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HADAMARD DIFFERENTIABILITY OF THE SOLUTION MAP IN THERMOVISCOPLASTICITY

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Abstract. We investigate the solution map of a quasistatic, thermoviscoplastic model at small strains with linear kinematic hardening, von Mises yield condition and mixed boundary conditions. The local Lipschitz continuity as well as the directional differentiability of the solution map are proved by the reformulation of the viscoplastic flow rule as a Banach space-valued ODE with nonsmooth right hand side, and employing maximal parabolic regularity theory. As a consequence it follows that the solution map is Hadamard differentiable.

Elastoplastic deformations play a tremendous role in industrial forming. Moreover, many of these processes happen at non-isothermal conditions. Therefore, the investigation of such systems is of interest not only mathematically but also with regard to applications.

The aim of this work is to prove the Hadamard differentiability of the solution map related to the following quasistatic, thermovisco(elasto)plastic model at small strains with linear kinematic hardening and von Mises yield condition:

$$\text{stress-strain relation:} \quad \boldsymbol{\sigma} = \mathbb{C}\left(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)\right), \tag{0.1}$$

$$\text{conjugate forces:} \quad \boldsymbol{\chi} = -\mathbb{H}\,\boldsymbol{p}, \tag{0.2}$$

$$\text{viscoplastic flow rule:} \quad \epsilon\,\dot{\boldsymbol{p}} + \partial_{\dot{\boldsymbol{p}}} D(\dot{\boldsymbol{p}}, \theta) \ni [\boldsymbol{\sigma} + \boldsymbol{\chi}], \tag{0.3}$$

$$\text{balance of momentum:} \quad -\operatorname{div}\left(\boldsymbol{\sigma} + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\right) = \boldsymbol{\ell}, \tag{0.4}$$

$$\begin{aligned}\text{and the heat equation:} \quad & \varrho\, c_p\, \dot{\theta} - \operatorname{div}(\kappa\,\nabla\theta) = r + \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) \\ & + (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}} - \theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}}).\end{aligned} \tag{0.5}$$

The unknowns are the stress $\boldsymbol{\sigma}$, back-stress $\boldsymbol{\chi}$, plastic strain $\boldsymbol{p}$, displacement $\boldsymbol{u}$ and temperature θ. Further, $\mathbb{C}$ and $\mathbb{H}$ denote the elastic and hardening moduli, respectively. $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the symmetrized gradient or linearized strain associated with $\boldsymbol{u}$. The temperature dependent term $\boldsymbol{t}(\theta)$ expresses thermally induced strains. D denotes the dissipation function. The right hand sides $\boldsymbol{\ell}$ and r represent mechanical and thermal loads, respectively, which may act in the volume or on the boundary or both. The constants ϱ, c_p and κ describe the density, specific heat capacity and thermal conductivity of the material. The positive parameters ϵ and γ represent viscous effects in the evolution of the plastic strain and in the balance of momentum.

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For the derivation of the system (0.1)–(0.5) and more on its physical background, we refer the reader to [13, Chapter 22 and 23].

The analysis of thermoplastic models poses numerous mathematical challenges, mainly due to the low integrability of the nonlinear terms on the right hand side of the heat equation. Several approaches have been considered in the literature to deduce the existence and uniqueness of a solution, and we mention [3–5,14] and [10]. We follow in this work the techniques developed in [10] which make use of the theory of maximal parabolic regularity. This will also be of major importance in order to show the local Lipschitz continuity of the solution map in proposition 3.1.

Additionally, the main difficulty w.r.t. the differentiability properties of the solution map is the non-smooth dissipation function D appearing in the viscoplastic flow rule (0.3). The aim of this work is to prove the Hadamard differentiability of the solution map as a consequence of its local Lipschitz continuity and directional differentiability, see corollary 4.6. We recall the following definition of Hadamard differentiability.

Definition 0.1 (Hadamard differentiability)**.** Let V, W be normed vector spaces. A function $f : V \to W$ is said to be Hadamard differentiable in $u \in V$ if

$$f'_H(u;h) = \lim_{t \downarrow 0} \frac{f(u + th + r(t)) - f(u)}{t} \in W$$

exists for every $h \in V$ and every function $r(t) : (0, \infty) \to V$ satisfying $r(t) = o(t)$, i.e., $\lim_{t\downarrow 0} \frac{r(t)}{t} = 0$. A function $f : V \to W$ is said to be Hadamard differentiable if it is Hadamard differentiable in all $u \in V$.

We remark that the following equivalent definition for Hadamard differentiability is often found in the literature, which requires that

$$f'_H(u;h) = \lim_{t\downarrow 0,\ \tilde{h}\to h} \frac{f(u + t\tilde{h}) - f(u)}{t} \in W$$

exists for every $h \in V$, independently of the sequence $\tilde{h} \to h$ in V; see for example in [16, eq.(6)]. The main advantage of definition 0.1 is that we have to handle just one limit process.

Since the Hadamard differentiability of the solution map is a direct consequence of its local Lipschitz continuity and directional differentiability, compare lemma 2.9, the main theorem 4.2 of this work is to establish the directional differentiability of the solution map. We remark that our approach is close to [12] where the Hadamard differentiability of the solution map related to a semilinear parabolic equation with directionally differentiable semilinear part has been proven with a similar strategy.

Nevertheless, our system is more complicated compared to the semilinear parabolic equation in [12]. The proof of the directional differentiability of the solution map is based on a reformulation of the viscoplastic flow rule as a Banach space-valued ODE (2.2), see proposition 2.5. Moreover, we will exploit the property of Hadamard differentiability since there exists a chain rule, and Lebesgue's dominated convergence theorem in order to show that all nonlinear term appearing in the thermoviscoplastic system (0.1)–(0.5) are directionally differentiability. This will be the other key element in the proof of the directional differentiability of the solution map.

The motivation behind our study is two-fold. On the one hand, weakened differentiability properties of non-smooth maps between infinite-dimensional spaces are of intrinsic interest. On the other hand, these properties are the basis for the derivation of first-order optimality conditions for optimization problems subject to the forward system (0.1)–(0.5), as well as tailored non-smooth optimization algorithms.

The paper is organized as follows. We start in section 1 with some notation and general assumptions in order to garantuee the existence of the solution map related to the thermoviscoplastic system (0.1)–(0.5). Next, in section 2 we give the notion of a weak solution and summarize some facts related to the thermoviscoplastic system and the property Hadamard differentiability. We prove in section 3 the local Lipschitz continuity of the solution map, see proposition 3.1. Finally, we obtain the directional differentiability of the solution map in section 4, see theorem 4.2. By combining this with the local Lipschitz continuity, Hadamard differentiability follows, see corollary 4.6.

1. Notation and general assumptions

In what follows, Ω denotes a bounded domain in $\mathbb{R}^3$ and $T > 0$ is a given final time. The spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$ denote Lebesgue and Sobolev spaces, respectively. For a Banach space X and its dual space X', we denote the duality product as $\langle \cdot, \cdot \rangle_{X',X}$ or simply $\langle \cdot, \cdot \rangle$ if no ambiguity arises. The norm of X is denoted as $\| \cdot \|_X$. In the case $X = W^{1,p}(\Omega)$ we denote the dual by $W_\diamond^{-1,p'}(\Omega)$ where $1/p + 1/p' = 1$.

The space $\mathrm{Lin}(X)$ denotes the space of bounded linear functions from X into itself. Furthermore the space $L^p(0,T;X)$ denotes a Bochner space and the space $W^{1,p}(0,T;X)$ is the subset of $L^p(0,T;X)$ such that distributional time derivative of the elements are again in $L^p(0,T;X)$, see, e.g., [17, Chapter III]. The space $W_0^{1,p}(0,T;X)$ denotes the subspace of functions which vanish at $t = 0$.

Vector-valued and matrix-valued functions, and spaces containing such functions are written in bold-face notation. The spaces $\mathbb{R}^{3\times3}$ and $\mathbb{R}^{3\times3}_{\mathrm{sym}}$ represent the (symmetric) 3×3 matrices. Furthermore, $\mathbb{R}^{3\times3}_{\mathrm{dev}}$ denotes the symmetric and trace-free (deviatoric) 3×3 matrices. For $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^{3\times3}$, the inner product and the associated Frobenius norm are denoted by $\boldsymbol{p} : \boldsymbol{q} = \mathrm{trace}(\boldsymbol{p}^\top \boldsymbol{q})$ and $|\boldsymbol{p}|$, respectively. The symmetrized gradient of a vector-valued function $\boldsymbol{u}$ is defined as $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$. The distributional time derivative of a function f defined on $\Omega \times (0,T)$ is denoted by $\dot{f}$. Further, we denote by g' the (Fréchet) derivative of a function g defined on $\mathbb{R}$. Moreover, the directional derivative of a function f in direction h is denoted by $f'(\cdot;h)$ and the Hadamard derivative is written as $f'_H(\cdot;h)$. The symbol $\partial_{\boldsymbol{q}} D$ stands for the partial convex subdifferential of the dissipation function $D(\boldsymbol{q}, \theta)$ w.r.t. $\boldsymbol{q}$ and we will simply denote it by ∂D in the sequel. The convex conjugate of a function $f : X \to (-\infty, +\infty]$ on a normed vector space X is denoted by $f^* : X' \to (-\infty, +\infty]$, see [6, Chapter I, Definition 5.1]. Finally, C denotes a generic non-negative constant and it is written as $C(\cdot)$ to indicate dependencies.

Now we are able to state our assumptions on the quantities in the thermoviscoplastic model (0.1)–(0.5). We begin with the physical constants and functions.

We then proceed to make precise the assumptions on the initial conditions and mechanical and thermal loads, respectively. We conclude the section with the assumptions on the domain Ω.

Assumption 1.1.

(1) *The moduli* $\mathbb{C}, \mathbb{H} : \Omega \to \operatorname{Lin}(\mathbb{R}^{3\times3}_{\text{sym}})$ *are*
   (a) *elements of* $L^\infty(\Omega, \operatorname{Lin}(\mathbb{R}^{3\times3}_{\text{sym}}))$,
   (b) *symmetric in the sense that*
$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij} \quad \textit{and} \quad \mathbb{H}_{ijkl} = \mathbb{H}_{jikl} = \mathbb{H}_{klij},$$
   (c) *coercive on* $\mathbb{R}^{3\times3}_{\text{sym}}$ *with coercivity constants* $\underline{c}, \underline{h} > 0$, *i.e.*
$$\boldsymbol{\varepsilon} : \mathbb{C}(\boldsymbol{x})\, \boldsymbol{\varepsilon} \geq \underline{c}\, |\boldsymbol{\varepsilon}|^2 \quad \textit{and} \quad \boldsymbol{p} : \mathbb{H}(\boldsymbol{x})\, \boldsymbol{p} \geq \underline{h}\, |\boldsymbol{p}|^2$$
   for all $\boldsymbol{\varepsilon}, \boldsymbol{p} \in \mathbb{R}^{3\times3}_{\text{sym}}$ *and almost all* $\boldsymbol{x} \in \Omega$.

