T}) = (DH_0^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})(\delta\mathbf{x^T},\delta\mathbf{u^T}), \ldots, DH_T^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})(\delta\mathbf{x^T},\delta\mathbf{u^T}))$$

where $DH_0^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})(\delta\mathbf{x^T},\delta\mathbf{u^T}) = -\delta x_1 + D_2 f_0(\eta, \hat{u}_0)\delta u_0$,
$DH_t^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})(\delta\mathbf{x^T},\delta\mathbf{u^T}) = -\delta x_{t+1} + D_1 f_t(\hat{x}_t, \hat{u}_t)\delta x_t + D_2 f_t(\hat{x}_t, \hat{u}_t)\delta u_t$, *for* $0<t<T$ *and* $DH_T^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})(\delta\mathbf{x^T},\delta\mathbf{u^T}) = D_1 f_T(\hat{x}_T, \hat{u}_T)\delta x_T + D_2 f_T(\hat{x}_T, \hat{u}_T)\delta u_T$.

Proof. H^T is of class C^1 as a composition of mappings of class C^1. □

Lemma 4.3. *Under the assumptions of Theorem 3.1, $ImDH^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ is closed in X^{T+1}.*

Proof. The proof can be found in [1]. It is done by showing first that $ImD_1H^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ is closed in X^{T+1} and then using assumptions (iv) or (v) to obtain that $ImDH^T(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ is closed in X^{T+1}. □

We shall use the following theorem for multiobjective abstract optimization in Banach spaces, which is a reduced form of Theorem 7.4 in Jahn's book [14].

Theorem 4.4. *Let $\mathcal{X}$ and $\mathcal{Z}$ be real Banach spaces and Y a partially ordered normed space. Let $\hat{\xi} \in \mathcal{X}$.*

Let C_Y denote the ordering cone in Y, which is assumed to have a nonempty interior.

Let $\hat{S}$ be a nonempty convex subset of $\mathcal{X}$ which has a nonempty interior.

Let $I : \mathcal{X} \to Y$ be Fréchet differentiable at $\hat{\xi}$ and $H : \mathcal{X} \to \mathcal{Z}$ be continuously Fréchet differentiable at $\hat{\xi}$.

Let $S := \{\xi \in \hat{S} /\ H(\xi) = 0_{\mathcal{Z}}\}$ and assume that S is nonempty.

Let $ImDH(\hat{\xi})$ be closed.

If $\hat{\xi}$ is a weak solution of the following problem

$$Minimize\ I(\xi) \quad when \quad \xi \in S$$

Then there exist $y \in C_{Y^*}$ and $w \in \mathcal{Z}^*$ with $(y, w) \neq (0, 0)$ such that

$$(y \circ DI(\hat{\xi}) + w \circ DH(\hat{\xi}))(\xi - \hat{\xi}) \geq 0 \text{ for all } \xi \in S.$$

Proof of Theorem 3.1. Since $(\underline{\hat{x}}, \underline{\hat{u}})$ is a weak Pareto optimal solution of Problem (PM^1) (respectively, a solution of $(PM^{2'})$, $(PM^{3'})$), Lemma 2.3 implies that the restriction $(\mathbf{\hat{x}^T},\mathbf{\hat{u}^T})$ is a weak Pareto optimal solution of the finite-horizon problem (FM^T).

Problem (FM^T) is in the form of the problem studied in Theorem 4.4 . Set $\mathcal{X} := X^T \times U^{T+1}$, $Y := R^\ell$, $\mathcal{Z} := X^{T+1}$, $C_Y := R_+^\ell$, $\hat{S} := \prod_{t=1}^T X_t \times \prod_{t=0}^T U_t$, $I := -J^T, H := -H^T$ and $\xi := (\mathbf{x^T},\mathbf{u^T})$. All assumptions of Jahn's Theorem

4.4 are satisfied by Lemma 4.1, Lemma 4.2 and Lemma 4.3. So we can apply this theorem to obtain $y = (\theta_1^T, .., \theta_\ell^T) \in \mathbb{R}_+^\ell$ and $w = (p_t^T)_{1 \le t \le T+1} \in (X^*)^{T+1}$ non simultaneously equal to zero such that:

$$(y \circ DI(\hat{\xi}) + w \circ DH(\hat{\xi}))(\xi - \hat{\xi}) \geq 0$$

for all ξ, which can be written

$$(\, y \circ DJ^T(\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T}) + w \circ DH^T(\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T}) \,)((\mathbf{x}^\mathbf{T}, \mathbf{u}^\mathbf{T}) - (\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T})) \leq 0$$

for all $(\mathbf{x}^\mathbf{T}, \mathbf{u}^\mathbf{T})$.

