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INVERSE PROBLEMS IN VARIATIONAL INEQUALITIES BY MINIMIZING ENERGY

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Abstract. This paper develops an abstract framework for the inverse problem of parameter estimation in elliptic variational inequalities. Motivated by practical applications, in the variational inequality, the parameter appears at three different places, namely, in the primary operator, on the right-hand side, and in the functional. Besides employing the commonly used output least-squares (OLS) approach, we propose and use a new modified output least-squares (MOLS) method that minimizes a parameter dependent energy norm. We provide existence results for the considered optimization problems. Using penalization, we obtain a new variational inequality defined on the whole space and consider OLS/MOLS based optimization problems with the new variational inequality as the constraint. Using smoothing of the penalty map, we explore differentiability of the parameter-to-solution map for the smooth penalized equation. We consider the two optimization problems with the smooth penalized equation as the constraint and derive necessary optimality conditions. As the penalty parameter diminishes, we recover necessary optimality conditions for the original OLS/MOLS based optimization problems. We devise a finite element based computational framework and present a numerical example showing the feasibility of the proposed framework.

1. Introduction

In recent years, the theory of variational inequalities emerged as one of the most promising branches of pure, applied, and industrial mathematics. Variational inequalities provide us powerful mathematical tools for studying a broad range of problems arising in diverse fields such as structural mechanics, elasticity, economics, optimization, financial mathematics, and others. See [2, 5, 11, 8, 9, 29, 32].

In this work, we study the inverse problem of identifying variable parameters in an elliptic variational inequality when a measurement of a solution of the variational inequality is available. Inverse problems have attracted lots of attention in recent years. However, a bulk of the available literature has only been devoted to identification in variational equations emerging from linear partial differential equations. The material dealing with the estimation in variational inequalities is somewhat limited and even more so when the emphasis is on the theory as well as the numerics. The tools for identification in variational equations are quite well

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developed. For example, there are existence theorems, results on the differentiability of the parameter-to-solution map, stability aspects, error analysis, etc. On the other hand, most of these issues have not been adequately addressed for the inverse problems in variational inequalities. Furthermore, although there are different optimization formulations for inverse problems in variational equations, the only available optimization framework for variational inequalities is through the well-known output least-squares objective.

Before discussing our main contribution, in the following, we briefly review some of the related research. A significant inverse problem in variational inequalities appears in the elastohydrodynamic lubrication problem (EHL). The EHL problem results in a variational inequality in which the unknown is the pressure u, and the coefficient a is known. However, due to the major theoretical and computational difficulties in solving the EHL problem, an efficient two-step procedure is typically designed. In this process, the first step comprised of an inverse problem of parameter identification in a variational inequality where the sought parameter is in the main operator and on the right-hand side of the inequality, see [3]. Inspired by the EHL problem, Hintermüller [22] studied the inverse problem of parameter identification for a certain variational inequality and besides a rigorous treatment of the analytical aspects, also presented a detailed computational framework. In the same vein, Gonzalez [15] explored the inverse problem of identifying multiple parameters in an elliptic variational inequality and provided an existence result. In an earlier work, Hasanov [20] presented useful results for the boundary inverse problem for elliptic variational inequalities. In another contribution [36], the authors gave a detailed numerical treatment of the inverse elasticity problem with Signorini's condition. In [35], the authors focused on the theoretical aspects of the identification inverse problem in a nonlinear variational inequality. Recently, Kupenko and Manzo [31] investigated the inverse problem of parameter identification for a variational inequality with anisotropic p-Laplacian. We also note that K. H. Hoffmann and J. Sprekels [23] were among the first ones to study parameter identification in variational inequalities. However, in [23], in contrast to the most papers on inverse problems where an optimization framework is a preferred choice, the authors developed an iterative scheme that is based on the construction of certain regularized time-dependent problems containing the original problem as the asymptotic steady state. Finally, we note that recently in [17], we studied identification in a quasi-variational inequality and in a variational inequality which is a particular case of the variational inequality studied here. There are many interesting articles on the optimal control of variational inequalities which rely on similar techniques as used in the inverse problem, see [1, 7, 6, 16, 18, 21, 30], and the cited references.

Motivated by some of the gaps discussed above in the available literature, in this work, we develop an abstract framework to identify variable parameters in variational inequalities. The main contributions of this research are as follows:

(1) We consider a general variational inequality and develop an abstract framework for identifying variable parameters appearing at three different places, namely, in the primary operator, on the right-hand side, and in the functional.

(2) Resorting to optimization formulations, we pose two optimization problems. The first approach is based on using the classical output least-squares (OLS) objective and another proposing a new energy norm based modified output least-squares (MOLS); its analog for variational equations have been studied extensively in recent years.
(3) We give existence results for the considered optimization problems. We penalize the variational inequality to obtain a variational inequality which is defined on the whole space and consider analogs of the two optimization problems with the new variational inequality as the constraint. We introduce smoothing of the penalty map and study differentiability of the parameter-to-smooth map for the penalized variational equation. We then consider the two optimization problems with the smooth penalized equation as the constraint and derive necessary optimality conditions depending on the penalty parameter. By a limit process sending the penalty parameter to zero, we recover necessary optimality conditions for the original optimization problems.
(4) Using a finite element based discretization approach, we devise a computation framework and present a numerical example showing the feasibility of the approach.

We organize this paper into seven sections. In Section 2, we introduce the inverse problem and propose two optimization formulations to derive an estimation of the solution. We also study the penalization of the variational inequality. Smoothing of the penalty map and its consequences are given in Section 3. We provide optimality conditions for the OLS approach in Section 4 and optimality conditions for the MOLS approach in Section 5. We give a numerical example in Section 6, and the paper concludes with some remarks.

2. Optimization frameworks for the inverse problem

Let B be a real Banach space, let $S \subset B$ be an open set, let $A \subset S$ be a closed and convex set, and let $\ell : S \to B$, with $\ell(S) \subseteq A$, be a continuously differentiable map which is bounded on A. To prove the existence of some derivatives, we will also assume that A has a nonempty interior. Let V be a real Hilbert space which we identify with its topological dual V^*. We denote the strong convergence and the weak convergence by $\to$ and $\rightharpoonup$, respectively. By $\|\cdot\|_N$, we denote the norm of space N. Let $K \subset V$ be a closed and convex set with $0 \in K$, and let $m(a) := Ma + m$, where $M : B \to V^*$ is a linear and continuous map and $m \in V^*$. Let $\Phi : B \times V \to \mathbb{R}$ be a nonnegative functional which is linear and continuous in the first argument, and convex and continuous in the second argument with $\Phi(a, 0) = 0$, for each $a \in B$. Let $T : B \times V \times V \to \mathbb{R}$ be a trilinear form with $T(\cdot, u, v)$ symmetric in u, v. Assume that there are constants $\alpha > 0$ and $\beta > 0$ such that the following continuity and coercivity conditions hold

$$T(a, u, v) \le \beta \|a\|_B \|u\|_V \|v\|_V, \quad \text{for all } u, v \in V,\ a \in B. \tag{2.1}$$

$$T(a, u, u) \ge \alpha \|u\|_V^2, \ \text{for all } u \in V,\ a \in A. \tag{2.2}$$

Consider the variational inequality: Given $a \in A$, find $u = u(a) \in K$ with

$$T(\ell(a), u, v-u) \geq \langle m(a), v-u \rangle_V + \Phi(a,u) - \Phi(a,v), \text{ for all } v \in K. \tag{2.3}$$

Variational inequality (2.3), which is uniquely solvable by standard arguments (see [33]), is the direct problem in this study. Our goal, however, is on the identification of the parameter a from a measurement z of $u(a)$. As typically done, to study this inverse problem, we will resort to an optimization framework.

In some applications, in (2.3), it is advantages to have different parameters in T, m, and Φ which belong to different function spaces. Such modifications, however, require minor changes and for the sake of simplicity in presentation, are not pursued.

It is well-known that inverse problems are ill-posed and a regularization is needed. For this, let H be a real Hilbert space compactly embedded in B with $A \subset H$. With this preparation, we consider the following two regularized optimization problems:

Find $a \in A$ by the output least-squares (OLS) minimization problem

$$\min_{a \in A} \widehat{J}_\kappa(a) := \frac{1}{2}\|u(a) - z\|_Z^2 + \frac{\kappa}{2}\|a\|_H^2. \tag{2.4}$$

Find $a \in A$ by the modified output least-squares (MOLS) minimizaing problem

$$\min_{a \in A} J_\kappa(a) := \frac{1}{2} T(a, u(a) - z, u(a) - z) + \frac{\kappa}{2}\|a\|_H^2. \tag{2.5}$$

In the above optimization problems, $\kappa > 0$ is the regularization parameter, $u(a)$ is the unique solution of (2.3), z is a measurement of $u(a)$, and Z is a Hilbert space with $V \subset Z$. The OLS functional (2.4) attempts to minimize the gap between the computed and the observed solution in the norm of the observation space Z, whereas the MOLS functional (2.5) aims to minimize the energy associated to the trilinear form. Evidently, (2.5) requires that $z \in V$. The MOLS objective has been used extensively in the inverse problem of identifying variable parameters in variational equations, see [12, 13, 14, 19, 25, 24, 24, 26].

