Pure and Applied Functional Analysis
Volume 4, Number 2, 2019, 205–217

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

SUFFICIENCY OF THE PONTRYAGIN MAXIMUM PRINCIPLE FOR L_1 MINIMIZERS AND AFFINE CONTROL SYSTEMS

M. MARGARIDA A. FERREIRA AND GEORGI V. SMIRNOV

Dedicated to Professor Boris Mordukhovich on the occasion of his 70th birthday

Abstract. This paper follows a previous work where a *refined maximum principle condition* is introduced to guarantee L_∞ local optimality of control processes for affine control systems with a polyhedral set of controls. Here we consider a different strengthened maximum principle condition that allows to extend the class of control processes and obtain sufficient conditions for a local minimizer in L_1 sense.

1. Introduction

In this work we focus on the Pontryagin Maximum Principle (MP) and give a strengthened form of this principle to obtain sufficient conditions of optimality. It is well known that if the optimal control (OC) problem is a linear convex problem, then the necessary conditions are also sufficient (see, for example [6]). Otherwise, sufficient conditions of optimality are traditionally of second order. There is by now a vast literature on this subject, see, for example, [1, 5, 7] and the references therein, to name but a few.

An interesting question to rise is how first order conditions for nonlinear and nonconvex problems became also sufficient. Such question is not new, it had already captured the attention of other researchers. In [2], the author considers a general optimal control problem and establishes some additional condition under which the MP is also sufficient for a local minimum, with respect to some particular metric called L_0 metric. However, the imposed condition is not satisfied if the optimal control problem involves an affine control system and a polyhedron as a control set, a situation that occurs frequently. In [4], the authors analyse a particular class of optimal control problems, involving an affine control system and a polyhedral set of controls. A *refined* maximum principle was introduced and sufficient conditions of optimality were obtained for a local minimum with respect to L_∞ norm in the space of controls.

In this paper we adjust the results obtained in [4] in order to guarantee the sufficiency of the MP for local optimality in the L_1 norm, in the space of controls. To achieve that, a strengthening of the MP, more restrictive than the previous *refined* maximum principle, is introduced. In the space of controls, the class of L_∞

2010 *Mathematics Subject Classification.* 49K15.
Key words and phrases. Pontryagin maximum principle, sufficient conditions.

local minimizers is larger than the class of L_1 local minimizers. On the other hand, the latter are more natural in the OC problems setting. The sufficient conditions developed here can be said to filter in a better way more interesting minimizers.

We introduce some notation that will be used throughout the paper. The closed unit ball in $\mathbb{R}^n$, centred at the origin, is denoted by $\bar{B}$. The Euclidean norm of a point x and the inner product between x, y are denoted by $|x|$ and $\langle x, y\rangle$. The norm $|\cdot|_p$ means the L_p-norm, with $1 \leq p \leq \infty$. The set of nonnegative real numbers is represented by $\mathbb{R}_+$ and $C([a,b]; D)$ denotes the set of continuous functions $f : [a,b] \to D$. Given a matrix A, the transpose of A is represented by A^* and the identity matrix is represented by I. The Lebesgue measure of a given set C is denoted by $\text{meas}(C)$.

2. Problem formulation and background notes

We consider the following optimal control problem (P), as we will call it from now on:

$$\phi(x(T)) \to \inf, \tag{2.1}$$

$$\dot{x} = f(t,x) + g(t,x)u, \tag{2.2}$$

$$u \in U, \tag{2.3}$$

$$x(0) = x_0, \tag{2.4}$$

where T is fixed and x_0 is a given point in $\mathbb{R}^n$. Here, and throughout the paper, $\phi : \mathbb{R}^n \to \mathbb{R}$, $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are twice continuously differentiable functions. The set $U \subset \mathbb{R}^m$ is a bounded polyhedron, i.e., $U = \{x \in \mathbb{R}^m : \langle x, c_i\rangle \leq \alpha_i,\ i = \overline{1,k}\}$, where c_i is a fixed vector in $\mathbb{R}^m$ and α_i is a fixed real constant, for every $i = \overline{1,k}$. We denote by $\bar{\omega}$ the *diameter* of U, i. e., a positive constant such that $\max_{u_1,u_2\in U} |u_1 - u_2| = \bar{\omega}$.

As usual in the optimal control framework, a *control function* $u(\cdot)$ is a measurable function $u : [0,T] \to \mathbb{R}^m$ that satisfies $u(t) \in U$, *a.e.*. A state *trajectory* $x(\cdot)$ corresponding to $u(\cdot)$ is an absolutely continuous function that satisfies the differential equation $\dot{x} = f(t,x) + g(t,x)u(t)$, *a.e.*. A *control process* $(u(\cdot), x(\cdot))$ (some times referred simply as *process* and/or represented for shortness by (u,x)) comprises a control function $u(\cdot)$ and a state trajectory $x(\cdot)$. An *optimal process* is a control process that minimizes the cost over all admissible processes. Here, we also make reference to a local minimum with respect to L_∞ norm in the space of controls. Such minimum was designated in [4] by *weak local minimum* and it is associated to a control process $(\hat{u}(\cdot), \hat{x}(\cdot))$ that minimizes the cost when compared with admissible processes $(u(\cdot), x(\cdot))$ satisfying $|u(\cdot) - \hat{u}(\cdot)|_\infty < \epsilon$, for some $\epsilon > 0$. Changing that norm by the L_1 norm, we obtain a definition for local minimizer with respect to L_1 norm, in the space of controls.