Since $\prod_{t=1}^T X_t$ is open we have

$$(y \circ D_1 J^T(\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T})) + (w \circ D_1 H^T(\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T})) = 0 \tag{4.1}$$

and we have

$$(y \circ D_2 J^T(\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T}) + w \circ D_2 H^T(\hat{\mathbf{x}}^\mathbf{T}, \hat{\mathbf{u}}^\mathbf{T}))(\mathbf{u}^\mathbf{T} - \hat{\mathbf{u}}^\mathbf{T}) \leq 0. \tag{4.2}$$

(4.1) can be written as

$$\begin{aligned} &\sum_{j=1}^{\ell} \theta_j^T \cdot \sum_{t=0}^{T} D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \\ &\quad + \sum_{t=0}^{T} \langle p_{t+1}^T, (D_1 f_t(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) - (x_{t+1} - \hat{x}_{t+1}) \rangle = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\sum_{j=1}^{\ell} \theta_j^T \cdot \sum_{t=0}^{T} D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \\ &\quad + \sum_{t=0}^{T} \langle p_{t+1}^T, D_1 f_t(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \rangle - \sum_{t=0}^{T} \langle p_{t+1}^T, (x_{t+1} - \hat{x}_{t+1}) \rangle = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &\sum_{j=1}^{\ell} \theta_j^T \cdot \sum_{t=1}^{T} D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \\ &\quad + \sum_{t=1}^{T} \langle p_{t+1}^T, D_1 f_t(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \rangle - \sum_{t=1}^{T} \langle p_t^T, (x_t - \hat{x}_t) \rangle = 0, \end{aligned} \tag{4.5}$$

For all $t \in \{1, \ldots, T\}$, for all $x_t \in X_t$ consider $(x_s)_{0 \le s \le T}$ such that $x_s = \hat{x}_s$ for $s \neq t$ we obtain

$$\begin{aligned} &\sum_{j=1}^{\ell} \theta_j^T . D_1 \phi_t^j(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \\ &\quad + \langle p_{t+1}^T, D_1 f_t(\hat{x}_t, \hat{u}_t)(x_t - \hat{x}_t) \rangle - \langle p_t^T, (x_t - \hat{x}_t) \rangle = 0, \end{aligned} \tag{4.6}$$

So

$$\sum_{j=1}^{\ell} \theta_j^T . D_1 \phi_t^j(\hat{x}_t, \hat{u}_t) + p_{t+1}^T \circ D_1 f_t(\hat{x}_t, \hat{u}_t) - p_t^T = 0, \text{ for all } t \in \{1, \ldots, T\} \tag{4.7}$$

which is conclusion (c) of Theorem 3.1 .

(4.2) can be written as

$$\sum_{j=1}^{\ell}\theta_j^T.\sum_{t=0}^{T}D_2\phi_t^j(\hat{x}_t,\hat{u}_t)(u_t-\hat{u}_t)+\sum_{t=0}^{T}\langle p_{t+1}^T,D_2f_t(\hat{x}_t,\hat{u}_t)(u_t-\hat{u}_t)\rangle\leq 0. \tag{4.8}$$

For all $t\in\{0,\dots,T\}$, for all $u_t\in U_t$ consider $(u_s)_{0\leq s\leq T}$ such that $u_s=\hat{u}_s$ for $s\neq t$ we obtain

$$\Big\langle\sum_{j=1}^{\ell}\theta_j^T.D_2\phi_t^j(\hat{x}_t,\hat{u}_t)+p_{t+1}^T\circ D_2f_t(\hat{x}_t,\hat{u}_t),u_t-\hat{u}_t\Big\rangle\leq 0, \tag{4.9}$$

for all $t\in\{0,\dots,T\}$, for all $u_t\in U_t$ which is conclusion (d) of Theorem 3.1 .

