



FINITE ELEMENT ERROR ESTIMATES FOR OPTIMAL CONTROL PROBLEMS WITH POINTWISE TRACKING

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ABSTRACT. We consider a linear-quadratic elliptic optimal control problem with point evaluations of the state variable in the cost functional. The state variable is discretized by conforming linear finite elements. For control discretization, three different approaches are considered. The main goal of the paper is to significantly improve known a priori discretization error estimates for this problem. We prove optimal error estimates for cellwise constant control discretizations in two and three space dimensions. Further, in two space dimensions, optimal error estimates for variational discretization and for the post-processing approach are derived.

1. INTRODUCTION

In this article, we develop a priori error estimates for the discretization of a linear-quadratic elliptic optimal control problem with point evaluation of the state variable in the cost functional. That is, we consider the following problem:

$$(1.1a) \quad \text{Minimize } \frac{1}{2} \sum_{i \in I} (u(x_i) - \xi_i)^2 + \frac{\alpha}{2} \|q\|_{L^2(\Omega)}^2$$

subject to

$$(1.1b) \quad \begin{aligned} -\Delta u &= q && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$(1.1c) \quad a \leq q(x) \leq b \quad \text{for a.a } x \in \Omega.$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded, convex, polygonal/polyhedral domain with $d \in \{2, 3\}$, $\alpha > 0$, $a, b \in \mathbb{R}$, and $a < b$. Problem (1.1) seeks to minimize the distance of the state u to prescribed values $\xi_i \in \mathbb{R}$ at fixed points x_i in the interior of Ω for $i \in I = \{1, 2, \dots, N\}$.

The consideration of such a cost functional involving pointwise evaluation is motivated by parameter identification problems with pointwise measurements, see, e.g., [23] for numerical analysis of a problem with a similar cost functional and finite dimensional control (parameter) variable. The optimal control problem (1.1)

2010 *Mathematics Subject Classification.* 49N10, 49M25, 65N15, 65N30.

Key words and phrases. Elliptic equations, optimal control, pointwise tracking, finite elements, error estimates.

and its finite element discretization are considered and analyzed in recent publications [1, 2, 6]. The goal of this paper is to significantly improve the a priori error estimates from these papers, see Table 1 and the detailed discussion below.

TABLE 1. Comparison of the orders of convergence of $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$

Discretization		variational	cellwise constant	post processing
Known results	$d = 2$	h	$h \ln h $	
	$d = 3$	$C(\varepsilon)h^{1-\varepsilon}$	$h^{\frac{1}{2}} \ln h ^2$	
Our approach	$d = 2$	$h^2 \ln h ^2$	$h \ln h $	$h^2 \ln h ^2$
	$d = 3$	$h \ln h $	$h \ln h $	

As in the above mentioned papers, we discretize the state equation (1.1b) with conforming linear finite elements, see Section 3 for details. With $h > 0$, we denote the discretization parameter describing the maximal mesh size. For the control discretization, we consider the following three approaches. For each of the approaches error estimates for the error between the optimal control \bar{q} and the discrete optimal control \bar{q}_h in terms of the discretization parameter h are derived.

- *Variational discretization:* In this case, the control variable is not discretized explicitly. It is implicitly discretized through the optimality system leading to an optimal discrete control being not a mesh function, see [12]. For this case, estimates of order $\mathcal{O}(h)$ for $d = 2$ and $\mathcal{O}(h^{1-\varepsilon})$ for $d = 3$ are provided in [1, 6]. For the two-dimensional case we prove a quasi-optimal (up to a logarithmic term) estimate of order $\mathcal{O}(h^2|\ln h|^2)$, see Theorem 7.5.
- *Cellwise constant discretization:* In this case, the control variable is discretized by cellwise constant functions on the same mesh as used for the state variable. For this choice, error estimates of order $\mathcal{O}(h|\ln h|)$ and $\mathcal{O}(h^{\frac{1}{2}}|\ln h|^2)$ for the dimensions $d = 2$ and $d = 3$, respectively, are derived in [1]. We prove an error estimate $\mathcal{O}(h|\ln h|)$ for both $d = 2$ and $d = 3$ significantly improving the known result for the three-dimensional case, see Theorem 6.6.
- *Post processing approach:* In this case, we use the approach suggested in [18] for an optimal control problem with L^2 tracking. That is, we discretize the control variable by cellwise constant functions and define a post-processed control through a projection formula (7.9). To our best knowledge, there are no results for this approach in the context of pointwise tracking in the literature. For the two-dimensional case we prove the same rate of convergence as for the variational discretization, i.e., $\mathcal{O}(h^2|\ln h|^2)$, see Theorem 7.12.

These results in the case of cellwise constant discretizations for both $d = 2$ and $d = 3$ as well as in the cases of variational discretization and of the post-processing approach for $d = 2$ can not be further improved, see Table 1. To our best knowledge, the question is open, if it is possible to prove second order error estimates (up to logarithmic terms) for the three-dimensional case ($d = 3$) on general quasi-uniform meshes. A possible way out is to use graded meshes locally refined towards the points $\{x_i\}$. By the techniques from [13], it seems to be directly possible to prove

second order estimates on such meshes. On such meshes also the case of absence of one or both of the control bounds (i.e. $a = -\infty$ or/and $b = \infty$) can be covered.

The structure of the paper is as follows. In the next section, we discuss the functional-analytic setting of the problem, provide optimality conditions and derive regularity results. It turns out, that although the adjoint state \bar{z} possesses in general only $W^{1,s}(\Omega)$ regularity with $s < d/(d-1)$, the optimal control \bar{q} is Lipschitz continuous due the presence of control constraints. In Section 3, the discrete problem for different control discretizations is introduced and the corresponding optimality conditions are stated. After some estimates for an auxiliary equation in Section 4, we discuss in Section 5 properties of the continuous and the discrete Green's functions. Thanks to recent results from [14], we are able to show that the discrete Green's function has similar growth behavior close to the singularity as the continuous one. This property is an important ingredient to prove our main results but is also of an independent interest. Then, we prove error estimates of order $\mathcal{O}(h|\ln h|)$ (see Table 1) in Section 6 and finally estimates of order $\mathcal{O}(h^2|\ln h|^2)$ in Section 7. The last section is devoted to numerical results illustrating our error estimates. For both, the cellwise constant discretization and the post-processing approach in two dimensions, the numerical results are fully in agreement with the presented theory. We present also a three-dimensional example for the post-processing approach and observe second order convergence, which is not covered by our theory, see the discussion above and in Section 8 of this issue.

2. CONTINUOUS PROBLEM

In this section, we give a rigorous definition of the continuous optimal control problem (1.1) and derive an optimality system as well as first regularity results for the optimal control. To this end, let $Q = L^2(\Omega)$ and

$$Q_{\text{ad}} = \{ q \in Q \mid a \leq q \leq b \text{ a.e. in } \Omega \}.$$

Further, let $\frac{2d}{d+2} < s < \frac{d}{d-1}$ and $\frac{1}{s} + \frac{1}{s'} = 1$. Then, in particular there holds $s' > d$, $W^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$, and $W^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$. The weak formulation of the state equation (1.1b) reads as: For a given control $q \in Q$ find the state $u \in W_0^{1,s'}(\Omega)$ such that

$$(2.1) \quad (\nabla u, \nabla \varphi) = (q, \varphi) \quad \forall \varphi \in W_0^{1,s}(\Omega),$$

where here and in the sequel, (\cdot, \cdot) denotes the $L^2(\Omega)$ inner product.

For the state equation one obtains (cf. [11], Theorem 3.2.1.2) the existence and uniqueness of a solution $u \in H^2(\Omega) \cap W_0^{1,s'}(\Omega) \hookrightarrow C(\bar{\Omega})$. This allows us to define a linear control-to-state mapping $S: Q \rightarrow C(\bar{\Omega})$ as $Sq = u$ where u is the solution of (2.1). We have the following standard estimate for Sq :

$$(2.2) \quad \|Sq\|_{L^\infty(\Omega)} \leq C\|Sq\|_{H^2(\Omega)} \leq C\|q\|_{L^2(\Omega)}.$$

For a mutually disjoint set of points $\{x_i \mid i \in I\} \subset \Omega$ with an index set $I = \{1, 2, \dots, N\} \subset \mathbb{N}$ and prescribed target values $\{\xi_i\}_{i \in I} \subset \mathbb{R}$ at these points, we

define the cost functional $J: Q \times C(\bar{\Omega}) \rightarrow \mathbb{R}$ as

$$J(q, u) = \frac{1}{2} \sum_{i \in I} (u(x_i) - \xi_i)^2 + \frac{\alpha}{2} \|q\|_{L^2(\Omega)}^2.$$

We then aim at solving the following optimal control problem:

$$(2.3) \quad \text{Minimize } J(q, u) \text{ subject to (2.1) and } (q, u) \in Q_{\text{ad}} \times C(\bar{\Omega}).$$

Theorem 2.1. *Problem (2.3) has the unique solution $\bar{q} \in Q_{\text{ad}}$.*

Proof. The proof can be done by standard arguments, cf., e.g., [27]. \square

With the reduced cost functional $j: Q \rightarrow \mathbb{R}$ given by means of the control-to-state mapping S as

$$j(q) = J(q, Sq),$$

it is straightforward to see that problem (2.3) is equivalent to the problem

$$(2.4) \quad \text{Minimize } j(q) \text{ subject to } q \in Q_{\text{ad}}.$$

Lemma 2.2. *For $q, \delta q \in Q$, the first Fréchet derivative of the reduced function j is given by*

$$j'(q)(\delta q) = (\alpha q + z, \delta q),$$

where $z \in W_0^{1,s}(\Omega)$ solves

$$(2.5) \quad (\nabla z, \nabla \varphi) = \sum_{i \in I} (Sq(x_i) - \xi_i) \varphi(x_i) \quad \forall \varphi \in W_0^{1,s'}(\Omega).$$

The second Fréchet derivative is given for $q, \delta q, \tau q \in Q$ by

$$j''(q)(\delta q, \tau q) = \sum_{i \in I} S \delta q(x_i) S \tau q(x_i) + \alpha(\delta q, \tau q).$$

Proof. The proof is standard. The regularity of z can be found, e.g., in [3, Theorem 4]. \square

For $i \in I$, let $z_i \in W_0^{1,s}(\Omega)$ be given as the solution of

$$(2.6) \quad (\nabla z_i, \nabla \varphi) = \varphi(x_i) \quad \forall \varphi \in W_0^{1,s'}(\Omega).$$

Then, it holds by construction that the solution z of (2.5) can be expressed as

$$(2.7) \quad z = \sum_{i \in I} (Sq(x_i) - \xi_i) z_i.$$

Theorem 2.3. *A control $\bar{q} \in Q_{\text{ad}}$ with associated state $\bar{u} = S\bar{q} \in W_0^{1,s'}(\Omega)$ is an optimal solution to the problem (2.3) if and only if there exists an adjoint state $\bar{z} \in W_0^{1,s}(\Omega)$ such that*

$$(2.8) \quad \begin{aligned} (\nabla \bar{u}, \nabla \varphi) &= (\bar{q}, \varphi) & \forall \varphi \in W_0^{1,s}(\Omega), \\ (\nabla \bar{z}, \nabla \varphi) &= \sum_{i \in I} (\bar{u}(x_i) - \xi_i) \varphi(x_i) & \forall \varphi \in W_0^{1,s'}(\Omega), \end{aligned}$$

$$(2.9) \quad (\alpha \bar{q} + \bar{z}, \delta q - \bar{q}) \geq 0 \quad \forall \delta q \in Q_{\text{ad}}.$$

Proof. It holds by a standard result (cf. [16, Theorem 1.4]) that \bar{q} is a solution to (1.1) if and only if $j'(\bar{q})(\delta q - \bar{q}) \geq 0$ for all $\delta q \in Q_{\text{ad}}$. Then, Lemma 2.2 implies the assertion. \square

Proposition 2.4. *A control $\bar{q} \in Q_{\text{ad}}$ is the solution to the optimal control problem (2.3) if and only if \bar{q} and the solution \bar{z} of (2.8) fulfill the projection formula*

$$\bar{q} = P_{[a,b]} \left(-\frac{1}{\alpha} \bar{z} \right).$$

Here the projection $P_{[a,b]}$ is given by

$$P_{[a,b]}(g(x)) = \min(b, \max(a, g(x)))$$

for $g(x) \in \mathbb{R} \cup \{-\infty, \infty\}$ at $x \in \Omega$.