Well known existence theorems in optimal control theory guarantee the existence of solution to problem (P). The Pontryagin maximum principle establishes necessary conditions of optimality for an optimal process. When applied to (P), it asserts that if $(\hat{u}, \hat{x})$ is an optimal process, then there exists an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ such that,

$$-\dot{p}(t) = (\nabla_x(f(t,\hat{x}) + g(t,\hat{x})\hat{u}))^* p(t) \tag{2.5}$$

$$\max_{u \in U} \langle p(t), g(t,\hat{x}(t))u \rangle = \langle p(t), g(t,\hat{x}(t))\hat{u}(t) \rangle \quad a.e.\, t \in [0,T] \tag{2.6}$$

$$-p(T) = \nabla \phi(\hat{x}(T)).$$

Observe that the cost multiplier is set to 1. This is a normal form of the MP which is valid in our setting. In [4] the above conditions were scrutinized with the aim of analysing the information that could be extracted for an admissible process that satisfies them. Take an admissible process $(\hat{u}(\cdot), \hat{x}(\cdot))$ and let $\bar{u}$ be a measurable function such that $\hat{u}(t) + \alpha \bar{u}(t) \in U$, $a.e.\, t \in [0,T]$, $\alpha \in [0, \alpha_0]$, for some $\alpha_0 > 0$. It was proved that if the maximum principle uniquely defines the control, i.e.,

$$\langle p(t), g(t,\hat{x}(t))u \rangle < \langle p(t), g(t,\hat{x}(t))\hat{u}(t) \rangle, \quad u \in U, \ u \neq \hat{u}(t), \tag{2.7}$$

then, $\hat{u}(\cdot)$ is a directional minimizer, in the sense that

$$\phi(x(T, \alpha\bar{u}(\cdot))) > \phi(\hat{x}(T))$$

for all $\alpha > 0$ sufficiently small. Here, $x(\cdot, \bar{u}(\cdot))$ denotes the solution to the Cauchy problem

$$\dot{x} = f(t,x) + g(t,x)(\hat{u} + \bar{u}), \quad x(0) = x_0. \tag{2.8}$$

The conditions of the MP where then enforced in order to guarantee local optimality of $\hat{u}$ and not merely directional optimality. Condition (2.7) was supplemented in such a way that some extra regularity on the adjoint variable $p(\cdot)$ is present. Sufficient conditions of optimality were then deduced for a local minimum with respect to L_∞ norm in the space of controls. We now adjust this result and derive sufficient optimality conditions with respect to L_1 norm in the space of controls. This will be done with a more demanding enforcement of condition (2.7).

We proceed with some estimates that will be of use.

Let $(\hat{u}, \hat{x})$ be a process for (P). Using the definition of $x(\cdot, \bar{u}(\cdot))$ given in (2.8), the trajectory $\hat{x}(\cdot)$ can be expressed as $x(\cdot, 0)$. Now, define $\bar{x}(\cdot)$ as the solution to the Cauchy problem

$$\dot{\bar{x}} = (\nabla_x f(t,\hat{x}) + \nabla_x(g(t,\hat{x})\hat{u}))\bar{x} + g(t,\hat{x})\bar{u}, \quad \bar{x}(0) = 0. \tag{2.9}$$

Taking a general solution of (2.8), we can write

$$x(t, \bar{u}(\cdot)) = \hat{x}(t) + \bar{x}(t) + r(t, \bar{u}(\cdot)), \tag{2.10}$$

for some *difference* function $r : [0,T] \times \mathbb{R}^m \to \mathbb{R}^n$. For this function r, the Filippov theorem [3] asserts that

$$|r(t, \bar{u}(\cdot))| \leq C_1 \int_0^T \rho(t, \bar{u}(\cdot)) dt, \tag{2.11}$$

where $C_1 = e^{\int_0^T k_1(t)\, dt}$ and $k_1(t)$ is the Lipschitz constant associated to the function $F(t,x) = f(t,x) + g(t,x)(\hat{u} + \bar{u})$. The function $\rho(t, \bar{u}(\cdot))$ represents the following distance

$$\begin{aligned}\rho(t,\bar u(\cdot)) &= |\dot{\hat x}(t)+\dot{\bar x}(t)-f(t,\hat x(t)+\bar x(t))-g(t,\hat x(t)+\bar x(t))(\hat u(t)+\bar u(t))|\\ &\le \frac12\max|\nabla_x^2 f(t,x)||\bar x(t)|^2+\frac12\max|\nabla_x^2 g(t,x)||\bar x(t)|^2(|\hat u(t)|+|\bar u(t)|)\\ &\quad+\max|\nabla_x g(t,x)||\bar x(t)||\bar u(t)|.\end{aligned}$$

We assume that x is in some *tube* around $\hat x$, i.e., the graph of x is in a set $\Omega = \{(t,x):\ x\in\hat x(t)+\omega\bar B,\ t\in[0,T]\}$, for some $\omega>0$.