(2) *The temperature dependent uni-axial yield stress* $\sigma_0 : \mathbb{R} \to \mathbb{R}$ *is positive and is of class* $C^1_b(\mathbb{R}, \mathbb{R}^{3\times3}_{\text{sym}})$ *(the space of bounded* C^1 *functions with bounded derivatives).*

(3) *The temperature dependent dissipation function* $D : \mathbb{R}^{3\times3}_{\text{dev}} \times \mathbb{R} \to \mathbb{R}$ *is defined as*
$$D(\boldsymbol{q}, \theta) := \tilde{\sigma}(\theta)\, |\boldsymbol{q}|, \quad \textit{where } \tilde{\sigma}(\theta) := \sqrt{2/3}\, \sigma_0(\theta).$$

(4) *The temperature dependent thermal strain function* $\boldsymbol{t} : \mathbb{R} \to \mathbb{R}^{3\times3}_{\text{sym}}$ *is*
   (a) *of class* $C^2_b(\mathbb{R}, \mathbb{R}^{3\times3}_{\text{sym}})$ *(the space of bounded* C^2 *functions with bounded derivatives),*
   (b) *such that* $\mathbb{R} \ni \theta \mapsto \theta\, \boldsymbol{t}'(\theta) \in \mathbb{R}^{3\times3}_{\text{sym}}$ *is Lipschitz continuous and bounded.*

(5) *The density* ϱ, *specific heat capacity* c_p, *thermal conductivity* κ *and heat transfer coefficient* β *are positive constants independent of the temperature. W.l.o.g. we set* $\varrho\, c_p = 1$ *in the analysis.*

(6) *The viscosity parameters* ϵ *and* γ *are positive.*

Remark 1.2. If the thermal strain $\boldsymbol{t}$ fulfills assumption 1.1 (4a) and satisfies
$$\boldsymbol{t}(\theta) = \boldsymbol{t}_{-\infty} \text{ for } \theta \leq -M \quad \text{and} \quad \boldsymbol{t}(\theta) = \boldsymbol{t}_{\infty} \text{ for } \theta \geq M$$
for some $M > 0$ and matrices $\boldsymbol{t}_{-\infty}$ and $\boldsymbol{t}_\infty$ in $\mathbb{R}^{3\times3}_{\text{sym}}$, then the product $\theta\, \boldsymbol{t}'(\theta)$ is Lipschitz continuous and bounded.

Next we introduce suitable function spaces for the displacement and the plastic strain.

Definition 1.3.

(1) We define for $p \geq 2$ the (vector-valued) Sobolev space
$$\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega) := \left\{ \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega; \mathbb{R}^3) : \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_{\mathfrak{D}} \right\}.$$
   Here $\Gamma_{\mathfrak{D}}$ denotes the Dirichlet part of the boundary, see assumption 1.6 (1).

(2) We denote the dual space of $\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)$ by $\boldsymbol{W}^{-1,p'}_{\mathfrak{D}}(\Omega)$, where $1/p + 1/p' = 1$.

(3) We define for $p \geq 2$ the (matrix-valued) Lebesgue space
$$\boldsymbol{Q}^p(\Omega) := \boldsymbol{L}^p(\Omega; \mathbb{R}^{3\times3}_{\text{dev}}).$$

The following regularities for the initial conditions and the mechanical and thermal loads are assumed.

Assumption 1.4. *Let $p, q \geq 2$ be fixed and define*

$$v(p) \begin{cases} = 3p/(6-p) & \text{if } p < 6 \\ \in (\frac{3p}{3+p}, \infty) \text{ arbitrary} & \text{if } p \geq 6. \end{cases} \tag{1.1}$$

(1) *The initial conditions $\boldsymbol{u}_0$, $\boldsymbol{p}_0$ and θ_0 have regularity*

$$\boldsymbol{u}_0 \in \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega), \quad \boldsymbol{p}_0 \in \boldsymbol{Q}^p(\Omega) \quad \text{and} \quad \theta_0 \in W^{1,v(p)}(\Omega).$$

(2) *The loads $\boldsymbol{\ell}$ and r belong to the spaces*

$$\boldsymbol{\ell} \in L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)) \quad \text{and} \quad r \in L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)).$$

(They may represent volume or boundary loads or both.)

Remark 1.5. The distinction of cases in the definition of $v(p)$ is due to the Sobolev embedding $L^{\frac{p}{2}}(\Omega) \hookrightarrow W^{-1,v(p)}_{\diamond}(\Omega)$ which becomes saturated for $p \geq 6$.

Finally, we present the assumptions on the domain.

Assumption 1.6.

(1) *$\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ, see, e.g., [7, Definition 1.2.1.1]. The boundary Γ is divided into disjoint measurable parts $\Gamma_{\mathfrak{N}}$ and $\Gamma_{\mathfrak{D}}$ such that $\Gamma = \Gamma_{\mathfrak{N}} \,\dot\cup\, \Gamma_{\mathfrak{D}}$. Furthermore, $\Gamma_{\mathfrak{N}}$ is an open and $\Gamma_{\mathfrak{D}}$ is a closed subset of Γ with positive measure.*

(2) *The set $\Omega \cup \Gamma_{\mathfrak{N}}$ is regular in the sense of [8], which will be necessary to obtain $\boldsymbol{W}^{1,p}$ regularity (for some $p > 2$) of a solution of* (0.4)*, as well as for the following assumption on maximal parabolic regularity.*

(3) *In addition, the domain Ω is assumed to be smooth enough such that the operator related to*

$$\langle \dot\vartheta, z\rangle + \int_\Omega \kappa\, \nabla\vartheta \cdot \nabla z \,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\, \vartheta\, z \,\mathrm{d}s = f, \quad \vartheta(0) = 0 \tag{1.2}$$

for all $z \in W^{1,v(p)'}(\Omega)$ and almost all $t \in (0,T)$ enjoys maximal parabolic regularity in $W^{-1,v(p)}_{\diamond}(\Omega)$. In other words, there exists a solution operator of (1.2)

$$\Pi : L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \to W^{1,\frac{q}{2}}_0(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))$$

defined by $\Pi f = \vartheta$, which is linear and bounded, i.e., the following estimate holds:

$$\|\vartheta\|_{W^{1,\frac{q}{2}}_0(0,T;W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))} \leq L_\Pi \, \|f\|_{L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega))}.$$

Remark 1.7.

(1) In 3D, there is no simple characterization for assumption 1.6 (2); cf. [9, Theorem 5.4]. For example, $\Omega \cup \Gamma_{\mathfrak{N}}$ is regular in the sense of [8] if $\Omega \subset \mathbb{R}^3$ is a Lipschitzian polyhedron and $\overline{\Gamma}_{\mathfrak{N}} \cap \Gamma_{\mathfrak{D}}$ is a finite union of line segments; see [9, Corollary 5.5].

(2) assumption 1.6 (3) is not very restrictive because there exists $\hat{v} > 2$ such that the operator related to (1.2) satisfies maximal parabolic regularity in $W_\diamond^{-1,v(p)}(\Omega)$ for $\hat{v}' \leq v(p) \leq \hat{v}$ (where $\hat{v}'$ is the conjugate exponent of $\hat{v}$); cf. [10, Lemmata 41 and 42].

2. Weak formulation of the thermoviscoplastic system and some useful results

Before we start with our analysis, we give a precise notion of (weak) solutions to the thermoviscoplastic system (0.1)–(0.5) and summarize some uselful results regarding the thermoviscoplastic system and the property of Hadamard differentiability.

Weak Formulation. (Weak) Solutions to the thermoviscoplastic system satisfy the system (0.1)–(0.5) in a weak sense as follows.

Definition 2.1 (Weak solution of the thermoviscoplastic system)**.** Let $p, q > 2$. Given initial data and inhomogeneities according to assumption 1.4, we say that a quintuple

$$\begin{gathered}
\boldsymbol{u} \in W^{1,q}(0,T; \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)), \quad \boldsymbol{p} \in W^{1,q}(0,T; \boldsymbol{Q}^p(\Omega)), \\
\boldsymbol{\sigma} \in W^{1,q}(0,T; \boldsymbol{L}^p(\Omega)), \quad \boldsymbol{\chi} \in W^{1,q}(0,T; \boldsymbol{L}^p(\Omega)), \\
\theta \in W^{1,\frac{q}{2}}(0,T; W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega))
\end{gathered}$$

is a *weak solution* of the thermoviscoplastic system (0.1)–(0.5), if it fulfills, for almost all $t \in (0,T)$, the

(0.1) stress-strain relation: $\boldsymbol{\sigma} = \mathbb{C}\big(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)\big)$

(0.2) conjugate forces: $\boldsymbol{\chi} = -\mathbb{H}\,\boldsymbol{p}$

(0.3') viscoplastic flow rule:
$$\epsilon \int_\Omega \dot{\boldsymbol{p}} : (\boldsymbol{q} - \dot{\boldsymbol{p}}) \,\mathrm{d}\boldsymbol{x} - \int_\Omega (\boldsymbol{\sigma} + \boldsymbol{\chi}) : (\boldsymbol{q} - \dot{\boldsymbol{p}}) \,\mathrm{d}\boldsymbol{x} + \int_\Omega D(\boldsymbol{q}, \theta) \,\mathrm{d}\boldsymbol{x} - \int_\Omega D(\dot{\boldsymbol{p}}, \theta) \,\mathrm{d}\boldsymbol{x} \geq 0$$
for all $\boldsymbol{q} \in \boldsymbol{Q}^p(\Omega)$

(0.4') balance of momentum:
$$\int_\Omega (\boldsymbol{\sigma} + \gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{\ell}, \boldsymbol{v} \rangle$$
for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p'}_{\mathfrak{D}}(\Omega)$

(0.5') and the heat equation:
$$\langle \dot{\theta}, z \rangle + \int_\Omega \kappa \nabla\theta \cdot \nabla z \,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\, \theta\, z \,\mathrm{d}s = \langle r, z \rangle + \int_\Omega (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{p}}\, z \,\mathrm{d}\boldsymbol{x} - \int_\Omega \theta\, \boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}})\, z \,\mathrm{d}\boldsymbol{x}$$

$$+\,\gamma \int_\Omega \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\, z \,\mathrm{d}\boldsymbol{x} \quad \text{for all } z \in W^{1,v(p)'}(\Omega),$$

along with the initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{p}(0) = \boldsymbol{p}_0$, and $\theta(0) = \theta_0$.

Notice that the associated stress $\boldsymbol{\sigma}$ and back-stress $\boldsymbol{\chi}$ are determined through $\boldsymbol{u}$, $\boldsymbol{p}$, and θ and can directly be calculated from the pointwise equations in (0.1) and (0.2). Their regularity then follows immediately from assumption 1.1.

We remark that $W^{1,v(p)'}(\Omega)$ is the dual space to $W_\diamond^{-1,v(p)}(\Omega)$. Note that the balance of momentum (0.4) is equipped with mixed boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathfrak{D}} \qquad \text{and} \qquad (\gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \boldsymbol{\sigma})\, \boldsymbol{n} = \boldsymbol{s} \quad \text{on } \Gamma_{\mathfrak{N}},$$

where $\boldsymbol{n}$ is the outwards unit normal of Ω. The surface traction forces $\boldsymbol{s}$, together with additional volume loads, are summarized in $\boldsymbol{\ell}$. Moreover, the heat equation (0.5) is endowed with Robin boundary conditions, whose left hand side is given by $\kappa\, \frac{\partial \theta}{\partial n} + \beta\, \theta$ and whose right hand side enters r.