Proof. A proof of the equivalence of (2.9) and this projection formula can be found in [27, Theorem 2.28]. \square

Proposition 2.5. *For the solution $\bar{q} \in Q_{\text{ad}}$ of the optimal control problem (2.3), it holds*

$$\bar{q} \in H^1(\Omega).$$

Proof. This result follows directly from [5, Lemma 3.3]. \square

3. DISCRETE PROBLEM

We approximate the continuous state equation (2.1) using a Galerkin finite element discretization. For this discretization, we use a family of triangulations $\{\mathcal{T}_h\}$. A cell $K \in \mathcal{T}_h$ has the diameter h_K . The discretization parameter h is given as $h = \max_{K \in \mathcal{T}_h} h_K$. We also require the triangulation to be regular and quasi-uniform.

For discretizing the state and adjoint equations, we consider the conforming space $V_h \subset W^{1,\infty}(\Omega)$ of linear finite elements on the triangulation \mathcal{T}_h

$$V_h = \{ v_h \in C(\bar{\Omega}) \mid v_h|_K \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}_h \text{ and } v_h|_{\partial\Omega} = 0 \}.$$

We consider two types of discretizations for the control variable. The first type is the so-called variational discretization introduced by [12]. Here, the control variable is not explicitly discretized. As second possibility, we consider a piecewise constant control discretization on the family of triangulations $\{\mathcal{T}_h\}$ introduced for the discretization of the states. Then, we define the space of discrete controls as

$$Q_h^c = \{ q_h \in Q \mid q_h|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathcal{T}_h \}.$$

where $\mathcal{P}_0(K)$ denotes the space of piecewise constant polynomials on a cell K . The discrete admissible set is then defined as $Q_{h,\text{ad}}^c = Q_h^c \cap Q_{\text{ad}}$.

In the following, we introduce properties for the discrete problem similar to the continuous case in Section 2 before. To this end, $Q_{h,\text{ad}}$ serves as placeholder for either Q_{ad} (for variational discretization) or $Q_{h,\text{ad}}^c$ (for cellwise constant discretization).

The discrete state equation for $u_h \in V_h$ with given $q \in Q$ reads as

$$(3.1) \quad (\nabla u_h, \nabla \varphi_h) = (q, \varphi_h) \quad \forall \varphi_h \in V_h$$

and the discrete analog to (2.3) has the form

$$(3.2) \quad \text{Minimize } J(q_h, u_h) \text{ subject to (3.1) and } (q_h, u_h) \in Q_{h,ad} \times V_h.$$

Again, we can define the discrete control-to-state mapping with $S_h: Q \rightarrow V_h$ as $S_h q = u_h$. We define the discrete reduced cost functional $j_h: Q \rightarrow \mathbb{R}$ by

$$j_h(q) = J(q, S_h q).$$

We start with the following stability result for the discrete solution operator S_h .

Lemma 3.1. *For $q \in Q$, it holds*

$$\|S_h q\|_{L^\infty(\Omega)} \leq C \|q\|_{L^2(\Omega)}.$$

Proof. The proof is standard. The sub-optimal L^∞ error estimate [7, p. 168] implies with (2.2) that

$$\|Sq - S_h q\|_{L^\infty(\Omega)} \leq Ch^{2-\frac{d}{2}} \|Sq\|_{H^2(\Omega)} \leq Ch^{2-\frac{d}{2}} \|q\|_{L^2(\Omega)}.$$

Hence,

$$\|S_h q\|_{L^\infty(\Omega)} \leq \|Sq\|_{L^\infty(\Omega)} + \|Sq - S_h q\|_{L^\infty(\Omega)} \leq C(1 + h^{2-\frac{d}{2}}) \|q\|_{L^2(\Omega)}$$

implies the assertion. \square

Theorem 3.2. *Problem (3.2) admits the unique solution $\bar{q}_h \in Q_{h,ad}$.*

Proof. As Theorem 2.1, this can be shown by standard arguments. \square

As in the continuous case, we have the following expressions for the first and second derivatives of j_h .

Lemma 3.3. *For $q, \delta q \in Q$, the first Fréchet derivative of the reduced function j is given by*

$$j'_h(q)(\delta q) = (\alpha q + z_h, \delta q),$$

where $z_h \in V_h$ solves

$$(3.3) \quad (\nabla z_h, \nabla \varphi_h) = \sum_{i \in I} (S_h q(x_i) - \xi_i) \varphi_h(x_i) \quad \forall \varphi_h \in V_h.$$

The second Fréchet derivative is given for $q, \delta q, \tau q \in Q$ by

$$j''_h(q)(\delta q, \tau q) = \sum_{i \in I} S_h \delta q(x_i) S_h \tau q(x_i) + \alpha(\delta q, \tau q).$$

For $i \in I$, let $z_{h,i} \in V_h$ be given as the solution of

$$(3.4) \quad (\nabla z_{h,i}, \nabla \varphi_h) = \varphi_h(x_i) \quad \forall \varphi_h \in V_h.$$

Then, it holds by construction that the solution z_h of (3.3) can be expressed as

$$(3.5) \quad z_h = \sum_{i \in I} (S_h q(x_i) - \xi_i) z_{h,i}.$$

In the following theorem, we formulate the optimality conditions for the discrete problem.

Theorem 3.4. *A control $\bar{q}_h \in Q_{h,ad}$ with the associated state $\bar{u}_h = S_h \bar{q}_h \in V_h$ is an optimal solution to the problem (3.2) if and only if there exists an adjoint state $\bar{z}_h \in V_h$ such that*

$$(3.6) \quad \begin{aligned} (\nabla \bar{u}_h, \nabla \varphi_h) &= (\bar{q}_h, \varphi_h) & \forall \varphi_h \in V_h, \\ (\nabla \bar{z}_h, \nabla \varphi_h) &= \sum_{i \in I} (\bar{u}_h(x_i) - \xi_i) \varphi_h(x_i) & \forall \varphi_h \in V_h, \end{aligned}$$

$$(3.7) \quad (\alpha \bar{q}_h + \bar{z}_h, \delta q_h - \bar{q}_h) \geq 0 \quad \forall \delta q_h \in Q_{h,ad}.$$

Proof. The assertion of the theorem can be proved in the same way as its continuous analogue of Theorem 2.3. \square

Remark 3.5. Similar to Proposition 2.4 on the continuous level, condition (3.7) can be formulated by the means of the projection $P_{[a,b]}$. For $Q_{h,ad} = Q_{ad}$, it holds

$$\bar{q}_h = P_{[a,b]} \left(-\frac{1}{\alpha} \bar{z}_h \right).$$

In the case $Q_{h,ad} = Q_{h,ad}^c$, the corresponding formula reads as

$$\bar{q}_h = P_{[a,b]} \left(-\frac{1}{\alpha} \pi_h \bar{z}_h \right),$$

where π_h denotes the L^2 projection on Q_h^c .

4. FINITE ELEMENT ERROR ANALYSIS FOR AN AUXILIARY EQUATION

For the numerical analysis carried out later, we first need to bound the error $u(x_i) - u_h(x_i)$ for given $q \in Q_{ad}$. There are multiple results available for the state equation in case of a bounded right-hand side, see e.g., [19, 21, 22] but mostly for C^2 smooth boundaries in contrast to our case.

For given a $f \in L^2(\Omega)$ let $w \in H_0^1(\Omega)$ be the solution of the auxiliary problem

$$(4.1) \quad (\nabla w, \nabla \varphi) = (f, \varphi) \quad \forall \varphi \in H_0^1(\Omega).$$

Lemma 4.1. *Let $w \in H_0^1(\Omega)$ be the solution of (4.1). Provided that $f \in L^p(\Omega)$ with $p > d$, it holds $w \in W^{1,\infty}(\Omega)$ with*

$$\|\nabla w\|_{L^\infty(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

Proof. For $d = 2$, the assertion follows from [11, Theorems 4.3.2.4 and 4.4.3.7] using the convexity of Ω which ensures the existence of $p > 2$ with $w \in W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$.

For $d = 3$ and given $y \in \Omega$, let $G(\cdot, y)$ be the Green's function associated to y . Then, there holds by the convexity of Ω and [17, Theorem 5.1.8] that

$$|\nabla_y G(x, y)| \leq |x - y|^{-2} \quad \forall x \in \Omega, x \neq y.$$

This allows us to estimate by the Hölder inequality for $1 = \frac{1}{p} + \frac{1}{p'}$ that

$$\begin{aligned} |\nabla w(y)| &= \left| \int_{\Omega} \nabla_y G(x, y) f(x) dx \right| \leq \int_{\Omega} |f(x)| |x - y|^{-2} dx \\ &\leq \|f\|_{L^p(\Omega)} \| |x - y|^{-2} \|_{L^{p'}(\Omega)}. \end{aligned}$$

Since $p > 3$, the second factor is bounded, which completes the proof. \square

We continue with a regularity result for the solution w of (4.1) in a subdomain $\Omega_1 \Subset \Omega_0 \Subset \Omega$, provided that the right-hand side f has higher regularity in Ω_0 .

Lemma 4.2. *Let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with Ω_0 smooth and w be the solution of (4.1).*

(i) *Provided that $f|_{\Omega_0} \in L^p(\Omega_0)$ for some $2 \leq p < \infty$, it holds $w \in W^{2,p}(\Omega_1)$ and*

$$\|w\|_{W^{2,p}(\Omega_1)} \leq C_p \{ \|f\|_{L^p(\Omega_0)} + \|f\|_{L^2(\Omega)} \}$$

where $C_p \sim Cp$ for $p \rightarrow \infty$.