Analysing now the solutions of (2.9), Gronwall's inequality allow us to deduce that

$$|\bar x(t)|\le C_2\int_0^T|\bar u(t)|dt, \tag{2.12}$$

where

$$C_2=e^{\int_0^T k_2(t)\,dt}\max_{\substack{|v|\le1\\0\le t\le T}}|g(t,\hat x(t))v|,\quad k_2(t)=|\nabla_x f(t,\hat x)+\nabla_x(g(t,\hat x)\hat u(t))|.$$

Going back to $\rho(t,\bar u(\cdot))$, we can write

$$\begin{aligned}\rho(t,\bar u(t)) &\le L_1|\bar x(t)|^2+L_2|\bar x(t)|.|\bar u(t)|+L_3|\bar x(t)|^2\\ &\le (L_1+L_3)C_2^2\left(\int_0^T|\bar u(t)|\,dt\right)^2+L_2C_2|\bar u(t)|\int_0^T|\bar u(t)|\,dt\end{aligned}$$

where

$$L_1=\frac12\max|\nabla_x^2 f(t,x)|,\quad L_2=\max|\nabla_x g(t,x)|$$

$$L_3=\frac12\max|\nabla_x^2 g(t,x)|(|\hat u(t)|+|\bar u(t)|).$$

The above maximums are taken on a closed tube around $\hat x$ with radius $C_2\cdot\bar\omega T$ which is an upper bound for $|\bar x(t)|$.

Therefore, from (2.11), we have

$$|r(t,\bar u(\cdot))|\le C_3\left(\int_0^T|\bar u(t)|dt\right)^2, \tag{2.13}$$

where

$$C_3=C_1(L_1+L_3)C_2^2T+C_1L_2C_2.$$

The above estimates can be used to compare the cost function at $\hat x$ and related points. Since

$$\phi(x(T,\bar u(\cdot)))=\phi(\hat x(T))+\langle\nabla\phi(\hat x(T)),\bar x(T)+r(T,\bar u(\cdot))\rangle+$$
$$+\frac12\langle\bar x(T)+r(T,\bar u(\cdot)),\nabla^2\phi(x_\theta)(\bar x(T)+r(T,\bar u(\cdot)))\rangle,$$

where $x_\theta=(1-\theta)\hat x(T)+\theta x(T,\bar u(\cdot))$ for some $\theta\in[0,1]$, we conclude that

$$\phi(x(T,\bar u(\cdot)))\ge\phi(\hat x(T))+\langle\nabla\phi(\hat x(T)),\bar x(T)\rangle-c\left(\int_0^T|\bar u(t)|dt\right)^2, \tag{2.14}$$

where

$$c \geq C_3 \left|\nabla\phi(\hat{x}(T))\right| + (C_2^2 + C_3^2) \max_{x\in\hat{x}(T)+(C_2+C_3T\bar{\omega})T\bar{\omega}\bar{B}} |\nabla^2\phi(x)|. \tag{2.15}$$

3. Main result

We start this section with the definition of the strengthened maximum principle.

Definition 3.1. Strengthened maximum principle.
Let $c > 0$ and $\bar{\omega} > 0$. We say that the control process $(\hat{u}(\cdot), \hat{x}(\cdot))$ satisfies a *strengthened maximum principle*, if there exists an absolutely continuous function $p : [0,T] \to \mathbb{R}^n$ such that,

$$-\dot{p}(t) = \left(\nabla_x(f(t,\hat{x}) + g(t,\hat{x})\hat{u})\right)^* p(t) \tag{3.1}$$

$$-p(T) = \nabla\phi(\hat{x}(T)) \tag{3.2}$$

and there exist a non-negative measurable function $\sigma : [0,T] \to \mathbb{R}_+$ and a constant $a_0 > 0$, such that

(1) $\max_{u\in U}(\langle g(t,\hat{x}(t))(u - \hat{u}(t)), p(t)\rangle + \sigma(t)|u - \hat{u}(t)|) \leq 0$;
(2) $\mathrm{meas}\{t \in [0,T] \mid \sigma(t) < 2ca\} < a/\bar{\omega}$, whenever $a \in]0, a_0]$,
(3) $\mathrm{meas}\{t \in [0,T] \mid \sigma(t) = a\} = 0$, whenever $a > 0$.

Observe that condition (1) in the above definition implies the maximum principle condition

$$\max_{u\in U}(\langle g(t,\hat{x}(t))(u - \hat{u}(t)), p(t)\rangle \leq 0. \tag{3.3}$$

Recall that the MP establishes necessary conditions of optimality for the control process $(\hat{u}(\cdot), \hat{x}(\cdot))$, defined by (3.1), (3.2) and (3.3). The strengthened maximum principle we propose here is an enforcement of condition (3.3) which is replaced by conditions (1), (2) and (3) of Definition 3.1. As we shall prove in Theorem 3.3, this strengthened maximum principle is a sufficient condition for optimality in L_1 norm, in the space of controls.