For simplicity, we will refer to (0.3') in the sequel as (0.3) and similarly for (0.4) and (0.5) but always have in mind the weak form of the respective equation.

Reformulation of the Balance of Momentum. We apply [11, Theorem 1.1] in order to solve the balance of momentum; see [10, Lemma 11] for a proof.

Lemma 2.2. *There exists $\hat{p} > 2$ such that for all $2 \le p \le \hat{p}$ and $\boldsymbol{F} \in \boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega)$, there exists a unique solution $\boldsymbol{u} \in \boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)$ of*

$$\int_\Omega \gamma\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \langle \boldsymbol{F},\, \boldsymbol{v} \rangle \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{W}_{\mathfrak{D}}^{1,p'}(\Omega).$$

The corresponding solution operator $\Phi^{\boldsymbol{u}} : \boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega) \to \boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)$, $\boldsymbol{F} \mapsto \boldsymbol{u}$ is linear and bounded and satisfies the following estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)} = \|\Phi^{\boldsymbol{u}}(\boldsymbol{F})\|_{\boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)} \le C\, \gamma^{-1} \|\boldsymbol{F}\|_{\boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega)}.$$

The Lipschitz constant $C\,\gamma^{-1}$ is independent of $p \in [2, \hat{p}]$.

Remark 2.3 (Reformulation of the balance of momentum)**.** We apply the linear and bounded solution operator $\Phi^{\boldsymbol{u}} : \boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega) \to \boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)$ established in lemma 2.2 to the balance of momentum (0.4) and obtain the Banach space-valued ODE

$$\dot{\boldsymbol{u}} = \Phi^{\boldsymbol{u}}(\boldsymbol{F}_1(\boldsymbol{\ell}, \boldsymbol{u}, \boldsymbol{p}) + \boldsymbol{F}_2(\theta)) \tag{2.1}$$

where $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are given by

$$\langle \boldsymbol{F}_1(\boldsymbol{\ell}, \boldsymbol{u}, \boldsymbol{p}),\, \boldsymbol{v} \rangle := \int_\Omega \boldsymbol{\ell} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} - \int_\Omega \mathbb{C}\,(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x},$$
$$\langle \boldsymbol{F}_2(\theta),\, \boldsymbol{v} \rangle := \int_\Omega \mathbb{C}\,(\boldsymbol{t}(\theta)) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}\boldsymbol{x}.$$

Reformulation of the Viscoplastic Flow Rule. The reformulation of the viscoplastic flow rule (0.3) is based on the following result from convex analysis; see [6, Chapter I, Corollary 5.2].

Lemma 2.4. *Let X be a normed vector space and $f : X \to (-\infty, \infty]$ convex, lower semicontinuous and proper. Then*

$$x^* \in \partial f(x) \qquad \Leftrightarrow \qquad x \in \partial f^*(x^*).$$

With lemma 2.4 at hand we are able to derive a reformulation of the viscoplastic flow rule (0.3).

Proposition 2.5. *The viscoplastic flow rule* (0.3) *can be equivalently reformulated as*

$$\dot{\boldsymbol{p}} = -\epsilon^{-1} \min\left(\frac{\tilde{\sigma}(\theta)}{|\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta)|} - 1,\ 0\right) \boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) \quad \textit{a.e. in } (0,T)\times\Omega, \tag{2.2}$$

where $\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) := [\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p},\theta) + \boldsymbol{\chi}(\boldsymbol{u},\boldsymbol{p},\theta)]^D$.

The right hand side of (2.2) is understood to be zero by continuous extension when $\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) = \boldsymbol{0}$.

Proof. We can understand the viscoplastic flow rule (0.3) in a pointwise sense, see [10, Remark 14]. Therefore we fix an arbitrary $(t,\boldsymbol{x}) \in (0,T)\times\Omega$ and prove the equivalence of (0.3) and (2.2) pointwise. For brevity we omit in the following the argument $(t,\boldsymbol{x})$ for all functions.

Since the mapping $\boldsymbol{q} \mapsto D(\boldsymbol{q},\theta) = \tilde{\sigma}(\theta)|\boldsymbol{q}|$ is proper, convex and lower semicontinuous, we apply lemma 2.4 and obtain

$$\text{viscoplastic flow rule (0.3)} \qquad \Leftrightarrow \qquad \dot{\boldsymbol{p}} \in \partial D^*(-\epsilon\,\dot{\boldsymbol{p}} + \boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta),\theta),$$

where $\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) := [\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p},\theta) + \boldsymbol{\chi}(\boldsymbol{u},\boldsymbol{p},\theta)]^D$. It remains to show that the subdifferential of $D^*(\cdot\,,\theta)$ is a singleton and can be characterized as in the assertion.

We start by calculating $D^*(\cdot\,,\theta) : \left(\mathbb{R}^{3\times3}_{\text{dev}}\right)^* = \mathbb{R}^{3\times3}_{\text{dev}} \to (-\infty,\infty]$ explicitly using the definition of the convex conjugate,

$$\begin{aligned} D^*(\boldsymbol{q}^*,\theta) &= \sup_{\boldsymbol{q}\in\mathbb{R}^{3\times3}_{\text{dev}}} \{\boldsymbol{q}^* : \boldsymbol{q} - D(\boldsymbol{q},\theta)\} = \sup_{\boldsymbol{q}\in\mathbb{R}^{3\times3}_{\text{dev}}} \{\boldsymbol{q}^* : \boldsymbol{q} - \tilde{\sigma}(\theta)\,|\boldsymbol{q}|\} \\ &= \left\{ \begin{array}{ll} 0 & \text{if } |\boldsymbol{q}^*| \le \tilde{\sigma}(\theta) \\ \infty & \text{if } |\boldsymbol{q}^*| > \tilde{\sigma}(\theta) \end{array} \right\} = I_{\boldsymbol{B}(\theta)}(\boldsymbol{q}^*), \end{aligned}$$

where $I_{\boldsymbol{B}(\theta)}$ is the indicator function of the set $\boldsymbol{B}(\theta) = \{\boldsymbol{q} \in \mathbb{R}^{3\times3}_{\text{dev}} : |\boldsymbol{q}| \le \tilde{\sigma}(\theta)\}$. Therefore, we have to determine the subdifferential of the indicator function $I_{\boldsymbol{B}(\theta)}$ which is nonempty only for $\boldsymbol{q}^* \in \boldsymbol{B}(\theta)$:

$$\boldsymbol{q} \in \partial D^*(\boldsymbol{q}^*,\theta) = \partial I_{\boldsymbol{B}(\theta)}(\boldsymbol{q}^*) \quad \Leftrightarrow \quad 0 \ge \boldsymbol{q} : (\boldsymbol{v} - \boldsymbol{q}^*) \quad \forall \boldsymbol{v} \in \boldsymbol{B}(\theta). \tag{2.3}$$

We multiply (2.3) with $\beta > 0$ and add a zero term in order to exploit the projection theorem, see [1, 2.3 Projektionssatz].

$$\begin{aligned} &\boldsymbol{q} \in \partial D^*(\boldsymbol{q}^*,\theta) \\ &\qquad \Leftrightarrow (\boldsymbol{q}^* - (\boldsymbol{q}^* + \beta\,\boldsymbol{q})) : (\boldsymbol{v} - \boldsymbol{q}^*) \ge 0 \quad \forall \boldsymbol{v} \in \boldsymbol{B}(\theta),\ \beta > 0 \\ &\qquad \Leftrightarrow \boldsymbol{q}^* = \text{proj}_{\boldsymbol{B}(\theta)}(\boldsymbol{q}^* + \beta\,\boldsymbol{q}) = \min\left(\tilde{\sigma}(\theta), |\boldsymbol{q}^* + \beta\,\boldsymbol{q}|\right) \frac{\boldsymbol{q}^* + \beta\,\boldsymbol{q}}{|\boldsymbol{q}^* + \beta\,\boldsymbol{q}|}, \end{aligned} \tag{2.4}$$

where we used the fact that the orthogonal projection w.r.t. the Frobenius norm onto the ball $\boldsymbol{B}(\theta)$ can be calculated explicitly. Note that (2.4) implies $|\boldsymbol{q}^*| =$

$\min(\tilde{\sigma}(\theta), |\boldsymbol{q}^* + \beta\, \boldsymbol{q}|) \le \tilde{\sigma}(\theta)$. Therefore, the equivalence (2.4) extends to the case $\boldsymbol{q}^* \notin \boldsymbol{B}(\theta)$, when $\partial D^*(\boldsymbol{q}^*, \theta) = \emptyset$. Finally we insert $\boldsymbol{q} \equiv \dot{\boldsymbol{p}} \in \mathbb{R}^{3\times 3}_{\text{dev}}$ and $\boldsymbol{q}^* \equiv -\epsilon\, \dot{\boldsymbol{p}} + \boldsymbol{\tau}(\boldsymbol{u}, \boldsymbol{p}, \theta) \in \mathbb{R}^{3\times 3}_{\text{dev}}$ into (2.4), choose $\beta = \epsilon$ and obtain the assertion. □

Properties of the Solution to the Forward Problem. We recall the existence result for the thermoviscoplasitc system (0.1)–(0.5) and the boundedness of the solution map.

Theorem 2.6 (Existence and uniqueness of a weak solution; [10, Theorem 10])**.** *Suppose that assumption 1.1 and assumption 1.6 hold. There exists* $\bar{p} > 2$ *such that for all* $2 < p \le \bar{p}$*, there exists* $\bar{q} > 2$ *(depending on* p*) such that for all* $\bar{q} \le q < \infty$ *and right hand sides* $(\boldsymbol{\ell}, r)$ *and initial conditions* $(\boldsymbol{u}_0, \boldsymbol{p}_0, \theta_0)$ *as in assumption 1.4, there exists a unique weak solution* $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ *of* (0.1)*–*(0.5) *according to definition 2.1.*

Lemma 2.7 (Boundedness of the solution map; [10, Lemma 27])**.** *Under the assumptions of theorem 2.6, the solution map*

$$
\begin{aligned}
(2.5)\quad \mathcal{G} : L^q(0,T; \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)) \times L^{\frac{q}{2}}(0,T; W^{-1,v(p)}_{\diamond}(\Omega)) \\
\to W^{1,q}(0,T; \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times W^{1,q}(0,T; \boldsymbol{Q}^p(\Omega)) \\
\times W^{1,\frac{q}{2}}(0,T; W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega)),
\end{aligned}
$$

defined by $\mathcal{G}(\boldsymbol{\ell}, r) := (\boldsymbol{u}, \boldsymbol{p}, \theta)$ *is bounded, i.e., the images of bounded sets are bounded.*

In section 4, we will need to address the individual components of $\mathcal{G}$, which will be written as $\mathcal{G}^{\boldsymbol{u}}$, $\mathcal{G}^{\boldsymbol{p}}$, and $\mathcal{G}^{\theta}$, respectively. Moreover, the first two components of $\mathcal{G}$ will be written as $\mathcal{G}^{\boldsymbol{u},\boldsymbol{p}}$.

