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OPTIMALITY CONDITIONS FOR STRONG SEMIVECTORIAL BILEVEL PROGRAMMING PROBLEMS VIA A CONJUGATE DUALITY

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Abstract. We are concerned with a strong semivectorial nonlinear bilevel programming problem where the upper and lower levels are vectorial and scalar respectively. For such a problem we give a duality approach via scalarization, regularization and a conjugate duality. Then, via this duality approach, we provide necessary and sufficient optimality conditions for the initial semivectorial bilevel programming problem. This duality approach extends the one given in [1] from the scalar case to the semivectorial one.

1. Introduction

In this paper, we are concerned with the following strong semivectorial bilevel programming problem where the upper and lower levels are vectorial and scalar respectively

$$(S) \qquad \mathrm{v}-\min_{\substack{x\in X\\ y\in\mathcal{M}(x)}} F(x,y),$$

with $\mathcal{M}(x)$ is the solution set of the scalar lower level problem

$$\mathcal{P}(x) \qquad \min_{\substack{z\in Y\\ g(x,z)\leq 0}} f(x,z),$$

$F=(F_1,...,F_k)^T:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^k, f:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}, g=(g_1,...,g_m)^T:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^m, m\geq 1$, are convex functions, and X and Y are two nonempty compact convex subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. The notation "$\mathrm{v}-\min$" means that we are concerned with a vectorial minimization problem.

A semivectorial bilevel programming problem is a bilevel problem where exatly one of the two levels is vectorial. Note that several papers dealing with such problems have appeared in the last decade. Let us summarize some recent interesting works of them.

In [2], Ankhili and Mansouri considered a semivectorial bilevel optimization problem with a linear vectorial lower level. For such a problem the authors gave an exact penalty method. They showed that any accumulation point of a sequence of solutions of the penalized problems is a solution of the initial problem. An algorithm

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is presented with a numerical example. In [4], Bonnel and Morgan were interested with a semivectorial bilevel programming problem with a scalar upper level. Then, using weakly efficient solutions of the lower level problem, they proposed a penalty approach to solve the bilevel programming problem. In [7], the authors considered a semivectorial bilevel programming problem with a linear vectorial lower level and the constraint sets of both levels are polyhedral. Considering a reformulation of the problem, and dealing with efficient and weakly efficient solutions of the lower level problem, they showed that a solution exists among the extreme points of the feasible set. In [8], Dempe et al. were interested in strong semivectorial bilevel programming problem where the upper level is scalar. Using a scalarization technique, they considered a transformation that leads to a problem with inequality constraints by means of the value function of the parameterized lower level problem. Then, using the notion of generalized differentiation, they derived first order necessary optimality conditions for both smooth and nonsmooth cases. In [16], the authors proposed a penalty method for a semivectorial bilevel programming problem with a linear vectorial lower level. Using the duality theory of linear programming and the Benson's method, the authors transformed the problem into a single level problem. Then, via the penalty method a solution of the initial problem is obtained. Finally, an algorithm with numerical examples are given. In [21], the authors considered a semivectorial bilevel programming problem with a linear vectorial lower level. The authors presented an exact penalty method with an algorithm that gives a solution of the initial problem via a projection on its feasible set. For more reading on semivectorial bilevel optimization, the reader can consult the references cited in the above works.

In this paper, the approach that we consider for the semivectorial bilevel programming problem (S) is different from those considered in the literature and is based on the use of three operations : a regularization, a scalarization and a conjugate duality. One of the classical constraint qualification that we will use in our investigation is the so-called Slater condition. Unfortunately, as easily remarked, due to the constraint $y \in \mathcal{M}(x)$, the problem (S) and its scalarized problem in the sense of Geoffrion ([10]) do not satisfy this condition. In order to avoid this situation, we first proceed to a regularization of problem (S). More precisely, for $\epsilon > 0$, we consider the following regularized problem (S_ϵ) of (S) ([13])

$$(S_\epsilon) \qquad \text{v} - \min_{\substack{x \in X \\ y \in \mathcal{M}_\epsilon(x)}} F(x, y)$$

with $\mathcal{M}_\epsilon(x)$ is the set of $\epsilon-$approximate solutions of problem $\mathcal{P}(x)$. As a main stability result, we show that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution of the initial problem (S). Then, by construction, the regularized problem (S_ϵ) and its scalarized problem (S_ϵ^s) satisfy the Slater condition. The conjugate duality that we adopt in our study is the so-called Fenchel-Lagrange duality. Such a duality was first introduced for the scalar convex case by Wanka and Boţ in [19] and then extended to the vectorial case in [6] for optimization problems with convex vectorial objective function and d.c. constraints. In order to start our procedure of dualization, and since the problem (S_ϵ) has a non convex constraint set, we first consider a decomposition of problem

(S_ϵ) into a family of vectorial convex minimization problems (S_{ϵ,x^*}), $x^* \in \mathbb{R}^p$. For every $x^* \in \mathbb{R}^p$, and via scalarization, we give the scalarized problem (S^s_{ϵ,x^*}) of (S_{ϵ,x^*}). Since the scalarization preserves the convexity, this allows us to give the Fenchel-Lagrange dual $(\mathcal{D}^s_{\epsilon,x^*})$ of the scalarized convex subproblem (S^s_{ϵ,x^*}). Afterwards, and under the Slater constraint qualification condition, we establish strong duality and provide optimality conditions for the scalar primal-dual pair (S^s_{ϵ,x^*})-$(\mathcal{D}^s_{\epsilon,x^*})$. Finally, via this duality approach we provide necessary and sufficient optimality conditions for problem (S). The obtained results extend those given in [1] from the scalar case to the semivectorial one.

The outline of this paper is as follows. In section 2, we give some preliminaries related to convex analysis and multiobjective optimization. In section 3, we define the regularized problem (S_ϵ) of (S) and establish some stability results. In section 4, we decompose the problem (S_ϵ) into the family of convex semivectorial bilevel programming problems $(S_{\epsilon,x^*})_{x^*\in\mathbb{R}^p}$. In section 5, for a given $x^* \in \mathbb{R}^p$ we define the Fenchel-Lagrange dual $(\mathcal{D}^s_{\epsilon,x^*})$ of the scalarized problem (S^s_{ϵ,x^*}) of (S_{ϵ,x^*}) and provide optimality conditions for problem (S^s_{ϵ,x^*}). In section 6, we provide necessary and sufficient optimality conditions for the initial problem (S). The necessary optimality conditions are given for a class of properly efficient solutions of problem (S).

2. Preliminaries

In this section, we recall some fundamental definitions and results related essentially to convex analysis and multiobjective optimization that we will use in the sequel. For two vectors $x = (x_1, ..., x_n)^\top$, and $y = (y_1, ..., y_n)^\top$ in $\mathbb{R}^n$, $\langle x, y\rangle$ will denote their inner product, i.e., $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Let A be a nonempty subset of $\mathbb{R}^n$. We will denote by ψ_A and σ_A the indicator and the support functions of the set A, respectively, defined on $\mathbb{R}^n$ by

$$\psi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A, \end{cases} \qquad \text{and} \qquad \sigma_A(x) = \sup_{y\in A}\langle x, y\rangle.$$

Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function.The conjugate function of h relative to the set A is denoted by h^*_A and defined on $\mathbb{R}^n$ by

$$h^*_A(p) = \sup_{x\in A}\{\langle p, x\rangle - h(x)\}.$$

When $A = \mathbb{R}^n$, we get the usual Legendre-Fenchel conjugate function of h, denoted by h^*. We denote by $\mathrm{dom}h$ the effective domain of h, i.e., the set defined by

$$\mathrm{dom}h = \{x \in \mathbb{R}^n \ / \ h(x) < +\infty\}.$$

We say that h is proper if $h(x) > -\infty$, for all $x \in \mathbb{R}^n$, and $\mathrm{dom}h \neq \emptyset$.

When A is a nonempty convex subset of $\mathbb{R}^n$ and $\bar{x} \in A$, the normal cone $\mathcal{N}_A(\bar{x})$ of A at $\bar{x}$ in the sense of convex analysis is the set defined by

$$\mathcal{N}_A(\bar{x}) = \big\{x^* \in \mathbb{R}^n / \langle x^*, x - \bar{x}\rangle \leq 0, \forall x \in A\big\}.$$

Now let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Let $\bar{x} \in \mathrm{dom}h$. The subdifferential $\partial h(\bar{x})$ in the sense of convex analysis of h at $\bar{x}$, is the set defined by

$$\partial h(\bar{x}) = \big\{x^* \in \mathbb{R}^n \ / \ h(x) \geq h(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \ \forall x \in \mathbb{R}^n\big\}.$$

An element x^* in $\partial h(\bar{x})$ is called a subgradient of h at $\bar{x}$.

Remark 2.1. *We have the following properties*

i) $x^* \in \partial h(\bar{x}) \Longleftrightarrow \langle x^*, \bar{x}\rangle = h(\bar{x}) + h^*(x^*)$,

ii) $h(x) + h^*(x^*) \geq \langle x^*, x\rangle,\ \forall x, x^* \in \mathbb{R}^n$ *(Fenchel's inequality).*

Theorem 2.2 ([17]). *Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function, and C be a compact subset of int(domh). Then, the set $\bigcup_{x\in C} \partial h(x)$ is compact.*

Theorem 2.3 ([17]). *Let $h_1, ..., h_m : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper convex functions. Assume that $\bigcap_{i=1}^m$ ri(domh$_i$) $\neq \emptyset$. Then, for any $x \in \mathbb{R}^n$, we have*

$$\left(\sum_{i=1}^m h_i\right)^*(x) = \inf_{\substack{x_1,...,x_m\in\mathbb{R}^n \\ x_1+...+x_m=x}} \left\{\sum_{i=1}^m h_i^*(x_i)\right\}$$

and the infimum is attained, where for a subset A of $\mathbb{R}^n$, riA denotes the relative interior of A, i.e., the topological interior of A relative to the smallest affine set containing A.

Let us recall some definitions and fundamental results relating to multiobjective optimization that we will need in our study. Consider the following vector optimization problem

$$(\mathcal{P}) \qquad \mathrm{v} - \min_{x\in\tilde{X}} \tilde{f}(x)$$

where $\tilde{X} \subset \mathbb{R}^n$, $\tilde{f}(x) = (\tilde{f}_1(x), ..., \tilde{f}_r(x))^\top$ and $\tilde{f}_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., r$, are functions. We will consider the following partial order on $\mathbb{R}^r$ defined as follows. For $y = (y_1, ..., y_r)^\top, y' = (y'_1, ..., y'_r)^\top \in \mathbb{R}^r$

$$y \leq y' \quad \text{if and only if} \quad y_i \leq y'_i \ \text{ for all}\, i = 1, ..., r.$$

The relation $y < y'$ means that $\quad y_i < y'_i$ for all $i = 1, ..., r$.

In what follows, we will adopt the following definitions in our study.

Definition 2.4. An element $\bar{x} \in \tilde{X}$ is said to be efficient (or Pareto-efficient) for problem $(\mathcal{P})$ if $\tilde{f}(x) \leq \tilde{f}(\bar{x})$, for some $x \in \tilde{X}$, then $\tilde{f}(\bar{x}) = \tilde{f}(x)$.

Definition 2.5 ([10]). An element $\bar{x} \in \tilde{X}$ is said to be properly efficient for problem $(\mathcal{P})$ if it is efficient and if there exists a real number $M > 0$ such that for each $i \in \{1, ..., r\}$ and $x \in \tilde{X}$ satisfying $\tilde{f}_i(x) < \tilde{f}_i(\bar{x})$, there exists at least one $j \in \{1, ..., r\}$ such that $\tilde{f}_j(\bar{x}) < \tilde{f}_j(x)$ and

$$\frac{\tilde{f}_i(\bar{x}) - \tilde{f}_i(x)}{\tilde{f}_j(x) - \tilde{f}_j(\bar{x})} \leq M.$$

We recall the following result in the convex case that characterizes the proper efficiency via scalarization.

Theorem 2.6 ([10]). *Assume that $\tilde{X}$ and $\tilde{f}$ are convex. A feasible point $\tilde{x}$ is properly efficient for problem $(\mathcal{P})$ if and only if there exists $\lambda = (\lambda_1, ...\lambda_r)^\top \in$ int$(\mathbb{R}^r_+)$*

with $\sum_{i=1}^{r} \lambda_i = 1$, *such that* $\tilde{x}$ *solves the following scalar convex minimization problem*

$$(\mathcal{P}^s) \qquad \inf_{x \in \tilde{X}} \sum_{i=1}^{r} \lambda_i \tilde{f}_i(x).$$

3. Regularization and stability results

As mentioned in the introduction, we will need the Slater constraint qualification condition for the application of the Fenchel-Lagrange duality in our study. Since (S) does not satisfy this condition, we will first proceed to its regularization. Then, unlike (S), by construction the obtained regularized problem will satisfy this condition. This regularization uses ϵ-approximate solutions of the lower level problem ([13]). As a main stability result, we show that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution of the initial problem (S).