The following lemma is homologous with Lemma 3.2 from [4]. It provides an inequality which will be of particular relevance to prove sufficient conditions of optimality in Theorem 3.3.

Lemma 3.2. *Let σ, c and $\bar{\omega}$ be as in the strengthened maximum principle condition in Definition 3.1. Then*

$$\int_0^T \sigma(t)w(t)dt - c\left(\int_0^T w(t)dt\right)^2 \geq 0, \tag{3.4}$$

for all $w(\cdot)$ such that $\int_0^1 w(t)\,dt < \epsilon$, $w(t) \in [0,\bar{\omega}]$, whenever ϵ is sufficiently small.

Proof. Consider the following optimal control problem

$$\text{Minimize } \int_0^T \sigma(t)w(t)dt - cy^2(T)$$

over processes (w, y) satisfying

$$\dot{y} = w, \quad w \in [0, \bar{\omega}],$$
$$y(0) = 0, \; y(T) \le \epsilon.$$

Known existence theorems for optimal control problems guarantee that an optimal solution to the above problem exists. Denote the optimal process by $(\hat{w}, \hat{y})$. Since $(w, y) = (0, 0)$ is an admissible process with cost function equal to zero, the minimum cost must be non positive. Assume that the optimal control $\hat{w}(\cdot)$ is different from zero. Then, $\hat{y}(T) > 0$. Moreover, if $\hat{y}(T) < \epsilon$, we can reset $\epsilon = \hat{y}(T)$. In fact, observe that $\hat{y}(T) < \epsilon$ implies that the set of admissible processes for the problem with the constraint $y(T) \le \hat{y}(T)$ is a subset of the corresponding set for the problem with the constraint $y(T) \le \epsilon$ and $(\hat{w}, \hat{y})$ is admissible for both. So, $(\hat{w}, \hat{y})$ is still an optimal process when ϵ is replaced by $\hat{y}(T)$. Without loss of generality, assume then that $\hat{y}(T) = \epsilon$. Known necessary conditions applied to this problem guarantee the existence of $\lambda \ge 0$, $\mu \ge 0$ and an absolutely continuous function $\psi(\cdot)$ such that

$$\dot{\psi} = 0, \quad \psi(T) = 2\lambda c\hat{y}(T) - \mu, \quad \mu(\hat{y}(T) - \epsilon) = 0;$$
$$\max_{w \in [0,\bar{w}]} (\psi(t) - \lambda\sigma(t))w = (\psi(t) - \lambda\sigma(t))\hat{w}(t),$$
$$\lambda + ||\psi(\cdot)||_\infty > 0.$$

If $\lambda = 0$, then $\psi(t) \equiv \psi(T) = -\mu$ must be negative. From the maximum condition we deduce that in such case $\hat{w} = 0$ and so $\hat{y} = 0$, a contradiction.

Set $\lambda = 1$. Then, we have

$$\psi(t) \equiv 2c\hat{y}(T) - \mu,$$

and

$$\hat{w}(t) = \begin{cases} \bar{\omega} & \text{if } 2c\hat{y}(T) - \mu - \sigma(t) > 0, \\ 0 & \text{if } 2c\hat{y}(T) - \mu - \sigma(t) < 0. \end{cases}$$

Now, observe that $\Gamma_1 = \{t : \; 2c\epsilon - \mu > \sigma(t)\} \subseteq \Gamma_2 = \{t : \; \sigma(t) < 2c\epsilon\}$. Moreover, from the properties of σ, we obtain $\text{meas}\,(\Gamma_2) < \epsilon/\bar{\omega}$ and $\text{meas}\,\{t : \; \sigma(t) = 2c\epsilon - \mu\} = 0$. Thus, we have

$$\epsilon = \hat{y}(T) = \int_0^T \hat{w}(t)\,dt = \text{meas}\,(\Gamma_1) \cdot \bar{\omega} \le \text{meas}\,(\Gamma_2) \cdot \bar{\omega} < \frac{\epsilon}{\bar{\omega}} \cdot \bar{\omega} = \epsilon,$$

a contradiction. □

We now present our main result stating that the strengthened maximum principle in Definition 3.1 is sufficient for optimality.

Theorem 3.3. *Let $(\hat{u}(\cdot), \hat{x}(\cdot))$ be an admissible control process for problem (P), satisfying the strengthened maximum principle, with c given by* (2.15) *and $\bar{\omega}$ being the diameter of U.*

Then $(\hat{u}(\cdot), \hat{x}(\cdot))$ is a local minimizer in L_1 norm, in the space of controls, i.e., there exists $\epsilon > 0$ such that, for any admissible control process $(u(\cdot), x(\cdot))$ satisfying $\int_0^T |u(t) - \hat{u}(t)|\, dt < \epsilon$, the inequality $\phi(x(T)) \geq \phi(\hat{x}(T))$ holds.