Embedding. Using [18, Corollary 8 and Lemma 12] one can show the following embedding, compare also [10, Corollary 44].

Lemma 2.8.

(1) *Suppose* $2 < p < 6$ *and thus* $v(p) = 3p/(6-p)$*; cf.* (1.1)*. Choose*

$$
q > \frac{2}{b} \text{ and } 0 < b < \begin{cases} 1 - \frac{3}{2p} & \text{if } p < 3 \\ \frac{1}{2} & \text{otherwise.} \end{cases}
$$

(2) *Suppose* $p \ge 6$ *and thus* $v(p) \in (\frac{3p}{3+p}, \infty)$*; cf.* (1.1)*. Choose*

$$
q > \frac{2}{b} \text{ and } 0 < b < \begin{cases} 1 - \frac{3}{2v(p)} + \frac{3}{2p} & \text{if } v(p) < p \\ \frac{1}{2} & \text{otherwise.} \end{cases}
$$

Then the following embedding is compact:

$$
W^{1,\frac{q}{2}}_0(0,T; W^{-1,v(p)}_{\diamond}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0,T]; L^p(\Omega)).
$$

Hadamard Differentiability. We recall that locally Lipschitz continuous and directionally differentiable functions are Hadamard differentiable, see [16, Proposition 3.5].

Lemma 2.9. *Let V, W be normed vector spaces and $f : V \to W$. If the mapping f is directionally differentiable in $u \in V$ and locally Lipschitz continuous in $u \in V$, then f is Hadamard differentiable in $u \in V$. Moreover, the Hadamard derivative $f'_H(u;\cdot)$ and the directional derivative $f'(u;\cdot)$ coincide.*

A major advantage of Hadamard differentiable functions—compared to merely directionally differentiable functions—is that there exists a chain rule; see [16, Proposition 3.6].

Lemma 2.10 (Chain rule)**.** *Let V, W, X be normed vector spaces, $f : V \to W$ and $g : W \to X$ given functions. If f is Hadamard/directionally differentiable in $u \in V$ and g is Hadamard differentiable in $f(u) \in W$ then $g \circ f$ is Hadamard/directionally differentiable in $u \in V$ with*

$$(g \circ f)'_H(u; h) = g'_H(f(u); f'_H(u; h)).$$

Notice that we can exploit the chain rule in order to obtain that compositions (addition, multiplication, division) of Hadamard differentiable functions are again Hadamard differentiable.

We close this subsection with some examples of Hadamard differentiable function which will be the key elements in order to prove the directional differentiability of the solution map related to the thermoviscoplastic system, see section 4.2. This result is a direct consequence of lemma 2.9.

Lemma 2.11. *The following functions are Hadamard differentiable.*

(1) $f : \mathbb{R} \to \mathbb{R}$, $f(\theta) = \boldsymbol{t}(\theta)$ *with* $f'_H(\theta; \delta\theta) = \boldsymbol{t}'(\theta)\,\delta\theta$.
(2) $f : \mathbb{R} \to \mathbb{R}$, $f(\theta) = \tilde{\sigma}(\theta)$ *with* $f'_H(\theta; \delta\theta) = \tilde{\sigma}'(\theta)\,\delta\theta$.
(3) $f : \mathbb{R}^{3\times3}_{\text{sym}} \times \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}$, $f(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} : \boldsymbol{b}$ *with* $f'_H(\boldsymbol{a}, \boldsymbol{b}; \boldsymbol{\delta a}, \boldsymbol{\delta b}) = \boldsymbol{\delta a} : \boldsymbol{b} + \boldsymbol{a} : \boldsymbol{\delta b}$.
(4) $f : \mathbb{R} \to \mathbb{R}$, $f(\theta) = \theta\,\boldsymbol{t}'(\theta)$ *with* $f'_H(\theta; \delta\theta) = \delta\theta\,\boldsymbol{t}'(\theta) + \theta\,\boldsymbol{t}''(\theta)\,\delta\theta$.
(5) $f : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{sym}}$, $f(\boldsymbol{a}) = \boldsymbol{a}^D$ *with* $f'_H(\boldsymbol{a}; \boldsymbol{\delta a}) = \boldsymbol{\delta a}^D$.
(6) $f : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}$, $f(\boldsymbol{a}) = |\boldsymbol{a}|$ *with*

$$f'_H(\boldsymbol{a}; \boldsymbol{\delta a}) = \begin{cases} \frac{\boldsymbol{a}:\boldsymbol{\delta a}}{|\boldsymbol{a}|} & \text{for } \boldsymbol{a} \neq \boldsymbol{0} \\ |\boldsymbol{\delta a}| & \text{for } \boldsymbol{a} = \boldsymbol{0}. \end{cases}$$

(7) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \min(x, 0)$ *with*

$$f'_H(x; \delta x) = \left\{ \begin{array}{ll} \delta x & \text{for } x < 0 \\ \min(0, \delta x) & \text{for } x = 0 \\ 0 & \text{for } x > 0 \end{array} \right\} =: \min{}'(x; \delta x).$$

3. Local Lipschitz continuity of the solution map

In this section we prove that the solution map $\mathcal{G}$ is locally Lipschitz continuous by adopting ideas of the proof of [10, Proposition 15] and by exploiting the boundedness of the solution map, see lemma 2.7, in order to handle the nonlinear terms of the right hand side of the heat equation. On the other hand, the use of lemma 2.7 does not allow a proof of a global Lipschitz property, see for example estimate (3.3). Whether or not $\mathcal{G}$ is globally Lipschitz is an open question.

Proposition 3.1 (Local Lipschitz continuity of the solution map)**.** *Under the assumptions of theorem 2.6, the solution map defined in* (2.5) *is locally Lipschitz continuous.*

Proof. We choose two loads

$$(\boldsymbol{\ell}_1, r_1), (\boldsymbol{\ell}_2, r_2) \in L^q(0,T; \boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega)) \times L^{\frac{q}{2}}(0,T; W_{\diamond}^{-1,v(p)}(\Omega))$$

with $\|(\boldsymbol{\ell}_1, r_1)\|$, $\|(\boldsymbol{\ell}_2, r_2)\| \leq M$ and denote the corresponding states by $(\boldsymbol{u}_i, \boldsymbol{p}_i, \theta_i) := \mathcal{G}(\boldsymbol{\ell}_i, r_i)$ for $i = 1, 2$.

Balance of momentum and plastic flow rule. We follow the ideas of the proof of [10, Proposition 15] and reformulate the balance of momentum (0.4) and the plastic flow rule (0.3) for $i = 1, 2$ as the Banach space-valued ODE system

$$\begin{pmatrix} \dot{\boldsymbol{u}}_i \\ \dot{\boldsymbol{p}}_i \end{pmatrix} = \begin{pmatrix} \Phi^{\boldsymbol{u}}(\boldsymbol{\ell} + \operatorname{div}(\Phi^{\boldsymbol{\sigma}}(\boldsymbol{u}_i, \boldsymbol{p}_i, \theta_i))) \\ \Phi^{\boldsymbol{p}}(\theta_i, \Phi^{\boldsymbol{\sigma}}(\boldsymbol{u}_i, \boldsymbol{p}_i, \theta_i) + \Phi^{\boldsymbol{\chi}}(\boldsymbol{u}_i, \boldsymbol{p}_i, \theta_i)) \end{pmatrix}$$

with the solution operators $\Phi^{\boldsymbol{u}}$ (defined in lemma 2.2) and $\Phi^{\boldsymbol{p}}$ (defined by the right hand side in (2.2), see also [10, Lemma 13] for an alternative representation), and the maps $\Phi^{\boldsymbol{\sigma}}$ and $\Phi^{\boldsymbol{\chi}}$ given by the algebraic relations (0.1) and (0.2).

Similarly to the proof of the Lipschitz property required for the application of a Picard-Lindel 目 f argument in [10, Proposition 15], we obtain with minor modifications (due to the loads being variable) the estimate

$$\begin{aligned} &\|(\boldsymbol{u}_1, \boldsymbol{p}_1) - (\boldsymbol{u}_2, \boldsymbol{p}_2)\|_{W^{1,q}(0,T;\boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)) \times W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega))} \\ &\qquad \leq C\,\|\theta_1 - \theta_2\|_{L^q(0,T;L^p(\Omega))} + C\,\|\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2\|_{L^q(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega))}. \end{aligned} \tag{3.1}$$

Heat equation. We apply the embedding, cf. lemma 2.8,

$$W_0^{1,\frac{q}{2}}(0,T; W_{\diamond}^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T; W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0,T]; L^p(\Omega)).$$

and the maximal parabolic regularity, assumption 1.6 (3), to the difference of the temperatures $\theta_1 - \theta_2$, where $[\theta_1 - \theta_2](0) = 0$ holds. We obtain the following chain of inequalities,

$$\begin{aligned} \|\theta_1(t) - \theta_2(t)\|_{L^p(\Omega)} &\leq C\,\|\theta_1 - \theta_2\|_{W_0^{1,\frac{q}{2}}(0,t;W_{\diamond}^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,t;W^{1,v(p)}(\Omega))} \\ &\leq C\,\|f_1 - f_2\|_{L^{\frac{q}{2}}(0,t;W_{\diamond}^{-1,v(p)}(\Omega))}, \end{aligned} \tag{3.2}$$

where $f_i \in L^{\frac{q}{2}}(0,T; W_{\diamond}^{-1,v(p)}(\Omega))$ for $i = 1, 2$ are defined as the right hand sides of the heat equation (0.5) related to the loads $(\boldsymbol{\ell}_i, r_i)$. It remains to bound the right hand side of (3.2) in a suitable way to exploit Gronwall's lemma. We estimate

$$\begin{aligned} &\|f_1 - f_2\|_{L^{\frac{q}{2}}(0,t;W_{\diamond}^{-1,v(p)}(\Omega))} \\ &\quad \leq \|r_1 - r_2\|_{L^{\frac{q}{2}}(0,t;W_{\diamond}^{-1,v(p)}(\Omega))} \\ &\qquad + \gamma\|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2)\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\ &\qquad + \|(\boldsymbol{\sigma}_1 + \boldsymbol{\chi}_1) : \dot{\boldsymbol{p}}_1 - (\boldsymbol{\sigma}_2 + \boldsymbol{\chi}_2) : \dot{\boldsymbol{p}}_2\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\ &\qquad + \|\theta_1 \boldsymbol{t}'(\theta_1) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1) - \dot{\boldsymbol{p}}_1) - \theta_2 \boldsymbol{t}'(\theta_2) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2) - \dot{\boldsymbol{p}}_2)\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\ &\quad =: \|r_1 - r_2\|_{L^{\frac{q}{2}}(0,t;W_{\diamond}^{-1,v(p)}(\Omega))} + \gamma B_1 + B_2 + B_3, \end{aligned}$$

where we used the embedding $L^{\frac{p}{2}}(\Omega) \hookrightarrow W_\diamond^{-1,v(p)}(\Omega)$, cf. remark 1.5. We estimate the individual terms as follows,

$$
\begin{aligned}
B_1 &\le \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \\
&\quad + \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \|\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))}, \\
B_2 &\le \|\boldsymbol{\sigma}_1 + \boldsymbol{\chi}_1\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \|\dot{\boldsymbol{p}}_1 - \dot{\boldsymbol{p}}_2\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \\
&\quad + \|(\boldsymbol{\sigma}_1 + \boldsymbol{\chi}_1) - (\boldsymbol{\sigma}_2 + \boldsymbol{\chi}_2)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \|\dot{\boldsymbol{p}}_2\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))},
\end{aligned}
$$

where $\boldsymbol{\sigma}_i := \Phi^{\boldsymbol{\sigma}}(\boldsymbol{u}_i, \boldsymbol{p}_i, \theta_i)$ and $\boldsymbol{\chi}_i := \Phi^{\boldsymbol{\chi}}(\boldsymbol{u}_i, \boldsymbol{p}_i, \theta_i)$, respectively. Finally, we apply the Lipschitz continuity of $\theta \mapsto \theta\, \boldsymbol{t}'(\theta)$, see assumption 1.1, and estimate

$$
\begin{aligned}
B_3 &\le \|\theta_1 \boldsymbol{t}'(\theta_1)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \|\mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_1) - \dot{\boldsymbol{p}}_1) - \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2) - \dot{\boldsymbol{p}}_2)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \\
&\quad + C\, \|\theta_1 - \theta_2\|_{L^q(0,t;L^p(\Omega))} \|\mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_2) - \dot{\boldsymbol{p}}_2)\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))}.
\end{aligned}
$$