We have the following existence result for the regularized optimization problems:

Theorem 2.1. *Optimization problems* (2.4) *and* (2.5) *have nonempty solution sets.*

Proof. We will only prove the solvability of (2.5), and the solvability of (2.4) can then be shown by analogous arguments. For every $a \in A$, $J_\kappa(a)$ is bounded from below, and hence there is a minimizing sequence $\{a_n\}$ in A such that $\lim_{n\to\infty} J_\kappa(a_n) = \inf\{J_\kappa(a) |\, a \in A\}$. Due to the regularizer in the definition of J_κ, it follows that sequence $\{a_n\}$ is bounded in $\|\cdot\|_H$, and due to the compact embedding of H into B, $\{a_n\}$ has a strongly convergent subsequence in B. Keeping the same notation for the subsequences as well, let $\{a_n\}$ be the subsequence converging to some $\bar{a} \in B$. Since A is closed, we have $\bar{a} \in A$. Let $\{u_n\}$ be the sequence of solutions of (2.3) for $\{a_n\}$, that is, $u_n = u(a_n)$. By the definition of u_n, for all $v \in K$, we have

$$T(\ell(a_n), u_n, v - u_n) \geq \langle m(a_n), v - u_n \rangle_V + \Phi(a_n, u_n) - \Phi(a_n, v). \tag{2.6}$$

We set $v = 0$ in the above inequality, use the positivity of Φ and the fact that $\Phi(a_n, 0) = 0$ to get $\alpha\|u_n\|_V^2 \leq \|m(a_n)\|_{V^*}\|u_n\|_V$ which confirms that $\{u_n\}$ is bounded, and hence contains a weakly convergent subsequence. Using the same notation for the subsequences as well, assume that $\{u_n\}$ is the subsequence that converges weakly to some $\bar{u} \in V$. We claim that $\bar{u} = u(\bar{a})$. For this, we consider (2.6) (for the subsequence) and rearrange it as follows

$$\begin{aligned} &T(\ell(\bar{a}), v, v - u_n) + T(\ell(a_n) - \ell(\bar{a}), v, v - u_n) \\ &\quad \geq T(\ell(a_n), u_n - v, u_n - v) + \langle m(a_n), v - u_n \rangle_V + \Phi(a_n, u_n) - \Phi(a_n, v) \\ &\quad \geq \langle m(a_n), v - u_n \rangle_V + \Phi(a_n, u_n) - \Phi(a_n, v) \end{aligned}$$

by using the ellipticity of T. We pass the above inequality to the limit $n \to \infty$ to obtain

$$T(\ell(\bar{a}), v, v - \bar{u}) \geq \langle m(\bar{a}), v - \bar{u} \rangle_V + \Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v), \quad \text{for every } v \in K.$$

We set $v := \bar{u} + t(v - \bar{u}) \in K$, for $t > 0$, in the above and use the convexity of Φ to obtain

$$tT(\ell(\bar{a}), \bar{u}, v - \bar{u}) + t^2 T(\ell(\bar{a}), v - \bar{u}, v - \bar{u}) \geq t\langle m(\bar{a}), v - \bar{u} \rangle_V + t[\Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v)]$$

which implies that

$$T(\ell(\bar{a}), \bar{u}, v - \bar{u}) + tT(\ell(\bar{a}), v - \bar{u}, v - \bar{u}) \geq \langle m(\bar{a}), v - \bar{u} \rangle_V + \Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v).$$

We now pass the above inequality to the limit $t \to 0$ to obtain

$$T(\ell(\bar{a}), \bar{u}, v - \bar{u}) \geq \langle m(\bar{a}), v - \bar{u} \rangle_V + \Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v),$$

which, in view of the fact that $v \in K$ was chosen arbitrarily, confirms that $\bar{u}$ solves (2.3). However, since variational inequality (2.3) is uniquely solvable, we conclude that $\bar{u} = u(\bar{a})$.

It turns out that indeed $\{u_n\}$ converges strongly to $\bar{u}$. For this we note that (2.6), taking $v = \bar{u}$, yields

$$T(\ell(a_n), u_n, \bar{u} - u_n) \geq \langle m(a_n), \bar{u} - u_n \rangle_V + \Phi(a_n, u_n) - \Phi(a_n, \bar{u}),$$

or equivalently

$$\begin{aligned} T(\ell(a_n), u_n - \bar{u}, u_n - \bar{u}) \leq\ & T(\ell(a_n), \bar{u}, \bar{u} - u_n) - \langle m(a_n), \bar{u} - u_n \rangle_V \\ & + \Phi(a_n, \bar{u}) - \Phi(a_n, u_n) \end{aligned}$$

which, in view of the ellipticity of T, confirms that $\|u_n - \bar{u}\|_V \to 0$ as $n \to \infty$.

To prove the continuity of the MOLS functional, let $\{a_n\}$ and $\{u_n\}$ be the sequences such that $a_n \to \bar{a}$ and $u_n \to \bar{u} = u(\bar{a})$. The following rearrangement of terms

$$\begin{aligned} T(\ell(a_n), u_n - z, u_n - z) =\ & T(\ell(a_n), u_n - \bar{u}, u_n - z) + T(\ell(a_n), \bar{u} - z, u_n - \bar{u}), \\ & + T(\ell(a_n) - \ell(\bar{a}), \bar{u} - z, \bar{u} - z) + T(\ell(\bar{a}), \bar{u} - z, \bar{u} - z), \end{aligned}$$

due to the properties of T yields $T(\ell(a_n), u_n - z, u_n - z) \to T(\ell(\bar{a}), \bar{u} - z, \bar{u} - z)$ and $n \to \infty$.

Consequently,

$$\begin{aligned} J_\kappa(\bar a) &= \frac{1}{2}T(\ell(\bar a), \bar u - z, \bar u - z) + \frac{\kappa}{2}\|\bar a\|_H^2 \\ &\le \lim_{n\to\infty} \frac{1}{2}T(\ell(a_n), u_n - z, u_n - z) + \liminf_{n\to\infty} \frac{\kappa}{2}\|a_n\|_H^2 \\ &\le \liminf_{n\to\infty} \left\{ \frac{1}{2}T(\ell(a_n), u_n - z, u_n - z) + \frac{\kappa}{2}\|a_n\|_H^2 \right\} \\ &= \inf\{J_\kappa(a) \mid a \in A\}, \end{aligned}$$

which confirms that $\bar a$ is a solution of (2.5). The proof is complete. □

We now replace the constraint (2.3) for the optimization problems (2.4) and (2.5) by a variational inequality defined on the whole space V. This new variational inequality, under some smoothness hypothesis on the data, will then be converted to an operator equation to derive optimality conditions. We define a penalty map $P : V \to V^*$ which is bounded, hemi-continuous and monotone map with

$$K = \{v \in V \mid P(v) = 0\}. \tag{2.7}$$

A simple example is constituted by $P = (I - P_K)$, where I is the identity map and P_K is the projection map defined from V onto K.

For a penalty parameter $\varepsilon > 0$ and the penalty map P, we consider the following penalized variational inequality: Given $a \in A$, find $u_\varepsilon = u_\varepsilon(a) \in V$ such that for every $v \in V$, we have

$$T(\ell(a), u_\varepsilon, v - u_\varepsilon) + \frac{1}{\varepsilon}\langle P(u_\varepsilon), v - u_\varepsilon\rangle_V \ge \langle m(a), v - u_\varepsilon\rangle_V + \Phi(a, u_\varepsilon) - \Phi(a, v). \tag{2.8}$$

In view of the ellipticity of T and monotonicity of P, for any $a \in A$, variational inequality (2.8) has a unique solution $u_\varepsilon(a)$.

We now consider analogues of (2.4) and (2.5) where the constraint variational inequality (2.3) has been replaced by variational inequality (2.8) which is defined on the whole space.

Find $a \in A$ by solving the following penalized OLS based optimization problem:

$$\min_{a\in A} \widehat{J}_\sigma(a) = \frac{1}{2}\|u_\varepsilon(a) - z\|_Z^2 + \frac{\kappa}{2}\|a\|_H^2, \tag{2.9}$$

where $\kappa > 0$ is the regularization parameter, $\varepsilon > 0$ is the penalization parameter, and for $a \in A$, the element $u_\varepsilon(a)$ solves (2.8). Find $a \in A$ by solving the following penalized MOLS based optimization problem:

$$\min_{a\in A} J_\sigma(a) = \frac{1}{2}T(\ell(a), u_\varepsilon(a) - z, u_\varepsilon(a) - z) + \frac{\kappa}{2}\|a\|_H^2, \tag{2.10}$$

where $\kappa > 0$ is the regularization parameter, $\varepsilon > 0$ is the penalization parameter, and for $a \in A$, the element $u_\varepsilon(a)$ solves (2.8).

We give the following existence and convergence result:

Theorem 2.2. *For every $\varepsilon > 0$, optimization problem* (2.10) *has a solution a_ε. Furthermore, there exists a sequence $\{(a_\varepsilon, u_\varepsilon)\}$, where $u_\varepsilon = u_\varepsilon(a_\varepsilon)$ is the unique solution of penalized variational inequality* (2.8)*, such that for $\varepsilon \to 0$, we have*

$a_\varepsilon \to \bar{a}$ in B, and $u_\varepsilon \to \bar{u}$ in V, where $\bar{a}$ is a solution of (2.5) *and $\bar{u} = u(\bar{a})$ is the unique solution of* (2.3).