Proof. Let (u, x) be an admissible process. Set $\bar{u} = u - \hat{u}$. Recall that $x(\cdot, \bar{u})$ and $\bar{x}$ denote the solution of the Cauchy problem (2.8) and (2.9).

If Φ represents the fundamental matrix of the system

$$\dot{y}(t) = (\nabla_x f(t, \hat{x}) + \nabla_x (g(t, \hat{x})\hat{u}))y(t)$$

then

$$\bar{x}(t) = \int_0^t \Phi(t, s) g(s, \hat{x}(s))\bar{u}(s)\, ds$$

and

$$p(t) = \Phi^*(T, t)p(T).$$

From (2.14), (3.1), (3.2) and the strengthened maximum principle we have

$$\begin{aligned}
\phi(x(T, \bar{u}(\cdot))) &\geq \phi(\hat{x}(T)) + \langle \nabla\phi(\hat{x}(T)), \bar{x}(T)\rangle - c\left(\int_0^T |\bar{u}(t)| dt\right)^2 \\
&= \phi(\hat{x}(T)) - \int_0^T \langle p(T), \Phi(T, t)g(t, \hat{x}(t))\bar{u}(t)\rangle\, dt - c\left(\int_0^T |\bar{u}(t)| dt\right)^2 \\
&= \phi(\hat{x}(T)) - \int_0^T \langle \Phi^*(T, t)p(T), g(t, \hat{x}(t))\bar{u}(t)\rangle\, dt - c\left(\int_0^T |\bar{u}(t)| dt\right)^2 \\
&= \phi(\hat{x}(T)) - \int_0^T \langle p(t), g(t, \hat{x}(t))\bar{u}(t)\rangle dt - c\left(\int_0^T |\bar{u}(t)| dt\right)^2 \\
&\geq \phi(\hat{x}(T)) + \int_0^T \sigma(t)|\bar{u}(t)| dt - c\left(\int_0^T |\bar{u}(t)| dt\right)^2.
\end{aligned}$$

Applying Lemma 3.2, we obtain the result. □

The strengthened maximum principle is crucial for sufficiency in L_1 norm. We recover an example in [4] where, under the weaker conditions of the refined maximum principle established in [4] (see, there, Theorem 3.3), we can guarantee optimality in the L_∞ norm but not in the L_1 norm:

$$\begin{aligned}
&x_1(1) - x_2^2(1) \to \min, \\
&\dot{x}_1 = x_2, \\
&\dot{x}_2 = u, \\
&u \in [0, 1], \\
&x_i(0) = 0, i = 1, 2.
\end{aligned}$$

The zero control process satisfies the refined maximum principle and its optimality in the L_∞ sense is then deduced (see [4]). However, if we take the control functions sequence

$$u_n(t) = \begin{cases} 0, & t \in [0, 1 - 1/n[\\ 1, & t \in [1 - 1/n, 1], \end{cases}$$

then $\int_0^1 |u_n(t) - 0|\, dt = \frac{1}{n}$ and for the corresponding trajectories, we have

$$x_1(1) - x_2^2(1) = \int_{1-1/n}^{1} \int_{1-1/n}^{t} ds dt - \left(\int_{1-1/n}^{1} dt \right)^2 = -\frac{1}{2n^2}.$$

The zero control process is not optimal in the L_1 norm.

The following result establishes conditions that are useful to verify properties (1)-(3) in Definition 3.1 of the strengthened maximum principle. Let $U = \mathrm{co}\{u_1, \dots, u_M\}$, $0 = t_0 < t_1 \dots < t_L = T$, $\hat{u}(t) = u_{m_l}$, $t \in]t_l, t_{l+1}[$, $l = \overline{0, L-1}$, $q(t) = (g(t, \hat{x}(t)))^* p(t)$, $\mathcal{M}_l = \{m \mid \langle q(t_l), u_m \rangle = \max_{u \in U} \langle q(t_l), u \rangle\}$, $l = \overline{0, L}$.

Lemma 3.4. *Assume that $q(\cdot)$ is a continuous and piece-wise continuously differentiable function, that the maximum principle uniquely defines the control $\hat{u}(\cdot)$ (in the sense that $\langle q(t), u - u_{m_l} \rangle < 0$, $\forall t \in]t_l, t_{l+1}[, \forall u \in U$, $u \neq u_{m_l}$), and*

$$\max_{m \notin \mathcal{M}_l} \langle q(t_l), u_m - u_{m_l} \rangle < -6c\bar{\omega}L|u_m - u_{m_l}|, \quad l = \overline{0, L-1}, \tag{3.5}$$

$$\max_{\substack{m \in \mathcal{M}_l \\ m \neq m_l}} \langle \dot{q}(t_l + 0), u_m - u_{m_l} \rangle < -6c\bar{\omega}L|u_m - u_{m_l}|, \quad l = \overline{0, L-1}, \tag{3.6}$$

$$\min_{\substack{m \in \mathcal{M}_l \\ m \neq m_{l-1}}} \langle \dot{q}(t_l - 0), u_m - u_{m_{l-1}} \rangle > 6c\bar{\omega}L|u_m - u_{m_l}|, \quad l = \overline{1, L}. \tag{3.7}$$

where c and $\bar{\omega}$ are as in Theorem 3.2.