Throughout the paper, we set $I = \{1, ..., k\}$ and $J = \{1, ..., m\}$. For $x \in \mathbb{R}^p$, let

$$Z(x) = \Big\{ y \in Y / \, g_i(x, y) \leq 0, \forall i \in J \Big\} \quad \text{and} \quad v(x) = \inf_{y \in Z(x)} f(x, y)$$

denote respectively the feasible set and the infimal value of problem $\mathcal{P}(x)$. The graph of the multifunction $Z(\cdot)$ relative to the set $\mathbb{R}^p \times Y$ denoted for simplicity by $GrZ(\cdot)$ is defined by

$$GrZ(\cdot) = \Big\{ (x, y) \in \mathbb{R}^p \times Y / \, y \in Z(x) \Big\}.$$

Throughout the paper, we assume that the following assumption is satisfied

$(\mathcal{H})$ For every $x \in X$, the set $Z(x)$ is nonempty.

Remark 3.1. 1) *The functions f, F and g are continuous on $\mathbb{R}^p \times \mathbb{R}^q$ as finite convex functions.*

2) *The function $v(\cdot)$ is convex on $\mathbb{R}$ (see [17]).*

3) *Let $x \in \mathbb{R}^p$. We distinguish the following cases:*

   i) *If $Z(x) \neq \emptyset$, then, since the function $f(x, .)$ is continuous on the compact set $Z(x)$ (a closed subset of Y compact), it follows that $v(x)$ is a finite real number. Moreover, according to assumption $(\mathcal{H})$, we have $X \subset domv(\cdot) = \{x \in \mathbb{R}^p \; / \; Z(x) \neq \emptyset\}$. In particular, $\{v(x), x \in X\} \subset \mathbb{R}$.*

   ii) *If $Z(x) = \emptyset$, then, $v(x) = +\infty$.*
   So that, from the two cases we deduce that $\{v(x), x \in \mathbb{R}^p\} \subset \mathbb{R} \cup \{+\infty\}$.

For $\epsilon > 0$, we consider the following multiobjective regularized problem of (S)

$$(S_\epsilon) \qquad \text{v} - \min_{\substack{x \in X \\ y \in \mathcal{M}_\epsilon(x)}} F(x, y)$$

with $\mathcal{M}_\epsilon(x) = \{y \in Z(x) / \; f(x, y) \leq v(x) + \epsilon\}$ is the set of $\epsilon-$approximate solutions of problem $\mathcal{P}(x)$. This expression of $\mathcal{M}_\epsilon(x)$ results from the fact that $v(x)$ is a finite real number [see 3)-i) of Remark 3.1]. Then, the problem (S_ϵ) can be rewritten in its value form as ([15])

$$(S_\epsilon) \qquad \text{v}-\min_{\substack{(x,y)\in X\times Y\\ f(x,y)-v(x)\le\epsilon\\ g_i(x,y)\le 0, i\in J}} F(x,y).$$

Throughout the paper, $(\epsilon_n)_n$ is a sequence of positive real numbers such that $\epsilon_n \searrow 0^+$, and we denote the problem (S_{ϵ_n}) and the multifunction $\mathcal{M}_{\epsilon_n}(\cdot)$ by (S_n) and $\mathcal{M}_n(\cdot)$ respectively.

Proposition 3.2 ([13])**.** *Let assumption $(\mathcal{H})$ hold. Then, for any $x \in X$ and any sequence $(x_n)_n$ converging to x in X, we have $\limsup_{n\to+\infty} \mathcal{M}_n(x_n) \subset \mathcal{M}(x)$.*

Lemma 3.3. *Assume that assumption $(\mathcal{H})$ and the following assumption are satisfied*

$(\mathcal{Q})$ *For any $x \in X$, and any nonempty subset J' of J, there exists $y_{x,J'} \in Y$, such that $g_i(x, y_{x,J'}) < 0,\ \forall i \in J'$.*

Then, for any $x \in X$ and any sequence (x_n) converging to x in X, we have $Z(x) \subset \liminf_{n\to+\infty} Z(x_n)$.

Proof. The result is deduced from a general case considered in [12]. □

Let $\text{Gr}(\mathcal{M})$ and $\text{Gr}(\mathcal{M}_\epsilon)$ $(\epsilon > 0)$ denote the graphs of the multifunctions $\mathcal{M}(\cdot)$ and $\mathcal{M}_\epsilon(\cdot)$ respectively relative to the set $X \times Y$, i.e.,

$\text{Gr}(\mathcal{M}) = \{(x,y) \in X \times Y / \, y \in \mathcal{M}(x)\}$ and $\text{Gr}(\mathcal{M}_\epsilon) = \{(x,y) \in X \times Y / \, y \in \mathcal{M}_\epsilon(x)\}$.

In what follows, for two subsets A and B of $\mathbb{R}^n$, the relation $A \subsetneq B$ means that $A \subset B$ and there exists $x \in B$ such that $x \notin A$. The following theorem establishes that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution of problem (S).

Theorem 3.4. *Let assumptions $(\mathcal{H})$ and $(\mathcal{Q})$ hold. For $n \in \mathbb{N}$, let (x_n, y_n) be a properly efficient solution of problem (S_n). Let $(\bar{x}, \bar{y})$ be an accumulation point of the sequence $(x_n, y_n)_n$. Then, $(\bar{x}, \bar{y})$ is a properly efficient solution of the initial problem (S).*

Proof. Feasibility. We obviously have $(\bar{x}, \bar{y}) \in X \times Y$. Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to +\infty, n \in \mathcal{N}$. For $n \in \mathcal{N}$, we have (x_n, y_n) is a feasible point of (S_n). It follows that for all $n \in \mathcal{N}$

$$f(x_n, y_n) - v(x_n) \le \epsilon_n, \quad \text{and} \quad g_i(x_n, y_n) \le 0, \text{ for all } j \in J. \tag{3.1}$$

Let us show that $\limsup_{\substack{n\to+\infty\\ n\in\mathcal{N}}} v(x_n) \le v(\bar{x})$. Since $\bar{x} \in X \subset \text{dom}v(\cdot)$, then $v(\bar{x}) \in \mathbb{R}$. Moreover, from the continuity of g_i, $i \in J$, we have $g_i(\bar{x}, \bar{y}) \le 0$, for all $i \in J$. Let $y \in Z(\bar{x})$. From Lemma 3.3, there exists $\bar{y}_n \in Z(x_n), n \in \mathcal{N}$ and $\bar{y}_n \to y$ as $n \to +\infty, n \in \mathcal{N}$. So that $v(x_n) \le f(x_n, \bar{y}_n)$. Using the continuity of the function f, we obtain

$$\limsup_{\substack{n\to+\infty\\ n\in\mathcal{N}}} v(x_n) \le \lim_{\substack{n\to+\infty\\ n\in\mathcal{N}}} f(x_n, \bar{y}_n) = f(\bar{x}, y).$$

Since y is arbitrary in $Z(\bar{x})$, it follows that $\limsup_{\substack{n\to+\infty\\ n\in\mathcal{N}}} v(x_n) \leq v(\bar{x})$. Hence, passing to the limit in (3.1) as $n \to +\infty, n \in \mathcal{N}$, we obtain

$$\limsup_{\substack{n\to+\infty\\ n\in\mathcal{N}}} f(x_n, y_n) = f(\bar{x}, \bar{y}) \leq v(\bar{x}).$$

Therefore, $\bar{y} \in \mathcal{M}(\bar{x})$ and hence $(\bar{x}, \bar{y})$ is a feasible point of (S).

Optimality

Efficiency. Let $(x, y) \in \mathrm{Gr}(\mathcal{M})$ such that

$$F_i(x,y) \leq F_i(\bar{x},\bar{y}), \quad \forall i \in I. \tag{3.2}$$

Let us show that for all $i \in I$, we have $F_i(x,y) = F_i(\bar{x},\bar{y})$. We have $(x,y) \in \mathrm{Gr}(\mathcal{M}) \subset \mathrm{Gr}(\mathcal{M}_n)$ and $(x_n, y_n) \in \mathrm{Gr}(\mathcal{M}_n)$. Let $i \in I$. We distinguish the following cases:

1) Assume that there exists $n_0 \in \mathcal{N}$ such that $F_i(x,y) \leq F_i(x_n,y_n)$ for all $n \in \mathcal{N}, n \geq n_0$. For $n \in \mathcal{N}$, (x_n, y_n) is a Pareto efficient solution of (S_n). Then,

$$F_i(x,y) = F_i(x_n, y_n).$$

Passing to the limit as $n \to \infty, n \in \mathcal{N}$, we obtain $F_i(x,y) = F_i(\bar{x},\bar{y})$.

2) Assume that there exists an infinite subset $\mathcal{N}'$ of $\mathcal{N}$, such that

$$F_i(x,y) > F_i(x_n,y_n) \quad \text{for all } n \in \mathcal{N}'.$$

Then, passing to the limit as $n \to \infty, n \in \mathcal{N}'$, we obtain

$$F_i(x,y) \geq F_i(\bar{x},\bar{y}).$$

Using (3.2), we deduce that $F_i(x,y) = F_i(\bar{x},\bar{y})$.

By means of the two cases, and since i is arbitrary in I, we deduce that $F(x,y) = F(\bar{x},\bar{y})$.

Proper efficiency. Now, let us show that $(\bar{x},\bar{y})$ is a properly efficient solution of (S). Assume the contrary. Let $M > 0$ be arbitrary. Then, there exist $(x^*, y^*) \in \mathrm{Gr}(\mathcal{M})$ and $i \in I$, such that

$$F_i(x^*, y^*) < F_i(\bar{x},\bar{y})$$

and

$$\frac{F_i(\bar{x},\bar{y}) - F_i(x^*,y^*)}{F_j(x^*,y^*) - F_j(\bar{x},\bar{y})} > M$$

for all $j \in I \setminus \{i\}$, verifying $F_j(\bar{x},\bar{y}) < F_j(x^*,y^*)$. Set

$$I(\bar{x},\bar{y}) = \{j \in I \setminus \{i\} / \, F_j(\bar{x},\bar{y}) < F_j(x^*,y^*)\}.$$

Therefore, since I is finite, then we easily deduce the following property:

$(\mathcal{L})$ There exists $n_3 \in \mathcal{N}$, such that for all $n \geq n_3$, $n \in \mathcal{N}$, we have

i) $F_i(x^*,y^*) < F_i(x_n,y_n)$,

ii) $F_j(x_n,y_n) < F_j(x^*,y^*)$, $\forall j \in I(\bar{x},\bar{y})$,

iii) $\frac{F_i(x_n,y_n) - F_i(x^*,y^*)}{F_j(x^*,y^*) - F_j(x_n,y_n)} > M$, $\forall j \in I(\bar{x},\bar{y})$.

Set

$$I_{n_3} = \{j \in I \setminus \{i\} / \, F_j(x_n, y_n) < F_j(x^*, y^*), \forall n \geq n_3, n \in \mathcal{N}\}.$$

Let us show that the third assertion in property $(\mathcal{L})$ is also true for all $j \in I_{n_3}$. Then, let $j \in I_{n_3}$. We distinguish the following cases:

3) If $j \in I(\bar{x}, \bar{y})$, then, there is nothing to prove.

4) If $j \notin I(\bar{x}, \bar{y})$, then $F_j(\bar{x}, \bar{y}) \geq F_j(x^*, y^*)$. Two subcases arise.

Subcase 1: Assume $F_j(\bar{x}, \bar{y}) > F_j(x^*, y^*)$. This implies that there exists $n_4 \in \mathcal{N}$ such that

$$F_j(x_n, y_n) > F_j(x^*, y^*) \qquad \forall n \geq n_4, n \in \mathcal{N}. \tag{3.3}$$

Set $n_5 = \max\{n_3, n_4\}$. Then, for all $n \geq n_5$, $n \in \mathcal{N}$, we get a contradiction between (3.3) and the fact that $j \in I_{n_3}$.