□

4.2. **Proof of Theorem 3.2.** The following lemma will be useful in the proof.

Lemma 4.5. *Set $Z:=T_{U_1}(\hat{u}_1)$. Under the hypotheses of Theorem 3.2 we have, for all $T\geq 2$:*

$$(\theta_1^T,\dots,\theta_\ell^T,\ p_2^T\circ D_2f_1(\hat{x}_1,\hat{u}_1)|_Z)\neq(0,\dots,0). \tag{4.10}$$

Proof. Under the hypotheses of Theorem 3.2, Theorem 3.1 applies and under hypothese (vii) of Theorem 3.2, Z is a closed vector space so a Banach space. We shall first show that we have, for all $T\geq 2$

$$(\theta_1^T,\dots,\theta_\ell^T,p_1^T)\neq(0,\dots,0). \tag{4.11}$$

So consider conclusion (c) of Theorem 3.1 :

$$p_t^T=p_{t+1}^T\circ D_1f_t(\hat{x}_t,\hat{u}_t)+\sum_{j=1}^{\ell}\theta_j^T.D_1\phi_t^j(\hat{x}_t,\hat{u}_t),$$

for all $t\in\{1,\dots,T\}$. Assume that there exists $T\geq 2$, such that $(\theta_1^T,\dots,\theta_\ell^T,p_1^T)=(0,\dots,0)$. Then using the invertibility assumption (vi) of Theorem 3.2 and conclusion (c) of Theorem 3.1 with $t=1$ we obtain $p_2^T=0$. Proceeding similarly for $2\leq t\leq T$, we obtain $(\theta_1^T,..,\theta_\ell^T,(p_t^T)_{1\leq t\leq T+1})=(0,\dots,0)$ which contradicts conclusion (a) of Theorem 3.1. So we have for all $T\geq 2$, $(\theta_1^T,\dots,\theta_\ell^T,p_1^T)\neq(0,\dots,0)$.

Now suppose there exists $T\geq 2$, such that

$$(\theta_1^T,\dots,\theta_\ell^T,\ p_2^T\circ D_2f_1(\hat{x}_1,\hat{u}_1)|_Z)=(0,\dots,0).$$

Conclusion (c) of Theorem 3.1 with $t=1$ gives:

$$p_1^T=p_2^T\circ D_1f_1(\hat{x}_1,\hat{u}_1)+\sum_{j=1}^{\ell}\theta_j^T.D_1\phi_1^j(\hat{x}_1,\hat{u}_1),$$

and assumption $(viii)$ of Theorem 3.2 states: $ImD_1f_1(\hat{x}_1,\hat{u}_1)\subset D_2f_1(\hat{x}_1,\hat{u}_1)(Z)$ Thus $p_1^T=0$ which contradicts (4.11). So conclusion (4.10) follows. □

Proof of Theorem 3.2. We now have to prove the existence of multipliers $\theta_1, .., \theta_\ell \in \mathbb{R}$, $(p_t)_{t\in\mathbb{N}_*} \in (X^*)^{\mathbb{N}_*}$, satisfying conclusions (a)-(d) of Theorem 3.2.

Theorem 3.1 provides for all $T \in \mathbb{N}$, $T \geq 2$, multipliers $(\theta_1^T, \dots, \theta_\ell^T) \in \mathbb{R}^\ell$, $(p_t^T)_{1\leq t\leq T+1} \in (X^*)^{T+1}$, satisfying conclusions (a)-(d) of Theorem 3.1. Moreover we obtained in the above lemma that $(\theta_1^T, \dots, \theta_\ell^T,\ p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)|_Z) \neq (0, \dots, 0)$. So we can normalize $(\theta_1^T, \dots, \theta_\ell^T,\ p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)|_Z)$ by writing

$$\sum_{j=1}^{\ell} \theta_j^T + ||p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)||_{Z^*} = 1 \tag{4.12}$$

since the set of multipliers is a cone. Hence the Banach-Alaoglu-Bourbaki theorem, in a separable Banach space [8] provides sequential compactness so there exists a subsequence, also denoted $(\theta_1^T, \dots, \theta_\ell^T,\ p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)|_Z)_{T\geq 2}$ that converges weakly star to a limit $(\theta_1, \dots, \theta_\ell, q)$.