We benefit from the boundedness of the solution map (see lemma 2.7), the boundedness of the mapping $\theta \mapsto \theta\, \boldsymbol{t}'(\theta)$ and the Lipschitz continuity of $\boldsymbol{t}$ (see assumption 1.1) to obtain

$$
\begin{aligned}
&\|\theta_1(t) - \theta_2(t)\|_{L^p(\Omega)} \\
&\quad \le C\, \|r_1 - r_2\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))} + C(M)\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,q}(0,t;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))} \\
&\qquad + C(M)\, \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_{W^{1,q}(0,t;\boldsymbol{L}^p(\Omega))} + C(M)\, \|\theta_1 - \theta_2\|_{L^q(0,t;L^p(\Omega))}.
\end{aligned}
\tag{3.3}
$$

Putting everything together. Now we combine the results from estimates (3.1) and (3.3) to obtain

$$
\begin{aligned}
\|\theta_1(t) - \theta_2(t)\|_{L^p(\Omega)} &\le C\, \|r_1 - r_2\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))} \\
&\quad + C(M)\, \|\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2\|_{L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega))} + C(M)\, \|\theta_1 - \theta_2\|_{L^q(0,t;L^p(\Omega))}.
\end{aligned}
\tag{3.4}
$$

We abbreviate

$$
D(t) := C\, \|r_1 - r_2\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))} + C(M)\, \|\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2\|_{L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega))}
$$

and obtain, using the convexity of $z \mapsto z^q$ for $z \ge 0$ for the right hand side, the inequality

$$
\|\theta_1(t) - \theta_2(t)\|^q_{L^p(\Omega)} \le C(M) \int_0^t \|\theta_1 - \theta_2\|^q_{L^p(\Omega)} \,\mathrm{d}\xi + D(t)^q.
$$

Now we can employ Gronwall's lemma to estimate

$$
\|\theta_1(t) - \theta_2(t)\|^q_{L^p(\Omega)} \le C(M,T)\, D(T)^q \quad \text{for all } t \in [0,T]
$$

and therefore

$$
\|\theta_1 - \theta_2\|_{L^\infty(0,T;L^p(\Omega))} \le C(M,T)\, D(T).
\tag{3.5}
$$

In addition we obtain from inequality (3.2) and the calculations above,

$$
\|\theta_1 - \theta_2\|_{W_0^{1,\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega)) \cap L^{\frac{q}{2}}(0,t;W^{1,v(p)}(\Omega))} \le C(M)\, \|\theta_1 - \theta_2\|_{L^q(0,t;L^p(\Omega))} + D(t),
$$

compare (3.4). Together with (3.5) we obtain

$$
\|\theta_1 - \theta_2\|_{W_0^{1,\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))}
\tag{3.6}
$$

$$\leq C(M,T)\,\|r_1 - r_2\|_{L^{\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega))} + C(M,T)\,\|\boldsymbol{\ell}_1 - \boldsymbol{\ell}_2\|_{L^q(0,T;\boldsymbol{W}_{\mathfrak{D}}^{-1,p}(\Omega))}.$$

Finally, we combine (3.1) with (3.5). Together with (3.6), this establishes the assertion. □

The proof of the previous proposition indeed shows that the solution map $\mathcal{G}$ is Lipschitz continuous on bounded sets, which is slightly stronger than local Lipschitz continuity.

Remark 3.2 (Fréchet differentiability of the solution map)**.** The result of [15] states that locally Lipschitz continuous functions defined on Asplund spaces (see [2]) are Fréchet differentiable on a dense subset of their domain. Therefore we conclude that the solution map $\mathcal{G}$ is Fréchet differentiable on a dense subset.

4. Directional differentiability of the solution map

In this section we provide the proof of the main theorem 4.2, which states the directional differentiability of the solution map $\mathcal{G}$. We start with defining the linearized system related to our thermoviscoplastic system (0.1)–(0.5) in definition 4.1. Then we present our main theorem followed by a detailed roadmap of its proof. The major part of this section is a rigorous proof of the main theorem.

In deriving the linearized thermoviscoplastic system and its weak solution, we consider the thermoviscoplastic flow rule in its representation as a Banach space-valued ODE (2.2), instead of the formulation as a variational inequality (0.3).

Definition 4.1 (Weak solution of the linearized thermoviscoplastic system)**.** Let $p, q > 2$ and $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ be a weak solution of the thermoviscoplastic system (0.1)–(0.5) with regularity

$$\boldsymbol{u} \in W^{1,q}(0,T;\boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)), \quad \boldsymbol{p} \in W^{1,q}(0,T;\boldsymbol{Q}^p(\Omega)),$$
$$\boldsymbol{\sigma} \in W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)), \quad \boldsymbol{\chi} \in W^{1,q}(0,T;\boldsymbol{L}^p(\Omega)),$$
$$\theta \in W^{1,\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega)),$$

where $v(p)$ is defined in (1.1). Given inhomogeneities according to assumption 1.4, we say that a quintuple $(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \boldsymbol{\delta\sigma}, \boldsymbol{\delta\chi}, \delta\theta)$ with the same regularities as above is a *weak solution* of the linearized thermoviscoplastic system, if it fulfills, for almost all $t \in (0,T)$, the

(4.1) stress-strain relation: $\boldsymbol{\delta\sigma} = \mathbb{C}\left(\boldsymbol{\varepsilon}(\boldsymbol{\delta u}) - \boldsymbol{\delta p} - \boldsymbol{t}'(\theta)\,\delta\theta\right)$

(4.2) conjugate forces: $\boldsymbol{\delta\chi} = -\mathbb{H}\,\boldsymbol{\delta p}$

(4.3) viscoplastic flow rule: $\dot{\boldsymbol{\delta p}} = -\epsilon^{-1}\min\left(\frac{\tilde{\sigma}(\theta)}{|\boldsymbol{\tau}|} - 1,\ 0\right)\boldsymbol{\delta\tau}$

$$- \epsilon^{-1}\min'\left(\frac{\tilde{\sigma}(\theta)}{|\boldsymbol{\tau}|} - 1;\ \frac{\tilde{\sigma}'(\theta)\,\delta\theta}{|\boldsymbol{\tau}|} - \tilde{\sigma}(\theta)\frac{\boldsymbol{\tau} : \boldsymbol{\delta\tau}}{|\boldsymbol{\tau}|^3}\right)\boldsymbol{\tau},$$

where $\min'(x;\cdot)$ is defined as in lemma 2.11

and $\boldsymbol{\delta\tau} = [\boldsymbol{\delta\sigma} + \boldsymbol{\delta\chi}]^D$, $\boldsymbol{\tau} = [\boldsymbol{\sigma} + \boldsymbol{\chi}]^D$

(4.4) balance of momentum: $\dot{\boldsymbol{\delta u}} = \Phi^{\boldsymbol{u}}(\boldsymbol{F}_1(\boldsymbol{\delta\ell}, \boldsymbol{\delta u}, \boldsymbol{\delta p}) + \boldsymbol{F}_2'(\theta;\delta\theta)),$

where $\Phi^{\boldsymbol{u}}$ is given in lemma 2.2 and $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ in remark 2.3

(4.5) and the heat equation: $$\langle \dot{\delta\theta},\, z\rangle + \int_\Omega \kappa\, \nabla\delta\theta \cdot \nabla z \,\mathrm{d}\boldsymbol{x} + \int_\Gamma \beta\, \delta\theta\, z \,\mathrm{d}s$$
$$= \langle \delta r,\, z\rangle + \int_\Omega 2\,\gamma\, \boldsymbol{\varepsilon}(\dot{\boldsymbol{\delta u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})\, z \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\boldsymbol{\sigma} + \boldsymbol{\chi}) : \dot{\boldsymbol{\delta p}}\, z \,\mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega (\boldsymbol{\delta\sigma} + \boldsymbol{\delta\chi}) : \dot{\boldsymbol{p}}\, z \,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{t}'(\theta)\, \delta\theta : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}})\, z \,\mathrm{d}\boldsymbol{x}$$
$$- \int_\Omega \theta\, \boldsymbol{t}''(\theta)\, \delta\theta : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) - \dot{\boldsymbol{p}})\, z \,\mathrm{d}\boldsymbol{x} - \int_\Omega \theta\, \boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{\delta u}}) - \dot{\boldsymbol{\delta p}})\, z \,\mathrm{d}\boldsymbol{x}$$
for all $z \in W^{1,v(p)'}(\Omega)$,

along with the initial conditions $\boldsymbol{\delta u}(0) = \boldsymbol{0}$, $\boldsymbol{\delta p}(0) = \boldsymbol{0}$, and $\delta\theta(0) = 0$.

Note that as in the ODE formulation of the flow rule (2.2), the right hand side of (4.3) is understood to be zero by continuous extension when $\boldsymbol{\tau}(\boldsymbol{u}, \boldsymbol{p}, \theta) = \boldsymbol{0}$. For detailed information about boundary conditions which are defined in the weak setting above implicitly, we refer the reader to the remarks following definition 2.1.

With definition 4.1 at hand we formulate our main theorem concerning the directional differentiability of the solution map as follows.

Theorem 4.2 (Directional differentiablity of the solution map)**.** *Under the assumptions of theorem 2.6, the solution map defined in* (2.5) *is directionally differentiable. When* $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ *is the (weak) solution of the thermoviscoplastic system* (0.1)–(0.5) *according to the control* $(\boldsymbol{\ell}, r)$*, the directional derivative is given by*

$$\mathcal{G}'(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) = (\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta),$$

where $(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta)$ *is the (weak) solution of the linearized thermoviscoplastic system* (4.1)–(4.5) *in the sense of definition 4.1.*

Since the proof of theorem 4.2 is quite involved, we first present a roadmap of the proof.