Subcase 2: $F_j(\bar{x}, \bar{y}) = F_j(x^*, y^*)$. Assume that there exists an infinite subset $\mathcal{N}' \subset \{n \in \mathcal{N} / \, n \geq n_3\}$, such that

$$\frac{F_i(x_n, y_n) - F_i(x^*, y^*)}{F_j(x^*, y^*) - F_j(x_n, y_n)} \leq M \qquad \forall n \in \mathcal{N}'. \tag{3.4}$$

According to property *iii)* above, we have $F_j(x^*, y^*) - F_j(x_n, y_n) \to 0^+$, as $n \to +\infty$, $n \in \mathcal{N}'$. Then, since $F_i(\bar{x}, \bar{y}) > F_i(x^*, y^*)$, we have

$$\lim_{\substack{n \to +\infty \\ n \in \mathcal{N}'}} \frac{F_i(x_n, y_n) - F_i(x^*, y^*)}{F_j(x^*, y^*) - F_j(x_n, y_n)} = +\infty$$

which leads to a contradiction in (3.4). On the other hand, for all $n \in \mathbb{N}$, we have $\mathcal{M}(x^*) \subset \mathcal{M}_n(x^*)$. So that, $(x^*, y^*) \in \mathrm{Gr}(\mathcal{M}_n)$. That is (x^*, y^*) is a feasible point of problem (S_n), for all $n \in \mathbb{N}$. Then, in summary, we have shown the following property :

$(\hat{\mathcal{L}})$ For $M > 0$ arbitrary, there exist $(x^*, y^*) \in \mathbb{R}^p \times \mathbb{R}^q$, $i \in I$, and $n_3 \in \mathcal{N}$, such that for all $n \geq n_3$, $n \in \mathcal{N}$, we have

i) $(x^*, y^*) \in \mathrm{Gr}(\mathcal{M}_n)$,

ii) $F_i(x^*, y^*) < F_i(x_n, y_n)$,

iii) $\frac{F_i(x_n, y_n) - F_i(x^*, y^*)}{F_j(x^*, y^*) - F_j(x_n, y_n)} > M, \forall j \in I_{n_3}$.

Therefore, the property $(\hat{\mathcal{L}})$ gives a contradiction with the fact that (x_n, y_n) is a properly efficient solution of (S_n), $n \geq n_3$, $n \in \mathcal{N}$. Then, we conclude that $(\bar{x}, \bar{y})$ is a properly efficient solution of (S). □

4. Decomposition of the regularized problem (S_ϵ)

In order to apply the Fenchel-Lagrange duality in our study, in this section we will first decompose the regularized problem (S_ϵ) $(\epsilon > 0)$ into a family of convex semivectorial programming subproblems $(S_{\epsilon, x^*})_{x^* \in \mathbb{R}^p}$.

Define on $X \times Y$ the following functions

$$h_{1,\epsilon}(x, y) = 0 \quad \text{and} \quad h_{2,\epsilon}(x, y) = v(x) + \epsilon.$$

Then, the regularized multiobjective problem (S_ϵ) can be rewritten in the following form

$$(S_\epsilon) \qquad \text{v}-\min_{\substack{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q\\ \psi_{X\times Y}(x,y)-h_{1,\epsilon}(x,y)\le 0\\ f(x,y)-h_{2,\epsilon}(x,y)\le 0\\ g_i(x,y)-h_{1,\epsilon}(x,y)\le 0,\ \forall i\in J}} F(x,y)$$

which under the data is a multiobjective minimization problem with convex vectorial objective function and d.c. constraints. This formulation will allow us to apply some existing results in the literature in [6] and [14]. As mentioned in the introduction our duality is based on the use of conjugacy. So that, we will express the constraints of the regularized problem in terms of the conjugates of the functions involved.

Proposition 4.1. *Let assumption $(\mathcal{H})$ hold. Then, for all $(x^*,y^*)\in\mathbb{R}^p\times\mathbb{R}^q$, we have*

i) $h^*_{1,\epsilon}(x^*,y^*)=\psi_{\{(0,0)\}}(x^*,y^*)$,

ii) $h^*_{2,\epsilon}(x^*,y^*)=\begin{cases}+\infty, & \text{if } y^*\neq 0,\\ f^*_{GrZ(\cdot)}(x^*,0)-\epsilon, & \text{if } y^*=0.\end{cases}$

Proof. The results are obvious. So that the proofs are omitted. □

For $\epsilon>0$, let $\mathcal{A}^\epsilon$ denote the feasible set of problem (S_ϵ), i.e.,

$$\mathcal{A}^\epsilon=\Big\{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q/\psi_{X\times Y}(x,y)-h_{1,\epsilon}(x,y)\le 0, f(x,y)-h_{2,\epsilon}(x,y)\le 0,\\ g_i(x,y)-h_{1,\epsilon}(x,y)\le 0,\forall i\in J\Big\}.$$

We obtain the following expression of $\mathcal{A}^\epsilon$ using the conjugate of functions.

Proposition 4.2. *Let assumption $(\mathcal{H})$ hold. We have*

$$\mathcal{A}^\epsilon=\bigcup_{\substack{(x^*,y^*)\in\mathbb{R}^p\times\mathbb{R}^q\\ (t^*,z^*)\in\mathbb{R}^p\times\mathbb{R}^q\\ (u_i^*,v_i^*)\in\mathbb{R}^p\times\mathbb{R}^q,\forall i\in J\\ h^*_{1,\epsilon}(x^*,y^*)-\psi^*_{X\times Y}(x^*,y^*)\le 0\\ h^*_{2,\epsilon}(t^*,z^*)-f^*(t^*,z^*)\le 0\\ h^*_{1,\epsilon}(u_i^*,v_i^*)-g_i^*(u_i^*,v_i^*)\le 0,\forall i\in J}}\Big\{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q/\,h^*_{1,\epsilon}(x^*,y^*)+\psi_{X\times Y}(x,y)-\langle\tbinom{x^*}{y^*},\tbinom{x}{y}\rangle\le 0\\ h^*_{2,\epsilon}(t^*,z^*)+f(x,y)-\langle\tbinom{t^*}{z^*},\tbinom{x}{y}\rangle\le 0\\ h^*_{1,\epsilon}(u_i^*,v_i^*)+g_i(x,y)-\langle\tbinom{u_i^*}{v_i^*},\tbinom{x}{y}\rangle\le 0,\forall i\in J\Big\}.$$

Proof. The proof is a direct application of Lemma 2.1 in [14]. So that, it is omitted. □

For $x^*\in\mathbb{R}^p$ and $\epsilon>0$, set

$$\mathcal{A}^\epsilon_{x^*}=\Big\{(x,y)\in X\times Y\ /\ f^*_{GrZ(\cdot)}(x^*,0_{\mathbb{R}^q})+f(x,y)-\langle x^*,x\rangle\le\epsilon,\\ g_i(x,y)\le 0,\forall i\in J\Big\}.$$

Then, using the results of Proposition 4.1 and proceeding to a simplification in the formula of Proposition 4.2, we obtain the following expression of $\mathcal{A}^\epsilon$ in terms of the sets $\mathcal{A}^\epsilon_{x^*}$, $x^*\in\mathbb{R}^p$.

Proposition 4.3. *Let assumption $(\mathcal{H})$ hold. Then $\mathcal{A}^\epsilon=\bigcup_{x^*\in\mathbb{R}^p}\mathcal{A}^\epsilon_{x^*}$.*

Proof. The result is obvious. So that, the proof is omitted. □

For given $x^* \in \mathbb{R}^p$ and $\epsilon > 0$, consider the following convex semivectorial programming problem

$$(S_{\epsilon,x^*}) \qquad \text{v} - \min_{(x,y)\in\mathcal{A}^{\epsilon}_{x^*}} F(x,y).$$

Proposition 4.4. *Let $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ be a properly efficient solution of (S_ϵ). Then, there exists $x^*_\epsilon \in \mathbb{R}^p$ such that $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ is a properly efficient solution of problem $(S_{\epsilon,x^*_\epsilon})$.*

Proof. We have

$$(S_\epsilon) \quad \text{v} - \min_{(x,y)\in\mathcal{A}^{\epsilon}} F(x,y).$$

Since $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ is a feasible point of (S_ϵ), then according to Proposition 4.3 there exists $x^*_\epsilon \in \mathbb{R}^p$ such that $(\bar{x}_\epsilon, \bar{y}_\epsilon) \in \mathcal{A}^{\epsilon}_{x^*_\epsilon}$. Then, $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ is a feasible point of problem $(S_{\epsilon,x^*_\epsilon})$.

Let us show that $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ is a properly efficient solution of $(S_{\epsilon,x^*_\epsilon})$.

Efficiency. Let $(\tilde{x}_\epsilon, \tilde{y}_\epsilon) \in \mathcal{A}^{\epsilon}_{x^*_\epsilon}$ such that $F(\tilde{x}_\epsilon, \tilde{y}_\epsilon) \leq F(\bar{x}_\epsilon, \bar{y}_\epsilon)$. Since $(\tilde{x}_\epsilon, \tilde{y}_\epsilon) \in \mathcal{A}^{\epsilon}$ and $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ is efficient for (S_ϵ), it follows that $F(\tilde{x}_\epsilon, \tilde{y}_\epsilon) = F(\bar{x}_\epsilon, \bar{y}_\epsilon)$, and the result follows.

Proper efficiency. Let $M > 0$ be the constant given by the proper efficiency of $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ to problem (S_ϵ). Let $(x,y) \in \mathcal{A}^{\epsilon}_{x^*_\epsilon}$ and $i \in I$ such that $F_i(x,y) < F_i(\bar{x}_\epsilon, \bar{y}_\epsilon)$. Since $(x,y) \in \mathcal{A}^{\epsilon}$, then

$$\begin{cases} \exists j \in \{1,...,k\}\backslash\{i\} \quad \text{such that} \quad F_j(\bar{x}_\epsilon, \bar{y}_\epsilon) < F_j(x,y) \\ \text{and} \\ \frac{F_i(\bar{x}_\epsilon,\bar{y}_\epsilon) - F_i(x,y)}{F_j(x,y) - F_j(\bar{x}_\epsilon,\bar{y}_\epsilon)} \leq M. \end{cases}$$

Hence, $(\bar{x}_\epsilon, \bar{y}_\epsilon)$ is a properly efficient solution of problem (S_{ϵ,x^*}). □

Then, according to Proposition 4.4, we obtain a decomposition of problem (S_ϵ) into the family of convex semivectorial programming problems $(S_{\epsilon,x^*})_{x^*\in\mathbb{R}^p}$.

5. Duality and optimality conditions for the scalarized problem of (S_{ϵ,x^*})

In this section, for given $\epsilon > 0$ and $x^* \in \mathbb{R}^p$, we first consider the scalarized problem (S^s_{ϵ,x^*}) of (S_{ϵ,x^*}) in the sense of Geoffrion ([10]). Then, we give the Fenchel-Lagrange dual $(\mathcal{D}^s_{\epsilon,x^*})$ of (S^s_{ϵ,x^*}), establish strong duality between them, and finally provide necessary and sufficient optimality conditions for problem (S^s_{ϵ,x^*}).

For $\epsilon > 0$ and $x^* \in \mathbb{R}^p$, we consider the following scalar convex minimization problem associated to the multiobjective problem (S_{ϵ,x^*})

$$(S^s_{\epsilon,x^*}) \qquad \min_{\substack{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q \\ \tilde{g}_\epsilon(x,y)\leq 0}} \sum_{j=1}^{k} \lambda_j F_j(x,y)$$

where $\lambda = (\lambda_1, ..., \lambda_k)^\top \in \text{int}\mathbb{R}^k_+$ is fixed, and $\tilde{g}_\epsilon = (\psi_{X\times Y}, \tilde{g}_{0,\epsilon}, \tilde{g}_{1,\epsilon}, ...\tilde{g}_{m,\epsilon})^T$, with

$$\begin{cases} \tilde{g}_{0,\epsilon}(x,y) = f^*_{GrZ(\cdot)}(x^*, 0) + f(x,y) - \langle x^*, x\rangle - \epsilon, \\ \tilde{g}_{i,\epsilon}(x,y) = g_i(x,y), \forall i \in J. \end{cases}$$

We will use the following constraint qualification (the Slater condition)

$(\mathcal{CQ})_{\epsilon,\mathbf{x}^*}$ There exists $(x'_{\epsilon,x^*}, y'_{\epsilon,x^*}) \in X \times Y$ such that

$$\begin{cases} f(x'_{\epsilon,x^*}, y'_{\epsilon,x^*}) + f^*_{GrZ(\cdot)}(x^*, 0_{\mathbb{R}^q}) - \langle x^*, x'_{\epsilon,x^*}\rangle < \epsilon, \\ g_i(x'_{\epsilon,x^*}, y'_{\epsilon,x^*}) < 0, \forall i \in J. \end{cases}$$

Consider the following dual problem of (S^s_{ϵ,x^*}) called the Fenchel-Lagrange dual ([19])

$$(\widehat{\mathcal{D}}^s_{\epsilon,x^*}) \qquad \sup_{\substack{(a,b)\in\mathbb{R}^p\times\mathbb{R}^q \\ \tilde{q}=(\tilde{q}_0,\dots,\tilde{q}_{m+1})^\top\in\mathbb{R}^{m+2}_+}} \Big\{ -\Big(\sum_{j=1}^k \lambda_j F_j\Big)^*(a,b) - (\tilde{q}^\top \tilde{g}_\epsilon)^*(-a,-b)\Big\}.$$

Let us give an explicit formulation of the objective function of problem $(\widehat{\mathcal{D}}^s_{\epsilon,x^*})$. Using Theorem 2.3, we obtain

$$\left(\sum_{j=1}^k \lambda_j F_j\right)^*(a,b) = \min_{\substack{(a,b)\in\mathbb{R}^p\times\mathbb{R}^q \\ (a,b)=\sum_{j=1}^k (a_j,b_j)}} \sum_{j=1}^k (\lambda_j F_j)^*(a_j,b_j).$$