Let us first show that $(\theta_1, \dots, \theta_\ell, q) \neq (0, \dots, 0)$. Since $T_{U_1}(\hat{u}_1)$ is a vector space, for all $z \in Z = T_{U_1}(\hat{u}_1)$, conclusion (d) of Theorem 3.1 applied at $t = 1$ becomes:

$$\langle p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1), z\rangle = -\Big\langle \sum_{j=1}^{\ell} \theta_j^T . D_2 \phi_1^j(\hat{x}_1, \hat{u}_1), z \Big\rangle. \tag{4.13}$$

Thus,

$$\begin{aligned} &|\langle p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1), z\rangle| \\ &\qquad \leq ||(\theta_1^T, .., \theta_\ell^T)||\ ||(\langle D_2\phi_1^1(\hat{x}_1, \hat{u}_1), z\rangle, \dots, \langle D_2\phi_1^\ell(\hat{x}_1, \hat{u}_1), z\rangle)\ ||. \end{aligned} \tag{4.14}$$

Using (4.14) and applying a result of Bachir and Blot ([2], Lemma 3.3) with $Z := T_{U_1}(\hat{u}_1)$, $K := T_{U_1}(\hat{u}_1) = Z$, (so $\overline{Aff(K)} = Z$);
$c_z := ||(\langle D_2\phi_1^1(\hat{x}_1, \hat{u}_1), z\rangle, \dots, \langle D_2\phi_1^\ell(\hat{x}_1, \hat{u}_1), z\rangle)\ ||$;
$a := 0 \in K$ and $B := B_Z(0, 1)$ the closed unit ball of Z,
there exist a positive real number R_B and $b \in Z$ such that

$$||p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)||_{Z^*} \leq R_B\big(\ ||(\theta_1^T, .., \theta_\ell^T)|| + \langle p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1), b\rangle\ \big). \tag{4.15}$$

If $(\theta_1, \dots, \theta_\ell, q) = (0, \dots, 0)$, then taking the limit in (4.15), implies that $\lim_{T\to+\infty} ||p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)||_{Z^*} = 0$ which leads to $\lim_{T\to+\infty}(\sum_{j=1}^{\ell} \theta_j^T + ||p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)||_{Z^*}) = 0$. But this contadicts (4.12). So $(\theta_1, \dots, \theta_\ell, q) \neq (0, \dots, 0)$. Now since $p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1)|_Z$ converges weakly star to q and $Im D_1 f_1(\hat{x}_1, \hat{u}_1) \subset D_2 f_1(\hat{x}_1, \hat{u}_1)(Z)$, it follows that $p_2^T \circ D_1 f_1(\hat{x}_1, \hat{u}_1)$ converges weakly star to some $\overline{q}$. By the invertibility of $D_1 f_1(\hat{x}_1, \hat{u}_1)$ we have

$$p_2^T \circ D_2 f_1(\hat{x}_1, \hat{u}_1) = p_2^T \circ D_1 f_1(\hat{x}_1, \hat{u}_1) \circ [D_1 f_t(\hat{x}_t, \hat{u}_t)]^{-1} \circ D_2 f_1(\hat{x}_1, \hat{u}_1).$$

It follows, after taking the weak star limit, that

$$q = \overline{q} \circ [D_1 f_t(\hat{x}_t, \hat{u}_t)]^{-1} \circ D_2 f_1(\hat{x}_1, \hat{u}_1).$$