(ii) *Provided that $f|_{\Omega_0} = 0$, it holds $w \in W^{2,\infty}(\Omega_1)$ and*

$$\|w\|_{W^{2,\infty}(\Omega_1)} \leq C \|f\|_{L^2(\Omega)}.$$

Proof. The proof of the first assertion follows along the lines of the proof of [15, Lemma 2.5]. We choose a smooth $\tilde{\Omega}_1$ such that $\Omega_1 \Subset \tilde{\Omega}_1 \Subset \Omega_0$. We first prove that $w \in W^{2,6}(\tilde{\Omega}_1)$. To this end, let $\tilde{\omega} \in [0, 1]$ be a smooth cutoff function with the properties

$$\begin{aligned} \tilde{\omega} &= 1 && \text{in } \tilde{\Omega}_1, \\ \tilde{\omega} &= 0 && \text{in } \Omega \setminus \Omega_0. \end{aligned}$$

Set v as $v = \tilde{\omega}w$. Then, there holds by the product rule

$$\begin{aligned} (\nabla v, \nabla \varphi) &= (\nabla(\tilde{\omega}w), \nabla \varphi) = (\tilde{\omega} \nabla w, \nabla \varphi) + (w \nabla \tilde{\omega}, \nabla \varphi) \\ &= (\nabla w, \nabla(\tilde{\omega} \varphi)) - (\nabla w, \varphi \nabla \tilde{\omega}) - (\nabla \cdot (w \nabla \tilde{\omega}), \varphi) \\ &= (\tilde{\omega} f, \varphi) - 2(\nabla \tilde{\omega} \nabla w, \varphi) - (w \Delta \tilde{\omega}, \varphi) \end{aligned}$$

and therefore v satisfies the following equation

$$\begin{aligned} -\Delta v &= \tilde{\omega} f - 2\nabla \tilde{\omega} \nabla w - w \Delta \tilde{\omega} && \text{in } \Omega_0, \\ v &= 0 && \text{on } \partial \Omega_0. \end{aligned}$$

By [10, Corollary 9.10], the $W^{2,6}(\Omega_0)$ norm of v is bounded by the $L^6(\Omega_0)$ norm of the right-hand side above. Using the smoothness of $\tilde{\omega}$, it follows

$$\begin{aligned} \|w\|_{W^{2,6}(\tilde{\Omega}_1)} &= \|v\|_{W^{2,6}(\tilde{\Omega}_1)} \leq \|v\|_{W^{2,6}(\Omega_0)} \leq C \|\tilde{\omega} f - 2\nabla \tilde{\omega} \nabla w - w \Delta \tilde{\omega}\|_{L^6(\Omega_0)} \\ &\leq C \{ \|f\|_{L^6(\Omega_0)} + \|\nabla w\|_{L^6(\Omega_0)} + \|w\|_{L^6(\Omega_0)} \}. \end{aligned}$$

By (2.2) and the continuous embedding $w \in H^2(\Omega) \hookrightarrow W^{1,6}(\Omega)$ we get

$$\|w\|_{W^{2,6}(\tilde{\Omega}_1)} \leq C \{ \|f\|_{L^6(\Omega_0)} + \|f\|_{L^2(\Omega)} \}.$$

For $p \leq 6$, this already states the assertion.

For $p > 6$, we iterate the previous steps with

$$\begin{aligned} \omega(x) &= 1 && \text{in } \Omega_1, \\ \omega(x) &= 0 && \text{in } \Omega \setminus \tilde{\Omega}_1, \end{aligned}$$

and use the smoothness of $\tilde{\Omega}_1$ to estimate

$$\begin{aligned} \|w\|_{W^{2,p}(\Omega_1)} &= \|v\|_{W^{2,p}(\Omega_1)} \leq \|v\|_{W^{2,p}(\tilde{\Omega}_1)} \\ &\leq C_p \|\omega f - 2\nabla \omega \nabla w - w \Delta \omega\|_{L^p(\tilde{\Omega}_1)} \\ &\leq C_p \{ \|f\|_{L^p(\tilde{\Omega}_1)} + \|\nabla w\|_{L^p(\tilde{\Omega}_1)} + \|w\|_{L^p(\tilde{\Omega}_1)} \}, \end{aligned}$$

where C_p can be traced from the proof of [10, Theorem 9.8, Theorem 9.9]. Exploiting $W^{2,6}(\tilde{\Omega}_1) \hookrightarrow W^{1,p}(\tilde{\Omega}_1)$ for all $1 < p < \infty$, we have

$$\|w\|_{L^p(\tilde{\Omega}_1)} + \|\nabla w\|_{L^p(\tilde{\Omega}_1)} \leq \|w\|_{W^{2,6}(\tilde{\Omega}_1)} \leq C\{\|f\|_{L^6(\Omega_0)} + \|f\|_{L^2(\Omega)}\},$$

which concludes the proof of the first assertion.

The second assertion follows similarly. Noting that $\tilde{\omega}f = 0$ on the whole domain Ω and the smoothness of Ω_0 , the first step implies $v \in H^3(\Omega_0)$ and hence $w \in H^3(\tilde{\Omega}_1)$. Then, since also $\omega f = 0$ on the whole Ω and $\tilde{\Omega}_1$ is smooth, the next step yields $v \in H^5(\tilde{\Omega}_1)$ and consequently $w \in H^5(\Omega_1)$. This implies the second assertion. \square

Let $w_h \in V_h$ be the Ritz projection of w given by

$$(4.2) \quad (\nabla w_h, \nabla \varphi_h) = (f, \varphi_h) \quad \forall v \in V_h.$$

We will use the following Schatz-Wahlbin-type estimate to bound the error between the solutions w of (4.1) and w_h of (4.2) on a subset of Ω .

Proposition 4.3. *Let be $\Omega_1 \Subset \Omega_0 \Subset \Omega$, $v \in H_0^1(\Omega) \cap C(\bar{\Omega})$, and $v_h \in V_h$ satisfying*

$$(\nabla(v - v_h), \nabla \varphi_h) = 0 \quad \forall \varphi_h \in V_h.$$

Then, there are constants $C, C' > 0$, $0 < h_0 < 1$, and $r > 0$ such that $C'h \leq r$, $\text{dist}(\Omega_1, \partial\Omega_0) \geq r$, and $\text{dist}(\Omega_0, \partial\Omega) \geq r$. For $0 < h \leq h_0$ and any $\varphi_h \in V_h$, there holds

$$\|v - v_h\|_{L^\infty(\Omega_1)} \leq C\{\ln rh\|v - \varphi_h\|_{L^\infty(\Omega_0)} + r^{-\frac{d}{2}}\|v - v_h\|_{L^2(\Omega_0)}\}.$$

Proof. See [26, Corollary 5.1]. The corollary there is stated in a slightly different form. Applying it to $v - \varphi_h - v_h + \varphi_h$ yields the stated assertion. \square

Lemma 4.4. *Let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with smooth Ω_0 . Further, let $w \in H_0^1(\Omega)$ be the solution of (4.1) and $w_h \in V_h$ be the solution of (4.2).*

(i) *Provided that $f|_{\Omega_0} \in L^\infty(\Omega_0)$ there is $h_0 > 0$ such that*

$$\|w - w_h\|_{L^\infty(\Omega_1)} \leq Ch^2 |\ln h|^2 \{\|f\|_{L^\infty(\Omega_0)} + \|f\|_{L^2(\Omega)}\}.$$

holds for $h \leq h_0$.

(ii) *Provided that $f|_{\Omega_0} = 0$, it holds*

$$\|w - w_h\|_{L^\infty(\Omega_1)} \leq Ch^2 |\ln h| \|f\|_{L^2(\Omega)}.$$

Proof. We start with the first assertion. By Proposition 4.3, we have for a suitable chosen smooth $\tilde{\Omega}_1$ with $\Omega_1 \Subset \tilde{\Omega}_1 \Subset \Omega_0$ and any $\varphi_h \in V_h$

$$\|w - w_h\|_{L^\infty(\Omega_1)} \leq C\{\ln rh\|w - \varphi_h\|_{L^\infty(\tilde{\Omega}_1)} + r^{-\frac{d}{2}}\|w - w_h\|_{L^2(\tilde{\Omega}_1)}\}.$$

Since r is constant, we can use a standard result to estimate the second term, i.e.,

$$r^{-\frac{d}{2}}\|w - w_h\|_{L^2(\tilde{\Omega}_1)} \leq r^{-\frac{d}{2}}\|w - w_h\|_{L^2(\Omega)} \leq Ch^2 \|f\|_{L^2(\Omega)}.$$

For the first term, we note that Lemma 4.2 ensures $w \in W^{2,p}(\tilde{\Omega}_1)$ for all $2 \leq p < \infty$. For $\varphi_h = i_h w$ being the standard nodal interpolant of w , it holds

$$\|w - i_h w\|_{L^\infty(\tilde{\Omega}_1)} \leq Ch^{2-\frac{d}{p}} \|\nabla^2 w\|_{L^p(\tilde{\Omega}_1)} \leq Cph^{2-\frac{d}{p}} \{\|f\|_{L^\infty(\Omega_0)} + \|f\|_{L^2(\Omega)}\}.$$

Choosing $p = |\ln h|$ for small h , we get $ph^{2-\frac{d}{p}} \leq Ch^2|\ln h|$, which implies the stated estimate.

The second assertion follows similarly by using the second assertion of Lemma 4.2 ensuring $w \in W^{2,\infty}(\tilde{\Omega}_1)$:

$$\|w - i_h w\|_{L^\infty(\tilde{\Omega}_1)} \leq Ch^2 \|\nabla^2 w\|_{L^\infty(\tilde{\Omega}_1)} \leq Ch^2 \|f\|_{L^2(\Omega)}. \quad \square$$

5. ESTIMATES FOR THE CONTINUOUS AND DISCRETE GREEN'S FUNCTIONS

In this section, we consider a point $x_0 \in \Omega$ and the associated Green's function solving

$$(5.1) \quad \begin{aligned} -\Delta g &= \delta_{x_0} && \text{in } \Omega, \\ g &= 0 && \text{on } \partial\Omega. \end{aligned}$$

It is well-known that $g \in W_0^{1,s}(\Omega)$ for any $s < \frac{d}{d-1}$ with

$$(5.2) \quad \|\nabla g\|_{L^s(\Omega)} \leq C,$$

see, e.g., [3, Theorem 4].

Lemma 5.1. *Let g be the solution of (5.1). Then, for every $M > 0$ there exists an open ball $B \subset \Omega$ containing x_0 such that*

$$g(x) \geq M \quad \forall x \in B.$$

Proof. It is well-known, see, e.g., [4, (3.11)] that the asymptotic behavior of g for $x \rightarrow x_0$ is of type

$$(5.3) \quad g(x) \approx \begin{cases} C_1 \ln \frac{1}{|x-x_0|} + C_2 & \text{for } d = 2, \\ C_1 \frac{1}{|x-x_0|} + C_2 & \text{for } d = 3 \end{cases}$$

with $C_1 > 0$. This directly implies the assertion. □

Lemma 5.2. *For the solution g of (5.1) and any open ball $B \subset \Omega$ containing x_0 , it holds*

$$g \in W^{1,\infty}(\Omega \setminus \bar{B}) \cap H^2(\Omega \setminus \bar{B}).$$

Proof. There exists an open ball $B' \Subset B$ with $x_0 \in B'$. As before, we consider a smooth cutoff function $\omega: \Omega \rightarrow [0, 1]$ with the following properties

$$\begin{aligned} \omega &= 1 && \text{in } \Omega \setminus \bar{B}, \\ \omega &= 0 && \text{in } B'. \end{aligned}$$

Let $\varphi \in W_0^{1,s'}(\Omega)$. As in the proof of Lemma 4.2, it holds for $v = \omega g$ that

$$\begin{aligned} (\nabla v, \nabla \varphi) &= \omega(x_0)\varphi(x_0) - 2(\nabla \tilde{\omega} \nabla g, \varphi) - (g \Delta \tilde{\omega}, \varphi) \\ &= -2(\nabla \omega \nabla g, \varphi) - (g \Delta \omega, \varphi), \end{aligned}$$

since $\omega(x_0) = 0$. Therefore, v satisfies the following equation

$$\begin{aligned} -\Delta v &= -g \Delta \omega - 2 \nabla \omega \nabla g && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega. \end{aligned}$$

For any $s < \frac{d}{d-1}$, we have $g \in W^{1,s}(\Omega)$ from (5.2) and it follows $v \in W^{2,s}(\Omega)$. Hence, by the smoothness of ω , we obtain $g \in W^{2,s}(\Omega \setminus \bar{B}) \hookrightarrow H^1(\Omega \setminus \bar{B})$.

Iterating the previous steps using $g \in H^1(\Omega \setminus \bar{B})$, we obtain $v \in H^2(\Omega)$ and hence $g \in H^2(\Omega \setminus \bar{B}) \hookrightarrow W^{1,6}(\Omega \setminus \bar{B})$.