Then properties (1)-(3) in Definition 3.1 of the strengthened maximum principle are satisfied for some non-negative measurable function $\sigma : [0, T] \to \mathbb{R}_+$ and some constant $a_0 > 0$.

Proof. Observe that

$$\langle q(t_l), u_{m_l} - u_{m_{l-1}} \rangle = 0, \qquad \forall l = \overline{0, L-1}, \tag{3.8}$$

$$\max_u \langle q(t_L), u \rangle = \langle q(t_L), u_{m_{L-1}} \rangle. \tag{3.9}$$

$$\langle q(t_l), u_m - u_{m_l} \rangle = 0, \text{ for } m \in \mathcal{M}_l,\ l = \overline{0, L-1}, \tag{3.10}$$

and

$$\langle q(t_l), u_m - u_{m_l} \rangle < 0 \text{ for } m \notin \mathcal{M}_l,\ l = \overline{0, L-1}. \tag{3.11}$$

The function σ of the strengthened maximum principle will now be constructed. To do that some analysis of $\langle q(t), u(t) - \hat{u}(t) \rangle$ is done in three different cases. In right neighbourhoods and in left neighbourhoods of t_l and also in the interval $]t_l, t_{l+1}[$.

Let $0 \leq l \leq L-1$, $\Delta t > 0$, $u = \sum_{m=1}^{M} \lambda_m u_m$, $u \neq u_{m_l}$, $\lambda_m \geq 0$, $\sum_{m=1}^{M} \lambda_m = 1$.

Using (3.6) and (3.10) we can write,

$$\begin{aligned}
\langle q(t_l+\Delta t), u-u_{m_l}\rangle &= \langle q(t_l+\Delta t), \sum_{m\neq m_l} \lambda_m(u_m-u_{m_l})\rangle \\
&= \sum_{\substack{m\in\mathcal{M}_l\\ m\neq m_l}} \lambda_m\langle q(t_l+\Delta t), u_m-u_{m_l}\rangle \\
&\quad + \sum_{m\notin\mathcal{M}_l} \lambda_m\langle q(t_l+\Delta t), u_m-u_{m_l}\rangle \\
&= \sum_{\substack{m\in\mathcal{M}_l\\ m\neq m_l}} \lambda_m(\langle q(t_l), u_m-u_{m_l}\rangle + \Delta t\langle \dot q(t_l+0), u_m-u_{m_l}\rangle + o(\Delta t)) \\
&\quad + \sum_{m\notin\mathcal{M}_l} \lambda_m\langle q(t_l+\Delta t), u_m-u_{m_l}\rangle \\
&\leq -\sum_{\substack{m\in\mathcal{M}_l\\ m\neq m_l}} \lambda_m\Delta t\left(6c\bar\omega L|u_m-u_{m_l}| - \frac{o(\Delta t)}{\Delta t}\right) \\
&\quad + \sum_{m\notin\mathcal{M}_l} \lambda_m\langle q(t_l+\Delta t), u_m-u_{m_l}\rangle \\
&\leq -4c\bar\omega L\,\Delta t \sum_{\substack{m\in\mathcal{M}_l\\ m\neq m_l}} \lambda_m|u_m-u_{m_l}| \\
&\quad + \sum_{m\notin\mathcal{M}_l} \lambda_m\langle q(t_l+\Delta t), u_m-u_{m_l}\rangle.
\end{aligned}$$

for Δt small enough. From (3.11) and the continuity of q, we also deduce that

$$\langle q(t_l+\Delta t), u_m-u_{m_l}\rangle < -\tau_1, \quad \forall\Delta t<\delta_1,\ l=\overline{0,L-1},\ m\notin\mathcal{M}_l$$

for some $\tau_1>0$, $\delta_1>0$. Reducing δ_1, if necessary, to have

$$\frac{4c\bar\omega L|u_m-u_{m_l}|\Delta t}{\tau_1} < 1, \quad \forall m\notin\mathcal{M}_l, \forall\Delta t<\delta_1,$$

we can write

$$\begin{aligned}
\langle q(t_l+\Delta t), u-u_{m_l}\rangle &\leq -\Delta t 4c\bar\omega L\left(\sum_{\substack{m\in\mathcal{M}_l\\ m\neq m_l}} \lambda_m|u_m-u_{m_l}| + \sum_{m\notin\mathcal{M}_l} \lambda_m|u_m-u_{m_l}|\right) \\
&= -4c\bar\omega L\left|\sum_{m\neq m_l} \lambda_m(u_m-u_{m_l})\right|\Delta t \\
&= -4c\bar\omega L|u-u_{m_l}|\Delta t.
\end{aligned}$$

In conclusion,

$$\langle q(t_l+\Delta t), u-u_{m_l}\rangle \leq -4c\bar\omega L|u-u_{m_l}|\Delta t, \quad \forall\Delta t<\delta_1, \quad l=\overline{0,L-1}. \tag{3.12}$$

Following similar arguments, using now (3.7), we can write

$$\langle q(t_l - \Delta t), u - u_{m_{l-1}} \rangle \leq -4c\bar{\omega}L\Delta t|u - u_{m_{l-1}}|, \quad \forall \Delta t < \delta_2, \quad l = \overline{1, L} \tag{3.13}$$

for some $\delta_2 > 0$.