Proof. For a fixed $\varepsilon > 0$, the solvability of optimization problem (2.10) follows by repeating the arguments used in the proof of Theorem 2.1. Furthermore, the sequence $\{a_\varepsilon\} \subset A$ of solutions of (2.10) is uniformly bounded in H. Therefore, due to the compact embedding, there is a subsequence, denoted by the same notation, that converges strongly to some $\bar{a} \in A$. Let u_ε be the sequence of solutions of (2.8) corresponding to $\{a_\varepsilon\}$. By taking $v = 0$ in (2.8), and using the coercivity, we notice that $\{u_\varepsilon\}$ remains bounded as well. Therefore, there is a subsequence $\{u_\varepsilon\}$ (again keeping the same notation), which converges weakly to some $\bar{u} \in V$. We claim that $\bar{u} \in K$. It follows from (2.8) that

$$\begin{aligned}\langle P(u_\varepsilon),&u_\varepsilon - v\rangle_V \\ &\leq \varepsilon\left[\Phi(a_\varepsilon, v) - \Phi(a_\varepsilon, u_\varepsilon) + \langle m(a_\varepsilon), u_\varepsilon - v\rangle_V - T(\ell(a_\varepsilon), u_\varepsilon, u_\varepsilon - v)\right],\end{aligned}$$

which yields

$$\limsup_{\varepsilon\to 0} \langle P(u_\varepsilon), u_\varepsilon - v\rangle_V \leq 0.$$

Furthermore, by using the monotonicity of the penalty map P, for every $v \in V$, we have

$$0 \leq \limsup_{\varepsilon\to 0}\langle P(v) - P(u_\varepsilon), v - u_\varepsilon\rangle_V \leq \langle P(v), v - \bar{u}\rangle_V.$$

By setting $v := \bar{u} + tz$, where $t > 0$, and $z \in V$ is arbitrary, we get that $\langle P(\bar{u} + tz), z\rangle_V \geq 0$, and, by passing $t \to 0$, it follows from the hemicontinuity of P that $\langle P(\bar{u}), z\rangle_V \geq 0$. Since $z \in V$ is arbitrary, we have $P(\bar{u}) = 0$ which confirms that $\bar{u} \in K$.

From (2.8), for every $v \in K$, we have

$$\begin{aligned}T(\ell(a_\varepsilon), u_\varepsilon, v - u_\varepsilon) + \frac{1}{\varepsilon}\langle P(u_\varepsilon), v - u_\varepsilon\rangle_V \geq \langle m(a_\varepsilon), v - u_\varepsilon\rangle_V& \\ + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, v)&,\end{aligned}$$

or equivalently,

$$\begin{aligned}T(\ell(a_\varepsilon), u_\varepsilon, v - u_\varepsilon) &- \frac{1}{\varepsilon}\langle P(v) - P(u_\varepsilon), v - u_\varepsilon\rangle_V + \frac{1}{\varepsilon}\langle P(v), v - u_\varepsilon\rangle_V \\ &\geq \langle m(a_\varepsilon), v - u_\varepsilon\rangle_V + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, v),\end{aligned}$$

and by using the monotonicity of P and the fact that $P(v) = 0$, for any $v \in K$, we deduce that for all $v \in K$, we have

$$T(\ell(a_\varepsilon), u_\varepsilon, v - u_\varepsilon) \geq \langle m(a_\varepsilon), v - u_\varepsilon\rangle_V + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, v). \tag{2.11}$$

By using the ellipticity of T, it follows from (2.11) that

$$\begin{aligned}T(\ell(a_\varepsilon), v, v - u_\varepsilon) &\geq T(\ell(a_\varepsilon), v - u_\varepsilon, v - u_\varepsilon) + \langle m(a_\varepsilon), v - u_\varepsilon\rangle_V \\ &\quad + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, v) \\ &\geq \langle m(a_\varepsilon), v - u_\varepsilon\rangle_V + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, v),\end{aligned}$$

which further implies that

$$T(\ell(\bar{a}), v, v - u_\varepsilon) + T(\ell(a_\varepsilon) - \ell(\bar{a}), v, v - u_\varepsilon) \geq \langle m(a_\varepsilon), v - u_\varepsilon \rangle_V + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, v),$$

and by passing to the limit $\varepsilon \to 0$, we obtain

$$T(\ell(\bar{a}), v, v - \bar{u}) \geq \langle m(\bar{a}), v - \bar{u} \rangle_V + \Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v),$$

and since $v \in K$ is arbitrary, the above inequality holds for every $v \in K$. To obtain (2.3) from this inequality, we set $v = \bar{u} + t(v - \bar{u}) \in K$, where $t \in (0, 1]$ and obtain

$$T(\ell(\bar{a}), \bar{u}, v - \bar{u}) + tT(\ell(\bar{a}), v - \bar{u}, v - \bar{u}) \geq \langle m(\bar{a}), v - \bar{u} \rangle_V + \Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v),$$

and by taking $t \to 0$, we get

$$T(\ell(\bar{a}), \bar{u}, v - \bar{u}) \geq \langle m(\bar{a}), v - \bar{u} \rangle_V + \Phi(\bar{a}, \bar{u}) - \Phi(\bar{a}, v), \quad \text{for every } v \in K, \tag{2.12}$$

verifying that $\bar{u} = u(\bar{a})$.

Let $\tilde{a} \in A$ be an arbitrary solution of (2.5) and let $\tilde{u} = u(\tilde{a})$ be the corresponding unique solution of (2.3). For $\tilde{a}$, let $\tilde{u}_\varepsilon := u_\varepsilon(\tilde{a})$ be the unique solution of the variational inequality such that for each $v \in V$, we have

$$T(\ell(\tilde{a}), \tilde{u}_\varepsilon, v - \tilde{u}_\varepsilon) + \frac{1}{\varepsilon}\langle P(\tilde{u}_\varepsilon), v - \tilde{u}_\varepsilon \rangle_V \geq \langle m(\tilde{a}), v - \tilde{u}_\varepsilon \rangle_V + \Phi(\tilde{a}, \tilde{u}_\varepsilon) - \Phi(\tilde{a}, v).$$

Note that, firstly, $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$ as $\varepsilon \to 0$, and secondly, $(\tilde{a}, \tilde{u}_\varepsilon)$ is feasible for (2.10). For the time being assume that $\tilde{u}_\varepsilon \to \tilde{u}$. Then, we have

$$\begin{aligned} J_\kappa(\bar{a}) &= \frac{1}{2}T(\ell(\bar{a}), \bar{u} - z, \bar{u} - z) + \frac{\kappa}{2}\|\bar{a}\|_H^2 \\ &\leq \lim_{\varepsilon \to 0} \frac{1}{2}T(\ell(a_\varepsilon), u_\varepsilon - z, u_\varepsilon - z) + \liminf_{\varepsilon \to 0} \frac{\kappa}{2}\|a_\varepsilon\|_H^2 \\ &= \liminf_{\varepsilon \to 0} \left(\frac{1}{2}T(\ell(a_\varepsilon), u_\varepsilon - z, u_\varepsilon - z) + \frac{\kappa}{2}\|a_\varepsilon\|_H^2 \right) \\ &\leq \lim_{\varepsilon \to 0} \left(\frac{1}{2}T(\ell(\tilde{a}), \tilde{u}_\varepsilon - z, \tilde{u}_\varepsilon - z) + \frac{\kappa}{2}\|\tilde{a}\|_H^2 \right) = J_\kappa(\tilde{a}), \end{aligned}$$

and since $\tilde{a} \in A$ was chosen arbitrarily, we deduce that $\bar{a} \in A$ solves (2.5).

By now we know that $\{u_\varepsilon\}$ converges weakly to $\bar{u}$. We conclude this proof by showing that $\{u_\varepsilon\}$ converges strongly to $\bar{u}$. Note that by the definition of u_ε, we have

$$T(\ell(a_\varepsilon), u_\varepsilon, \bar{u} - u_\varepsilon) + \frac{1}{\varepsilon}\langle P(u_\varepsilon), \bar{u} - u_\varepsilon \rangle_V \geq \langle m(a_\varepsilon), \bar{u} - u_\varepsilon \rangle_V + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, \bar{u}),$$

and because the above inequality can be written as follows

$$T(\ell(a_\varepsilon), u_\varepsilon, \bar{u} - u_\varepsilon) + \frac{1}{\varepsilon}\langle P(u_\varepsilon) - P(\bar{u}), \bar{u} - u_\varepsilon \rangle_V \geq \langle m(a_\varepsilon), \bar{u} - u_\varepsilon \rangle_V + \Phi(a_\varepsilon, u_\varepsilon) - \Phi(a_\varepsilon, \bar{u}),$$

we obtain by the monotonicity of P and the ellipticity of T that

$$\alpha \|u_\varepsilon - \bar{u}\|_V^2 \leq T(\ell(a_\varepsilon), u_\varepsilon - \bar{u}, u_\varepsilon - \bar{u}) + \frac{1}{\varepsilon} \langle P(u_\varepsilon) - P(\bar{u}), u_\varepsilon - \bar{u}\rangle_V$$
$$\leq T(\ell(a_\varepsilon) - \ell(\bar{a}), \bar{u}, \bar{u} - u_\varepsilon) + T(\ell(\bar{a}), \bar{u}, \bar{u} - u_\varepsilon) + \langle m(a_\varepsilon), u_\varepsilon - \bar{u}\rangle_V$$
$$+ \Phi(a_\varepsilon, \bar{u}) - \Phi(a_\varepsilon, u_\varepsilon),$$

and by passing the above inequality to limit $\varepsilon \to 0$ and using the properties of the trilinear map T, and the fact that $u_\varepsilon \rightharpoonup \bar{u}$, we deduce that $\|u_\varepsilon - \bar{u}\|_V \to 0$ and hence proving the desired strong convergence. The proof is complete. □

3. Smoothness of the parameter-to-solution map

We now additionally assume that for every $a \in B$, the map $\Phi(a, \cdot) : V \to \mathbb{R}$ is second-order Fréchet differentiable and that the derivative

$$\frac{\partial^2 \Phi(\cdot, v)}{\partial a \partial u} := \partial^2_{(a,u)} \Phi(\cdot, v) : B \to V$$

exists and is linear and continuous.