(1) We establish in proposition 4.3 that the thermoviscoplastic linearized system (4.1)–(4.5) has a unique solution in the same space as the thermoviscoplastic system (0.1)–(0.5).
(2) We prove that all nonlinear terms appearing in the thermoviscoplastic system (0.1)–(0.5) are directionally differentiable.
  (a) First we show the directional differentiability of the functions pointwise in a finite dimensional setting by exploiting the chain rule for Hadamard differentiable functions, see lemma 2.10.
  (b) Secondly, we apply Lebesgue's dominated convergence theorem, to obtain the property also in Bochner spaces.
(3) We finalize the proof of the main theorem 4.2 using Gronwall's lemma and the previous results.

The following three subsections are arranged according to the structure above.

4.1. **Existence of a Unique Solution to the System Related to the Directional Derivative.** In this subsection we show the existence of a unique solution of the system (4.1)–(4.5).

Proposition 4.3. *Under the assumptions of theorem 2.6, let* $(\boldsymbol{u}, \boldsymbol{p}, \theta, \boldsymbol{\sigma}, \boldsymbol{\chi})$ *be the unique weak solution of the thermoviscoplastic system* (0.1)–(0.5) *for given* $(\boldsymbol{\ell}, r) \in L^q(0,T; \boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)) \times L^{\frac{q}{2}}(0,T; W^{-1,v(p)}_{\diamond}(\Omega))$. *Then for all* $(\boldsymbol{\delta\ell}, \delta r)$ *sharing the same regularity, there exists a unique weak solution* $(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta, \boldsymbol{\delta\sigma}, \boldsymbol{\delta\chi})$ *of the linearized viscoplastic system* (4.1)–(4.5) *in the sense of definition 4.1.*

Proof. Since the structure of the linearized thermoviscoplastic system (4.1)–(4.5) is similar to the thermoviscoplastic system (0.1)–(0.5), the proof can be achieved with the same techniques developed in [10, Theorem 10] with the following modifications:

(1) In comparison to [10, Proposition 15], we have to assume more regularity for the temperature, viz. $\delta\theta \in L^q(0,T; L^p(\Omega))$ instead of $L^1(0,T; L^1(\Omega))$. This is since we have to estimate in the linearized system the term $\boldsymbol{t}'(\theta)\,\delta\theta$ appearing in the stress-strain relation (4.1) instead of only $\boldsymbol{t}(\theta)$, which is bounded by assumption.

(2) The image of the solution operator $\delta\theta \mapsto (\boldsymbol{\delta u}, \boldsymbol{\delta p}, \boldsymbol{\delta\sigma}, \boldsymbol{\delta\chi})$ is no longer bounded independently of the temperature $\delta\theta$, in contrast to [10, Proposition 15]. This is due to the term $\boldsymbol{t}'(\theta)\,\delta\theta$ appearing in the stress-strain relation (4.1) of the linearized system. Thanks to the linearity of the right hand side of the linearized heat equation (4.5), the boundedness property will not be needed in the analysis, compare [10, Lemma 16].

(3) The concatenation argument in the proof of [10, Proposition 24] can be simplified exploiting that the Lipschitz constant $L_{\boldsymbol{\delta}\mathcal{R}}$ is independent of the initial values $\boldsymbol{\delta u}(0)$ and $\boldsymbol{\delta p}(0)$.

□

4.2. **Directional Differentiability of the Nonlinear Terms in the Forward System.** The following nonlinear mappings appearing in the balance of momentum (0.4) and heat equation (0.5) of the thermoviscoplastic system are directionally differentiable.

Lemma 4.4 (Directional differentiability of nonlinear terms)**.**

(1) *The mapping*

$$\mathrm{Therm} : L^q(0,T; L^p(\Omega)) \to L^q(0,T; \boldsymbol{L}^p(\Omega)), \quad \mathrm{Therm}(\theta) = \boldsymbol{t}(\theta)$$

is directionally differentiable with directional derivative

$$\mathrm{Therm}'(\theta; \delta\theta) = \boldsymbol{t}'(\theta)\,\delta\theta.$$

(2) *The mapping*

$$\mathrm{Heat}_1 : L^q(0,T; \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \to L^{\frac{q}{2}}(0,T; L^{\frac{p}{2}}(\Omega)), \quad \mathrm{Heat}_1(\boldsymbol{u}) = \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u})$$

is directionally differentiable with directional derivative

$$\mathrm{Heat}_1{}'(\boldsymbol{u}; \boldsymbol{\delta u}) = 2\,\boldsymbol{\varepsilon}(\boldsymbol{\delta u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}).$$

(3) *The mapping*

$$\begin{aligned} \mathrm{Heat}_2 : L^q(0,T; \boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times W^{1,q}(0,T; \boldsymbol{L}^p(\Omega)) \times L^q(0,T; L^p(\Omega)) \\ \to L^{\frac{q}{2}}(0,T; L^{\frac{p}{2}}(\Omega)), \\ \mathrm{Heat}_2(\boldsymbol{u}, \boldsymbol{p}, \theta) = (\boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{p}, \theta) + \boldsymbol{\chi}(\boldsymbol{u}, \boldsymbol{p}, \theta)) : \dot{\boldsymbol{p}} \end{aligned}$$

is directionally differentiable with directional derivative

$$\mathrm{Heat}_2{}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) = (\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p},\theta)+\boldsymbol{\chi}(\boldsymbol{u},\boldsymbol{p},\theta)) : \dot{\boldsymbol{\delta p}} - (\boldsymbol{\delta\sigma}(\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)+\boldsymbol{\delta\chi}(\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)) : \dot{\boldsymbol{p}}.$$

where the mappings $\boldsymbol{\sigma}, \boldsymbol{\delta\sigma}, \boldsymbol{\chi}$ *and* $\boldsymbol{\delta\chi}$ *are defined by the algebraic relations* (0.1)–(0.2) *and* (4.1)–(4.2)*, respectively.*

(4) *The mapping*

$$\mathrm{Heat}_3 : L^q(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega)) \times L^q(0,T;L^p(\Omega)) \to L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega)),$$
$$\mathrm{Heat}_3(\boldsymbol{u},\boldsymbol{p},\theta) = \theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{p})$$

is directionally differentiable with directional derivative

$$\mathrm{Heat}_3{}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) = \delta\theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{p}) - \theta\,\boldsymbol{t}''(\theta)\,\delta\theta : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{p}) + \theta\,\boldsymbol{t}'(\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{\delta u})-\boldsymbol{\delta p}).$$

Proof. (1) Wie fix a point θ and direction $\delta\theta$. The sequence

$$\boldsymbol{f}_s(t,\boldsymbol{x}) := \frac{\boldsymbol{t}(\theta(t,\boldsymbol{x})+s\,\delta\theta(t,\boldsymbol{x}))-\boldsymbol{t}(\theta(t,\boldsymbol{x}))}{s} \to \boldsymbol{t}'(\theta(t,\boldsymbol{x}))\,\delta\theta(t,\boldsymbol{x})$$

converges pointwise for almost all $(t,\boldsymbol{x}) \in (0,T)\times\Omega$ since the mapping $\boldsymbol{t} : \mathbb{R} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ is directionally differentiable, see assumption 1.1. $\boldsymbol{f}_s$ is also bounded,

$$|\boldsymbol{f}_s(t,\boldsymbol{x})| \le C\,|\delta\theta(t,\boldsymbol{x})| \quad \text{with} \quad \delta\theta \in L^q(0,T;L^p(\Omega)),$$

where we used the Lipschitz continuity of $\boldsymbol{t}$. The dominated convergence theorem shows the assertion.

(2) We write $\mathrm{Heat}_1 =: \mathrm{heat}_1 \circ \boldsymbol{g}$ as the composition of a Hadamard and a directionally differentiable function, where

$$\mathrm{heat}_1 : L^q(0,T;\boldsymbol{L}^p(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega)) \to L^{\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega))$$
$$\mathrm{heat}_1(\boldsymbol{a},\boldsymbol{b}) := \boldsymbol{a} : \boldsymbol{b}$$

and

$$\boldsymbol{g} : L^q(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \to L^q(0,T;\boldsymbol{L}^p(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega)),$$
$$\boldsymbol{g}(\boldsymbol{u}) := \begin{pmatrix}\boldsymbol{\varepsilon}(\boldsymbol{u})\\ \boldsymbol{\varepsilon}(\boldsymbol{u})\end{pmatrix}.$$

Since $\boldsymbol{g}$ is linear, the map is obviously directionally differentiable. Moreover, we can prove that the mapping heat_1 is Hadamard differentiable with similar techniques as in (1); show the convergence of the difference quotient pointwise first, and then apply the dominated convergence theorem. Therefore, using the chain rule (lemma 2.10) we see that Heat_1 is directionally differentiable with the directional derivative as specified.

The proof of the two remaining assertions follows analogously. □

To cover all the nonlinearities in the thermoviscoplastic system, it remains to show that the right hand side of the flow rule (2.2) is Hadamard differentiable.

Lemma 4.5 (Directional differentiability of the flow rule)**.** *The right hand side of the flow rule* (2.2)*,*

$$\mathrm{Flow} : L^q(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega)) \times L^q(0,T;L^p(\Omega)) \to L^q(0,T;\boldsymbol{L}^p(\Omega)),$$

$$\mathrm{Flow}(\boldsymbol{u},\boldsymbol{p},\theta) = -\epsilon^{-1} \min\left(\frac{\tilde{\sigma}(\theta)}{|\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta)|} - 1,\ 0\right) \boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta),$$

where $\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) := [\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p},\theta) + \boldsymbol{\chi}(\boldsymbol{u},\boldsymbol{p},\theta)]^D = [\mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)) - \mathbb{H}\,\boldsymbol{p}]^D$*, is directionally differentiable with directional derivative*

$$\begin{aligned}
&\mathrm{Flow}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\\
&= -\epsilon^{-1} \min\left(\frac{\tilde{\sigma}(\theta)}{|\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta)|} - 1,\ 0\right) \boldsymbol{\delta\tau}(\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\\
&- \epsilon^{-1}\min'\left(\frac{\tilde{\sigma}(\theta)}{|\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta)|} - 1;\ \frac{\tilde{\sigma}'(\theta)\,\delta\theta}{|\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta)|} - \tilde{\sigma}(\theta)\frac{\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) : \boldsymbol{\delta\tau}(\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)}{|\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta)|^3}\right)\cdot\\
&\quad\cdot \boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta).
\end{aligned}$$

Here $\boldsymbol{\delta\tau}(\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) := [\mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{\delta u}) - \boldsymbol{\delta p} - \boldsymbol{t}'(\theta)\,\delta\theta - \mathbb{H}\,\boldsymbol{\delta p}]^D$ *and* $\min'$ *is the directional derivative of* $\min(\cdot,0)$*, see lemma 2.11.*

Note that we include in this formulation the case $\boldsymbol{\tau}(\boldsymbol{u},\boldsymbol{p},\theta) = \boldsymbol{0}$, which, by continuous extension, is understood as $\mathrm{Flow}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) := \boldsymbol{0}$, compare proposition 2.5.