Let us now proceed to define σ of the strengthened maximum principle. Taking $\delta = \min\{\delta_1, \delta_2\}$, we have (3.12) and (3.13) satisfied when $\Delta t < \delta$. Besides that, in each interval $I_l = [t_l + \delta, t_{l+1} - \delta]$, $l = \overline{0, L-1}$, we have

$$\langle q(t), u - u_{m_l} \rangle = \sum_{m=1}^{M} \lambda_m \langle q(t), u_m - u_{m_l} \rangle < -\tau_2 \sum_{m \neq m_l} \lambda_m,$$

where

$$\tau_2 = -\max_{t \in I_l, m \neq m_l, l = \overline{0, L-1}} \langle q(t), u_m - u_{m_l} \rangle > 0.$$

Since $\frac{|u - u_{m_l}|}{2\bar{\omega}} < 1$, $\forall u \in U$, we can write

$$\begin{aligned}
\langle q(t), u - u_{m_l} \rangle &< -\frac{\tau_2}{2\bar{\omega}} \sum_{m \neq m_l} \lambda_m |u_m - u_{m_l}| \\
&< -\frac{\tau_2}{2\bar{\omega}} \left| \sum_{m \neq m_l} \lambda_m (u - u_{m_l}) \right| \\
&= -\frac{\tau_2}{2\bar{\omega}} |u - \hat{u}(t)| \\
&< -\frac{\tau_2}{2\bar{\omega}} \frac{t - t_l}{\max_l (t_{l+1} - t_l)} |u - \hat{u}(t)|. \qquad (3.14)
\end{aligned}$$

Hence, $\sigma(t)$ can be defined in the following way:

$$\sigma(t) = \begin{cases} 4c\bar{\omega}L|t - t_l| & \text{if } t \in]t_l - \delta, t_l + \delta[, \; l = \overline{1, L-1}, \\ 4c\bar{\omega}Lt & \text{if } t \in [0, \delta[, \\ 4c\bar{\omega}L(T - t) & \text{if } t \in]T - \delta, T], \\ \frac{\tau_2}{2\bar{\omega}} \frac{t - t_l}{\max_l (t_{l+1} - t_l)} & \text{if } t \in [t_l + \delta, t_{l+1} - \delta], \; l = \overline{0, L-1}. \end{cases}$$

This function satisfies conditions (1)-(3) in Definition 3.1 of the strengthened maximum principle. Condition (1) results from (3.12), (3.13) and (3.14). Condition (3) is satisfied since here the set $\{t \in [0, T] \mid \sigma(t) = a\}$ is a finite set. In what concerns condition (2), choose a_0 defined by

$$a_0 = \frac{\tau_2 \delta}{2\bar{\omega} \max_l (t_{l+1} - t_l)}.$$

Then, for $a < a_0$, we have

$$\operatorname{meas}\{t \in [0, T] \mid \sigma(t) < 2ca\} = \operatorname{meas}\{t \in [0, T] \mid 4c\bar{\omega}L|t - t_l| < 2ca\}$$

and

$$\operatorname{meas}\{t \in [0, T] \mid |t - t_l| < a/(2\bar{\omega}L)\} = \frac{a}{\bar{\omega}}.$$

□

4. Example

Consider the optimal control problem:

$$\begin{aligned}
&-(x_1(1)-a)^2-(x_2(1)-b)^2 \to \min,\\
&\dot{x}_1 = x_2,\\
&\dot{x}_2 = u,\\
&u(t)\in[0,2],\\
&x_i(0)=0, i=1,2.
\end{aligned}$$

where $a=-196$ and $b=\frac{795}{8}$. This problem is nonconvex.

Put

$$\hat{u}(t)=\begin{cases} 2 & 0\le t\le 1/2\\ 0 & 1/2<t\le 1\end{cases}$$

$$\hat{x}_1(t)=\begin{cases} t^2 & 0\le t\le 1/2\\ t-1/4 & 1/2<t\le 1\end{cases} \qquad \hat{x}_2(t)=\begin{cases} 2t & 0\le t\le 1/2\\ 1 & 1/2<t\le 1\end{cases}$$

The pair $(\hat{u}(\cdot),\hat{x}(\cdot))$ is optimal in the sense described in Theorem 3.2, i.e., it is a local minimizer with respect to the L_1 norm, in the space of controls. That follows from Lemma 3.3 and Theorem 3.2. Next we present some calculus justifying application of those results.