Given $a \in A$, we recast (2.8) as a variational equation of finding $u_\varepsilon \in V$ such that

$$T(\ell(a), u_\varepsilon, v) + \frac{1}{\varepsilon} \langle P(u_\varepsilon), v\rangle_V + \langle \partial_u \Phi(a, u_\varepsilon), v\rangle_V = \langle m(a), v\rangle_V, \text{ for all } v \in V.$$

We now take the penalty map to be $P(u) = (I - P_K)(u)$ and approximate it by a family of smooth penalty maps $P_\varepsilon : V \to V$ satisfying the following conditions:

(1) For every $\varepsilon > 0$, the map P_ε is bounded, monotone, and hemi-continuous such that $K = \{v \in V | P_\varepsilon(v) = 0\}$. Moreover, for any $v \in V$, $P_\varepsilon(v) \to P(v)$, as $\varepsilon \to 0$, and for any sequence $\{u_\varepsilon\}$ converging weakly to some u, the following inequality holds

$$\langle P(u), v\rangle_V \leq \liminf_{\varepsilon \to 0} \langle P_\varepsilon(u_\varepsilon), v\rangle_V, \quad \text{for every } v \in V. \tag{3.1}$$

(2) For each $\varepsilon > 0$, P_ε has a derivative at each $v \in V$ such that

$$\left\langle P_\varepsilon'^*(u)v, v\right\rangle_V \geq 0, \quad \text{for every } u, v \in V. \tag{3.2}$$

$$\left\langle P_\varepsilon'^*(u)v, P_K(u)\right\rangle_V = 0, \quad \text{for every } u, v \in V. \tag{3.3}$$

The above conditions are motivated by Bayada and Talibi [3] where a concrete example can be found.

For a penalty parameter $\varepsilon > 0$ and a family of smooth penalty maps P_ε, we consider the smooth penalized variational equation: Given $a \in A$, find $u_\varepsilon := u_\varepsilon(a) \in V$ such that for every $v \in V$, we have

$$T(\ell(a), u_\varepsilon, v) + \frac{1}{\varepsilon} \langle P_\varepsilon(u_\varepsilon), v\rangle_V + \langle \partial_u \Phi(a, u_\varepsilon), v\rangle_V = \langle m(a), v\rangle_V. \tag{3.4}$$

Due the ellipticity of T and the monotonicity of P_ε and $\partial_u \Phi$, for any $\varepsilon > 0$, the penalized variational equation (3.4) has a unique solution $u_\varepsilon(a)$.

The next result sheds some light on the smoothness of the parameter-to-solution map:

Theorem 3.1. *For a fixed $\varepsilon > 0$, the map $a \to u_\varepsilon(a)$ is differentiable at any point a in the interior of A. For any direction δa, the derivative $\delta_a u_\varepsilon := D_a u_\varepsilon(a)(\delta a)$ exists and is the unique solution of the variational equation*

$$\begin{aligned} T(\ell(a), \delta_a u_\varepsilon, v) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)\delta_a u_\varepsilon, v\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a,u_\varepsilon)\delta_a u_\varepsilon, v\right\rangle_V \\ = \langle M\delta a, v\rangle_V - T(D\ell(a)(\delta a), u_\varepsilon, v) \\ - \left\langle \partial^2_{(a,u)}\Phi(a,u_\varepsilon)\delta a, v\right\rangle_V, \text{ for all } v \in V. \end{aligned} \tag{3.5}$$

Proof. For differentiability, we apply the implicit function theorem to the map $G : A \times V \to V$ given by

$$(a,u) \to \langle T(\ell(a), u) - m(a), \cdot\rangle_V + \frac{1}{\varepsilon}\langle P_\varepsilon(u), \cdot\rangle_V + \partial_u\Phi(a,u),$$

where $T(\ell(a), u)$ is viewed as the associated dual element given by the Riesz representation theorem. The derivative $D_u G(a,u) : V \to V$ is given by

$$D_u G(a,u)(\delta u) = T(\ell(a), \delta u, \cdot) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u)(\delta u), \cdot\right\rangle_V + \partial^2_{(u,u)}\Phi(a,u)\delta u.$$

By hypotheses (2.2), (3.2), and the convexity of $\Phi(a,\cdot)$, the map

$$T(\ell(a), \cdot, \cdot) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u)(\cdot), \cdot\right\rangle_V + \partial^2_{(u,u)}\Phi(a,u)(\cdot)$$

is coercive. Therefore, for every $w \in V$, the operator equation

$$T(\ell(a), \delta u, \cdot) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u)(\delta u), \cdot\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a,u)\delta u, \cdot\right\rangle_V = \langle w, \cdot\rangle_V$$

is uniquely solvable. The map $D_u G(a,\cdot)(u) : V \to V$ is surjective and the differentiability follows from the implicit function theorem. By differentiating (3.4) with respect to a, for every $v \in V$, we get

$$\begin{aligned} T(\ell(a), \delta_a u_\varepsilon, v) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)\delta_a u_\varepsilon, v\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a,u_\varepsilon)\delta_a u_\varepsilon, v\right\rangle_V \\ = \langle M\delta a, v\rangle_V - T(D\ell(a)(\delta a), u_\varepsilon, v) - \left\langle \partial^2_{(a,u)}\Phi(a,u_\varepsilon)\delta a, v\right\rangle_V, \end{aligned}$$

by recalling that $m(a) := Ma + m$. From the convexity of Φ, $\partial^2_{(u,u)}\Phi(a,u_\varepsilon)$ is positive semi-definite, and as a consequence (3.5) is uniquely solvable. The proof is complete. □

Remark 3.2. In the following, relying on perturbation arguments, we will assume that the closed and convex set A of feasible parameters, remains in the interior of the set on which the above differentiability result holds. This will permit us to use a variational inequality as a necessary optimality condition for optimizing OLS/MOLS.

4. Optimality conditions for the OLS formulation

We consider the following perturbed regularized optimization problem: Find $a_\varepsilon \in A$ by solving

$$\min_{a\in A} \widehat{J}_\sigma(a) = \frac{1}{2}\|u_\varepsilon(a) - z\|^2_Z + \frac{\kappa}{2}\|a\|^2_H. \tag{4.1}$$

where for $a \in A$, the element $u_\varepsilon(a)$ solves the smooth penalized variational equation (3.4). Here $\kappa > 0$ is the regularization parameter, $\varepsilon > 0$ is the penalization as well as the smoothing parameter.

The following result gives an optimality condition for the above optimization problem:

Theorem 4.1. *For each $\varepsilon > 0$ and $\kappa > 0$, (4.1) has a solution. Moreover, for any solution $a_\varepsilon \in A$ of (4.1), there is an element $p_\varepsilon \in V$, uniformly bounded in V with*

$$T(\ell(a_\varepsilon), p_\varepsilon, v) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* p_\varepsilon, v\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)p_\varepsilon, v\right\rangle_V = \langle z - u_\varepsilon, v\rangle_Z, \tag{4.2}$$

$$\begin{aligned} &T(D\ell(a_\varepsilon)(a - a_\varepsilon), u_\varepsilon, p_\varepsilon) + \left\langle \partial^2_{(a,u)}\Phi(a_\varepsilon, u_\varepsilon)(a - a_\varepsilon), p_\varepsilon\right\rangle_V \\ &+\kappa\langle a_\varepsilon, a - a_\varepsilon\rangle_H - \langle M^* p_\varepsilon, a - a_\varepsilon\rangle_V \geq 0, \end{aligned} \tag{4.3}$$

for every $v \in V$, for every $a \in A$.

Proof. Let $\varepsilon > 0$ be fixed. The existence of a solution a_ε of (4.1) follows by the arguments used above. A necessary condition for the optimality of a_ε is the variational inequality

$$D\widehat{J}_\varepsilon(a_\varepsilon)(a - a_\varepsilon) + \kappa\,\langle a - a_\varepsilon, a_\varepsilon\rangle_H \geq 0, \quad \text{for every}\, a \in A, \tag{4.4}$$

where

$$\begin{aligned} \widehat{J}_\varepsilon(a) &:= \frac{1}{2}\|u_\varepsilon(a) - z\|^2_Z, \\ D\widehat{J}_\varepsilon(a)(\delta a) &= \langle \delta_a u_\varepsilon, u_\varepsilon(a) - z\rangle_Z, \\ \delta_a u_\varepsilon &:= Du_\varepsilon(a_\varepsilon)(\delta a). \end{aligned}$$

We now define the adjoint equation (associated to (4.1)): Find $p_\varepsilon \in V$, such that for every $v \in V$, we have

$$T(\ell(a_\varepsilon), p_\varepsilon, v) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* p_\varepsilon, v\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)p_\varepsilon, v\right\rangle_V = \langle z - u_\varepsilon, v\rangle_Z. \tag{4.5}$$

Evidently, (4.5) is uniquely solvable, and let $p_\varepsilon \in V$ be its unique solution. Then,

$$\begin{aligned} \langle \delta_a u_\varepsilon, u_\varepsilon - z\rangle_Z &= -T(\ell(a_\varepsilon), p_\varepsilon, \delta_a u_\varepsilon) - \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* p_\varepsilon, \delta_a u_\varepsilon\right\rangle_V \\ &\quad - \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)p_\varepsilon, \delta_a u_\varepsilon\right\rangle_V \\ &= -T(\ell(a_\varepsilon), \delta_a u_\varepsilon, p_\varepsilon) - \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)\delta_a u_\varepsilon, p_\varepsilon\right\rangle_V \\ &\quad - \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)\delta_a u_\varepsilon, p_\varepsilon\right\rangle_V, \end{aligned}$$

where we used properties of trilinear form T, and the fact that $\partial^2_{(u,u)}\Phi(a_\varepsilon, \bar{u}_\varepsilon)$ is symmetric, $\partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)^* = \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)$. Using the derivative formula (3.5), we have

$$\begin{aligned} D\widehat{J}_\varepsilon(a_\varepsilon)(a - a_\varepsilon) &= T(D\ell(a_\varepsilon)(a - a_\varepsilon), u_\varepsilon, p_\varepsilon) \\ &\quad + \left\langle \partial^2_{(a,u)}\Phi(a_\varepsilon, \bar{u}_\varepsilon)(a - a_\varepsilon) - M(a - a_\varepsilon), p_\varepsilon\right\rangle_V, \end{aligned}$$

and (4.3) follows by substituting this expression in (4.4).