We have

$$\begin{aligned} -\Big(\sum_{j=1}^k \lambda_j F_j\Big)^*(a,b) &= -\min_{\substack{(a_j,b_j)\in\mathbb{R}^p\times\mathbb{R}^q \\ (a,b)=\sum_{j=1}^k (a_j,b_j)}} \textstyle\sum_{j=1}^k (\lambda_j F_j)^*(a_j,b_j) \\ &= -\min_{\substack{(a_j,b_j)\in\mathbb{R}^p\times\mathbb{R}^q \\ (a,b)=\sum_{j=1}^k (a_j,b_j)}} \textstyle\sum_{j=1}^k \lambda_j F_j^*(\frac{a_j}{\lambda_j}, \frac{b_j}{\lambda_j}) \\ &= \max_{\substack{(a_j,b_j)\in\mathbb{R}^p\times\mathbb{R}^q \\ (a,b)=\sum_{j=1}^k (a_j,b_j)}} \textstyle\sum_{j=1}^k (-\lambda_j F_j^*)(\frac{a_j}{\lambda_j}, \frac{b_j}{\lambda_j}). \end{aligned}$$

So that

$\mathcal{B} = \sup_{\substack{(a,b)\in\mathbb{R}^p\times\mathbb{R}^q \\ \tilde{q}\in\mathbb{R}^{m+2}_+}} \Big\{ -\Big(\sum_{j=1}^k \lambda_j F_j\Big)^*(a,b) - (\tilde{q}^\top\tilde{g}_\epsilon)^*(-a,-b)\Big\} =$
$\sup_{\substack{(a,b)\in\mathbb{R}^p\times\mathbb{R}^q \\ \tilde{q}\in\mathbb{R}^{m+2}_+}} \Big\{ \max_{\substack{(a_j,b_j)\in\mathbb{R}^p\times\mathbb{R}^q \\ (a,b)=\sum_{j=1}^k (a_j,b_j)}} \sum_{j=1}^k (-\lambda_j F_j^*)(\frac{a_j}{\lambda_j}, \frac{b_j}{\lambda_j}) - (\tilde{q}^\top\tilde{g}_\epsilon)^*(-a,-b)\Big\}$

Set $a'_j = \frac{a_j}{\lambda_j}$, $b'_j = \frac{b_j}{\lambda_j}$, $j = 1, ..., k$. Then, we obtain

$$\mathcal{B} = \sup_{\substack{(a'_j,b'_j)\in\mathbb{R}^p\times\mathbb{R}^q \\ \tilde{q}\in\mathbb{R}^{m+2}_+}} \Big\{ \textstyle\sum_{j=1}^k (-\lambda_j F_j^*)(a'_j, b'_j) - (\tilde{q}^\top\tilde{g}_\epsilon)^*(-\sum_{j=1}^k \lambda_j(a'_j, b'_j))\Big\}$$

On the other hand, we have

$$
\begin{aligned}
&(\tilde{q}^\top \tilde{g}_\epsilon)^*\Big(-\textstyle\sum_{j=1}^k \lambda_j a_j', -\sum_{j=1}^k \lambda_j b_j'\Big)\\
&= \sup_{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q} \Big\{\Big\langle \begin{pmatrix} -\sum_{j=1}^k \lambda_j a_j' \\ -\sum_{j=1}^k \lambda_j b_j' \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle - (\tilde{q}^\top \tilde{g}_\epsilon)(x,y)\Big\}\\
&= \sup_{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q} \Big\{\Big\langle \begin{pmatrix} -\sum_{j=1}^k \lambda_j a_j' \\ -\sum_{j=1}^k \lambda_j b_j' \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle - (\tilde{q}_0\psi_{X\times Y} + \sum_{i=0}^m \tilde{q}_{i+1}\tilde{g}_{i,\epsilon})(x,y)\Big\}\\
&= \sup_{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q} \Big\{\Big\langle \begin{pmatrix} -\sum_{j=1}^k \lambda_j a_j' \\ -\sum_{j=1}^k \lambda_j b_j' \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle - \psi_{X\times Y}(x,y) - \sum_{i=0}^m \tilde{q}_{i+1}\tilde{g}_{i,\epsilon}(x,y)\Big\}\\
&= \left(\sum_{i=0}^m \tilde{q}_{i+1}\tilde{g}_{i,\epsilon}\right)^*_{X\times Y}\Big(-\sum_{j=1}^k \lambda_j a_j', -\sum_{j=1}^k \lambda_j b_j'\Big).
\end{aligned}
$$

So that $(\widehat{\mathcal{D}}^s_{\epsilon,x^*})$ and the following problem

$$
(\mathcal{D}^s_{\epsilon,x^*}) \qquad \sup_{\substack{(a_j,b_j)\in\mathbb{R}^p\times\mathbb{R}^q\\ j=1,\dots,k,\\ (\tilde{q}_0,\dots,\tilde{q}_m)\in\mathbb{R}^{m+1}_+}} \Big\{-\textstyle\sum_{j=1}^k \lambda_j F_j^*(a_j,b_j) - \Big(\displaystyle\sum_{i=0}^m \tilde{q}_i\tilde{g}_{i,\epsilon}\Big)^*_{X\times Y}\Big(-\sum_{j=1}^k \lambda_j a_j, -\sum_{j=1}^k \lambda_j b_j\Big)\Big\}
$$

have the same optimal value. Finally, via simple calculations, we obtain the following simplified form

$$
(\mathcal{D}^s_{\epsilon,x^*}) \quad \sup_{\substack{(a_j,b_j)\in\mathbb{R}^p\times\mathbb{R}^q\\ j=1,\dots,k,\\ (\tilde{q}_0,\dots,\tilde{q}_m)\in\mathbb{R}^{m+1}_+}} \inf_{(x,y)\in X\times Y} \Big\{-\sum_{j=1}^k \lambda_j F_j^*(a_j,b_j) + \Big\langle \Big(\begin{smallmatrix} \sum_{j=1}^k \lambda_j a_j - \tilde{q}_0 x^* \\ \sum_{j=1}^k \lambda_j b_j \end{smallmatrix}\Big), \Big(\begin{smallmatrix} x \\ y \end{smallmatrix}\Big)\Big\rangle + \Big(\tilde{q}_0 f + \sum_{i=1}^m \tilde{q}_i g_i\Big)(x,y) + \tilde{q}_0(f^*_{GrZ(\cdot)}(x^*, 0_{\mathbb{R}^q}) - \epsilon)\Big\}.
$$

In what follows, we will use the problem $(\mathcal{D}^s_{\epsilon,x^*})$ instead of $(\widehat{\mathcal{D}}^s_{\epsilon,x^*})$, and without losing the sense of this duality, we will also call it the Fenchel-Lagrange dual of problem (S^s_{ϵ,x^*}).

In the rest of this section $\epsilon > 0$ and $x^* \in \mathbb{R}^n$ are given. The following proposition establishes that weak duality always holds between (S^s_{ϵ,x^*}) and $(\mathcal{D}^s_{\epsilon,x^*})$.

Proposition 5.1. *Let assumption $(\mathcal{H})$ hold. Then, we have* $\inf S^s_{\epsilon,x^*} \geq \sup \mathcal{D}^s_{\epsilon,x^*}$.

Proof. The result uses the fact that $\sup \widehat{\mathcal{D}}^s_{\epsilon,x^*} = \sup \mathcal{D}^s_{\epsilon,x^*}$ and the well-known result of weak Fenchel-Lagrange duality between (S^s_{ϵ,x^*}) and $(\widehat{\mathcal{D}}^s_{\epsilon,x^*})$(see [6] where duality for scalar problems is considered). □

The following theorem establishes strong duality between (S^s_{ϵ,x^*}) and $(\mathcal{D}^s_{\epsilon,x^*})$.

Theorem 5.2. *Assume that assumption $(\mathcal{H})$ and the constraint qualification $(\mathcal{CQ})_{\epsilon,x^*}$ are fulfilled. Then strong duality holds between problems (S^s_{ϵ,x^*}) and $(\mathcal{D}^s_{\epsilon,x^*})$, i.e.,* $\inf S^s_{\epsilon,x^*} = \sup \mathcal{D}^s_{\epsilon,x^*}$ *and $(\mathcal{D}^s_{\epsilon,x^*})$ admits a solution.*

Proof. The result follows from Theorem 3.3 in [6]. $\square$

The following theorems provide optimality conditions for the primal-dual pair $(S^s_{\epsilon,x^*}) - (\mathcal{D}^s_{\epsilon,x^*})$.

Theorem 5.3. *(Necessary optimality conditions) Assume that assumption $(\mathcal{H})$ and the constraint qualification $(\mathcal{CQ})_{\epsilon,x^*}$ are fulfilled. Let $(x_{\epsilon,x^*}, y_{\epsilon,x^*})$ be a solution of problem (S^s_{ϵ,x^*}). Then, there exists a solution $((a_\epsilon, b_\epsilon), \alpha_\epsilon)$ of $(\mathcal{D}^s_{\epsilon,x^*})$, with $a_\epsilon = (a_{1\epsilon}, ..., a_{k\epsilon}), b_\epsilon = (b_{1\epsilon}, ..., b_{k\epsilon}), a_{i\epsilon} \in \mathbb{R}^p, b_{i\epsilon} \in \mathbb{R}^q, i = 1, ..., k, \alpha_\epsilon = (\alpha_{0,\epsilon}, ..., \alpha_{m,\epsilon})^\top \in \mathbb{R}^{m+1}_+$, such that the following optimality conditions are satisfied*

i) $\binom{a_{j\epsilon}}{b_{j\epsilon}} \in \partial F_j(x_{\epsilon,x^*}, y_{\epsilon,x^*}), j = 1, ..., k,$

ii) $\left\{ \begin{array}{l} \alpha_{0,\epsilon}(f(x_{\epsilon,x^*}, y_{\epsilon,x^*}) + f^*_{GrZ(\cdot)}(x^*, 0_{\mathbb{R}^q}) - \langle x^*, x_{\epsilon,x^*}\rangle - \epsilon) = 0, \\ \alpha_{i,\epsilon} g_i(x_{\epsilon,x^*}, y_{\epsilon,x^*}) = 0, \forall i \in J, \end{array} \right.$

iii) $\begin{pmatrix} \alpha_{0,\epsilon}x^* - \sum_{i=1}^k \lambda_i a_{i\epsilon} \\ - \sum_{i=1}^k \lambda_i b_{i\epsilon} \end{pmatrix} \in \partial \left(\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i\right)(x_{\epsilon,x^*}, y_{\epsilon,x^*}) +$
$$\mathcal{N}_{X\times Y}(x_{\epsilon,x^*}, y_{\epsilon,x^*}).$$

Proof. The properties *i)* and *ii)* are directly obtained by application of Theorem 3.4 in [6]. Let us show the property *iii)*. From Theorem 3.4 in [6], we also have

$$\left(\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i\right)^*_{X\times Y}(\alpha_{0,\epsilon}x^* - \sum_{i=1}^k \lambda_i a_{i\epsilon}, - \sum_{i=1}^k \lambda_i b_{i\epsilon}) =$$
$$\left\langle \binom{-\sum_{i=1}^k \lambda_i a_{i\epsilon}}{-\sum_{i=1}^k \lambda_i b_{i\epsilon}}, \binom{x_{\epsilon,x^*}}{y_{\epsilon,x^*}} \right\rangle + \alpha_{0,\epsilon}(f^*_{GrZ(\cdot)}(x^*, 0_{\mathbb{R}^q}) - \epsilon).$$

Then, using the first property in *ii)* we obtain

$$\left(\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i\right)^*_{X\times Y}(\alpha_{0,\epsilon}x^* - \sum_{i=1}^k \lambda_i a_{i\epsilon}, - \sum_{i=1}^k \lambda_i b_{i\epsilon}) =$$
$$\left\langle \binom{-\sum_{i=1}^k \lambda_i a_{i\epsilon}}{-\sum_{i=1}^k \lambda_i b_{i\epsilon}}, \binom{x_{\epsilon,x^*}}{y_{\epsilon,x^*}} \right\rangle + \alpha_{0,\epsilon}\left(\langle x^*, x_{\epsilon,x^*}\rangle - f(x_{\epsilon,x^*}, y_{\epsilon,x^*})\right).$$

So that, using the second property in *ii)*, we obtain

$$\left\langle \binom{-\sum_{i=1}^k \lambda_i a_{i\epsilon} + \alpha_{0,\epsilon}x^*}{-\sum_{i=1}^k \lambda_i b_{i\epsilon}}, \binom{x_{\epsilon,x^*}}{y_{\epsilon,x^*}} \right\rangle - (\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i)(x_{\epsilon,x^*}, y_{\epsilon,x^*}) \geq$$
$$\left\langle \binom{-\sum_{i=1}^k \lambda_i a_{i\epsilon} + \alpha_{0,\epsilon}x^*}{-\sum_{i=1}^k \lambda_i b_{i\epsilon}}, \binom{x}{y} \right\rangle - (\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i)(x, y), \ \ \forall (x, y) \in X \times Y.$$

i.e., $(x_{\epsilon,x^*}, y_{\epsilon,x^*})$ solves the problem

$$\min_{(x,y)\in X\times Y} \left\{ \left\langle \binom{\sum_{i=1}^k \lambda_i a_{i\epsilon} - \alpha_{0,\epsilon}x^*}{\sum_{i=1}^k \lambda_i b_{i\epsilon}}, \binom{x}{y} \right\rangle + (\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i)(x, y) \right\}.$$