Iterating again using $g \in W^{1,6}(\Omega \setminus \bar{B})$ implies $v \in W^{1,\infty}(\Omega)$ by Lemma 4.1. Consequently, we obtain the assertion $g \in W^{1,\infty}(\Omega \setminus \bar{B}) \cap H^2(\Omega \setminus \bar{B})$. \square

Let $g_h \in V_h$ be the Ritz projection of g given as solution of

$$(5.4) \quad (\nabla g_h, \nabla \varphi_h) = \varphi_h(x_0) \quad \forall \varphi_h \in V_h.$$

Lemma 5.3. *For the solutions $g \in W_0^{1,s}(\Omega)$ of (5.1) and $g_h \in V_h$ of (5.4), it holds*

$$\|g - g_h\|_{L^1(\Omega)} \leq Ch^2 |\ln h|^2,$$

where C is independent of h .

Proof. The assertion is a direct consequence of [20, Lemma 3.3 (ii)], cf. also [8] for $d = 2$. However, the exponent of the log-term in [20, Lemma 3.3 (ii)] is different from 2 in the three-dimensional case. Therefore, we give a proof here, which is a direct consequence of Lemma 4.4, (i).

Let $f = \text{sgn}(g - g_h)$ and let $w \in H_0^1(\Omega)$ and $w_h \in V_h$ be the corresponding solutions of (4.1) and (4.2). There holds

$$\begin{aligned} \|g - g_h\|_{L^1(\Omega)} &= (f, g - g_h) = w(x_0) - w_h(x_0) \\ &\leq Ch^2 |\ln h|^2 \|f\|_{L^\infty(\Omega)} \leq Ch^2 |\ln h|^2, \end{aligned}$$

where we have used $\|f\|_{L^\infty(\Omega)} \leq 1$. \square

Lemma 5.4. *For the solution $g_h \in V_h$ of (5.4), it holds*

$$\|g_h\|_{L^2(\Omega)} \leq C$$

with C independent of h .

Proof. We write the discretization error as $e = g - g_h$ and define $w \in H_0^1(\Omega)$ as the solution of

$$(\nabla w, \nabla \varphi) = (e, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

and $w_h \in V_h$ as the solution of

$$(\nabla w_h, \nabla \varphi_h) = (e, \varphi_h) \quad \forall \varphi_h \in V_h.$$

Then, the L^2 error on Ω can be expressed as

$$\|e\|_{L^2(\Omega)}^2 = (e, g) - (e, g_h) = (\nabla w, \nabla g) - (\nabla w_h, \nabla g_h) = w(x_0) - w_h(x_0)$$

and the sub-optimal L^∞ error estimate [7, p. 168] implies

$$\|e\|_{L^2(\Omega)}^2 \leq Ch^{2-\frac{d}{2}} \|e\|_{L^2(\Omega)}.$$

Hence, we get

$$\|g_h\|_{L^2(\Omega)} \leq Ch^{2-\frac{d}{2}} + \|g\|_{L^2(\Omega)} \leq C$$

due to (5.2). \square

Lemma 5.5. *For the solutions $g \in W_0^{1,s}(\Omega)$ of (5.1) and $g_h \in V_h$ of (5.4) and any open ball $B \subset \Omega$ containing x_0 , there is $h_0 > 0$ such that*

$$\|g - g_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|$$

for all $h \leq h_0$.

Proof. We proceeded as in the proof of Lemma 5.4 by writing $e = g - g_h$ and introducing the indicator function χ_B of B . Further, we define $w \in H_0^1(\Omega)$ as the solution of

$$(5.5) \quad (\nabla w, \nabla \varphi) = ((1 - \chi_B)e, \varphi) \quad \forall \varphi \in H_0^1(\Omega)$$

and $w_h \in V_h$ the solution of

$$(\nabla w_h, \nabla \varphi_h) = ((1 - \chi_B)e, \varphi_h) \quad \forall \varphi_h \in V_h.$$

Then, the L^2 error on $\Omega \setminus \bar{B}$ can be expressed as

$$\begin{aligned} \|e\|_{L^2(\Omega \setminus \bar{B})}^2 &= ((1 - \chi_B)e, g) - ((1 - \chi_B)e, z_{h,i}) = (\nabla w, \nabla g) - (\nabla w_h, \nabla g_h) \\ &= w(x_0) - w_h(x_0). \end{aligned}$$

Since the right-hand side of (5.5) is zero on B , Lemma 4.4 implies by choosing a suitable subdomain $B' \Subset B$ containing x_0 that

$$\|e\|_{L^2(\Omega \setminus \bar{B})}^2 = w(x_0) - w_h(x_0) \leq \|w - w_h\|_{L^\infty(B')} \leq Ch^2 |\ln h| \|e\|_{L^2(\Omega \setminus \bar{B})},$$

which concludes the proof. \square

Lemma 5.6. *For the solution $g_h \in V_h$ of (5.4) and any open ball $B \subset \Omega$ containing x_0 there is $h_0 > 0$ such that*

$$\|g_h\|_{L^\infty(\Omega \setminus \bar{B})} \leq C \quad \text{and} \quad \|\nabla g_h\|_{L^2(\Omega \setminus \bar{B})} \leq C,$$

for all $h \leq h_0$ with a constant C independent of h .

Proof. On $\Omega \setminus \bar{B}$, we get by an inverse inequality (cf. [7, Theorem 3.2.6]), Lemma 5.5, and by inserting the nodal interpolant $i_h g$ of g that

$$\begin{aligned} \|g - g_h\|_{L^\infty(\Omega \setminus \bar{B})} &\leq \|g_h - i_h g\|_{L^\infty(\Omega \setminus \bar{B})} + \|i_h g - g\|_{L^\infty(\Omega \setminus \bar{B})} \\ &\leq Ch \|\nabla g\|_{L^\infty(\Omega \setminus \bar{B})} + Ch^{-\frac{d}{2}} \|i_h g - g_h\|_{L^2(\Omega \setminus \bar{B})} \\ &\leq Ch \|\nabla g\|_{L^\infty(\Omega \setminus \bar{B})} \\ &\quad + Ch^{-\frac{d}{2}} \{ \|i_h g - g\|_{L^2(\Omega \setminus \bar{B})} + \|g - g_h\|_{L^2(\Omega \setminus \bar{B})} \} \\ &\leq Ch^{2-\frac{d}{2}} |\ln h|. \end{aligned}$$

Here, we used that $g \in W^{1,\infty}(\Omega \setminus \bar{B}) \cap H^2(\Omega \setminus \bar{B})$ according to Lemma 5.2. Hence, we get

$$\|g_h\|_{L^\infty(\Omega \setminus \bar{B})} \leq \|g\|_{L^\infty(\Omega \setminus \bar{B})} + \|g - g_h\|_{L^\infty(\Omega \setminus \bar{B})},$$

which implies the first assertion again by means of Lemma 5.2. For the second assertion, we similarly obtain

$$\begin{aligned} \|\nabla(g - g_h)\|_{L^2(\Omega \setminus \bar{B})} &\leq Ch \|\nabla^2 g\|_{L^2(\Omega \setminus \bar{B})} \\ &\quad + Ch^{-1} \{ \|i_h g - g\|_{L^2(\Omega \setminus \bar{B})} + \|g - g_h\|_{L^2(\Omega \setminus \bar{B})} \} \\ &\leq Ch |\ln h|. \end{aligned}$$

Here, we again used for the interpolation estimates that $g \in H^2(\Omega \setminus \bar{B})$. Then,

$$\|\nabla g_h\|_{L^2(\Omega \setminus \bar{B})} \leq \|\nabla g\|_{L^2(\Omega \setminus \bar{B})} + \|\nabla(g - g_h)\|_{L^2(\Omega \setminus \bar{B})}$$

together with Lemma 5.2 implies the second assertion. \square

We close this section by a discrete analogue of Lemma 5.1 for $d = 2$ only.

Lemma 5.7. *Let $d = 2$ and $g_h \in V_h$ be the solution of (5.4). Then, for every $M > 0$ there exists an open ball $B \Subset \Omega$ containing x_0 and $h_0 > 0$ such that for all $h \leq h_0$, it holds*

$$g_h(x) \geq M \quad \forall x \in B.$$

Proof. The main idea for the proof stems from the proof of [14, Theorem 4.6]. While in [14], smoothness of the domain is required we do not need this for our particular result since x_0 lies in the interior of Ω and we are only interested in the behavior of g_h in a neighborhood of x_0 and not close to the boundary. Hence, case 3 of the proof of [14, Theorem 4.6] does not need to be considered here. We adapt the technique to our case.

We distinguish the following two cases:

$|x - x_0| \geq \kappa h |\ln h|^{\frac{1}{2}}$: We show, that for this case we have pointwise convergence of g_h . This setting fulfills the assumptions of [26, Theorem 6.1] which states the existence of constants κ and C_κ such that for h sufficiently small and $x, x_0 \in \Omega_0 \Subset \Omega$ with $|x - x_0| \geq \kappa h$, it holds

$$(5.6) \quad |g(x) - g_h(x)| \leq C_\kappa \frac{h^2}{|x - x_0|^2} \ln \frac{|x - x_0|}{h}.$$

Abbreviating $\eta = |x - x_0| h^{-1}$, the right-hand side of (5.6) becomes $C_\kappa \eta^{-2} \ln \eta$. Since $\kappa h |\ln h|^{\frac{1}{2}} \leq |x - x_0|$, we have that $\eta \geq \kappa |\ln h|^{\frac{1}{2}}$ which means that for h small enough $C_\kappa \eta^{-2} \ln \eta$ is maximal at $\eta = \kappa |\ln h|^{\frac{1}{2}}$. Combined, we have

$$|g(x) - g_h(x)| \leq C_\kappa \eta^{-2} \ln \eta \leq C_\kappa |\ln h|^{-1} \ln |\ln h|^{\frac{1}{2}}.$$

Hence, for $h \leq h_0$ sufficiently small, we obtain

$$g_h(x) = g(x) + (g_h(x) - g(x)) \geq \frac{1}{2}g(x),$$

which implies the assertion in this case by Lemma 5.1.

$|x - x_0| \leq \kappa h |\ln h|^{\frac{1}{2}}$: From [14, Lemma 4.8], we know that $g_h(x_0) \geq C(1 + |\ln h|)$. Due to [14, Lemma 4.9], the first derivative of g_h is bounded by $\|\nabla g_h\|_{L^\infty(\Omega)} \leq Ch^{-1}$. Then, by the mean value theorem, it holds

$$g_h(x) \geq g_h(x_0) - \|\nabla g_h\|_{L^\infty(\Omega)} |x - x_0| \geq C |\ln h| - C |\ln h|^{\frac{1}{2}}.$$

That implies $|g_h(x)| \geq M$ for $h \leq h_0$ sufficiently small.

Combination of these cases yields the assertion with a set B which can be chosen independently of $h \leq h_0$. \square

Remark 5.8. To our best knowledge, the question is open, if it is possible to prove a similar estimate for the discrete Green's function g_h in the three-dimensional case ($d = 3$) on general quasi-uniform meshes.

6. ERROR ANALYSIS FOR THE OPTIMAL CONTROL PROBLEM

In this section, we derive estimates for the discretization error between the continuous optimal control $\bar{q} \in Q_{\text{ad}}$ and the discrete optimal control $\bar{q}_h \in Q_{h,\text{ad}}$ for the case of variational control (i.e. $Q_{h,\text{ad}} = Q_{\text{ad}}$) and piecewise constant control discretization (i.e. $Q_{h,\text{ad}} = Q_{h,\text{ad}}^c$). In both cases, we derive error estimates of order $h|\ln h|$, cf. the Theorems 6.5 and 6.6.