We still need to show that $\{p_\varepsilon\}$ is uniformly bounded. For this we take $v = p_\varepsilon$ in adjoint equation (4.2). Now, by ellipticity of T, (3.2) and positiviness of $\partial^2_{(u,u)}\Phi(a_\varepsilon,\cdot)$, we obtain

$$\begin{aligned}\alpha \|p_\varepsilon\|_V^2 &\le T(\ell(a_\varepsilon),p_\varepsilon,p_\varepsilon) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* p_\varepsilon, p_\varepsilon\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon,u_\varepsilon)p_\varepsilon,p_\varepsilon\right\rangle_V \\ &= \langle z-u_\varepsilon, p_\varepsilon\rangle_Z \\ &\le c_1 \|p_\varepsilon\|_V \|z-u_\varepsilon\|_Z,\end{aligned}$$

and hence $\|p_\varepsilon\|_V \le c$, where c, c_1 are constants. The proof is complete. □

For the next result, we assume that the derivatives $\partial^2_{(u,u)}\Phi(\cdot,\cdot)$, and $\partial^2_{(a,u)}\Phi(\cdot,\cdot)$ are continuous, that is, for every sequence $\{(a_n,v_n)\} \subset A \times V$ converging to some (a,v), we have

$$\partial^2_{(u,u)}\Phi(a_n,v_n) \;\to\; \partial^2_{(u,u)}\Phi(a,v), \tag{4.6a}$$

$$\partial^2_{(a,u)}\Phi(a_n,v_n) \;\to\; \partial^2_{(a,u)}\Phi(a,v). \tag{4.6b}$$

We have the following optimality conditions:

Theorem 4.2. *There exist a solution* $\bar{a}$ *of* (2.4) *and elements* $\bar{u} \in V$, $\bar{p} \in V$, $\lambda \in V^*$ *such that for every* $a \in A$, *we have*

$$T(\ell(\bar{a}),\bar{p},v) + \left\langle \partial^2_{(u,u)}\Phi(\bar{a},\bar{u})\bar{p},v\right\rangle_V + \lambda(v) = \langle z-\bar{u},v\rangle_Z, \quad v \in V, \tag{4.7}$$

$$T(D\ell(\bar{a})(a-\bar{a}),\bar{u},\bar{p}) + \left\langle \partial^2_{(a,u)}\Phi(\bar{a},\bar{u})(a-\bar{a}),\bar{p}\right\rangle_V + \kappa\langle\bar{a},a-\bar{a}\rangle_H \ge \langle M^*\bar{p}, a-\bar{a}\rangle_V, \tag{4.8}$$

$$T(\ell(\bar{a}),\bar{u},v-\bar{u}) \ge \langle m(\bar{a}),v-\bar{u}\rangle_V + \Phi(\bar{a},\bar{u}) - \Phi(\bar{a},v), \quad v \in K, \tag{4.9}$$

$$\lambda(\bar{u}) = 0. \tag{4.10}$$

Proof. For $\varepsilon > 0$, let $a_\varepsilon \in A$ be a sequence of solutions (4.1), let u_ε be the solutions of (3.4), and let p_ε be the solutions of (4.2). By the definition of u_ε, for every $v \in K$, we have

$$T(\ell(a_\varepsilon),u_\varepsilon,v-u_\varepsilon) \ge \langle m(a_\varepsilon),v-u_\varepsilon\rangle_V + \Phi(a_\varepsilon,u_\varepsilon) - \Phi(a_\varepsilon,v). \tag{4.11}$$

Using similar arguments as used in the proof of Theorem 2.2, we can show that the sequence $\{a_\varepsilon\}$ converges strongly to $\bar{a}$, and the sequence $\{u_\varepsilon\}$ converges strongly to $\bar{u}$ as $\varepsilon \to 0$. As before, passing (4.11) to limit $\varepsilon \to 0$, we obtain

$$T(\ell(\bar{a}),\bar{u},v-\bar{u}) \ge \langle m(\bar{a}),v-\bar{u}\rangle_V + \Phi(\bar{a},\bar{u}) - \Phi(\bar{a},v), \quad \text{for every } v \in K.$$

Furthermore, since the sequence $\{p_\varepsilon\} \subset V$ is bounded, there exists a weakly convergent subsequence. By keeping the same notation for subsequences as well, let $\{p_\varepsilon\}$ be a subsequence that converges weakly to some $\bar{p} \in V$. We define two functionals

$\lambda_\varepsilon,\ \lambda : V \to \mathbb{R}$ by

$$\lambda_\varepsilon(v) := \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* p_\varepsilon, v\right\rangle_V = \langle z - u_\varepsilon, v\rangle_Z - T(\ell(a_\varepsilon), p_\varepsilon, v) - \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)p_\varepsilon, v\right\rangle_V, \tag{4.12}$$

$$\lambda(v) := \langle z - \bar{u}, v\rangle_Z - T(\ell(\bar{a}), \bar{p}, v) - \left\langle \partial^2_{(u,u)}\Phi(\bar{a}, \bar{u})\bar{p}, v\right\rangle_V. \tag{4.13}$$

Clearly, the above functionals are well defined, linear, and continuous, and hence $\lambda_\varepsilon,\ \lambda \in V^*$. Furthermore, due to the facts that $a_\varepsilon \to \bar{a}$ and $u_\varepsilon \to \bar{u}$, we have

$$\begin{aligned}\lambda^*_\varepsilon(v) &= \langle z - u_\varepsilon, v\rangle_Z - T(\ell(a_\varepsilon), p_\varepsilon, v) - \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)p_\varepsilon, v\right\rangle_V \\ &\to \langle z - \bar{u}, v\rangle_Z - T(\ell(\bar{a}), \bar{p}, v) - \left\langle \partial^2_{(u,u)}\Phi(\bar{a}, \bar{u})\bar{p}_0, v\right\rangle_V \\ &= \lambda(v)\end{aligned}$$

as $\varepsilon \to 0$, where we applied properties of T and (4.6).

Since this convergence holds for every $v \in V$, we deduce that the sequence $\{\lambda_\varepsilon\}$ converges weakly to λ. By taking $v = P_K(u_\varepsilon)$ in (4.12) and using (3.3), we get

$$\lambda_\varepsilon(P_K(u_\varepsilon)) = \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* p_\varepsilon, P_K(u_\varepsilon)\right\rangle_V = 0.$$

By using the continuity of the projection map, we get $0 = \lambda_\varepsilon(P_K(u_\varepsilon)) \to \lambda(\bar{u})$ for $\varepsilon \to 0$, and consequently, $\lambda(\bar{u}) = 0$. For (4.8), by using (4.3), we have

$$\begin{aligned}T(D\ell(a_\varepsilon)(a - a_\varepsilon), u_\varepsilon, p_\varepsilon) + \langle \partial_{(a,u)}\Phi(a_\varepsilon, u_\varepsilon)(a - a_\varepsilon), p_\varepsilon\rangle_V + \kappa\langle a_\varepsilon, a - a_\varepsilon\rangle_H \\ \geq \langle M^* p_\varepsilon, a - a_\varepsilon\rangle_V,\end{aligned}$$

for every $a \in A$. By using the properties of T, continuity of M, and property (4.6) of Φ, we can take limits $\varepsilon \to 0$ to obtain

$$T(D\ell(\bar{a})(a - \bar{a}), \bar{u}, \bar{p}) + \langle \partial_{(a,u)}\Phi(\bar{a}, \bar{u})(a - \bar{a}), \bar{p}\rangle_V + \kappa\langle \bar{a}, a - \bar{a}\rangle_H \geq \langle M^*\bar{p}, a - \bar{a}\rangle_V,$$

for every $a \in A$. The proof is complete. □

5. Optimality conditions for the MOLS functional

We consider the following MOLS-based regularized optimization problem: Find $a_\varepsilon \in A$ by solving

$$\min_{a\in A} J_\sigma(a) := \frac{1}{2}T(\ell(a), u_\varepsilon(a) - z, u_\varepsilon(a) - z) + \frac{\kappa}{2}\|a\|^2_H, \tag{5.1}$$

where for $a \in A$, the element $u_\varepsilon(a)$ solves the penalized variational equation (3.4). Here $\kappa > 0$ is the regularization parameter, $\varepsilon > 0$ is the penalization as well as the smoothing parameter.