Then

$$\binom{0_{\mathbb{R}^p}}{0_{\mathbb{R}^q}} \in \partial\left(\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i\right)(x_{\epsilon,x^*}, y_{\epsilon,x^*}) + \binom{\sum_{i=1}^k \lambda_i a_{i\epsilon} - \alpha_{0,\epsilon}x^*}{\sum_{i=1}^k \lambda_i b_{i\epsilon}} +$$
$$\mathcal{N}_{X\times Y}(x_{\epsilon,x^*}, y_{\epsilon,x^*}).$$

It follows that

$$\begin{pmatrix} \alpha_{0,\epsilon}x^* - \sum_{i=1}^k \lambda_i a_{i\epsilon} \\ - \sum_{i=1}^k \lambda_i b_{i\epsilon} \end{pmatrix} \in \partial(\alpha_{0,\epsilon} f + \sum_{i=1}^m \alpha_{i,\epsilon} g_i)(x_{\epsilon,x^*}, y_{\epsilon,x^*}) + \mathcal{N}_{X\times Y}(x_{\epsilon,x^*}, y_{\epsilon,x^*}).$$

$\square$

Theorem 5.4 (Sufficient optimality conditions)**.** *Assume that assumption* $(\mathcal{H})$ *and the constraint qualification* $(\mathcal{CQ})_{\epsilon,x^*}$ *are fulfilled. Let* $(x_{\epsilon,x^*}, y_{\epsilon,x^*})$ *and* $(a_\epsilon, b_\epsilon, \alpha_\epsilon)$ *be feasible points of problems* (S^s_{ϵ,x^*}) *and* $(\mathcal{D}^s_{\epsilon,x^*})$ *respectively satisfying the above properties* $i) - iii)$ *with* $a_\epsilon = (a_{1\epsilon}, ..., a_{k\epsilon}), b_\epsilon = (b_{1\epsilon}, ..., b_{k\epsilon}), a_{i\epsilon} \in \mathbb{R}^p, b_{i\epsilon} \in \mathbb{R}^q, i = 1, ..., k, \alpha_\epsilon = (\alpha_{0,\epsilon}, ..., \alpha_{m,\epsilon})^\top \in \mathbb{R}^{m+1}_+$. *Then,* $(x_{\epsilon,x^*}, y_{\epsilon,x^*})$ *and* $(a_\epsilon, b_\epsilon, \alpha_\epsilon)$ *solve the problems* (S^s_{ϵ,x^*}) *and* $(\mathcal{D}^s_{\epsilon,x^*})$ *respectively, and strong duality holds between them.*

Proof. The result is obtained by application of Theorem 3.4 in [6]. □

Remark 5.5. *It is not difficult to see that if* $\alpha_{0,\epsilon} > 0$*, then, the first complementary slackness condition* $ii)$ *in Theorem 5.3 yields the following property*

$$\begin{pmatrix} x^* \\ 0_{\mathbb{R}^q} \end{pmatrix} \in \partial_\epsilon (f + \psi_{GrZ(\cdot)})(x_{\epsilon,x^*}, y_{\epsilon,x^*}).$$

6. Optimality conditions for problem (S)

In this section, we provide necessary and sufficient optimality conditions for the semivectorial nonlinear bilevel programming problem (S). We will need the following additional assumptions

$(\mathcal{H}_1)$ For every $\epsilon > 0$ sufficiently small, there exists $(x_\epsilon, y_\epsilon) \in \operatorname{int}(X \times Y)$ such that $g_i(x_\epsilon, y_\epsilon) < 0$, $\forall i \in J$, and $f(x_\epsilon, y_\epsilon) \leq \inf_{y \in Z(x_\epsilon)} f(x_\epsilon, y) + \epsilon$,

$(\mathcal{H}_2)$ There exists $(\tilde{x}, \tilde{y}) \in \mathbb{R}^p \times \mathbb{R}^q$ such that we have

$$F_j(\tilde{x}, \tilde{y}) < F_j(x, y), \quad \forall (x, y) \in \mathrm{Gr}Z(\cdot),\ \forall j \in I,$$

$(\mathcal{H}_3)$ $\inf_{y \in \mathbb{R}^q} f(x, y) < \inf_{y \in Z(x)} f(x, y)$, $\forall x \in X$.

For $x \in \mathbb{R}^p$, we define the function $f_x(\cdot)$ on $\mathbb{R}^q$ by $f_x(y) = f(x, y)$.

Remark 6.1. 1) *Assumption* $(\mathcal{H}_1)$ *implies that* $y_\epsilon \in \mathcal{M}(\epsilon, x_\epsilon)$*. Hence,* (x_ϵ, y_ϵ) *is a feasible point of problem* (S_ϵ)*.*

2) *Assumptions* $(\mathcal{H}_2)$ *and* $(\mathcal{H}_3)$ *imply respectively, that*

i) $\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix} \notin \partial \left(\sum_{j=1}^k \lambda_j F_j \right)(x, y), \forall (x, y) \in GrZ(\cdot),$

ii) For every $x \in X$, $0_{\mathbb{R}^q} \notin \partial f_x(y), \forall y \in Z(x)$.

Let the following example where assumptions $(\mathcal{H}), (\mathcal{H}_1) - (\mathcal{H}_3)$ and assumptions of convexity and compactness are satisfied.

Example 6.1. *Let* $X = [0, 1]$, $Y = [-1, 2]$, $F_i, i = 1, 2, 3, f$ *and* g_1, g_2 *be the functions defined on* $\mathbb{R} \times \mathbb{R}$ *by*

$$F_1(x, y) = x^2 + y, \quad F_2(x, y) = y, \quad F_3(x, y) = 2x^2 + 2y,$$
$$f(x, y) = x + y, \quad g_1(x, y) = x^2 - y, \quad g_2(x, y) = y - x.$$

Then, X *and* Y *are compact convex sets and* F, f *and* g *are convex functions. For* $x \in \mathbb{R}$*, we have*

$$Z(x) = \left\{ y \in [-1, 2] / \, x^2 \leq y \leq x \right\} \quad \text{and} \quad v(x) = \inf_{y \in Z(x)} f(x, y) = x^2 + x.$$

The graph of $Z(\cdot)$ *relative to* $\mathbb{R} \times Y$ *is*

$$GrZ(\cdot) = \left\{ (x, y) \in \mathbb{R} \times [-1, 2] / \, x^2 \leq y \leq x \right\}.$$

It is easy to verify that assumption $(\mathcal{Q})$ in Lemma 3.3 is satisfied. Let us verify if assumptions $(\mathcal{H})$ and $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ are satisfied.

The verification of the fulfillment of assumption $(\mathcal{H})$ is trivial.

-Assumption $(\mathcal{H}_1)$: Let $\epsilon > 0$ sufficiently small and $x \in [0,1]$. We have $\mathcal{M}_\epsilon(x) = [x^2, x^2+\epsilon]$. Let $x_\epsilon = \frac{1}{2}$ and $y_\epsilon = \epsilon + \frac{1}{4}$. Then, $x_\epsilon \in intX$ and $y_\epsilon \in intY$. Moreover, we have

$$\begin{cases} g_1(x_\epsilon, y_\epsilon) = -\epsilon < 0, \\ g_2(x_\epsilon, y_\epsilon) = \epsilon - \frac{1}{4} < 0, \\ f(x_\epsilon, y_\epsilon) \leq v(x_\epsilon) + \epsilon. \end{cases}$$

So that, assumption $(\mathcal{H}_1)$ is satisfied.

-Assumption $(\mathcal{H}_2)$: Let $(\tilde{x}, \tilde{y}) = (0, -2)$. Then, for all $(x,y) \in GrZ(.)$, we have

$$\begin{cases} -2 = F_1(\tilde{x}, \tilde{y}) < F_1(x,y) = x^2 + y, \\ -2 = F_2(\tilde{x}, \tilde{y}) < F_2(x,y) = y, \\ -4 = F_3(\tilde{x}, \tilde{y}) < F_3(x,y) = 2x^2 + 2y. \end{cases}$$

So that, assumption $(\mathcal{H}_2)$ is satisfied.

-Assumption $(\mathcal{H}_3)$: Let $x \in [0,1]$. Then $\inf_{y\in\mathbb{R}}\{x+y\} = -\infty$ and $\inf_{y\in Z(x)}\{x+y\} = x^2 + x$. So that, assumption $(\mathcal{H}_3)$ is satisfied.

Let $\epsilon_n \searrow 0^+$ and (x_n, y_n) be a properly efficient solution of problem (S_n) given by assumption $(\mathcal{H}_1)$. The following theorem gives necessary optimality conditions for the properly efficient solutions of problem (S) which are accumulation points of the sequence $(x_n, y_n)_n$.

Theorem 6.2 (Necessary optimality conditions)**.** *Let assumptions $(\mathcal{H})$, $(\mathcal{Q})$, and $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ hold. Assume moreover that the following constraint qualification is satisfied for every $\epsilon > 0$:*

$(\mathcal{CQ})_\epsilon$ *For all $x^* \in \mathbb{R}^p$, there exists $(x'_{\epsilon,x^*}, y'_{\epsilon,x^*}) \in X \times Y$ such that*

$$\begin{cases} f(x'_{\epsilon,x^*}, y'_{\epsilon,x^*}) + f^*_{GrZ(.)}(x^*, 0_{\mathbb{R}^q}) - \langle x^*, x'_{\epsilon,x^*}\rangle < \epsilon, \\ g_i(x'_{\epsilon,x^*}, y'_{\epsilon,x^*}) < 0, \forall i \in J. \end{cases}$$

Let $(\bar{x}, \bar{y})$ be an accumulation point of the sequence $(x_n, y_n)_n$. Then, $(\bar{x}, \bar{y})$ is a properly efficient solution of problem (S), and there exist $x^* \in \mathbb{R}^p, \bar{a}_j \in \mathbb{R}^p, \bar{b}_j \in \mathbb{R}^q, i \in I, \bar{q} \in \mathbb{R}^*_+$, such that

i) $\binom{\bar{a}_j}{\bar{b}_j} \in \partial F_j(\bar{x}, \bar{y}),\ j = 1, ..., k,$ *ii)* $\binom{x^*}{0} \in \partial f(\bar{x}, \bar{y}) + \mathcal{N}_{GrZ(\cdot)}(\bar{x}, \bar{y}),$

iii) $$\begin{pmatrix} x^* - \frac{\sum_{j=1}^k \bar{\lambda}_j \bar{a}_j}{\bar{q}} \\ -\frac{\sum_{j=1}^k \bar{\lambda}_j \bar{b}_j}{\bar{q}} \end{pmatrix} \in \partial f(\bar{x}, \bar{y}).$$

Proof. We have $(x_n, y_n) \in \mathcal{A}_n$. By Proposition 4.4, there exists $x_n^* \in \mathbb{R}^p$ such that (x_n, y_n) is properly efficient to (S_{ϵ_n, x_n^*}). From Theorem 2.6, there exists

$\bar{\lambda}^n = (\bar{\lambda}_1^n, ..., \bar{\lambda}_k^n) \in \mathrm{int}(\mathbb{R}_+^k), \sum_{i=1}^k \bar{\lambda}_i^n = 1,$ such that (x_n, y_n) solves the scalar minimization problem

$$(S^s_{\epsilon_n, x_n^*}) \qquad \inf_{(x,y)\in \mathcal{A}^{\epsilon_n}_{x_n^*}} \sum_{j=1}^k \bar{\lambda}_j^n F_j(x,y).$$

On the other hand, the constraint qualification $(\mathcal{CQ})_{\epsilon_n}$ implies the constraint qualification $(\mathcal{CQ})_{\epsilon_n, x_n^*}$. Then, from Theorem 5.2 the problems $(S^s_{\epsilon_n,x_n^*})$ and $(\mathcal{D}^s_{\epsilon_n,x_n^*})$ are in strong duality. Furthermore, Theorem 5.3 implies that there exists a solution $((\tilde{a}_n, \tilde{b}_n), \tilde{q}_n)$ to the dual problem $(\mathcal{D}^s_{\epsilon_n,x_n^*}), \tilde{q}_n = (\tilde{q}_{0,n}, ..., \tilde{q}_{m,n}) \in \mathbb{R}_+^{m+1}$ such that the following optimality conditions are satisfied