Before doing so, we introduce an additional piece of notation. So far we have only discussed the parts of the adjoint equation related to the singular behavior. Now we also consider the coefficient $S\bar{q}(x_i) - \xi_i$ accompanying each z_i . If this coefficient becomes zero, the singularity at that point vanishes. For the rest of the paper we define

$$L = \{ i \in I \mid S\bar{q}(x_i) - \xi_i = 0 \}$$

to be the set of such indices.

Lemma 6.1. *Let $\bar{q} \in Q_{\text{ad}}$ be the solution of (2.3). Then, for each $i \in I \setminus L$ there is an open ball $B_i \subset \Omega$ containing x_i such that, depending on the sign of $S\bar{q}(x_i) - \xi_i$, either $\bar{q}(x) = a$ or $\bar{q}(x) = b$ holds for all $x \in B_i$. Moreover, it holds $\bar{q} \in W^{1,\infty}(\Omega)$ with*

$$\|\nabla \bar{q}\|_{L^\infty(\Omega)} \leq C.$$

Proof. For the solution \bar{z} of the adjoint equation (2.8), it holds

$$\bar{z} = \sum_{i \in I} (S\bar{q}(x_i) - \xi_i) z_i$$

with z_i being the solution of (2.6). The Lemmas 5.1 and 5.2 applied to z_i for any $i \in I \setminus L$ ensure that for such i and any $M > 0$ there are open balls B_i containing x_i such that

$$|z_i(x)| \geq M \quad \forall x \in B_i \quad \text{and} \quad \|z_j\|_{L^\infty(B_i)} \leq C \quad \text{for } j \in I \setminus \{i\}.$$

Hence, by using $S\bar{q}(x_i) - \xi_i \neq 0$, we can choose B_i for $i \in I \setminus L$ such that either

$$-\frac{1}{\alpha} \bar{z} = -\frac{1}{\alpha} \sum_{j \in I} (S\bar{q}(x_j) - \xi_j) z_j \leq a \quad \text{or} \quad -\frac{1}{\alpha} \bar{z} = -\frac{1}{\alpha} \sum_{j \in I} (S\bar{q}(x_j) - \xi_j) z_j \geq b$$

holds. Hence, by Proposition 2.4, we obtain the first assertion.

For the second assertion, we note that $\bar{q} \in H^1(\Omega)$ by Proposition 2.5. Hence, to prove $\bar{q} \in W^{1,\infty}(\Omega)$ it is sufficient to ensure $\bar{q} \in W^{1,\infty}(B_i)$ for every $i \in I \setminus L$ and $\bar{q} \in W^{1,\infty}(\Omega \setminus \bigcup_{i \in I \setminus L} \bar{B}_i)$. The first result follows directly from the previous discussion. Lemma 5.2 yields $\bar{q} \in W^{1,\infty}(\Omega \setminus \bigcup_{i \in I \setminus L} \bar{B}_i)$, which completes the proof. \square

As preparation for deriving the error estimates for $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$, we prove the following three lemmas.

Lemma 6.2. *For $p, q, q_h \in Q$, it holds*

$$\alpha \|q - q_h\|_{L^2(\Omega)}^2 \leq j_h''(p)(q - q_h, q - q_h).$$

Proof. This follows directly from Lemma 3.3. \square

Lemma 6.3. *For $q, \delta q \in Q_{ad}$, it holds*

$$|j'(q)(\delta q) - j'_h(q)(\delta q)| \leq Ch^2 |\ln h|^2 \|\delta q\|_{L^\infty(\Omega)}$$

with a constant C independent of h .

Proof. By the Lemmas 2.2 and 3.3, we arrive at

$$(6.1) \quad |j'(q)(\delta q) - j'_h(q)(\delta q)| = |(z - z_h, \delta q)| \leq |(z - \tilde{z}_h, \delta q)| + |(\tilde{z}_h - z_h, \delta q)|,$$

where $z \in W_0^{1,s}(\Omega)$ and $z_h \in V_h$ are the solutions of (2.5) and (3.3), respectively and $\tilde{z}_h \in V_h$ denotes the solution of

$$(\nabla \tilde{z}_h, \nabla \varphi_h) = \sum_{i \in I} (Sq(x_i) - \xi_i) \varphi_h(x_i) \quad \forall \varphi_h \in V_h.$$

By construction, it holds

$$\tilde{z}_h = \sum_{i \in I} (Sq(x_i) - \xi_i) z_{h,i}$$

with $z_{h,i}$ given by (3.4).

For the first term on the right-hand side of (6.1), we get by (2.7) and Lemma 5.3

$$\begin{aligned} |(z - \tilde{z}_h, \delta q)| &\leq \|z - \tilde{z}_h\|_{L^1(\Omega)} \|\delta q\|_{L^\infty(\Omega)} \\ &\leq \sum_{i \in I} |Sq(x_i) - \xi_i| \|z_i - z_{h,i}\|_{L^1(\Omega)} \|\delta q\|_{L^\infty(\Omega)} \\ &\leq Ch^2 |\ln h|^2 \{ \|Sq\|_{L^\infty(\Omega)} + |\xi| \} \|\delta q\|_{L^\infty(\Omega)} \\ &\leq Ch^2 |\ln h|^2 \{ \|q\|_{L^2(\Omega)} + |\xi| \} \|\delta q\|_{L^\infty(\Omega)} \end{aligned}$$

with $|\xi|^2 = \sum_{i \in I} \xi_i^2$.

For estimating the second term on the right-hand side of (6.1), let $\Omega_1 \Subset \Omega_0 \Subset \Omega$ with smooth Ω_0 such that $\{x_i \mid i \in I\} \subset \Omega_1$. Then, (3.5) and the Lemmas 5.4 and 4.4 yield

$$\begin{aligned} |(\tilde{z}_h - z_h, \delta q)| &\leq \|\tilde{z}_h - z_h\|_{L^2(\Omega)} \|\delta q\|_{L^2(\Omega)} \\ &\leq \sum_{i \in I} |Sq(x_i) - S_h q(x_i)| \|z_{h,i}\|_{L^2(\Omega)} \|\delta q\|_{L^2(\Omega)} \\ &\leq C \|Sq - S_h q\|_{L^\infty(\Omega_1)} \|\delta q\|_{L^2(\Omega)} \\ &\leq Ch^2 |\ln h|^2 \|q\|_{L^\infty(\Omega)} \|\delta q\|_{L^2(\Omega)}. \end{aligned}$$

Inserting this back in (6.1) proves the lemma. \square

Lemma 6.4. *Let $q, p, \delta q \in Q$. Then, it holds*

$$|j'_h(q)(\delta q) - j'_h(p)(\delta q)| \leq C \|q - p\|_{L^2(\Omega)} \|\delta q\|_{L^2(\Omega)}.$$

Proof. Due to Lemma 3.3, it holds by the mean value theorem for any $\rho \in Q$ that

$$\begin{aligned} j'_h(q)(\delta q) - j'_h(p)(\delta q) &= j''_h(\rho)(q - p, \delta q) \\ &= \sum_{i \in I} S_h(q - p)(x_i) S_h \delta q(x_i) + \alpha(q - p, \delta q). \end{aligned}$$

Therefore, using Lemma 3.1, we can estimate

$$\begin{aligned} |j'_h(q)(\delta q) - j'_h(p)(\delta q)| &\leq C \|S_h(q-p)\|_{L^\infty(\Omega)} \|S_h \delta q\|_{L^\infty(\Omega)} + \alpha |(q-p, \delta q)| \\ &\leq C \|q-p\|_{L^2(\Omega)} \|\delta q\|_{L^2(\Omega)}, \end{aligned}$$

which states the estimate. \square

6.1. Variational Discretization. As first discretization, we consider the variational discretization approach as introduced by [12], i.e., we choose $Q_{h,\text{ad}} = Q_{\text{ad}}$. Note, that the following error estimate extends [2, Theorem 5.2] to domains with polygonal or polyhedral boundary and for $d = 3$ improves convergence rate by $h^{\frac{1}{2}}$ compared to [2, Theorem 5.2] and [6, Theorem 3.2].

Theorem 6.5. *Let $\bar{q} \in Q_{\text{ad}}$ be the solution of the continuous problem (2.3) and $\bar{q}_h \in Q_{\text{ad}}$ the solution of the corresponding discrete problem (3.2) with $Q_{h,\text{ad}} = Q_{\text{ad}}$. Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq Ch |\ln h|$$

with a constant C independent of h .

Proof. Using Lemma 6.2, we have by the optimality conditions (2.9) and (3.7) that

$$\begin{aligned} \alpha \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq j''_h(p)(\bar{q} - \bar{q}_h, \bar{q} - \bar{q}_h) = j'_h(\bar{q})(\bar{q} - \bar{q}_h) - j'_h(\bar{q}_h)(\bar{q} - \bar{q}_h) \\ &\leq j'_h(\bar{q})(\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h) \end{aligned}$$

for arbitrary $p \in Q$. Since $\bar{q}, \bar{q}_h \in Q_{\text{ad}}$ are bounded, Lemma 6.3 implies

$$\begin{aligned} \alpha \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq j'_h(\bar{q})(\bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h) \leq Ch^2 |\ln h|^2 \|\bar{q} - \bar{q}_h\|_{L^\infty(\Omega)} \\ &\leq Ch^2 |\ln h|^2, \end{aligned}$$

which yields the result. \square

6.2. Cellwise Constant Control Discretization. We now consider the fully discretized case where the discrete state and adjoint are approximated by functions in V_h and the discrete control is searched for in $Q_{h,\text{ad}} = Q_{h,\text{ad}}^c$. Note, that the following error estimate improves for $d = 3$ the—to our knowledge—best known error estimate from [1, Theorem 4.3] by $h^{\frac{1}{2}}$.

Theorem 6.6. *Let $\bar{q} \in Q_{\text{ad}}$ be the solution of the continuous problem (2.3) and $\bar{q}_h \in Q_{h,\text{ad}}$ the corresponding solution of the discrete problem (3.2) with $Q_{h,\text{ad}} = Q_{h,\text{ad}}^c$. Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq Ch |\ln h|$$

with a constant C independent of h .