For this case we have the following result:

Theorem 5.1. *For each $\varepsilon > 0$, optimization problem* (5.1) *has a solution. Moreover, for any solution $a_\varepsilon \in A$ of* (5.1)*, there exists $q_\varepsilon \in V$, uniformly bounded in V,* *such that for every $a \in A$ and for every $v \in V$, we have*

$$T(\ell(a_\varepsilon), q_\varepsilon, v) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* q_\varepsilon, v\right\rangle_V + \langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon) q_\varepsilon, v\rangle_V = T(\ell(a_\varepsilon), z - u_\varepsilon, v). \tag{5.2}$$

$$\frac{1}{2}T(D\ell(a_\varepsilon)(a-a_\varepsilon), u_\varepsilon - z, u_\varepsilon - z) + T(D\ell(a_\varepsilon)(a-a_\varepsilon), u_\varepsilon, q_\varepsilon) + \left\langle \partial^2_{(a,u)}\Phi(a_\varepsilon, u_\varepsilon)(a-a_\varepsilon), q_\varepsilon\right\rangle_V + \kappa\langle a_\varepsilon, a-a_\varepsilon\rangle_H \geq \langle M^* q_\varepsilon, a - a_\varepsilon\rangle_V. \tag{5.3}$$

Proof. We shall follow the scheme of Theorem 4.1. For a fixed $\varepsilon > 0$, let $a_\varepsilon \in A$ be a solution of (5.1), and u_ε be the corresponding solution of the penalized equation. Then,

$$DJ(a_\varepsilon)(a - a_\varepsilon) + \kappa\,\langle a - a_\varepsilon, a_\varepsilon\rangle_H \geq 0, \quad \text{for every } a \in A,$$

where $J(a) := \frac{1}{2}T(\ell(a), u_\varepsilon(a) - z, u_\varepsilon(a) - z)$.

Evidently, by using the notation $\delta_a u_\varepsilon := D_a u_\varepsilon(a_\varepsilon)(a - a_\varepsilon)$, we have

$$DJ_\varepsilon(a_\varepsilon)(a - a_\varepsilon) = \frac{1}{2}T(D\ell(a_\varepsilon)(a-a_\varepsilon), u_\varepsilon - z, u_\varepsilon - z) + T(\ell(a_\varepsilon), \delta_a u_\varepsilon, u_\varepsilon - z). \tag{5.4}$$

We consider the adjoint equation: Find $q_\varepsilon \in V$ such that for all $v \in V$, we have

$$T(\ell(a_\varepsilon), q_\varepsilon, v) + \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* q_\varepsilon, v\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon) q_\varepsilon, v\right\rangle_V = T(\ell(a_\varepsilon), z - u_\varepsilon, v). \tag{5.5}$$

Clearly, (5.5) is uniquely solvable, and let q_ε be its unique solution. Then,

$$\begin{aligned}
T(\ell(a_\varepsilon), u_\varepsilon - z, \delta_a u_\varepsilon) &= -T(\ell(a_\varepsilon), q_\varepsilon, \delta_a u_\varepsilon) - \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)^* q_\varepsilon, \delta_a u_\varepsilon\right\rangle_V \\
&\quad - \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon) q_\varepsilon, \delta_a u_\varepsilon\right\rangle_V \\
&= -T(\ell(a_\varepsilon), \delta_a u_\varepsilon, q_\varepsilon) - \frac{1}{\varepsilon}\left\langle P'_\varepsilon(u_\varepsilon)\delta_a u_\varepsilon, q_\varepsilon\right\rangle_V \\
&\quad - \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon) q_\varepsilon, \delta_a u_\varepsilon\right\rangle_V \\
&= T(D\ell(a_\varepsilon)(a - a_\varepsilon), u_\varepsilon, q_\varepsilon) \\
&\quad + \left\langle \partial^2_{(a,u)}\Phi(a_\varepsilon, u_\varepsilon)(a - a_\varepsilon), q_\varepsilon\right\rangle_V - \langle a - a_\varepsilon, M^* q_\varepsilon\rangle_V,
\end{aligned}$$

by using Theorem 3.1. Combining this with (5.4), we have

$$\begin{aligned}
DJ_\varepsilon(a_\varepsilon)(a - \bar{a}_\varepsilon) &= \frac{1}{2}T(D\ell(a_\varepsilon)(a - a_\varepsilon), u_\varepsilon - z, u_\varepsilon - z) \\
&\quad + T(D\ell(a_\varepsilon)(a - a_\varepsilon), u_\varepsilon, q_\varepsilon) + \left\langle \partial^2_{(a,u)}\Phi(a_\varepsilon, u_\varepsilon)(a - a_\varepsilon), q_\varepsilon\right\rangle_V \\
&\quad - \langle a - a_\varepsilon, M^* q_\varepsilon\rangle_V,
\end{aligned}$$

which when combined with (5.4), results in the desired inequality

In order to prove that $\{q_\varepsilon\}$ is uniformly bounded, let us take $v = q_\varepsilon$ in adjoint equation (5.2). Now, by ellipticity of T, (3.2) and positivity of $\partial_u^2\Phi(a_\varepsilon,\cdot)$, we obtain

$$\begin{aligned}\alpha\left\|q_\varepsilon\right\|_V^2 &\le T(\ell(a_\varepsilon), q_\varepsilon, q_\varepsilon) + \frac{1}{\varepsilon}\left\langle P_\varepsilon'(a_\varepsilon)^* q_\varepsilon, q_\varepsilon\right\rangle_V + \left\langle \partial^2_{(u,u)}\Phi(a_\varepsilon, u_\varepsilon)q_\varepsilon, q_\varepsilon\right\rangle_V\\ &= T(\ell(a_\varepsilon), z-u_\varepsilon, q_\varepsilon)\\ &\le c_1\left\|q_\varepsilon\right\|_V\left\|z-u_\varepsilon\right\|_V\\ &\le c_2\left\|q_\varepsilon\right\|_V,\end{aligned}$$

and therefore $\|q_\varepsilon\|_V \le c$, where c, c_1, c_2 are positive constants. The proof is complete. □

Finally, we have the following optimality conditions for (5.1):

Theorem 5.2. *There exists a solution $\bar a$ of* (2.5) *and $\bar u \in V$, $\bar q \in V$, $\lambda \in V^*$ with*

$$T(\ell(\bar a), \bar q, v) + \left\langle \partial^2_{(u,u)}\Phi(\bar a, \bar u)\bar q, v\right\rangle_V + \lambda(v) = T(\ell(\bar a), z-\bar u, v)_Z \ \forall\ v \in V, \tag{5.6}$$

$$T(\ell(\bar a), \bar u, v-\bar u) \ge \langle m(\bar a), v-\bar u\rangle_V + \Phi(\bar a,\bar u) - \Phi(\bar a, v),\ \forall\ v\in K, \tag{5.7}$$

$$\frac{1}{2}T(D\ell(\bar a)(a-\bar a), \bar u - z, \bar u - z) + T(D\ell(\bar a)(a-\bar a), \bar u, \bar q) + \left\langle \partial^2_{(a,u)}\Phi(\bar a,\bar u)(a-\bar a), \bar q\right\rangle_V + \kappa\langle \bar a, a-\bar a\rangle_H \ge \langle M^*\bar q, a-\bar a\rangle_V,\ \forall\ a \in A. \tag{5.8}$$

$$\lambda(\bar u) = 0, \tag{5.9}$$

Proof. The proof follows from the arguments used in the proof of Theorem 4.2. □

6. A numerical example

We now test our theoretical results on the inverse problem of identifying a in the variational inequality of finding $u \in K \subset V := H_0^1(\Omega)$ such that

$$\int_\Omega a^3 \nabla u \nabla (v-u) \ge \int_\Omega \frac{da}{dx_2}(v-u), \quad \text{for every } v\in K, \tag{6.1}$$

where $\Omega \subset \mathbb{R}^2$ is a suitable domain and the constraint set

$$K := \{u \in H_0^1(\Omega)|\ u(x) \ge 0,\ \text{a.e. in } \Omega\}.$$

We choose $B = L_\infty(\Omega)$, $H = H^2(\Omega)$, and for given positive constants a_0, a_1, define

$$A := \{a \in H^2(\Omega)|\ 0 < a_0 \le a \le a_1 \text{ a.e. in } \Omega\}.$$

The variational inequality that we focus on in our numerical experiment, emerges from the elastohydrodynamic lubrication problem, see [3, 4] for more details. For the numerical experiment, we choose $\Omega = [0,1]\times[0,1]$, $\ell(a) = a^3$ and $m(a) = \frac{da}{dx_2}$.

The exact solution in this setting is given by

$$\bar a(x_1, x_2) = 1 + 0.5\cos(2\pi x_2)$$

We use finite element discretization to numerically solve the discrete analogs of the optimality systems for the OLS and the MOLS objectives. For simplicity, we keep the iterates for a in the interior of the discrete analog of A, and hence the corresponding inequality is replaced by an equation. The discrete optimality system is solved by using a Damped Gauss-Newton iteration with an Armijo rule line search (see [28]). We use a suitable complementarity function and smoothing techniques for solving the nonsmooth equations (see [10, 27, 34]).

Tables 1 and 2 show that the MOLS functional yields slightly better reconstruction than the OLS functional. Our general framework essentially collapses to the optimality conditions given in [22] for variational inequality for the OLS approach. However, in [22] an additional equation in the optimality system was considered $\bar{a}\bar{p} = 0$. We note that the performance of the OLS approach slightly improves, if we additionally impose this constraint $\bar{a}\bar{p} = 0$. See the Table 3.