$a)$ $\binom{\tilde{a}_{jn}}{\tilde{b}_{jn}} \in \partial F_j(x_n, y_n),\ j = 1, ..., k,$

$$b) \begin{cases} \tilde{q}_{0,n}(f(x_n,y_n) + f^*_{GrZ(\cdot)}(x_n^*, 0_{\mathbb{R}^q}) - \epsilon_n - \langle x_n^*, x_n\rangle) = 0, \\ \tilde{q}_{i,n} g_i(x_n,y_n) = 0,\ \forall i \in J, \end{cases}$$

$$c) \begin{pmatrix} \tilde{q}_{0,n}x_n^* - \sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ -\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix} \in \partial(\tilde{q}_{0,n} f + \textstyle\sum_{i=1}^m \tilde{q}_{i,n} g_i)(x_n,y_n) + \mathcal{N}_{X\times Y}(x_n,y_n),$$

First of all, let us show that $\tilde{q}_{0,n} > 0$, for large $n \in \mathbb{N}$. Assume that there exists an infinite subset $\mathcal{N}^*$ of $\mathbb{N}$ such that $\tilde{q}_{0,n} = 0$, for all $n \in \mathcal{N}^*$. Let $n \in \mathcal{N}^*$. From assumption $(\mathcal{H}_1)$ we have $g_i(x_n, y_n) < 0$, for all $i \in J$ and $(x_n, y_n) \in \mathrm{int}(X \times Y)$. This latter property implies that $\mathcal{N}_{X\times Y}(x_n,y_n) = \left\{\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q}\end{pmatrix}\right\}$. On the other hand, (x_n, y_n) satisfies $b)$. Hence, $\tilde{q}_{i,n} g_i(x_n, y_n) = 0$, for all $i \in J$. It follows that $\tilde{q}_{i,n} = 0$, for all $i \in J$. Then, property $c)$ becomes

$$\begin{pmatrix} \tilde{q}_{0,n}x_n^* - \sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ -\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix} \in \partial(\tilde{q}_{0,n} f)(x_n,y_n).$$

That is, for all $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, we have

$$\tilde{q}_{0,n} f(x,y) \geq \left\langle \begin{pmatrix} \tilde{q}_{0,n}x_n^* - \sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ -\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle + \tilde{q}_{0,n} f(x_n, y_n).$$

Since $\tilde{q}_{0,n} = 0$, then

$$\left\langle \begin{pmatrix} -\sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ -\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle \leq 0,\ \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q.$$

i.e.,

$$\begin{pmatrix} -\sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ -\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix} \in \mathcal{N}_{\mathbb{R}^p\times\mathbb{R}^q}(x_n,y_n) = \left\{\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q}\end{pmatrix}\right\}.$$

From property $a)$ we have $\binom{\tilde{a}_{jn}}{\tilde{b}_{jn}} \in \partial F_j(x_n, y_n)$, i.e.,

$$F_j(x,y) \geq F_j(x_n,y_n) + \left\langle \begin{pmatrix} \tilde{a}_{jn} \\ \tilde{b}_{jn} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle, \forall j \in I, \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q. \tag{6.1}$$

Then, for all $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$

$$\lambda_j^n F_j(x,y) \geq \lambda_j^n F_j(x_n,y_n) + \left\langle \begin{pmatrix} \lambda_j^n \tilde{a}_{jn} \\ \lambda_j^n \tilde{b}_{jn} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle, \ \forall j \in I.$$

Hence

$$\sum_{j=1}^{k} \lambda_j^n F_j(x,y) \geq \sum_{j=1}^{k} \lambda_j^n F_j(x_n,y_n) + \left\langle \begin{pmatrix} \sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ \sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle, \quad \forall j \in I, \ \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q.$$

So that

$$\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ \sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix} \in \partial \left(\sum_{j=1}^{k} \lambda_j^n F_j \right) (x_n, y_n). \tag{6.2}$$

Furthermore, assumption $(\mathcal{H}_2)$ implies that there exists $(\tilde{x}, \tilde{y}) \in \mathbb{R}^p \times \mathbb{R}^q$ verifying $F_j(\tilde{x}, \tilde{y}) < F_j(x,y), \ \forall j \in I, \ \forall (x,y) \in X \times Y$. Then,

$$\begin{aligned} \inf_{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q} \sum_{j=1}^{k} \lambda_j^n F_j(x,y) &\leq \sum_{j=1}^{k} \lambda_j^n F_j(\tilde{x}, \tilde{y}) \\ &< \sum_{j=1}^{k} \lambda_j^n F_j(x,y), \ \forall (x,y) \in X \times Y. \end{aligned}$$

Since the function $\sum_{j=1}^k \lambda_j^n F_j$ is lower semicontinuous on $\mathbb{R}^p \times \mathbb{R}^q$ and $X \times Y$ is compact, it follows that

$$\inf_{(x,y) \in \mathbb{R}^p \times \mathbb{R}^q} \sum_{j=1}^{k} \lambda_j^n F_j(x,y) < \min_{(x,y) \in X \times Y} \sum_{j=1}^{k} \lambda_j^n F_j(x,y).$$

Therefore

$$\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix} \notin \partial \Big(\sum_{j=1}^{k} \lambda_j^n F_j \Big) (x_n, y_n)$$

which gives a contradiction with (6.2). Hence $\tilde{q}_{0,n} > 0$, for a large $n \in \mathbb{N}$. We deduce that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, n \in \mathbb{N}, \tilde{q}_{0,n} > 0$. Now, let us show that the accumulation point $(\bar{x}, \bar{y})$ is a properly efficient solution of problem (S). Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ such that $(x_n, y_n) \to (\bar{x}, \bar{y})$, as $n \to +\infty, n \in \mathcal{N}, n \geq n_0$. Since (x_n, y_n) is a properly efficient solution of (S_n), then, from Theorem 3.4, $(\bar{x}, \bar{y})$

is a properly efficient solution of (S). In order to show the properties $i) - iii)$, set $\mathcal{N}_1 = \mathcal{N} \cap \{n \in \mathbb{N} \,/\, n \geq n_0\}$.

Property $i)$: For $n \in \mathcal{N}_1$, we have

$$\begin{pmatrix} \tilde{a}_{jn} \\ \tilde{b}_{jn} \end{pmatrix} \in \partial F_j(x_n, y_n) \subset \bigcup_{(x,y)\in X\times Y} \partial F_j(x,y).$$

Since $X \times Y \subset \operatorname{int}(\operatorname{dom} F) = \mathbb{R}^p \times \mathbb{R}^q$ and $X \times Y$ is compact, it follows that $\bigcup_{(x,y)\in X\times Y} \partial F(x,y)$ is compact (Theorem 2.2). Then, there exists an infinite subset $\mathcal{N}_2$ of $\mathcal{N}_1$ such that the sequence $(\tilde{a}_{jn}, \tilde{b}_{jn})_{n\in\mathcal{N}_2}$ converges to $(\bar{a}_j, \bar{b}_j)$. Then, passing to the limit in (6.1) as $n \to +\infty, n \in \mathcal{N}_2$, we deduce that

$$\begin{pmatrix} \bar{a}_j \\ \bar{b}_j \end{pmatrix} \in \partial F_j(\bar{x}, \bar{y}),\ \forall j \in I.$$

Hence, the property $i)$ is satisfied.

Property $ii)$: Let $n \in \mathcal{N}_2$. Since $\tilde{q}_{0,n} > 0$, the first complementary slackness condition in $b)$ yields

$$\begin{aligned} f^*_{GrZ(.)}(x^*_n, 0_{\mathbb{R}^q}) &= \Big\langle \begin{pmatrix} x^*_n \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \Big\rangle - f(x_n, y_n) + \epsilon_n \\ &= \sup_{(x,y)\in GrZ(.)} \Big\{ \Big\langle \begin{pmatrix} x^*_n \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \Big\rangle - f(x,y) \Big\}. \end{aligned}$$

So that

$$f(x,y) \geq f(x_n, y_n) + \langle x^*_n, x - x_n \rangle - \epsilon_n,\ \forall (x,y) \in \operatorname{Gr}Z(\cdot). \tag{6.3}$$

Hence, for all $x \in \mathbb{R}^p$, we have

$$\inf_{y\in Z(x)} f(x,y) = v(x) \geq v(x_n) + \langle x^*_n, x - x_n \rangle - \epsilon_n$$

i.e., $x^*_n \in \partial_{\epsilon_n} v(x_n)$. Let $\epsilon^* > 0$. Since $\epsilon_n \searrow 0^+, n \in \mathcal{N}_2$, then, there exists $n_1 \in \mathcal{N}_2$ such that $\epsilon_n < \epsilon^*,\ \forall n \geq n_1, n \in \mathcal{N}_2$. Hence

$$\partial_{\epsilon_n} v(x_n) \subset \partial_{\epsilon^*} v(x_n) \quad n \geq n_1,\ \forall n \in \mathcal{N}_2.$$

Since $x^*_n \in \partial_{\epsilon_n} v(x_n) \subset \bigcup_{x\in X} \partial_{\epsilon^*} v(x)$ which is compact (Theorem 2.2), then, there exists an infinite subset $\mathcal{N}_3$ of $\mathcal{N}_2$ such that $x^*_n \to x^*$, as $n \to +\infty, n \in \mathcal{N}_3$. After passing to the limit in (6.3) as $n \to +\infty, n \in \mathcal{N}_3$, we obtain

$$f(x,y) \geq f(\bar{x}, \bar{y}) + \langle x^*, x - \bar{x} \rangle,\ \forall (x,y) \in \operatorname{Gr}Z(\cdot).$$

Then

$$\Big\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \Big\rangle - f(x,y) \leq \Big\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \Big\rangle - f(\bar{x}, \bar{y}),\ \forall (x,y) \in \operatorname{Gr}Z(\cdot).$$

So that

$$\sup_{(x,y)\in GrZ(\cdot)} \Big\{ \Big\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \Big\rangle - f(x,y) \Big\} = \Big\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \Big\rangle - f(\bar{x}, \bar{y}),$$

i.e.,

$$f^*_{GrZ(\cdot)}(x^*, 0_{\mathbb{R}^q}) = \left\langle \begin{pmatrix} x^* \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle - f(\bar{x}, \bar{y}).$$

Which is equivalent to $\binom{x^*}{0} \in \partial f(\bar{x}, \bar{y}) + \mathcal{N}_{GrZ(\cdot)}(\bar{x}, \bar{y})$. That is property *ii)* is satisfied.

Property *iii)*: Let $n \in \mathcal{N}$. From property *c)* and the fact that $(x_n, y_n) \in \text{int}(X \times Y)$ we have

$$\begin{pmatrix} \tilde{q}_{0,n} x_n^* - \sum_{j=1}^k \lambda_j^n \tilde{a}_{jn} \\ -\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} \end{pmatrix} \in \partial(\tilde{q}_{0,n} f)(x_n, y_n).$$

That is for all $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$, we have

$$f(x, y) \geq f(x_n, y_n) + \left\langle \begin{pmatrix} x_n^* - \frac{\sum_{j=1}^k \lambda_j^n \tilde{a}_{jn}}{\tilde{q}_{0,n}} \\ -\frac{\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn}}{\tilde{q}_{0,n}} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle. \tag{6.4}$$

So that

$$\begin{pmatrix} x_n^* - \frac{\sum_{j=1}^k \lambda_j^n \tilde{a}_{jn}}{\tilde{q}_{0,n}} \\ -\frac{\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn}}{\tilde{q}_{0,n}} \end{pmatrix} \in \partial f(x_n, y_n).$$

Moreover, we have $\partial f(x_n, y_n) \subset \bigcup_{(x,y) \in X \times Y} \partial f(x, y)$. Since, $\bigcup_{(x,y) \in X \times Y} \partial f(x, y)$ is compact (Theorem 2.2), then, there exists $(r_1, r_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and an infinite subset $\mathcal{N}_4$ of $\mathcal{N}_3$, such that

$$r_{1n} = x_n^* - \frac{\sum_{j=1}^k \lambda_j^n \tilde{a}_{jn}}{\tilde{q}_{0,n}} \to r_1 \quad \text{and} \quad r_{2n} = -\frac{\sum_{j=1}^k \lambda_j^n \tilde{b}_{jn}}{\tilde{q}_{0,n}} \to r_2,$$

as $n \to +\infty, n \in \mathcal{N}_4$. Passing to the limit in (6.4) as $n \to +\infty, n \in \mathcal{N}_4$, we obtain

$$f(x, y) \geq f(\bar{x}, \bar{y}) + \left\langle \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} \right\rangle, \forall x \in \mathbb{R}^p, \forall y \in \mathbb{R}^q. \tag{6.5}$$