Proof. We follow the idea of the proof of lemma 4.4 and rewrite the mapping Flow := flow $\circ\, \boldsymbol{g}$ as the composition of a Hadamard and a directionally differentiable function, where

$$\mathrm{flow} : L^q(0,T;L^p(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega)) \to L^q(0,T;\boldsymbol{L}^p(\Omega)),$$

$$\mathrm{flow}(a,\boldsymbol{b}) := -\epsilon^{-1} \min\left(\frac{\tilde{\sigma}(a)}{|\boldsymbol{b}^D|} - 1,\ 0\right) \cdot \boldsymbol{b}^D$$

and $\mathrm{flow}(a,\boldsymbol{b}) := \boldsymbol{0}$ if $\boldsymbol{b}^D = \boldsymbol{0}$. The mapping

$$\begin{aligned}
\boldsymbol{g} : L^q(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega)) \times L^q(0,T;L^1(\Omega))&\\
\to L^q(0,T;L^1(\Omega)) \times L^q(0,T;\boldsymbol{L}^p(\Omega))&
\end{aligned}$$

is defined as

$$\boldsymbol{g}(\boldsymbol{u},\boldsymbol{p},\theta) := (\theta,\, \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{p} - \boldsymbol{t}(\theta)) - \mathbb{H}\,\boldsymbol{p})^\top.$$

The directional differentiability of the mapping $\boldsymbol{g}$ can be inferred with similar arguments as in the proof of (1) of lemma 4.4 using that $\boldsymbol{t}$ is pointwise Lipschitz continuous and that $\mathbb{C}$ is linear and bounded, see assumption 1.1. The directional derivative of $\boldsymbol{g}$ is given by

$$\boldsymbol{g}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) = (\delta\theta,\, \mathbb{C}(\boldsymbol{\varepsilon}(\boldsymbol{\delta u}) - \boldsymbol{\delta p} - \boldsymbol{t}'(\theta)\,\delta\theta) - \mathbb{H}\,\boldsymbol{\delta p})^\top.$$

It remains to show that flow is Hadamard differentiable. We fix $n(s) : (0,\infty) \to \mathbb{R}$ with $n(s) = o(s)$ and $\boldsymbol{m}(s) : (0,\infty) \to \mathbb{R}^{3\times3}_{\mathrm{sym}}$ with $\boldsymbol{m}(s) = o(s)$ respectively. Furthermore we choose an arbitrary point $(a,\boldsymbol{b})$ and direction $(\delta a,\boldsymbol{\delta b})$. For $\boldsymbol{b}^D(t,\boldsymbol{x}) \neq \boldsymbol{0}$ the sequence

$$\begin{aligned}\boldsymbol{f}_s(t,\boldsymbol{x}) &:= s^{-1}\big(\operatorname{flow}(a+s\,\delta a,\boldsymbol{b}+s\,\boldsymbol{\delta b})-\operatorname{flow}(a,\boldsymbol{b})\big)(t,\boldsymbol{x})\\ &\to -\epsilon^{-1}\min\left(\frac{\tilde\sigma(a(t,\boldsymbol{x}))}{|\boldsymbol{b}^D(t,\boldsymbol{x})|}-1;\ 0\right)\boldsymbol{\delta b}^D(t,\boldsymbol{x})-\epsilon^{-1}\min{}'\left(\frac{\tilde\sigma(a(t,\boldsymbol{x}))}{|\boldsymbol{b}^D(t,\boldsymbol{x})|}-1;\right.\\ &\qquad\left.\frac{\tilde\sigma'(a(t,\boldsymbol{x}))\,\delta a(t,\boldsymbol{x})}{|\boldsymbol{b}^D(t,\boldsymbol{x})|}-\tilde\sigma(a(t,\boldsymbol{x}))\frac{\boldsymbol{b}^D(t,\boldsymbol{x}):\boldsymbol{\delta b}^D(t,\boldsymbol{x})}{|\boldsymbol{b}^D(t,\boldsymbol{x})|^3}\right)\boldsymbol{b}^D(t,\boldsymbol{x})\end{aligned}$$

converges pointwise for almost all $(t,\boldsymbol{x})\in(0,T)\times\Omega$ using lemma 2.11 and the chain rule for Hadamard differentiable functions (lemma 2.10). In case $\boldsymbol{b}^D(t,\boldsymbol{x})=\boldsymbol{0}$, the sequence satisfies $\boldsymbol{f}_s(t,\boldsymbol{x})=\boldsymbol{0}$ for s small enough.

Next, we will estimate the difference quotient pointwise. Note that we are only interested in points $(t,\boldsymbol{x})$ with $\boldsymbol{b}^D(t,\boldsymbol{x})\neq\boldsymbol{0}$. We estimate

$$|\boldsymbol{f}_s(t,\boldsymbol{x})|\le C\epsilon^{-1}\left(|\delta a(t,\boldsymbol{x})|+|\boldsymbol{\delta b}^D(t,\boldsymbol{x})|\right)+C\epsilon^{-1}=:M(t,\boldsymbol{x})$$

with $M\in L^q(0,T;L^p(\Omega))$ for s small enough. We apply the dominated convergence theorem, and obtain that flow is Hadamard differentiable with

$$\begin{aligned}\operatorname{flow}'(a,\boldsymbol{b};\delta a,\boldsymbol{\delta b}) &= -\epsilon^{-1}\min\left(\frac{\tilde\sigma(a)}{|\boldsymbol{b}^D|}-1,\ 0\right)\boldsymbol{\delta b}^D\\ &\quad-\epsilon^{-1}\min{}'\left(\frac{\tilde\sigma(a)}{|\boldsymbol{b}^D|}-1;\ \frac{\tilde\sigma'(a)\,\delta a}{|\boldsymbol{b}^D|}-\tilde\sigma(a)\frac{\boldsymbol{b}^D:\boldsymbol{\delta b}^D}{|\boldsymbol{b}^D|^3}\right)\boldsymbol{b}^D.\end{aligned}$$

Therefore, the chain rule (lemma 2.10) shows that the mapping Flow is directionally differentiable with the related derivative claimed in the assertion. ☐

4.3. **Proof of the Directional Differentiability.** In this subsection we provide the proof of theorem 4.2. We emphasize that the structure of the proof is very close to the proof of local Lipschitz continuity in proposition 3.1. In the sequel we have to deal with many difference quotients. Therefore, we introduce the short-hand notation

$$\mathcal{D}_s f(v;\delta v):=\frac{f(v+s\,\delta v)-f(v)}{s},$$

where $f:V\to W$ and $v,\delta v\in V$ and $s>0$.

Proof of theorem 4.2. Let

$$(\boldsymbol{\ell},r),(\boldsymbol{\delta\ell},\delta r)\in L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega))\times L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega))$$

be arbitrary but fixed. We have to verify the definition of directional differentiability,

$$\lim_{s\downarrow 0}\frac{(\boldsymbol{u}^s,\boldsymbol{p}^s,\theta^s)-(\boldsymbol{u},\boldsymbol{p},\theta)}{s}=\lim_{s\downarrow 0}\mathcal{D}_s\mathcal{G}(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r)=(\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta),$$

where $(\boldsymbol{u}^s,\boldsymbol{p}^s,\theta^s):=\mathcal{G}(\boldsymbol{\ell}+s\,\boldsymbol{\delta\ell},r+s\,\delta r)$ solves the perturbed thermoviscoplastic system consisting of the

(4.6) stress-strain relation: $\boldsymbol{\sigma}^s=\mathbb{C}\left(\boldsymbol{\varepsilon}(\boldsymbol{u}^s)-\boldsymbol{p}^s-\boldsymbol{t}(\theta^s)\right)$

(4.7) conjugate forces: $\boldsymbol{\chi}^s=-\mathbb{H}\,\boldsymbol{p}^s$

(4.8) viscoplastic flow rule: $\dot{\boldsymbol{p}}^s=-\epsilon^{-1}\min\left(\frac{\tilde\sigma(\theta^s)}{|\boldsymbol{\tau}^s(\boldsymbol{u}^s,\boldsymbol{p}^s,\theta^s)|}-1,\ 0\right)\cdot$

$$\cdot\,\boldsymbol{\tau}^s(\boldsymbol{u}^s,\boldsymbol{p}^s,\theta^s),\quad\text{where } \boldsymbol{\tau}^s(\boldsymbol{u}^s,\boldsymbol{p}^s,\theta^s):=[\boldsymbol{\sigma}^s+\boldsymbol{\chi}^s]^D$$

(4.9) balance of momentum: $\dot{\boldsymbol{u}}^s = \Phi^{\boldsymbol{u}}(\boldsymbol{F}_1(\boldsymbol{\ell} + s\,\boldsymbol{\delta\ell}, \boldsymbol{u}^s, \boldsymbol{p}^s) + \boldsymbol{F}_2(\theta^s))$

(4.10) heat equation: $\dot{\theta}^s - \operatorname{div}(\kappa\,\nabla\theta^s) = r + s\,\delta r$
$$+ \gamma\,\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) + (\boldsymbol{\sigma}^s + \boldsymbol{\chi}^s) : \dot{\boldsymbol{p}}^s - \theta^s \boldsymbol{t}'(\theta^s) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) - \dot{\boldsymbol{p}}^s).$$

Moreover, $(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta)$ denotes the solution of the linearized thermoviscoplastic system (4.1)–(4.5) related to the weak solution of the thermoviscoplastic system (0.1)–(0.5) for the control $(\boldsymbol{\ell}, r)$. Note that proposition 4.3 ensures the existence of the weak solution of the linearized thermoviscoplastic system (4.1)–(4.5).

The strategy now is to estimate the three states separately and then to combine the results to obtain the assertion using Gronwall's lemma. We recall that $\mathcal{G}^{\boldsymbol{u}}$, $\mathcal{G}^{\boldsymbol{p}}$, $\mathcal{G}^{\boldsymbol{u},\boldsymbol{p}}$ and $\mathcal{G}^{\theta}$, denotes the individual components of the solution mapping $\mathcal{G}$, see after lemma 2.7.

Balance of momentum. We consider the difference quotient of (4.9) and (2.1), subtract (4.4), and integrate over time. Following the ideas of the proof of [10, Proposition 15] and using $\mathcal{D}_s\mathcal{G}^{\boldsymbol{u}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(0) - \boldsymbol{\delta u}(0) = \boldsymbol{0}$, we obtain

$$\begin{aligned}
(4.11)\quad & \|\mathcal{D}_s\mathcal{G}^{\boldsymbol{u}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \boldsymbol{\delta u}(t)\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} \\
&\le C\gamma^{-1} \int_0^t \big\|\mathcal{D}_s\mathcal{G}^{\boldsymbol{u},\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}\big)\big\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{L}^p(\Omega)}\,\mathrm{d}\xi \\
&\quad + C\gamma^{-1} \int_0^t \left\|\frac{\boldsymbol{t}(\theta^s) \pm \boldsymbol{t}(\theta + s\,\delta\theta) - \boldsymbol{t}(\theta)}{s} - \boldsymbol{t}'(\theta)\,\delta\theta\right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\
&\le C\gamma^{-1} \int_0^t \big\|\mathcal{D}_s\mathcal{G}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta\big)\big\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{L}^p(\Omega)\times L^p(\Omega)}\,\mathrm{d}\xi \\
&\quad + C\gamma^{-1} \int_0^t \big\|\mathcal{D}_s\boldsymbol{t}(\theta; \delta\theta) - \boldsymbol{t}'(\theta)\,\delta\theta\big\|_{\boldsymbol{L}^p(\Omega)}\,\mathrm{d}\xi,
\end{aligned}$$

where we used for the mapping $\Phi^{\boldsymbol{u}}$ the estimate given in lemma 2.2 and the Lipschitz continuity of $\boldsymbol{t}$. Note that the idea behind adding a zero term was to obtain one term whose Lipschitz properties we can exploit, and one term enjoying directional differentiablity, see lemma 4.4.