TABLE 1. Reconstruction Error for the MOLS.

h	$\frac{\|a^h - I_h\bar{a}\|_{L^2(\Omega)}}{\|I_h\bar{a}\|_{L^2(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^2(\Omega)}}{\|\bar{u}^h\|_{L^2(\Omega)}}$	$\frac{\|a^h - I_h\bar{a}\|_{L^\infty(\Omega)}}{\|I_h\bar{a}\|_{L^\infty(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^\infty(\Omega)}}{\|\bar{u}^h\|_{L^\infty(\Omega)}}$
0.0707107	0.012	0.009	0.026	0.020
0.0565685	0.011	0.009	0.023	0.017
0.0471405	0.009	0.006	0.021	0.011
0.0404061	0.009	0.006	0.019	0.010
0.0353553	0.008	0.005	0.018	0.008

TABLE 2. Reconstruction Error for the OLS.

h	$\frac{\|a^h - I_h\bar{a}\|_{L^2(\Omega)}}{\|I_h\bar{a}\|_{L^2(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^2(\Omega)}}{\|\bar{u}^h\|_{L^2(\Omega)}}$	$\frac{\|a^h - I_h\bar{a}\|_{L^\infty(\Omega)}}{\|I_h\bar{a}\|_{L^\infty(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^\infty(\Omega)}}{\|\bar{u}^h\|_{L^\infty(\Omega)}}$
0.0707107	0.050	0.147	0.106	0.203
0.0565685	0.062	0.180	0.113	0.220
0.0471405	0.051	0.150	0.096	0.189
0.0404061	0.047	0.139	0.092	0.177
0.0353553	0.042	0.124	0.089	0.172

TABLE 3. Reconstruction Error for the OLS with the Additional Constraint $\bar{a}\bar{p} = 0$.

h	$\frac{\|a^h - I_h\bar{a}\|_{L^2(\Omega)}}{\|I_h\bar{a}\|_{L^2(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^2(\Omega)}}{\|\bar{u}^h\|_{L^2(\Omega)}}$	$\frac{\|a^h - I_h\bar{a}\|_{L^\infty(\Omega)}}{\|I_h\bar{a}\|_{L^\infty(\Omega)}}$	$\frac{\|u^h - \bar{u}^h\|_{L^\infty(\Omega)}}{\|\bar{u}^h\|_{L^\infty(\Omega)}}$
0.0707107	0.029	0.090	0.083	0.139
0.0565685	0.041	0.146	0.093	0.177
0.0471405	0.039	0.143	0.088	0.171
0.0404061	0.032	0.123	0.072	0.144
0.0353553	0.026	0.110	0.051	0.144

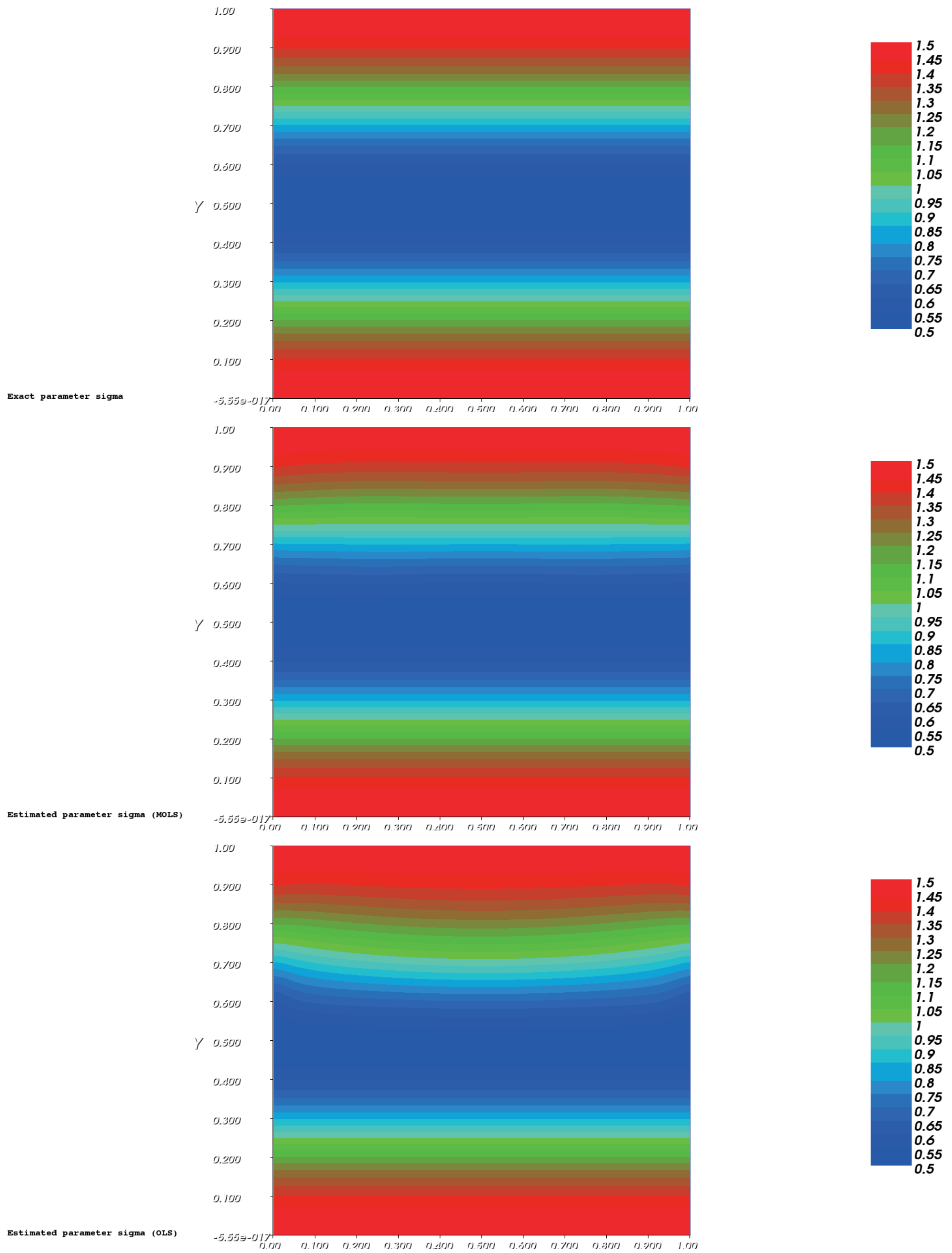

FIGURE 1. $h = 0.0353553$. The left figure shows the exact coefficient, the middle figure shows the reconstruction by MOLS, and the right figure shows the reconstruction by OLS.

7. Concluding remarks

We employed two objective functionals to investigate the inverse problem of parameter identification in an elliptic variational inequality. We provided necessary

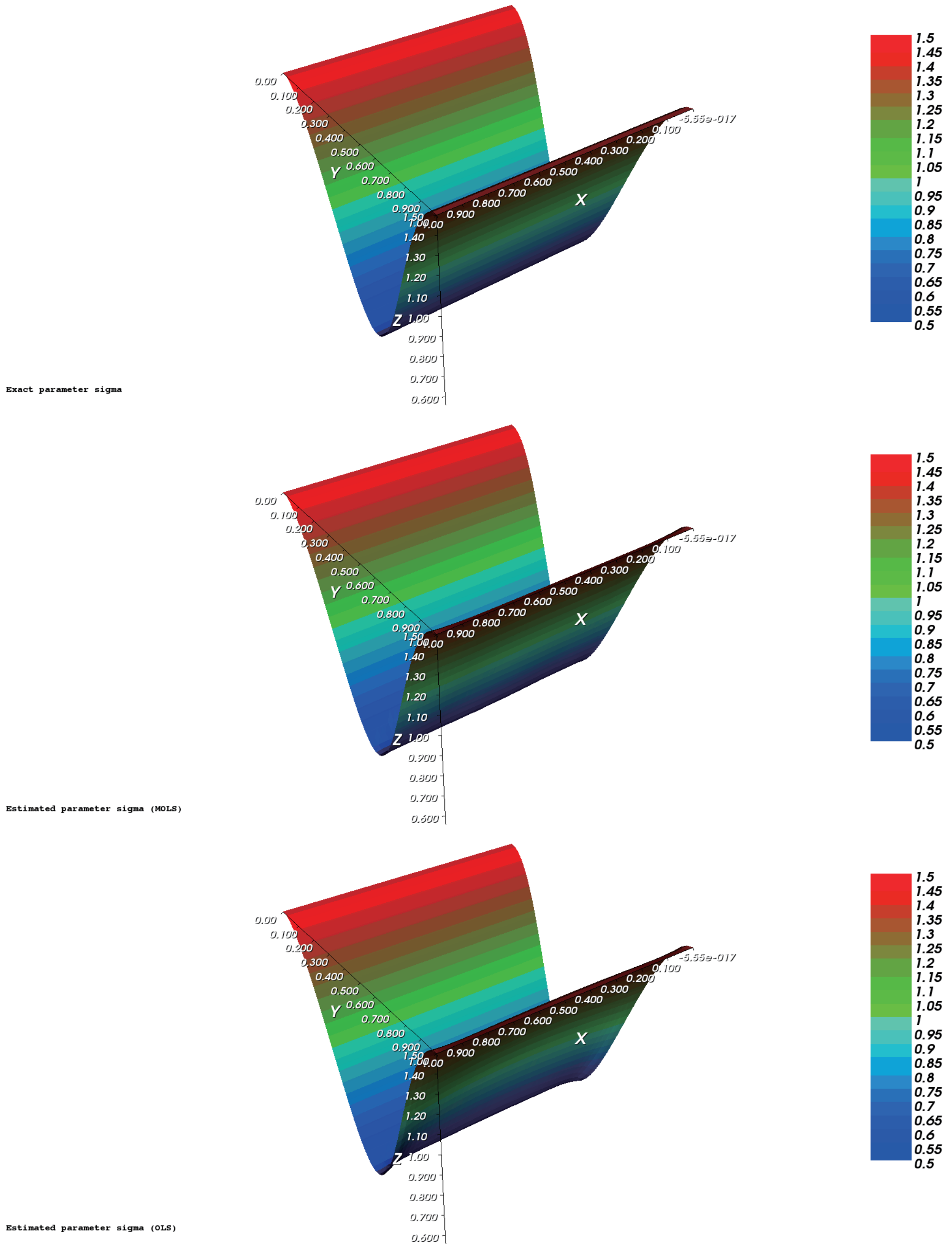

FIGURE 2. $h = 0.0353553$. The left figure shows the exact coefficient, the middle figure shows the reconstruction by MOLS, and the right figure shows the reconstruction by OLS.