That is $(r_1, r_2) \in \partial f(\bar{x}, \bar{y})$. For $x = \bar{x}$ in (6.5), we deduce that $r_2 \in \partial f_{\bar{x}}(\bar{y})$. Moreover, from assumption $(\mathcal{H}_1)$, we have $g_i(x_n, y_n) < 0, \ \forall i \in J$. Passing to the limit as $n \to +\infty, n \in \mathcal{N}_4$, and using the continuity of the functions $g_i, i \in J$ and the closedness of the set Y, we deduce that $\bar{y} \in Z(\bar{x})$. Moreover, assumption $(\mathcal{H}_3)$ implies that $0_{\mathbb{R}^q} \notin \partial f_{\bar{x}}(\bar{y})$. So that $r_2 \neq 0_{\mathbb{R}^q}$. We have $||r_{2n}|| \to ||r_2|| \quad$ as $n \to +\infty, n \in \mathcal{N}_4$. Since $r_2 \neq 0_{\mathbb{R}^q}$, then, there exists $n_2 \in \mathcal{N}_4$ such that

$$||r_{2n}|| > 0, \ \forall n \geq n_2, n \in \mathcal{N}_4.$$

Hence, for all $n \geq n_2, n \in \mathcal{N}_4$, we have $\tilde{q}_{0,n} = \frac{|| \sum_{j=1}^k \lambda_j^n \tilde{b}_{jn} ||}{||r_{2n}||}$. Since for each $n \in \mathbb{N}$ we have $\sum_{j=1}^k \lambda_j^n = 1$, and $\lambda_j^n > 0$, then, $\lambda_j^n \in [0, 1], j \in I$. From the compactness of the set $[0,1]$, there exists $\mathcal{N}_5 \subset \mathcal{N}_4$ such that $\lambda_j^n \to \bar{\lambda}_j$, as $n \to +\infty, n \in \mathcal{N}_5$. So that

$$\sum_{j=1}^{k} \lambda_j^n \tilde{b}_{jn} \to \sum_{j=1}^{k} \bar{\lambda}_j \bar{b}_j, \text{ as } n \to +\infty, \; n \in \mathcal{N}_5.$$

Hence

$$\tilde{q}_{0,n} = \frac{|| \sum_{j=1}^{k} \lambda_j^n \tilde{b}_{jn} ||}{||r_{2n}||} \to \bar{q} = \frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{b}_j}{||r_2||}, \text{ as } n \to +\infty, n \in \mathcal{N}_5.$$

Then

$$\begin{pmatrix} x_n^* - \frac{\sum_{j=1}^{k} \lambda_j^n \tilde{a}_{jn}}{\tilde{q}_{0,n}} \\ -\frac{\sum_{j=1}^{k} \lambda_j^n \tilde{b}_{jn}}{\tilde{q}_{0,n}} \end{pmatrix} \to \begin{pmatrix} x^* - \frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{a}_j}{\bar{q}} \\ -\frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{b}_j}{\bar{q}} \end{pmatrix}, \text{ as } n \to +\infty, n \in \mathcal{N}_5.$$

On the other hand, passing to the limit in (6.4) as $n \to +\infty, n \in \mathcal{N}_5$, we obtain

$$f(x,y) \geq f(\bar{x},\bar{y}) + \left\langle \begin{pmatrix} x^* - \frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{a}_j}{\bar{q}} \\ -\frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{b}_j}{\bar{q}} \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} \right\rangle, \quad \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q.$$

So that

$$\begin{pmatrix} x^* - \frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{a}_j}{\bar{q}} \\ -\frac{\sum_{j=1}^{k} \bar{\lambda}_j \bar{b}_j}{\bar{q}} \end{pmatrix} \in \partial f(\bar{x},\bar{y}).$$

That is the property *iii)* is satisfied. □

Theorem 6.3 (Sufficient optimality conditions)**.** *Assume that assumption* $(\mathcal{H})$ *is satisfied. Let* $(\bar{x},\bar{y}) \in X \times Y$*, satisfying* $g_i(\bar{x},\bar{y}) \leq 0$*, for all* $i \in J$*. Assume that there exists* $x^* \in \mathbb{R}^p, \bar{a}_i \in \mathbb{R}^p, i = 1,...,k, \bar{b}_i \in \mathbb{R}^q, i = 1,...,k, \bar{q} \in \mathbb{R}_+^*, \bar{\lambda} \in int(\mathbb{R}_+^k)$ *such that the following optimality conditions are satisfied*

i) $\binom{\sum_{i=1}^{k} \lambda_i \bar{a}_i}{\sum_{i=1}^{k} \lambda_i \bar{b}_i} \in \partial F_j(\bar{x},\bar{y}), \; j = 1,...,k,$

ii) $\binom{0}{0} \in \partial f(\bar{x},\bar{y}) + \mathcal{N}_{GrZ(\cdot)}(\bar{x},\bar{y}),$

iii) $\begin{pmatrix} x^* - \frac{\sum_{i=1}^{k} \lambda_i \bar{a}_i}{\bar{q}} \\ -\frac{\sum_{i=1}^{k} \lambda_i \bar{b}_i}{\bar{q}} \end{pmatrix} \in \partial f(\bar{x},\bar{y}),$

iv) $(\bar{x},\bar{y})$ *solves the problem* $(\mathcal{Q}_{x^*}) : \; \max_{(x,y) \in X \times Y} \{ f(x,y) - \langle x^*, x \rangle \}.$

Then, $(\bar{x},\bar{y})$ *is a properly efficient solution of problem* (S)*.*

Proof. Feasibility: Follows from *ii)*.
Efficiency: Let $x \in X$, and $y \in \mathcal{M}(x)$ such that

$$F(x,y) \leq F(\bar{x},\bar{y}).$$

Let us show that $F(x,y)=F(\bar{x},\bar{y})$. From $i)$ for $j\in I$, we have

$$F_j^*\left(\sum_{i=1}^k \lambda_i\bar{a}_i,\sum_{i=1}^k \lambda_i\bar{b}_i\right)=\left\langle\begin{pmatrix}\sum_{i=1}^k \lambda_i\bar{a}_i\\ \sum_{i=1}^k \lambda_i\bar{b}_i\end{pmatrix},\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\right\rangle-F_j(\bar{x},\bar{y}).$$

Hence, for all $(\tilde{x},\tilde{y})\in\mathbb{R}^p\times\mathbb{R}^q$, we have

$$\left\langle\begin{pmatrix}\sum_{i=1}^k \lambda_i\bar{a}_i\\ \sum_{i=1}^k \lambda_i\bar{b}_i\end{pmatrix},\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\right\rangle-F_j(\bar{x},\bar{y})\geq\left\langle\begin{pmatrix}\sum_{i=1}^k \lambda_i\bar{a}_i\\ \sum_{i=1}^k \lambda_i\bar{b}_i\end{pmatrix},\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\right\rangle-F_j(\tilde{x},\tilde{y}).$$

For $(\tilde{x},\tilde{y})=(x,y)$, we obtain

$$F_j(\bar{x},\bar{y})\leq F_j(x,y)+\left\langle\begin{pmatrix}\sum_{i=1}^k \lambda_i\bar{a}_i\\ \sum_{i=1}^k \lambda_i\bar{b}_i\end{pmatrix},\begin{pmatrix}\bar{x}-x\\ \bar{y}-y\end{pmatrix}\right\rangle,\ \forall j=1,...,k. \tag{6.6}$$

From $iii)$, for all $(x^{'},y^{'})\in\mathbb{R}^p\times\mathbb{R}^q$, we have

$$f(x^{'},y^{'})\geq f(\bar{x},\bar{y})+\left\langle\begin{pmatrix}x^*-\frac{\sum_{i=1}^k \lambda_i\bar{a}_i}{\bar{q}}\\ -\frac{\sum_{i=1}^k \lambda_i\bar{b}_i}{\bar{q}}\end{pmatrix},\begin{pmatrix}x^{'}-\bar{x}\\ y^{'}-\bar{y}\end{pmatrix}\right\rangle.$$

Then, for all $(x^{'},y^{'})\in\mathbb{R}^p\times\mathbb{R}^q$

$$\left\langle\begin{pmatrix}\frac{\sum_{i=1}^k \lambda_i\bar{a}_i}{\bar{q}}\\ \frac{\sum_{i=1}^k \lambda_i\bar{b}_i}{\bar{q}}\end{pmatrix},\begin{pmatrix}x^{'}-\bar{x}\\ y^{'}-\bar{y}\end{pmatrix}\right\rangle\geq f(\bar{x},\bar{y})-f(x^{'},y^{'})+\left\langle\begin{pmatrix}x^*\\ 0\end{pmatrix},\begin{pmatrix}x^{'}-\bar{x}\\ y^{'}-\bar{y}\end{pmatrix}\right\rangle.$$

For $(x^{'},y^{'})=(x,y)$, we obtain

$$\left\langle\begin{pmatrix}\frac{\sum_{i=1}^k \lambda_i\bar{a}_i}{\bar{q}}\\ \frac{\sum_{i=1}^k \lambda_i\bar{b}_i}{\bar{q}}\end{pmatrix},\begin{pmatrix}x-\bar{x}\\ y-\bar{y}\end{pmatrix}\right\rangle\geq f(\bar{x},\bar{y})-f(x,y)+\left\langle\begin{pmatrix}x^*\\ 0\end{pmatrix},\begin{pmatrix}x-\bar{x}\\ y-\bar{y}\end{pmatrix}\right\rangle. \tag{6.7}$$

On the other hand, property $iv)$ implies $f(\bar{x},\bar{y})-\langle x^*,\bar{x}\rangle\geq f(x,y)-\langle x^*,x\rangle$. Hence, from (6.7) we obtain $\left\langle\begin{pmatrix}\frac{\sum_{i=1}^k \lambda_i\bar{a}_i}{\bar{q}}\\ \frac{\sum_{i=1}^k \lambda_i\bar{b}_i}{\bar{q}}\end{pmatrix},\begin{pmatrix}x-\bar{x}\\ y-\bar{y}\end{pmatrix}\right\rangle\geq 0$. It follows, from (6.6) that $F_j(\bar{x},\bar{y})\leq F_j(x,y)$, $\forall j=1,...,k$. Hence $F(x,y)=F(\bar{x},\bar{y})$.

Proper efficiency: Let us show that $(\bar{x},\bar{y})$ is a properly efficient solution of (S). Assume the contrary. Let $M>0$ be arbitrary. Then, there exists $(x',y')\in\mathrm{Gr}\mathcal{M}$ and $i\in J$ such that

$$\begin{cases}F_i(x',y')<F_i(\bar{x},\bar{y})\\ \text{and}\\ \frac{F_i(\bar{x},\bar{y})-F_i(x',y')}{F_j(x',y')-F_j(\bar{x},\bar{y})}>M.\end{cases} \tag{6.8}$$

for all $j \in J$, verifying $F_j(\bar{x}, \bar{y}) < F_j(x', y')$. From i), for $j \in I$, we have

$$F_j^*\left(\sum_{i=1}^{k} \lambda_i \bar{a}_i, \sum_{i=1}^{k} \lambda_i \bar{b}_i\right) = \left\langle \begin{pmatrix} \sum_{i=1}^{k} \lambda_i \bar{a}_i \\ \sum_{i=1}^{k} \lambda_i \bar{b}_i \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle - F_j(\bar{x}, \bar{y}).$$

Hence, for all $(\tilde{x}, \tilde{y}) \in \mathbb{R}^p \times \mathbb{R}^q$, we have

$$\left\langle \begin{pmatrix} \sum_{i=1}^{k} \lambda_i \bar{a}_i \\ \sum_{i=1}^{k} \lambda_i \bar{b}_i \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle - F_j(\bar{x}, \bar{y}) \geq \left\langle \begin{pmatrix} \sum_{i=1}^{k} \lambda_i \bar{a}_i \\ \sum_{i=1}^{k} \lambda_i \bar{b}_i \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \right\rangle - F_j(\tilde{x}, \tilde{y}).$$

For $(\tilde{x}, \tilde{y}) = (x', y')$, we obtain

$$F_j(\bar{x}, \bar{y}) \leq F_j(x', y') + \left\langle \begin{pmatrix} \sum_{i=1}^{k} \lambda_i \bar{a}_i \\ \sum_{i=1}^{k} \lambda_i \bar{b}_i \end{pmatrix}, \begin{pmatrix} \bar{x} - x' \\ \bar{y} - y' \end{pmatrix} \right\rangle, \ \forall j = 1, ..., k. \tag{6.9}$$

From iii), for all $(\tilde{x}, \tilde{y}) \in \mathbb{R}^p \times \mathbb{R}^q$, we have

$$\left\langle \begin{pmatrix} \frac{\sum_{i=1}^{k} \lambda_i \bar{a}_i}{\bar{q}} \\ \frac{\sum_{i=1}^{k} \lambda_i \bar{b}_i}{\bar{q}} \end{pmatrix}, \begin{pmatrix} \tilde{x} - \bar{x} \\ \tilde{y} - \bar{y} \end{pmatrix} \right\rangle \geq f(\bar{x}, \bar{y}) - f(\tilde{x}, \tilde{y}) + \left\langle \begin{pmatrix} x^* \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{x} - \bar{x} \\ \tilde{y} - \bar{y} \end{pmatrix} \right\rangle.$$