Clearly $(\theta_1, \dots, \theta_\ell, \overline{q}) \neq (0, .., 0)$ (otherwise $(\theta_1, \dots, \theta_\ell, q) = (0, \dots, 0)$.)
Now using conclusion (c) of Theorem 3.1 at $t = 1$ with hypothesis $(viii)$ we obtain that $(p_1^T)_{T\geq 2}$ converges weakly star to p_1 where $p_1 = \overline{q} + \sum_{j=1}^{\ell} \theta_j . D_1 \phi_1^j(\hat{x}_1, \hat{u}_1)$. So $(\theta_1^T, \dots, \theta_\ell^T, p_1^T)_{T\geq 2}$ converges weakly star to $(\theta_1, \dots, \theta_\ell, p_1)$ and $(\theta_1, \dots, \theta_\ell, p_1) \neq$

$(0,..,0)$ since $(\theta_1,\dots,\theta_\ell,\overline{q}) \neq (0,\dots,0)$.

Now using hypothesis (vi) with conclusion (c) of Theorem 3.1 we have for all $T \geq 2$:

$$p^T_{t+1} = p^T_t \circ [D_1 f_t(\hat{x}_t,\hat{u}_t)]^{-1} - \sum_{j=1}^{\ell} \theta^T_j . D_1\phi^j_t(\hat{x}_t,\hat{u}_t) \circ [D_1 f_t(\hat{x}_t,\hat{u}_t)]^{-1}$$

and proceeding recursively we obtain that the sequence $(p^T_t)_{T\geq 2}$ converges weakly star to a limit p_t for all $t \in \mathbb{N}_*$ and so we obtain for all $t \in \mathbb{N}$,

$p_t = p_{t+1} \circ D_1 f_t(\hat{x}_t,\hat{u}_t) + \sum_{j=1}^{\ell} \theta_j . D_1\phi^j_t(\hat{x}_t,\hat{u}_t)$,
and
$\langle \sum_{j=1}^{\ell} \theta_j . D_2\phi^j_t(\hat{x}_t,\hat{u}_t) + p_{t+1} \circ D_2 f_t(\hat{x}_t,\hat{u}_t), u_t - \hat{u}_t \rangle \ \leq 0$, for all $u_t \in U_t$,

which are conclusions (c) and (d) of Theorem 3.2. Notice that conclusion (a) of Theorem 3.2 was obtained above and conclusion (b) of Theorem 3.2 is satisfied. Indeed for all $j = 1,\dots,\ell$, $\theta_j \geq 0$ since $\theta^T_j \geq 0$ for all $T \geq 2$. □

4.3. **Proof of Theorem 3.3.** The following Lemma generalizes Lemma 4.7 of [2].

Lemma 4.6. *Under the assumptions of Theorem 3.1 together with assumption (ix), for all $T \in \mathbb{N}$, $T \geq 2$, there exist $(\theta^T_1,..,\theta^T_\ell) \in \mathbb{R}^\ell_+$, $(p^T_t)_{1\leq t\leq T+1} \in (X^*)^{T+1}$, which satisfy the following conditions.*

(a) *For all $T \geq 2$, for all $s \in \{1,\dots,T\}$ and all $1 \leq t \leq T+1$ there exist $a_t, b_t \geq 0$ such that $||p^T_t|| \leq a_t||(\theta^T_1,..,\theta^T_\ell)|| + b_t||p^T_s||$*
(b) *$(\theta^T_1,..,\theta^T_\ell,p^T_s) \neq (0,\dots,0,0)$, for all $s \leq T$*
(c) *for all $s \in \{1,\dots,T\}$ and all $v \in D_2 f_{s-1}(\hat{x}_{s-1},\hat{u}_{s-1})(T_{U_{s-1}}(\hat{u}_{s-1}))$ there exist $d_v \in R$ such that for all $T \geq 2$, $p^T_s(v) \leq d_v||(\theta^T_1,..,\theta^T_\ell)||$*