Let $p:[0,1]\to\mathbb{R}^2$ be such that

$$-\dot{p}(t)=(0,p_1(t)) \;\Leftrightarrow\; p_1(t)=p_1(0),\forall t \text{ and } p_2(t)=p_2(0)-p_1(0)t$$
$$p_1(1)=2(\hat{x}_1(1)-a),\quad \text{and}\quad p_2(1)=2(\hat{x}_2(1)-b).$$

Verification of conditions of Lemma 3.4. In this case we have:

- $q(t)=g(t,\hat{x}(t))^*p(t)=p_2(t)=2(\hat{x}_2(1)-b)+2(\hat{x}_1(1)-a)(1-t)$.
- $U=\mathrm{co}\{u_1,u_2\}$, where $u_1=0$ and $u_2=2$.
- $L=2$, $u_{m_0}=2$ for $t\in]0,1/2[$ and $u_{m_1}=0$ for $t\in]1/2,1[$.
- $\mathcal{M}_0=\{2\}$, $\mathcal{M}_1=\{1,2\}$, $\mathcal{M}_2=\{1\}$.

Take $c=9$ (see (2.15)). We have

- $\nabla\phi(\hat{x}(1))=(-2(\hat{x}_1(1)-a),-2(\hat{x}_2(1)-b))$,
- $\max_{|v|\le 1}|v^*\nabla^2\phi(x)v|=2$,
- $\frac{1}{2}|\bar{x}(1)|^2\max_{|v|\le 1}|v^*\nabla^2\phi(x)v| \;\le\; \frac{9}{2}\cdot 2=9$,

and the conditions of Lemma 3.4 can be translated in:

(1) $\langle q(0), u_m - 2\rangle \ < \ -6c\bar{\omega}L|u_m - 2|, \ m \notin \mathcal{M}_0$

$$\Leftrightarrow \ p_2(0) > 24c \quad \Leftrightarrow \quad (\hat{x}_2(1) - b) + (\hat{x}_1(1) - a) > 12c,$$

(2) $\langle \dot{q}(1/2 + 0), u_m - 0\rangle \ < \ -6c\bar{\omega}L|u_m - 0|, \ m \in \mathcal{M}_1, m \neq m_1$

$$\Leftrightarrow \ \dot{q}(1/2 + 0).2 \ < \ -6c\bar{\omega}L2 \quad \Leftrightarrow \quad \hat{x}_1(1) - a > 12c,$$

(3) $\langle \dot{q}(1/2 - 0), u_m - 2\rangle \ > \ 6c\bar{\omega}L|u_m - 2|, \ m \in \mathcal{M}_1, m \neq m_0$

$$\Leftrightarrow -2\left(\hat{x}_2(1) - b + \hat{x}_1(1) - a\right)(-2) \ > \ 48c$$

$$\Leftrightarrow \quad \hat{x}_2(1) - b + \hat{x}_1(1) - a > 12c.$$

So, we must have

$$(1 - b) + \left(\frac{3}{4} - a\right) > 12c \ \text{ and } \ \frac{3}{4} - a > 12c.$$

For $a = -196$, $b = \frac{795}{8}$ and $c = 9$ these inequalities are satisfied.

The assumptions of Lemma 3.4 are satisfed, so $(\hat{u}, (\hat{x}_1, \hat{x}_2))$ is optimal in the sense of Theorem 3.3.

Acknowledgment

The authors gratefully acknowledge the support of FEDER funds through COMPETE and Portuguese funds through the Portuguese Foundation for Science and Technology (FCT), within the projects PTDC/EEI-AUT/2933/
2014 TOCCATTA and POCI-01-0145-FEDER-006933 - SYSTEC.

References

[1] A. V. Arutyunov, *Optimality Conditions: Abnormal and Degenerate Problems.* Kluwer Academic Publishers, 2000.

[2] A. V. Arutyunov, *The Pontryagin maximum principe and sufficient optimality conditions for nonlinear problems*, Differential Equations **39** (2003), 1671–1679.

[3] A. F. Filippov, *Classical solutions of differential equations with multivalued right-hand side*, SIAM J. Control **5** (1967), 609–621.

[4] M. Margarida A. Ferreira and Georgi V. Smirnov, *On the sufficiency of Pontryagin's maximum principle*, Pure Appl. Func. Anal. **1** (2016), 217–230.

[5] A. D. Ioffe and V. M. Tihomirov, *Theory of Extremal Problems*, North-Holland, Amsterdam, 1979.

[6] B.S. Mordukhovich and A.M. Sasonkin, *On the duality theory in control systems*, Diff. Equations **14** (1978), 975–981.

[7] N. Osmolovskii and H. Maurer, *Applications to regular and Bang-Bang Control: Second-Order Necessary and Sufficient Optimality Conditions in Calculus of Variations and Optimal Control*, SIAM, Philadelphia, 2012.

[8] L.S. Pontryagin, V.G.Boltyanski, R.V. Gamkrelidze, and E.F. Mischenko, *The Mathematical Theory of Optimal Processes*, Wiley-Interscience, New York, 1962.

Manuscript received December 4 2017
revised April 6 2018

M. M. A. Ferreira
DEEC-FEUP, University of Porto, Portugal
E-mail address: mmf@fe.up.pt

Georgi V. Smirnov
University of Minho, Portugal
E-mail address: smirnov@math.uminho.pt