optimality conditions and gave numerical results. We would like to note that in most applications of variational inequalities, the functional Φ is typically nonsmooth and the assumption on the smoothness of Φ is a simplification. However, the functional

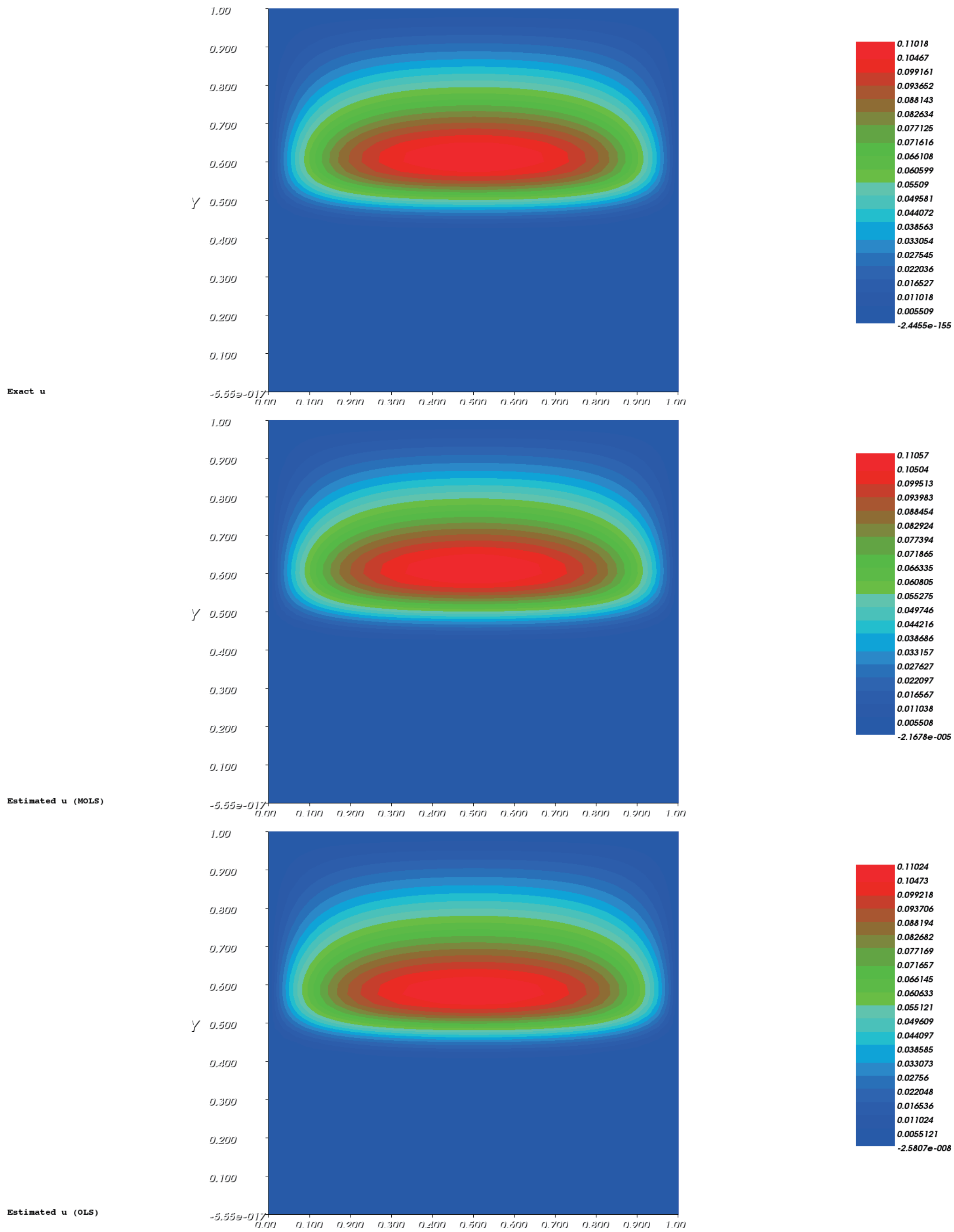

FIGURE 3. $h = 0.0353553$. The left figure shows the exact coefficient, the middle figure shows the reconstruction by MOLS, and the right figure shows the reconstruction by OLS.

Φ should be replaced by a sequence of its sufficiently smooth approximations, just like the smoothing of the projection map. We plan to carry out this in a forthcoming work where we intend to conduct detailed experiments.

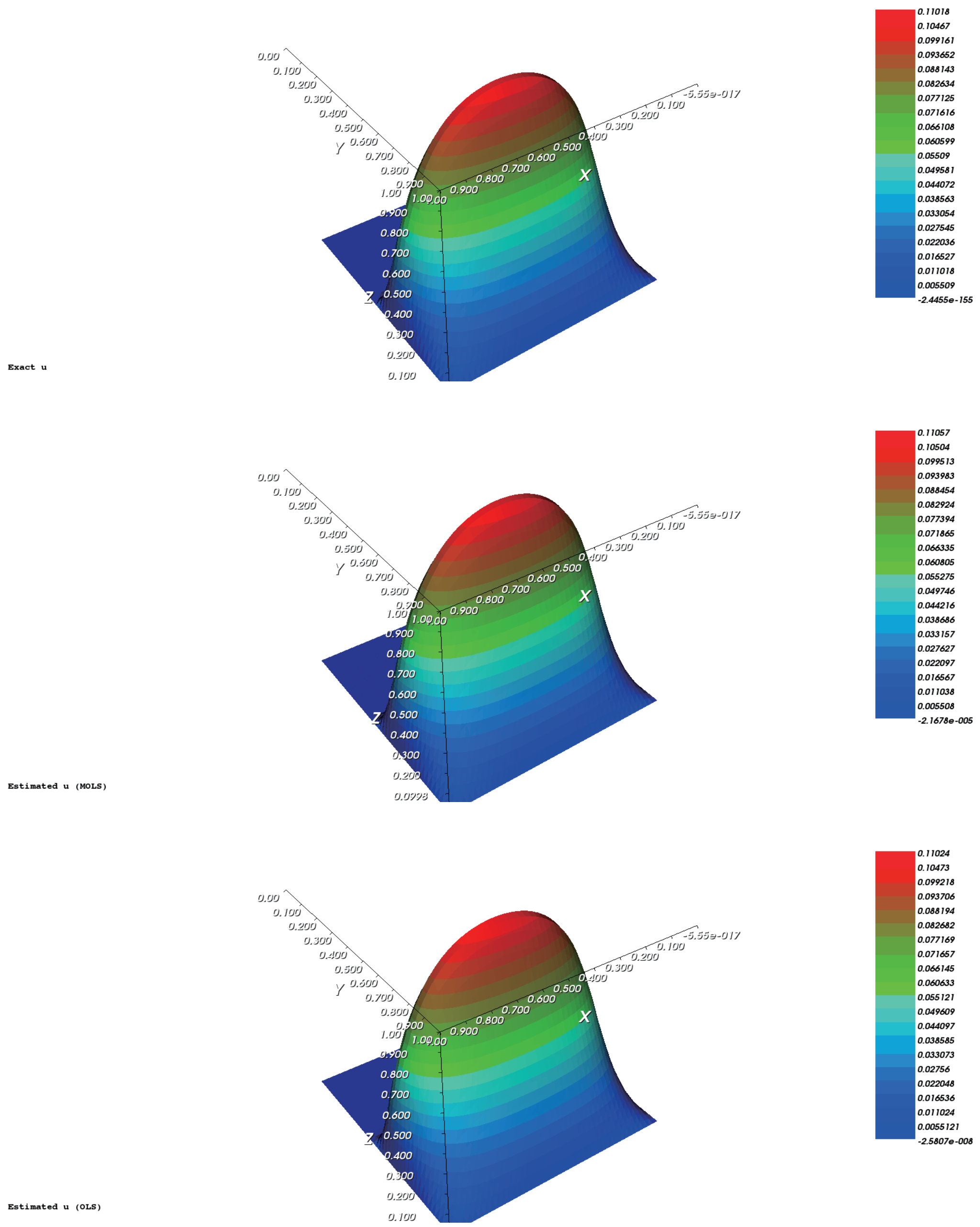

FIGURE 4. Reconstruction for $h = 0.0353553$. The left figure shows the exact solution, the middle figure shows the solution by the MOLS approach, and the right figure shows the solution by the OLS approach.

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