Proof. Applying Theorem 3.1 and adding conclusions (c) and (d) of Theorem 3.1 gives for all $t \in \{1,\dots,T\}$, for all $h \in X$ and for all $u_t \in U_t$:

$$\langle p^T_{t+1}, D_1 f_t(\hat{x}_t,\hat{u}_t)(h) + D_2 f_t(\hat{x}_t,\hat{u}_t).(u_t - \hat{u}_t)\rangle + \sum_{j=1}^{\ell} \theta^T_j .[D_1\phi^j_t(\hat{x}_t,\hat{u}_t)(h) + D_2\phi^j_t(\hat{x}_t,\hat{u}_t).(u_t-\hat{u}_t)] \leq p^T_t(h)$$

which can be written : for all $t \in \{1,\dots,T\}$, for all $(h,k) \in X \times T_{U_t}(\hat{u}_t)$

$$\langle p^T_{t+1}, Df_t(\hat{x}_t,\hat{u}_t)(h,k)\rangle \leq p^T_t(h) - \sum_{j=1}^{\ell} \theta^T_j D\phi^j_t(\hat{x}_t,\hat{u}_t)(h,k)$$

Hence for all $t \in \{1,\dots,T\}$, for all $(h,k) \in X \times T_{U_t}(\hat{u}_t)$

$$\langle p^T_{t+1}, Df_t(\hat{x}_t,\hat{u}_t)(h,k)\rangle \leq ||p^T_t||\ ||h||_X + \sum_{j=1}^{\ell} \theta^T_j ||D\phi^j_t(\hat{x}_t,\hat{u}_t)||\ ||(h,k)||_{X\times U}$$

Using (ix) we have for all $t \in \mathbb{N}$, $0 \in int[Df_t(\hat{x}_t,\hat{u}_t)((X \times T_{U_t}(\hat{u}_t)) \cap B_{X\times U})]$ so there exists a constant $r_t > 0$ such that $B_X(0,r_t) \subset Df_t(\hat{x}_t,\hat{u}_t)((X \times T_{U_t}(\hat{u}_t)) \cap B_{X\times U})$, thus

$$||p_{t+1}^T|| \le \frac{1}{r_t}\big(||p_t^T|| + \sum_{j=1}^{\ell}\theta_j^T ||D\phi_t^j(\hat{x}_t,\hat{u}_t)||\big)$$
$$\le \frac{1}{r_t}\big(||p_t^T|| + ||(\theta_1^T,..,\theta_\ell^T)||\;||(||D\phi_t^1(\hat{x}_t,\hat{u}_t)||,\dots,||D\phi_t^\ell(\hat{x}_t,\hat{u}_t)||)\;||\;\big)$$

Moreover, using (c) of Theorem (3.1) we obtain:

$$||p_t^T|| \le ||p_{t+1}^T||.||D_1 f_t(\hat{x}_t,\hat{u}_t)|| + \sum_{j=1}^{\ell}\theta_j^T ||D_1\phi_t^j(\hat{x}_t,\hat{u}_t)||$$
$$\le ||p_{t+1}^T||.||D_1 f_t(\hat{x}_t,\hat{u}_t)||$$
$$+ ||(\theta_1^T,..,\theta_\ell^T)||\;||(\;||D\phi_t^1(\hat{x}_t,\hat{u}_t)||,\dots,||D\phi_t^\ell(\hat{x}_t,\hat{u}_t)||\;)\;||$$

Combining the two inequalities garantees conclusion (a) of Lemma 4.6.
To prove conclusion (b) of Lemma 4.6, suppose that there exists $s \in \{1,\dots,T\}$ such that $(\theta_1^T,..,\theta_\ell^T,p_s^T) = (0,\dots,0,0)$, then by conclusion (a) of Lemma 4.6, $p_t^T = 0$ for all $t \in \{1,\dots,T+1\}$ which contradicts conclusion (a) of Theorem 3.1. So conclusion (b) of Lemma 4.6 is proved.