Proof. We split the error as

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq \|\bar{q} - \pi_h \bar{q}\|_{L^2(\Omega)} + \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)},$$

where π_h denotes the L^2 projection on Q_h^c given for $v \in L^1(\Omega)$ by

$$(6.2) \quad (\pi_h v)|_K = \frac{1}{|K|} \int_K v \, dx \quad \forall K \in \mathcal{T}_h.$$

Standard estimates yield by Lemma 6.1 that

$$(6.3) \quad \|\bar{q} - \pi_h \bar{q}\|_{L^2(\Omega)} \leq Ch \|\nabla \bar{q}\|_{L^2(\Omega)} \leq Ch.$$

To bound $\|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)}$, note that $\pi_h \bar{q} \in Q_{h,\text{ad}}^c$. We derive by means of Lemma 6.2, as well as the optimality conditions (2.9) and (3.7) that

$$\begin{aligned} \alpha \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq j_h''(p)(\pi_h \bar{q} - \bar{q}_h, \pi_h \bar{q} - \bar{q}_h) \\ &= j_h'(\pi_h \bar{q})(\pi_h \bar{q} - \bar{q}_h) - j_h'(\bar{q}_h)(\pi_h \bar{q} - \bar{q}_h) \\ &\leq j_h'(\pi_h \bar{q})(\pi_h \bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \bar{q}_h). \end{aligned}$$

By further splitting, we obtain

$$(6.4) \quad \begin{aligned} \alpha \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq j_h'(\pi_h \bar{q})(\pi_h \bar{q} - \bar{q}_h) \\ &\quad - j'(\bar{q})(\pi_h \bar{q} - \bar{q}_h) - j'(\bar{q})(\bar{q} - \pi_h \bar{q}) \\ &= \{j_h'(\pi_h \bar{q})(\pi_h \bar{q} - \bar{q}_h) - j_h'(\bar{q})(\pi_h \bar{q} - \bar{q}_h)\} \\ &\quad + \{j_h'(\bar{q})(\pi_h \bar{q} - \bar{q}_h) - j'(\bar{q})(\pi_h \bar{q} - \bar{q}_h)\} \\ &\quad - j'(\bar{q})(\bar{q} - \pi_h \bar{q}). \end{aligned}$$

For the first term in (6.4) we apply Lemma 6.4 and (6.3) to end up with

$$\begin{aligned} |j_h'(\pi_h \bar{q})(\pi_h \bar{q} - \bar{q}_h) - j_h'(\bar{q})(\pi_h \bar{q} - \bar{q}_h)| &\leq C \|\pi_h \bar{q} - \bar{q}\|_{L^2(\Omega)} \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)} \\ &\leq Ch^2 + \frac{\alpha}{2} \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2. \end{aligned}$$

The second difference in (6.4) can be dealt with as in Theorem 6.5. We apply Lemma 6.3 leading by $\|\pi_h \bar{q}\|_{L^\infty(\Omega)} \leq \|\bar{q}\|_{L^\infty(\Omega)}$ to

$$\begin{aligned} |j_h'(\bar{q})(\pi_h \bar{q} - \bar{q}_h) - j'(\bar{q})(\pi_h \bar{q} - \bar{q}_h)| &\leq Ch^2 |\ln h|^2 \|\pi_h \bar{q} - \bar{q}_h\|_{L^\infty(\Omega)} \\ &\leq Ch^2 |\ln h|^2 \{\|\bar{q}\|_{L^\infty(\Omega)} + \|\bar{q}_h\|_{L^\infty(\Omega)}\} \\ &\leq Ch^2 |\ln h|^2. \end{aligned}$$

By the L^2 orthogonality of the projection π_h , the last term in (6.4) amounts to

$$\begin{aligned} |j'(\bar{q})(\bar{q} - \pi_h \bar{q})| &= |(\alpha \bar{q} + \bar{z}, \bar{q} - \pi_h \bar{q})| = |((\alpha \bar{q} + \bar{z}) - \pi_h(\alpha \bar{q} + \bar{z}), \bar{q} - \pi_h \bar{q})| \\ &\leq \|(\alpha \bar{q} + \bar{z}) - \pi_h(\alpha \bar{q} + \bar{z})\|_{L^1(\Omega)} \|\bar{q} - \pi_h \bar{q}\|_{L^\infty(\Omega)}. \end{aligned}$$

Since \bar{z} is in $W^{1,1}(\Omega)$ (cf. [3, Theorem 4]), we can use standard estimates for the L^2 projection in $L^1(\Omega)$ and $L^\infty(\Omega)$ to get

$$\begin{aligned} |j'(\bar{q})(\bar{q} - \pi_h \bar{q})| &\leq Ch^2 \|\nabla(\alpha \bar{q} + \bar{z})\|_{L^1(\Omega)} \|\nabla \bar{q}\|_{L^\infty(\Omega)} \\ &\leq Ch^2 \{\|\nabla \bar{z}\|_{L^1(\Omega)}^2 + \|\nabla \bar{q}\|_{L^\infty(\Omega)}^2\} \leq Ch^2 \end{aligned}$$

by (5.2) applied to \bar{z} and Lemma 6.1. Collecting the estimated for the right-hand side of (6.4) and absorbing the term $\frac{\alpha}{2} \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2$ to the left-hand side, yields

$$\alpha \|\pi_h \bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 \leq Ch^2 |\ln h|^2,$$

which implies the assertion together with (6.3). \square

7. IMPROVED ERROR ANALYSIS FOR THE OPTIMAL CONTROL PROBLEM IN TWO DIMENSIONS

In this section, we improve the error estimates derived in Section 6 using a more detailed analysis of the behavior of the discrete optimal control in two space dimensions. Moreover, when discretizing the controls by cellwise constants (i.e. for $Q_{h,\text{ad}} = Q_{h,\text{ad}}^c$), we employ a post processing strategy to overcome the limitations to first order convergence in this case. For variational discretization and for cellwise constant control discretization with post processing, we derive error estimates of order $h^2 |\ln h|^2$, cf. the Theorems 7.5 and 7.12.

Throughout this section, the analysis is restricted to $d = 2$. We start by proving a result for the discrete optimal control \bar{q}_h similar to Lemma 6.1. This is possible due to the Lemmas 5.6 and 5.7.

Lemma 7.1. *Let $d = 2$. Further, let $\bar{q}_h \in Q_{h,\text{ad}}$ be the solution of (3.2). Then, for each $i \in I \setminus L$ there is an open ball $B_i \subset \Omega$ containing x_i such that, depending on the sign of $S_h \bar{q}(x_i) - \xi_i$, either $\bar{q}_h(x) = a$ or $\bar{q}_h(x) = b$ holds for all $x \in B_i$ and $h \leq h_0$.*

Proof. We first show that for $i \in I \setminus L$ the difference $|S_h \bar{q}_h(x_i) - \xi_i|$ is bounded away from zero provided that h is sufficiently small. For $i \in I \setminus L$, it holds

$$\begin{aligned} 0 &< |S \bar{q}(x_i) - \xi_i| \\ &\leq |S_h \bar{q}_h(x_i) - S \bar{q}_h(x_i)| + |S(\bar{q}_h - \bar{q})(x_i)| + |S_h \bar{q}_h(x_i) - \xi_i| \\ &\leq Ch^2 |\ln h| \|\bar{q}_h\|_{L^\infty(\Omega)} + C \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} + |S_h \bar{q}_h(x_i) - \xi_i| \\ &\leq Ch |\ln h| + |S_h \bar{q}_h(x_i) - \xi_i|. \end{aligned}$$

Here we used Lemma 4.4, (2.2), and Theorem 6.6. Thereby, we conclude that there is $h_0 > 0$ such that $S_h \bar{q}_h(x_i) - \xi_i \neq 0$ for $i \in I \setminus L$ and $h < h_0$.

For the solution $\bar{z}_h \in V_h$ of the discrete adjoint equation (3.6), it holds

$$\bar{z}_h = \sum_{i \in I} (S_h \bar{q}_h(x_j) - \xi_j) z_{h,i}$$

with the solutions $z_{h,i} \in V_h$ of (3.4). Lemma 5.7 in combination with Lemma 5.6 applied to $z_{h,i}$ for any $i \in I \setminus L$ ensures (by possibly reducing h_0) that for such i and any $M > 0$ there are open balls B_i containing x_i such that

$$|z_{h,i}(x)| \geq M \quad \forall x \in B_i \quad \text{and} \quad \|z_{h,j}\|_{L^\infty(B_i)} \leq C \quad \text{for } j \in I \setminus \{i\}$$

for all $h < h_0$. Hence, by using $S_h \bar{q}_h(x_i) - \xi_i \neq 0$, we obtain on B_i for $i \in I \setminus L$ that either

$$-\frac{1}{\alpha} \bar{z}_h = -\frac{1}{\alpha} \sum_{j \in I} (S_h \bar{q}_h(x_j) - \xi_j) z_{h,j} \leq a$$

or

$$-\frac{1}{\alpha} \bar{z}_h = -\frac{1}{\alpha} \sum_{j \in I} (S_h \bar{q}_h(x_j) - \xi_j) z_{h,j} \geq b$$

holds. In the case of variational discretization, i.e., for $Q_{h,\text{ad}} = Q_{\text{ad}}$, the assertion follows immediately from Remark 3.5 as in Lemma 6.1. For cellwise discrete control

discretization where $Q_{h,\text{ad}} = Q_{h,\text{ad}}^c$, let π_h be the L^2 projection on Q_h^c as given by (6.2). Then, again by Remark 3.5, it holds for $K \in \mathcal{T}_h$ that

$$\bar{q}_h|_K = P_{[a,b]} \left(-\frac{1}{\alpha} (\pi_h \bar{z}_h)|_K \right) = P_{[a,b]} \left(-\frac{1}{\alpha} \bar{z}_h(S_K) \right),$$

where S_K denotes the centroid of the cell K , cf. Section 7.2. This implies the assertion also in this case. \square

Lemma 7.2. *Let $d = 2$, $\bar{q} \in Q_{\text{ad}}$ be the solution of (2.3), and $\bar{q}_h \in Q_{h,\text{ad}}$ the solution of (3.2). Then, there exist $h_0 > 0$ and open balls $B_i \subset \Omega$ containing x_i for $i \in I \setminus L$, such that for all $h \leq h_0$, it holds either*

$$\bar{q}(x) = \bar{q}_h(x) = a \quad \text{or} \quad \bar{q}(x) = \bar{q}_h(x) = b \quad \forall x \in B_i.$$

Proof. The assertion is a direct consequence of the Lemmas 6.1 and 7.1. \square

Lemma 7.3. *Let $d = 2$ and $B_i \subset \Omega$ open balls with $x_i \in B_i$ for $i \in I \setminus L$. For a $q \in Q_{\text{ad}}$, let z be the solution of (2.5) and $\tilde{z}_h \in V_h$ be given as the solution of*

$$(7.1) \quad (\nabla \tilde{z}_h, \nabla \varphi_h) = \sum_{i \in I} (Sq(x_i) - \xi_i) \varphi_h(x_i) \quad \forall \varphi_h \in V_h.$$

Then, it holds for $B = \bigcup_{i \in I \setminus L} B_i$ that

$$\|z - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2,$$

with a constant C independent of h .

Proof. Let z_i be defined by (2.6) and $z_{h,i}$ be defined by (3.4). We begin with the splitting

$$(7.2) \quad \|z - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq \sum_{i \in L} \|(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \\ + \sum_{i \in I \setminus L} \|(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega \setminus \bar{B})}.$$

To discuss the first term of (7.2), we note that for $i \in L$ there holds $Sq(x_i) - \xi_i = 0$. Therefore, we have for such i that

$$(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i} = (S_h q(x_i) - \xi_i)z_{h,i} = (S_h q(x_i) - Sq(x_i))z_{h,i}.$$

Then, we get by the Lemmas 4.4 and 5.4 that

$$(7.3) \quad \|(Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \\ = \|(S_h q(x_i) - Sq(x_i))z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \\ \leq |S_h q(x_i) - Sq(x_i)| \|z_{h,i}\|_{L^2(\Omega)} \\ \leq Ch^2 |\ln h|^2 \|q\|_{L^\infty(\Omega)}.$$

For the second term of (7.2), we split

$$(7.4) \quad (Sq(x_i) - \xi_i)z_i - (S_h q(x_i) - \xi_i)z_{h,i} \\ = (Sq(x_i) - \xi_i)(z_i - z_{h,i}) + (Sq(x_i) - S_h q(x_i))z_{h,i}$$

and obtain for the first term by (2.2) and Lemma 5.5 applied to z_i and $z_{h,i}$ separately on B_i that

$$\begin{aligned} \|(Sq(x_i) - \xi_i)(z_i - z_{h,i})\|_{L^2(\Omega \setminus \bar{B})} &\leq \{\|q\|_{L^2(\Omega)} + |\xi|\} \|z_i - z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \\ &\leq Ch^2 |\ln h| \{\|q\|_{L^2(\Omega)} + |\xi|\}. \end{aligned}$$

For the second term of the right-hand side of (7.4), we can proceed as for (7.3) to obtain

$$\|(Sq(x_i) - S_h q(x_i))z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2 \|q\|_{L^\infty(\Omega)}.$$

Collecting the terms, we get for the second term on the right-hand side of (7.2)

$$\|(S\bar{q}(x_i) - \xi_i)z_i - (S_h \bar{q}(x_i) - \xi_i)z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2 \{\|q\|_{L^\infty(\Omega)} + |\xi|\}.$$

By adding up the two estimates for (7.2), we arrive at the result. □

Based on the previous lemma, we can improve Lemma 6.3 for the case $d = 2$. The improvement consists in the fact that the L^∞ norm of δq can be replaced by its L^2 norm provided that δq vanishes in the neighborhood of the points x_i for $i \in I \setminus L$.