For $(\tilde{x}, \tilde{y}) = (x', y')$, we obtain

$$\left\langle \begin{pmatrix} \frac{\sum_{i=1}^{k} \lambda_i \bar{a}_i}{\bar{q}} \\ \frac{\sum_{i=1}^{k} \lambda_i \bar{b}_i}{\bar{q}} \end{pmatrix}, \begin{pmatrix} x' - \bar{x} \\ y' - \bar{y} \end{pmatrix} \right\rangle \geq f(\bar{x}, \bar{y}) - f(x', y') + \left\langle \begin{pmatrix} x^* \\ 0 \end{pmatrix}, \begin{pmatrix} x' - \bar{x} \\ y' - \bar{y} \end{pmatrix} \right\rangle. \tag{6.10}$$

On the other hand, property iv) implies $f(\bar{x}, \bar{y}) - \langle x^*, \bar{x} \rangle \geq f(x', y') - \langle x^*, x' \rangle$. Hence, from (6.10) we obtain

$$\left\langle \begin{pmatrix} \frac{\sum_{i=1}^{k} \lambda_i \bar{a}_i}{\bar{q}} \\ \frac{\sum_{i=1}^{k} \lambda_i \bar{b}_i}{\bar{q}} \end{pmatrix}, \begin{pmatrix} x' - \bar{x} \\ y' - \bar{y} \end{pmatrix} \right\rangle \geq 0.$$

It follows, from (6.9) that $F_j(\bar{x}, \bar{y}) \leq F_j(x', y')$, $\forall j = 1, ..., k$. Then, for $j = i$, we obtain $F_i(\bar{x}, \bar{y}) \leq F_i(x', y')$. Hence, from (6.8) we obtain

$$\begin{cases} F_i(x', y') < F_i(\bar{x}, \bar{y}) \leq F_i(x', y') \\ 0 \geq \frac{F_i(\bar{x}, \bar{y}) - F_i(x', y')}{F_j(x', y') - F_j(\bar{x}, \bar{y})} > M. \end{cases}$$

for all $j \in J$ verifying $F_j(\bar{x}, \bar{y}) < F_j(x', y')$. Then, $F_i(x', y') < F_i(x', y')$ and $M < 0$, and we obtain a contradiction. $\square$

Example 6.2. *Let* $X = [-\frac{1}{2}, \frac{1}{2}]$, $Y = [0, 1] \times [-1, 1]$, $F = (F_1, F_2, F_3)^\top$, f *and* $g = (g_1, g_2)^\top$ *be the functions defined on* $\mathbb{R} \times \mathbb{R}^2$ *by*

$$F_1(x, y) = -2|x| - 2y_1 + y_2^2 - 2y_2, \qquad F_2(x, y) = -2x - 2y_1,$$
$$F_3(x, y) = -2|x| - 2y_1 - 1, \qquad f(x, y) = x^2 + y_1,$$
$$g_1(x, y) = x^2 - y_1 + \frac{3}{4}, \qquad g_2(x, y) = x^2 - y_2 + \frac{3}{4},$$

with $y = (y_1, y_2)^T$. *Then, the functions* F, f *and* g *are convex and* X *and* Y *are convex compact sets. We have*

$$Z(x) = [x^2 + \tfrac{3}{4}, 1]^2,\ v(x) = 2x^2 + \tfrac{3}{4},\ \mathcal{M}(x) = \{x^2 + \tfrac{3}{4}\} \times [x^2 + \tfrac{3}{4}, 1].$$

Then, the multiobjective bilevel programming problem that we consider is

$$(S) \qquad v - \min_{\substack{x \in [-\frac{1}{2}, \frac{1}{2}] \\ y \in \{x^2 + \frac{3}{4}\} \times [x^2 + \frac{3}{4}, 1]}} \big\{ -2|x| - 2y_1 + y_2^2 - 2y_2, \ -2x - 2y_1, -2|x| - 2y_1 - 1 \big\}.$$

Let us determine a point $(\bar{x}, \bar{y}) \in X \times Y$, *with* $\bar{x}^2 - \bar{y}_1 + \frac{3}{4} \leq 0$ *and* $\bar{x}^2 - \bar{y}_2 + \frac{3}{4} \leq 0$, *that satisfies the sufficient conditions in Theorem 6.3. Then, we are led to verify if there exist* $x^* \in \mathbb{R}$ *and* $\bar{a}_i \in \mathbb{R}, \bar{b}_i = (b_i, \tilde{b}_i) \in \mathbb{R}^2, i = 1, 2, 3, \bar{q} \in \mathbb{R}_+^*, \lambda = (\lambda_1, \lambda_2, \lambda_3)^T \in$ int$\mathbb{R}_+^3$ *such that the following optimality conditions are satisfied*

i) $\binom{\sum_{i=1}^3 \lambda_i \bar{a}_i}{\sum_{i=1}^3 \lambda_i \bar{b}_i} \in \partial F_j(\bar{x}, \bar{y}),\ j = 1, 2, 3,$

ii) $\binom{0}{0} \in \partial f(\bar{x}, \bar{y}) + \mathcal{N}_{GrZ(\cdot)}(\bar{x}, \bar{y}),$

iii) $\begin{pmatrix} x^* - \frac{\sum_{i=1}^3 \lambda_i \bar{a}_i}{\bar{q}} \\ -\frac{\sum_{i=1}^3 \lambda_i \bar{b}_i}{\bar{q}} \end{pmatrix} \in \partial f(\bar{x}, \bar{y}),$

iv) $(\bar{x}, \bar{y})$ *solves the problem* $(\mathcal{Q}_{x^*})$: $\max_{(x,y) \in X \times Y} \{x^2 + y_1 - x^* x\}$.

For $x \in X$ *and* $y \in Y$, *we have*

$$\partial F_1(x, y) = \begin{cases} \{-2\} \times \{(-2, 2y_2 - 2)^\top\} & \textit{if } x > 0 \textit{ and } y \in Y \\ [-2, 2] \times \{(-2, 2y_2 - 2)^\top\} & \textit{if } x = 0 \textit{ and } y \in Y \\ \{2\} \times \{(-2, 2y_2 - 2)^\top\} & \textit{if } x < 0 \textit{ and } y \in Y \end{cases}$$

$$\partial F_2(x, y) = \{-2\} \times \{(-2, 0)^\top\}$$

and

$$\partial F_3(x, y) = \begin{cases} \{-2\} \times \{(-2, 0)^\top\} & \textit{if } x > 0 \textit{ and } y \in Y \\ [-2, 2] \times \{(-2, 0)^\top\} & \textit{if } x = 0 \textit{ and } y \in Y \\ \{2\} \times \{(-2, 0)^\top\} & \textit{if } x < 0 \textit{ and } y \in Y \end{cases}$$

Assume that $\bar{x} > 0$ *(if it exists), then from the first property i), we have*

$$\begin{pmatrix} \sum_{i=1}^3 \lambda_i a_i \\ \sum_{i=1}^3 \lambda_i b_i \\ \sum_{i=1}^3 \lambda_i \tilde{b}_i \end{pmatrix} \in \{-2\} \times \{(-2, 2y_2 - 2)^\top\},\ \begin{pmatrix} \sum_{i=1}^3 \lambda_i a_i \\ \sum_{i=1}^3 \lambda_i b_i \\ \sum_{i=1}^3 \lambda_i \tilde{b}_i \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix}$$

$$\Big(\sum_{i=1}^3 \lambda_i a_i, \sum_{i=1}^3 \lambda_i b_i, \sum_{i=1}^3 \lambda_i \tilde{b}_i\Big)^T \in \{-2\} \times \{(-2, 0)^\top\}.$$

On the other hand, ii) implies that $(\bar{y}_1, \bar{y}_2) \in \mathcal{M}(\bar{x})$, *i.e.,* $\bar{y}_1 = \bar{x}^2 + \frac{3}{4}$ *and* $\bar{y}_2 \in [\bar{x}^2 + \frac{3}{4}, 1]$. *Moreover, from (iii), we have*

$$\Big(\bar{q}x^* - \sum_{i=1}^3 \lambda_i a_i, -\sum_{i=1}^3 \lambda_i b_i, -\sum_{i=1}^3 \lambda_i \tilde{b}_i\Big)^T \in \bar{q}\partial f(\bar{x}, \bar{y}) = \{2\bar{q}\bar{x}\} \times \{(\bar{q}, 0)^\top\}.$$

Then, we obtain the following system

$$\begin{cases} \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = -2 & (\mathcal{E}_1) \\ \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 = -2 & (\mathcal{E}_2) \\ \lambda_1 \tilde{b}_1 + \lambda_2 \tilde{b}_2 + \lambda_3 \tilde{b}_3 = 2\bar{y}_2 - 2 = 0 & (\mathcal{E}_3) \\ \bar{q}x^* - (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) = 2\bar{q}\bar{x} & (\mathcal{E}_4) \\ -(\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) = \bar{q} & (\mathcal{E}_5) \\ -(\lambda_1 \tilde{b}_1 + \lambda_2 \tilde{b}_2 + \lambda_3 \tilde{b}_3) = 0 & (\mathcal{E}_6) \end{cases}$$

We easily deduce that $\bar{y}_2 = 1$, $\bar{q} = 2$ and equation $(\mathcal{E}_4)$ yields

$$2x^* + 2 = 4\bar{x}. \tag{6.11}$$

Moreover, the problem $(\mathcal{Q}_{x^*})$ is reduced to $\max_{(x,y_1)\in[-\frac{1}{2},\frac{1}{2}]\times[0,1]} h(x,y_1)$, with $h(x,y_1) = x^2 + y_1 - x^*x$, which corresponds to a maximization of a convex function over the convex compact set $[-\frac{1}{2},\frac{1}{2}]\times[0,1]$. So that, the maximum is attained at an extreme point of $[-\frac{1}{2},\frac{1}{2}]\times[0,1]$. Let us verify if $(\bar{x},\bar{y}_1)$ can be chosen among these extreme points. The values of h at the extreme points are

$$h(-\frac{1}{2},0) = \frac{1}{4} + \frac{1}{2}x^*, h(\frac{1}{2},0) = \frac{1}{4} - \frac{1}{2}x^*, h(-\frac{1}{2},1) = \frac{5}{4} + \frac{1}{2}x^*, h(\frac{1}{2},1) = \frac{5}{4} - \frac{1}{2}x^*$$

Comparing the values $\frac{5}{4} + \frac{1}{2}x^*$ and $\frac{5}{4} - \frac{1}{2}x^*$, we easily verify that the maximum is attained at $(\frac{1}{2},1)$ in the case where $x^* = 0$. Furtheremore $\bar{x} = \frac{1}{2}$ and $x^* = 0$ satisfy (6.11)*. According to Theorem 6.3, $(\bar{x},(\bar{y}_1,\bar{y}_2))^T = (\frac{1}{2},(1,1))^T$ is a properly efficient solution of the multiobjective bilevel problem (S) (by letting $\lambda = (1,1,1), a_1 = -1, a_2 = 0, a_3 = -1, b_1 = 0, b_2 = -1, b_3 = 1, \tilde{b}_1 = \tilde{b}_2 = \tilde{b}_3 = 0$).*

7. Conclusion

As it is well known the most considered semivectoriel bilevel problems in the literature have a scalar upper level. In this paper we have considered a strong semivectorial bilevel programming problem (S) in which the upper level is vectorial. For such a problem we have provided necessary and sufficient conditions for global optimality. These optimality conditions are obtained via a procedure using three operations: regularization, scalarization and a conjugate duality. The necessity to involve these three operations results from the fact that our problem (S) is not convex and does not satisfy the classical Slater condition. The lack of convexity and the non fulfillment of this latter condition are due to the presence of the lower level's solution set in the constraints of (S). In order to avoid this situation, we have first associated a regularized problem (S_ϵ) to (S) ([13]) that satisfies the Slater condition. As a stability result, we showed that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution of (S). In order to use the Fenchel-Lagrange duality, we have first decomposed the non convex problem (S_ϵ) into a family of vectorial convex minimization subproblems $(S_{\epsilon,x^*})_{x^*\in\mathbb{R}^p}$. Then, for every subproblem (S_{ϵ,x^*}), we have associated its scalarized problem (S^s_{ϵ,x^*}). The preservation of the convexity by scalarization allows us to give for every convex subproblem (S^s_{ϵ,x^*}) its Fenchel-Lagrange dual $(\mathcal{D}^s_{\epsilon,x^*})$. Under a Slater constraint qualification condition, we have established strong duality and provided optimality conditions for the scalar primal-dual pair (S^s_{ϵ,x^*})-$(\mathcal{D}^s_{\epsilon,x^*})$. Finally, via this duality approach we have provided necessary and sufficient optimality conditions for problem (S). We note that these optimality conditions are new for the class of semivectorial strong bilevel programming problems where the upper level is vectorial. These optimality conditions are expressed in terms of subdifferentials and normal cones in the sense of convex analysis. Moreover, the obtained results extend those given in [1] from the scalar case to the semivectorial one. Our future research will consist in extending this approach to the case where the two levels are vectorial.

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