Now using conclusion (d) of Theorem 3.1 we can write for an arbitrary s

$$\langle p_s^T\circ D_2 f_{s-1}(\hat{x}_{s-1},\hat{u}_{s-1}), u_{s-1}-\hat{u}_{s-1}\rangle \;\le\; -\Big\langle \sum_{j=1}^{\ell}\theta_j^T D_2\phi_{s-1}^j(\hat{x}_{s-1},\hat{u}_{s-1}), u_{s-1}-\hat{u}_{s-1}\Big\rangle$$

For all $v \in A_{s-1} = D_2 f_{s-1}(\hat{x}_{s-1},\hat{u}_{s-1})(T_{U_{s-1}}(\hat{u}_{s-1}))$, by definition of $T_{U_{s-1}}(\hat{u}_{s-1})$, there exist $(u_{s-1}^{y_k})_k \in U_{s-1}^N$ and $(\alpha_k)_k \in (R^+)^N$ such that $y_v = \lim_{k\to+\infty}(\alpha_k(u_{s-1}^{y_k} - \hat{u}_{s-1}))$ and $v = D_2 f_{s-1}(\hat{x}_{s-1},\hat{u}_{s-1}).y_v$.

Lettig $k \to +\infty$ in the inequality gives:

$$p_s^T(v) \;\le\; -\Big\langle \sum_{j=1}^{\ell}\theta_j^T D_2\phi_{s-1}^j(\hat{x}_{s-1},\hat{u}_{s-1}), y_v\Big\rangle$$

Set $d_v^j := -\langle D_2\phi_{s-1}^j(\hat{x}_{s-1},\hat{u}_{s-1}), y_v\rangle$ and obtain

$$p_s^T(v) \;\le\; \sum_{j=1}^{\ell}\theta_j^T d_v^j \;\le d_v||(\theta_1^T,..,\theta_\ell^T)||$$

where $d_v := ||(d_v^1,\dots,d_v^\ell)||$ which proves conclusion (c) of Lemma 4.6. □

Proof of Theorem 3.3. The proof is in the spirit of the proof of [2]. Assumption (x) implies that there exists $s \in \mathbb{N}$ such that $A_s = D_2 f_s(\hat{x}_s,\hat{u}_s)(T_{U_s}(\hat{u}_s))$ contains a closed convex subset K with $ri(K) \neq \emptyset$ and such that $\overline{Aff(K)}$ is of finite codimension in X. Since the set of multipliers of a maximization problem is a cone, using the above consequences of Lemma 4.6 we can normalize $(\theta_1^T,\dots,\theta_\ell^T,p_s^T) \neq (0,\dots,0)$ by writing $\sum_{i=1}^{\ell}\theta_i^T + ||p_s^T||_{X^*} = 1$. Now using the above lemma and Proposition 3.9 of [2] we get a strictly increasing map $k \to T_k$ from $\mathbb{N}$ into $\mathbb{N}$, $(\theta_1,..,\theta_\ell) \in \mathbb{R}_+^\ell$, and $(p_t)_{t\in\mathbb{N}_*} \in (X^*)^{\mathbb{N}_*}$ such that

- $(\theta_1^{(T_k)}, .., \theta_\ell^{(T_k)}) \to (\theta_1, .., \theta_\ell) \geq (0, \dots, 0)$ when $k \to +\infty$.
- for each $t \in \mathbb{N}$, $p_t^{(T_k)} \to^{w^*} p_t$ when $k \to +\infty$.
- $(\theta_1, .., \theta_\ell, p_s) \neq (0, \dots, 0, 0)$.

By letting $k \to +\infty$ in conclusions (c) and (d) of Theorem 3.1 we obtain conclusions (c) and (d) of Theorem 3.3. The first point implies (b) of Theorem 3.3. Now if there exists $t > s$ such that $(\theta_1, .., \theta_\ell, p_t) = (0, \dots, 0, 0)$, we use (c) and proceed recursively to obtain that $(\theta_1, .., \theta_\ell, p_s) = (0, \dots, 0, 0)$ which is a contradiction. So (a) of Theorem 3.3 is satisfied. □

Remark: In the single-objective case, Bachir and Blot [2] provided an abstract result (Lemma 3.3 of [2]) which allows to avoid the Josefson-Nissenzweig phenomenon [10] which states that in an infinite dimensional Banach space Z, there always exists a sequence $(p_n)_n$ in the dual space Z^* that is weak null and $\inf_{n \in N} ||p_n|| > 0$. They looked for conditions on a sequence of norm one in Z^* such that this sequence does not converge to the origin in the weak star topology. Proposition 3.9 of [2] is a consequence of Lemma 3.3 of [2].

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