Lemma 7.4. *Let $d = 2$ and $B_i \subset \Omega$ open balls with $x_i \in B_i$ for $i \in I \setminus L$. For $B = \bigcup_{i \in I \setminus L} B_i$ and $q, \delta q \in Q_{ad}$ with $\delta q|_B = 0$, it holds*

$$|j'(q)(\delta q) - j'_h(q)(\delta q)| \leq Ch^2 |\ln h|^2 \|\delta q\|_{L^2(\Omega)}.$$

with a constant C independent of h .

Proof. The proof follows the lines of Lemma 6.3 obtaining

$$|j'(q)(\delta q) - j'_h(q)(\delta q)| \leq |(z - \tilde{z}_h, \delta q)| + |(\tilde{z}_h - z_h, \delta q)|.$$

with \tilde{z}_h solving (7.1). The second term is treated as in Lemma 6.3. Using Lemma 7.3 for the first term then implies the result by

$$|(z - \tilde{z}_h, \delta q)| \leq \|z - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} \|\delta q\|_{L^2(\Omega)}$$

since $\delta q|_B = 0$. □

7.1. Variational Discretization. The following result constitutes an improvement of Theorem 6.5 in the case $d = 2$. Since the result relies on Lemma 5.7 which is available for $d = 2$ only, an extension to $d = 3$ is not directly possible.

Theorem 7.5. *Let $d = 2$, $\bar{q} \in Q_{ad}$ be the solution of the continuous problem (2.3), and $\bar{q}_h \in Q_{ad}$ be the solution of the corresponding discrete problem (3.2) with $Q_{h,ad} = Q_{ad}$. Then, it holds*

$$\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)} \leq Ch^2 |\ln h|^2$$

with a constant C independent of h .

Proof. The assertion is proven as Theorem 6.5: Let $B = \bigcup_{i \in I \setminus L} B_i$ for the sets B_i given by Lemma 7.2. On B , it holds $\delta q = \bar{q} - \bar{q}_h = 0$. Hence, by using Lemma 7.4, we conclude

$$\begin{aligned} \alpha \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}^2 &\leq |j'(q)(\delta q) - j'_h(q)(\delta q)| \\ &\leq Ch^2 |\ln h|^2 \{\|\bar{q}\|_{L^\infty(\Omega)} + |\xi|\} \|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}. \end{aligned}$$

Dividing by $\|\bar{q} - \bar{q}_h\|_{L^2(\Omega)}$ proves the theorem. □

7.2. Cellwise Constant Control Discretization with Post Processing. We adapt the proof technique from [18]. We split up the mesh into subsets with respect to the regularity of \bar{q} . For a given mesh \mathcal{T}_h , we define three subsets of cells

$$\begin{aligned} \mathcal{T}_h^1 &= \{ K \in \mathcal{T}_h \mid \bar{q}|_K = a \text{ or } \bar{q}|_K = b \}, & \mathcal{T}_h^2 &= \{ K \in \mathcal{T}_h \mid a < \bar{q}|_K < b \}, \\ & & \text{and } \mathcal{T}_h^3 &= \mathcal{T}_h \setminus (\mathcal{T}_h^1 \cup \mathcal{T}_h^2). \end{aligned}$$

Then, \mathcal{T}_h^3 denotes the set of cells where \bar{q} is only Lipschitz continuous in contrast to $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$ which consists of cells K where \bar{q} is in $H^2(K)$. As in [18], we make the following assumption:

Assumption 7.6. We assume the existence of a constant C independent of h , such that for all h sufficiently small, it holds

$$\sum_{K \in \mathcal{T}_h^3} |K| \leq Ch.$$

Similar assumptions have been made in the case of cellwise linear discretization or a postprocessing approach in, e.g., [18, 25].

Remark 7.7. By (5.3) applied to \bar{z} , it seems likely that the active set $\{x \in \Omega \mid \bar{q}(x) = a \vee \bar{q}(x) = b\}$ has rectifiable boundary. This would imply that the boundary is a curve of finite length, i.e., it can be covered by a subset $\mathcal{T}_h^3 \subset \mathcal{T}_h$ fulfilling Assumption 7.6.

We denote by S_K the centroid of a cell $K \in \mathcal{T}_h$. For $w \in C(\Omega)$, we define the projection $R_h w \in Q_h^c$ by

$$(7.5) \quad R_h w|_K = w(S_K) \quad \forall K \in \mathcal{T}_h.$$

Further, we set

$$(7.6) \quad r_h = R_h \bar{q},$$

which is well-defined by Lemma 6.1, and note that by construction, it holds $r_h \in Q_{h,ad}^c$.

Proposition 7.8. *Let $f \in H^2(K)$ for a given cell $K \in \mathcal{T}_h$. Then, it holds*

$$\left| \int_K (f(x) - f(S_K)) dx \right| \leq Ch^2 |K|^{\frac{1}{2}} \|\nabla^2 f\|_{L^2(K)}.$$

Proof. See [18, Lemma 3.2]. □

Similar to [18, Lemma 3.3], we prove the following result adapted to the problem considered here:

Lemma 7.9. *Let $d = 2$, $\bar{q} \in Q_{ad}$ be the solution of (2.3), and $r_h \in Q_{h,ad}^c$ defined by (7.6). Then, there is $h_0 > 0$ such that for all $i \in I \setminus L$ and all $h \leq h_0$, it holds*

$$|S_h(\bar{q} - r_h)(x_i)| \leq Ch^2$$

with a constant C independent of h .

Proof. Let $z_{h,i} \in V_h$ be the solution of (3.4). Then, we can write

$$S_h(\bar{q} - r_h)(x_i) = (\nabla z_{h,i}, \nabla(S_h \bar{q} - S_h r_h)) = (z_{h,i}, \bar{q} - r_h).$$

For $K \in \mathcal{T}_h^1$, it holds $\bar{q}|_K \in \{a, b\}$. Hence, we have $r_h = \bar{q}$ there and $(z_{h,i}, \bar{q} - r_h)$ vanishes on $\bigcup \mathcal{T}_h^1$. Consequently, we get

$$(z_{h,i}, \bar{q} - r_h) = \sum_{K \in \mathcal{T}_h^2} \int_K z_{h,i}(\bar{q} - r_h) dx + \sum_{K \in \mathcal{T}_h^3} \int_K z_{h,i}(\bar{q} - r_h) dx.$$

Lemma 6.1 ensures the existence of open sets $B_i \subset \Omega$ for $i \in I \setminus L$ with $x_i \in B_i$ and $\bar{q}(x) \in \{a, b\}$ for $x \in B_i$. For $h \leq h_0$, the sets B_i can be chosen such that $B_i \subset \bigcup \mathcal{T}_h^1$.

We first consider the integral over $K \in \mathcal{T}_h^3$. Again by Lemma 6.1, we know that $\bar{q} \in W^{1,\infty}(K)$ which implies for $x \in K$ that

$$|\bar{q}(x) - r_h(x)| = |\bar{q}(x) - \bar{q}(S_K)| \leq C|x - S_K| \|\nabla \bar{q}\|_{L^\infty(\Omega)} \leq Ch \|\nabla \bar{q}\|_{L^\infty(\Omega)}.$$

For $B = \bigcup_{i \in I \setminus L} B_i$ it holds $B \subset \bigcup \mathcal{T}_h^1$. Hence, we have $\bigcup \mathcal{T}_h^3 \subset \Omega \setminus \bar{B}$ and one obtains

$$\begin{aligned} \left| \sum_{K \in \mathcal{T}_h^3} \int_K z_{h,i}(\bar{q} - r_h) dx \right| &\leq \sum_{K \in \mathcal{T}_h^3} \int_K |z_{h,i}(\bar{q} - r_h)| dx \\ &\leq Ch \|z_{h,i}\|_{L^\infty(\Omega \setminus \bar{B})} \|\nabla \bar{q}\|_{L^\infty(\Omega)} \sum_{K \in \mathcal{T}_h^3} |K| \\ &\leq Ch^2 \|\nabla \bar{q}\|_{L^\infty(\Omega)} \end{aligned}$$

by means of Assumption 7.6 and Lemma 5.6.

For a cell $K \in \mathcal{T}_h^2$, we have that

$$\int_K z_{h,i} r_h dx = \int_K z_{h,i} \bar{q}(S_K) dx = \int_K z_{h,i}(S_K) \bar{q}(S_K) dx.$$

Using this, we obtain by Proposition 7.8 and the Cauchy-Schwarz inequality that

$$\left| \sum_{K \in \mathcal{T}_h^2} \int_K z_{h,i}(\bar{q} - r_h) dx \right| \leq Ch^2 \left(\sum_{K \in \mathcal{T}_h^2} \|\nabla^2(z_{h,i} \bar{q})\|_{L^2(K)}^2 \right)^{\frac{1}{2}}.$$

On $\bigcup \mathcal{T}_h^2$, we have $\bar{q} = -\alpha^{-1} \bar{z}$. By the product rule, we get since $\mathcal{T}_h^2 \subset \Omega \setminus \bar{B}$ that

$$\begin{aligned} &\left(\sum_{K \in \mathcal{T}_h^2} \|\nabla^2(z_{h,i} \bar{q})\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \\ &\leq C \|z_{h,i}\|_{L^\infty(\Omega \setminus \bar{B})} \|\nabla^2 \bar{z}\|_{L^2(\Omega \setminus \bar{B})} + C \|\nabla z_{h,i}\|_{L^2(\Omega \setminus \bar{B})} \|\nabla \bar{z}\|_{L^\infty(\Omega \setminus \bar{B})}. \end{aligned}$$

Using Lemma 5.6 for the terms involving $z_{h,i}$ and Lemma 5.2 for

$$\bar{z} = \sum_{i \in I \setminus L} (S \bar{q}(x_i) - \xi_i) z_i,$$

we obtain the assertion by collecting the previous estimates. \square

Lemma 7.10. *Let be $d = 2$ and $B_i \subset \Omega$ open balls with $x_i \in B_i$ for $i \in I \setminus L$. For $\hat{z}_h \in V_h$ given as the solution of*

$$(7.7) \quad (\nabla \hat{z}_h, \nabla \varphi_h) = \sum_{i \in I} (S_h r_h(x_i) - \xi_i) \varphi_h(x_i) \quad \forall \varphi_h \in V_h.$$

for r_h defined by (7.6) and $B = \bigcup_{i \in I \setminus L} B_i$, it holds

$$\|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2$$

with a constant C independent of h .

Proof. We split

$$\|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq \|\bar{z} - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} + \|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})},$$

where $\tilde{z}_h \in V_h$ is the solution of (7.1). For the first term, Lemma 7.3 states

$$\|\bar{z} - \tilde{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2.$$

For the second term, we get from Lemma 5.4 that

$$\|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq \|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega)} \leq C \sum_{i \in I} |S_h(\bar{q} - r_h)(x_i)|.$$

Then, by Lemma 7.9, we get

$$\|\tilde{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2.$$

Collecting the estimates yields the assertion. \square

Lemma 7.11. *Let $d = 2$, $\bar{q}_h \in Q_{h,ad}^c$ be the solution of (3.2) with $Q_{h,ad} = Q_{h,ad}^c$, and r_h defined by (7.6). Then, there exists $h_0 > 0$ such that for $h \leq h_0$, it holds*

$$\|\bar{q}_h - r_h\|_{L^2(\Omega)} \leq Ch^2 |\ln h|^2$$

with a constant C independent of h .

Proof. As in [18], we first derive a variational inequality for r_h . To this end let $h_0 > 0$ be sufficiently small such that the sets $\{B_{0,i} \subset \Omega \mid i \in I \setminus L\}$ given by Lemma 7.2, subsets $\{B_{1,i} \subset B_{0,i} \mid i \in I \setminus L\}$, and a subset of cells $\tilde{\mathcal{T}}_{h_0} \subset \mathcal{T}_{h_0}$ fulfill the relation

$$B_1 \subset \Omega_{h_0} \subset B_0$$

for $B_0 = \bigcup_{i \in I \setminus L} B_{0,i}$, $\Omega_{h_0} = \bigcup \tilde{\mathcal{T}}_{h_0}$, and $B_1 = \bigcup_{i \in I \setminus L} B_{1,i}$.

By Lemma 5.2, we have that $\bar{z} \in C(\bar{\Omega} \setminus B_1)$. We now apply the optimality condition (2.9), which holds true for all $\delta q \in Q_{ad}$ and also pointwise on $\bar{\Omega} \setminus B_1$:

$$(\alpha \bar{q}(x) + \bar{z}(x))(\delta q - \bar{q}(x)) \geq 0 \quad \forall \delta q \in [a, b], x \in \bar{\Omega} \setminus B_1.$$

We apply this formula for $x = S_K$ with $K \in \mathcal{T}_h$ and $K \subset \Omega \setminus \Omega_{h_0}$ and $\delta q = \bar{q}_h(S_K)$ which gives

$$(\alpha r_h(S_K) + \bar{z}(S_K))(\bar{q}_h(S_K) - r_h(S_K)) \geq 0.$$

Integrating this inequality over K and summing this up over $K \in \mathcal{T}_h$ with $K \subset \Omega \setminus \Omega_{h_0}$ yields

$$(\alpha r_h + R_h \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega \setminus \Omega_{h_0})} \geq 0.$$

Noting that $\bar{q}_h - r_h = 0$ on B_0 implies

$$(\alpha r_h + R_h \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} \geq 0.$$

By testing the discrete optimality condition (3.7) with $\delta q_h = r_h$, we get

$$(\alpha \bar{q}_h + \bar{z}_h, r_h - \bar{q}_h)_{L^2(\Omega \setminus \bar{B}_0)} \geq 0$$

again by using $r_h - \bar{q}_h = 0$ on B_0 . Adding the last two inequalities results in the estimate

$$(7.8) \quad \alpha \|r_h - \bar{q}_h\|_{L^2(\Omega)}^2 = \alpha \|r_h - \bar{q}_h\|_{L^2(\Omega \setminus \bar{B}_0)}^2 \leq (R_h \bar{z} - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)}.$$

We split the right-hand side of the above inequality to get

$$\begin{aligned} (R_h \bar{z} - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} &= (R_h \bar{z} - \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} \\ &\quad + (\bar{z} - \hat{z}_h, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} + (\hat{z}_h - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)}, \end{aligned}$$

where $\hat{z}_h \in V_h$ solves (7.7). We separately estimate the three terms on the right-hand side.

Using Proposition 7.8, the fact that $\bar{q}_h = r_h$ on B_0 , and that \bar{q}_h and r_h are piecewise constant, one arrives for the first term at

$$\begin{aligned} (R_h \bar{z} - \bar{z}, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} &\leq \sum_{\substack{K \in \mathcal{T}_h \\ K \subset \Omega \setminus \Omega_{h_0}}} |\bar{q}_h(S_K) - r_h(S_K)| \left| \int_K (R_h \bar{z} - \bar{z}) dx \right| \\ &\leq Ch^2 \sum_{\substack{K \in \mathcal{T}_h \\ K \subset \Omega \setminus \Omega_{h_0}}} |\bar{q}_h(S_K) - r_h(S_K)| |K|^{\frac{1}{2}} \|\nabla^2 \bar{z}\|_{L^2(K)} \\ &\leq Ch^2 \|\bar{q}_h - r_h\|_{L^2(\Omega)} \|\bar{z}\|_{H^2(\Omega \setminus \bar{B}_1)}, \end{aligned}$$

where $\|\bar{z}\|_{H^2(\Omega \setminus \bar{B}_1)}$ is bounded due to Lemma 5.2.

For the second term, it follows by Lemma 7.10 that

$$(\bar{z} - \hat{z}_h, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} \leq Ch^2 |\ln h|^2 \|\bar{q}_h - r_h\|_{L^2(\Omega)}.$$

For the last term, we have since $\bar{q}_h - r_h = 0$ on B_0 that

$$\begin{aligned} (\hat{z}_h - \bar{z}_h, \bar{q}_h - r_h)_{L^2(\Omega \setminus \bar{B}_0)} &= (\nabla(\hat{z}_h - \bar{z}_h), \nabla S_h(\bar{q}_h - r_h)) \\ &= - \sum_{i \in I} (S_h(\bar{q}_h - r_h)(x_i))^2 \leq 0. \end{aligned}$$

By using the last three estimates and (7.8), we complete the proof. \square

Using the previous lemmas, we can conclude this section by formulating the error estimate for the post-processed control \hat{q}_h given by

$$(7.9) \quad \hat{q}_h = P_{[a,b]}(-\alpha^{-1} \bar{z}_h),$$

where $\bar{z}_h \in V_h$ is the adjoint state associated to the solution $\bar{q}_h \in Q_{h,\text{ad}}^c$ of the discrete problem (3.2) with $Q_{h,\text{ad}} = Q_{h,\text{ad}}^c$.

Theorem 7.12. *Let $d = 2$, $\bar{q} \in Q_{\text{ad}}$ be the solution of the continuous problem (2.3), and \hat{q}_h given by (7.9). Furthermore, let Assumption 7.6 hold. Then, it holds*

$$\|\bar{q} - \hat{q}_h\|_{L^2(\Omega)} \leq Ch^2 |\ln h|^2$$

with a constant C independent of h .

Proof. By Lemma 7.2 there is $B = \bigcup_{i \in I \setminus L} B_i$ such that $\bar{q} = \hat{q}_h$ on B . Then, the Lipschitz continuity of the projection operator $P_{[a,b]}$ in $L^2(\Omega)$ implies

$$\begin{aligned} \|\bar{q} - \hat{q}_h\|_{L^2(\Omega)} &= \|\bar{q} - \hat{q}_h\|_{L^2(\Omega \setminus B)} \\ &= \|P_{[a,b]}(-\alpha^{-1}\bar{z}) - P_{[a,b]}(-\alpha^{-1}\bar{z}_h)\|_{L^2(\Omega \setminus \bar{B})} \\ &\leq C\|\bar{z} - \bar{z}_h\|_{L^2(\Omega \setminus \bar{B})}. \end{aligned}$$

Then, we split

$$\|\bar{z} - \bar{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq \|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} + \|\hat{z}_h - \bar{z}_h\|_{L^2(\Omega \setminus \bar{B})},$$

where \hat{z}_h solves (7.7). For the first term, we have by Lemma 7.10

$$\|\bar{z} - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} \leq Ch^2 |\ln h|^2.$$

For the second term, we get by the Lemmas 3.1, 7.11, and 5.4.

$$\begin{aligned} \|\bar{z}_h - \hat{z}_h\|_{L^2(\Omega \setminus \bar{B})} &\leq \|\bar{z}_h - \hat{z}_h\|_{L^2(\Omega)} \leq C \sum_{i \in I} |S_h(\bar{q}_h - r_h)(x_i)| \\ &\leq C\|\bar{q}_h - r_h\|_{L^2(\Omega)} \leq Ch^2 |\ln h|^2. \end{aligned}$$

Combining the estimates implies the assertion. \square

8. NUMERICAL RESULTS

We give numerical results to confirm the results of the previous sections. To this end we consider different sample problems. The optimal control problems are solved by the optimization library RoDoBo [24] and the finite element toolkit GASCOIGNE [9].

We consider the optimal control problem (1.1) with the slightly modified state equation

$$-\Delta u = f + q$$

with given right-hand side f in a ball $\Omega = B_{0.5}(x_1) \subset \mathbb{R}^d$ with $d \in \{2, 3\}$. The cost functional consist of one point evaluation at $x_1 = (0.5, 0.5)^T$ for $d = 2$ and respectively $x_1 = (0.5, 0.5, 0.5)^T$ for $d = 3$. The choice of Ω allows to give an exact solution of $-\Delta z_1 = \delta_{x_1}$ by

$$z_1 = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x-x_1|} - \frac{\ln 2}{2\pi} & \text{for } d = 2, \\ \frac{1}{4\pi} \frac{1}{|x-x_1|} - \frac{1}{2\pi} & \text{for } d = 3, \end{cases}$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. We choose $\bar{u}(x) = \cos(\pi|x - x_1|)$ and $\xi_1 = \bar{u}(x_1) - 1$. Hence, the adjoint solution \bar{z} is then given as $\bar{z} = z_1$. By choosing $\alpha = 1$, the optimal control fulfills $\bar{q} = P_{[a,b]}(-\bar{z})$. The right-hand side f is chosen such that \bar{u} solves the state equation. Thus, we have $f = -\Delta \cos(\pi|x|) - q$.

First, we present results for cellwise constant discretization of the control. Here, the bounds are chosen as $-a = b = 1$. The numerical results depicted in Figure 1 (left) confirm the estimates of Theorem 6.6.

Further, for choosing a different value for the bounds, $-a = b = 0.2$, we present in Figure 1 (right) results for cellwise constant discretization of the control with post processing. They confirm the estimate of Theorem 7.12, which was proved in

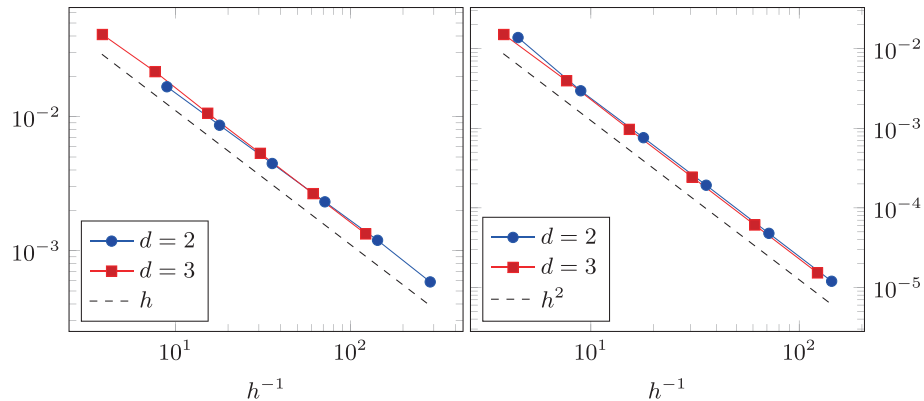


FIGURE 1. Errors $\|q - q_h\|_{L^2(\Omega)}$ for cellwise constant control discretization (left) and $\|q - \hat{q}_h\|_{L^2(\Omega)}$ for cellwise constant control discretization with post processing (right)

$d = 2$ dimensions only. However, the numerical results for $d = 3$ indicate that a similar convergence result may also hold in three dimensions.

ACKNOWLEDGEMENT

Supported by the DFG through the International Research Training Group IGDK 1754 “Optimization and Numerical Analysis for Partial Differential Equations with Nonsmooth Structures”.

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