Next, we consider again the difference quotient of (4.9) and (0.4), subtract (4.4), and calculate as above

$$\begin{aligned}
(4.12)\quad & \left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}_s\mathcal{G}^{\boldsymbol{u}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \dot{\boldsymbol{\delta u}}(t)\right\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)} \\
&\le C\gamma^{-1} \big\|\mathcal{D}_s\mathcal{G}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta\big)(t)\big\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)\times \boldsymbol{L}^p(\Omega)\times L^p(\Omega)} \\
&\quad + C\gamma^{-1} \big\|\mathcal{D}_s\boldsymbol{t}(\theta; \delta\theta)(t) - \boldsymbol{t}'(\theta(t))\,\delta\theta(t)\big\|_{\boldsymbol{L}^p(\Omega)}.
\end{aligned}$$

Plastic flow rule. For brevity we omit the arguments for $\boldsymbol{\tau}, \boldsymbol{\tau}^s$ and $\boldsymbol{\delta\tau}$ having in mind that their dependencies are given by the algebraic relations in (2.2), (4.8) and (4.3), respectively. Moreover, we define $\hat{\boldsymbol{\tau}} := \boldsymbol{\tau}(\boldsymbol{u} + s\,\boldsymbol{\delta u}, \boldsymbol{p} + s\,\boldsymbol{\delta p}, \theta + s\,\delta\theta)$.

Note that for points $(t, \boldsymbol{x}) \in (0, T) \times \Omega$ with $\boldsymbol{\tau}(t, \boldsymbol{x}) = \boldsymbol{0}$, the right hand side of the flow rule (2.2) and the linearized flow rule (4.3) are zero by definition (see the comments after proposition 2.5 and definition 4.1). The continuity property of $\mathcal{G}$, proposition 3.1, which means that $\boldsymbol{\tau}^s(t, \boldsymbol{x}) \to \boldsymbol{\tau}(t, \boldsymbol{x}) = \boldsymbol{0}$ for $s \to 0$ almost everywhere, leads for s small enough to the same result for the perturbed flow rule (4.8). Therefore, we can neglect in the following estimate the case $\boldsymbol{\tau}(t,\boldsymbol{x}) = \mathbf{0}$.

We consider the difference quotient of (4.8) and (2.2), subtract (4.3), integrate over time, and using $\frac{\boldsymbol{p}^s(0)-\boldsymbol{p}(0)}{s} - \boldsymbol{\delta p}(0) = \mathbf{0}$ we obtain

$$\begin{aligned}
&\|\mathcal{D}_s \mathcal{G}^{\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \boldsymbol{\delta p}(t)\|_{\boldsymbol{L}^p(\Omega)} \\
&\le \epsilon^{-1} \int_0^t \Bigg\| \frac{\min\left(\frac{\tilde\sigma(\theta^s)}{|\boldsymbol{\tau}^s|} - 1,\ 0\right)\boldsymbol{\tau}^s - \min\left(\frac{\tilde\sigma(\theta)}{|\boldsymbol{\tau}|} - 1,\ 0\right)\boldsymbol{\tau}}{s} \\
&- \min\left(\frac{\tilde\sigma(\theta)}{|\boldsymbol{\tau}|} - 1,\ 0\right)\boldsymbol{\delta\tau} - \min{}'\left(\frac{\tilde\sigma(\theta)}{|\boldsymbol{\tau}|} - 1;\ \frac{\tilde\sigma'(\theta)\,\delta\theta}{|\boldsymbol{\tau}|} - \tilde\sigma(\theta)\frac{\boldsymbol{\tau} : \boldsymbol{\delta\tau}}{|\boldsymbol{\tau}|^3}\right)\boldsymbol{\tau}\Bigg\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi.
\end{aligned}$$

Now we use the equality $\min(\frac{a}{b} - c, 0) = b^{-1}\min(a - b\,c, 0)$ for $a, b, c \in \mathbb{R}$ and $b > 0$ and add, with the same strategy as in the balance of momentum above, suitable zero terms. We end up with

$$\|\mathcal{D}_s \mathcal{G}^{\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \boldsymbol{\delta p}(t)\|_{\boldsymbol{L}^p(\Omega)} \le \epsilon^{-1}(A_1 + A_2 + A_3) + A_4,$$

where A_1, A_2 and A_3 are given by

$$\begin{aligned}
A_1 &:= \int_0^t s^{-1} \left\| \left[\min\left(\tilde\sigma(\theta^s) - |\boldsymbol{\tau}^s|,\ 0\right) - \min\left(\tilde\sigma(\theta + s\,\delta\theta) - |\hat{\boldsymbol{\tau}}|,\ 0\right)\right] \frac{\boldsymbol{\tau}^s}{|\boldsymbol{\tau}^s|} \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\
A_2 &:= \int_0^t s^{-1} \left\| \min\left(\tilde\sigma(\theta + s\,\delta\theta) - |\hat{\boldsymbol{\tau}}|,\ 0\right) \left[\frac{\boldsymbol{\tau}^s}{|\boldsymbol{\tau}^s|} - \frac{\boldsymbol{\tau}^s}{|\hat{\boldsymbol{\tau}}|}\right] \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\
&\le \int_0^t s^{-1} \left\| \min\left(\frac{\tilde\sigma(\theta + s\,\delta\theta)}{|\hat{\boldsymbol{\tau}}|} - 1,\ 0\right) \left[|\hat{\boldsymbol{\tau}}| - |\boldsymbol{\tau}^s|\right] \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\
A_3 &:= \int_0^t s^{-1} \left\| \min\left(\tilde\sigma(\theta + s\,\delta\theta) - |\hat{\boldsymbol{\tau}}|,\ 0\right) \left[\frac{\boldsymbol{\tau}^s}{|\hat{\boldsymbol{\tau}}|} - \frac{\hat{\boldsymbol{\tau}}}{|\hat{\boldsymbol{\tau}}|}\right] \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\
&= \int_0^t s^{-1} \left\| \min\left(\frac{\tilde\sigma(\theta + s\,\delta\theta)}{|\hat{\boldsymbol{\tau}}|} - 1,\ 0\right) \left[\boldsymbol{\tau}^s - \hat{\boldsymbol{\tau}}\right] \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\
A_4 &:= \int_0^t \left\| \mathcal{D}_s \operatorname{Flow}(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) - \operatorname{Flow}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi.
\end{aligned}$$

For A_1 we exploit the Lipschitz continuity of the mapping $\min(\cdot, 0)$, the yield function $\tilde\sigma$ and the thermal strain $\boldsymbol{t}$. For A_2 and A_3 we make also use of the Lipschitz continuity of the thermal strain $\boldsymbol{t}$ and

$$-1 < \min\left(\frac{\tilde\sigma(\theta)}{|\boldsymbol{\tau}|} - 1,\ 0\right) \le 0 \qquad \text{for all } \boldsymbol{\tau} \in \mathbb{R}^{3\times 3}_{\text{sym}} \text{ and } \theta \in \mathbb{R}.$$

It follows that

$$\begin{aligned}
(4.13)\quad &\|\mathcal{D}_s \mathcal{G}^{\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \boldsymbol{\delta p}(t)\|_{\boldsymbol{L}^p(\Omega)} \\
&\le \epsilon^{-1} C \int_0^t \left\| \mathcal{D}_s \mathcal{G}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \left(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta\right) \right\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega) \times \boldsymbol{L}^p(\Omega) \times L^p(\Omega)} \mathrm{d}\xi
\end{aligned}$$

$$+ \int_0^t \|\mathcal{D}_s \operatorname{Flow}(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \| \\ - \operatorname{Flow}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta)_{\boldsymbol{L}^p(\Omega)} \, \mathrm{d}\xi.$$

In addition we consider again the difference quotient of (4.8) and (0.3), subtract (4.3) and estimate as above

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{D}_s \mathcal{G}^{\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \dot{\boldsymbol{\delta p}}(t) \right\|_{\boldsymbol{L}^p(\Omega)} \\ \leq \epsilon^{-1} C \, \left\| \mathcal{D}_s \mathcal{G}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta\big)(t) \right\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega) \times \boldsymbol{L}^p(\Omega) \times L^p(\Omega)} \\ + \|\mathcal{D}_s \operatorname{Flow}(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta)(t) \| \\ - \operatorname{Flow}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta)(t)_{\boldsymbol{L}^p(\Omega)}. \tag{4.14}$$

Combination of balance of momentum and plastic flow rule. Now we can add (4.11) and (4.13) and obtain with the Gronwall lemma

$$\left\| \mathcal{D}_s \mathcal{G}^{\boldsymbol{u},\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}\big)(t) \right\|_{\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega) \times \boldsymbol{L}^p(\Omega)} \\ \leq C(\epsilon, \gamma, T) \int_0^t \left\| \mathcal{D}_s \mathcal{G}^{\theta}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \delta\theta \right\|_{L^p(\Omega)} \mathrm{d}\xi \\ + C(\epsilon, \gamma, T) \int_0^t \big\| \mathcal{D}_s \operatorname{Flow}(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \\ - \operatorname{Flow}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \big\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi \\ + C(\epsilon, \gamma, T) \int_0^t \left\| \mathcal{D}_s \boldsymbol{t}(\theta; \delta\theta) - \boldsymbol{t}'(\theta) \, \delta\theta \right\|_{\boldsymbol{L}^p(\Omega)} \mathrm{d}\xi. \tag{4.15}$$

Owing to the convexity of $z \mapsto z^q$ for $z \geq 0$, and using (4.12) and (4.14), this results in

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{D}_s \mathcal{G}^{\boldsymbol{u},\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \big(\dot{\boldsymbol{\delta u}}, \dot{\boldsymbol{\delta p}}\big)(t) \right\|_{L^q(0,t;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega)) \times L^q(0,t;\boldsymbol{L}^p(\Omega))} \\ \leq C(\gamma, \epsilon, T) \, \left\| \mathcal{D}_s \mathcal{G}^{\theta}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \delta\theta \right\|_{L^q(0,t;L^p(\Omega))} \\ + C(\gamma, \epsilon, T) \, \left\| \mathcal{D}_s \boldsymbol{t}(\theta; \delta\theta) - \boldsymbol{t}'(\theta) \, \delta\theta \right\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))} \\ + C(\epsilon, \gamma, T) \, \big\| \mathcal{D}_s \operatorname{Flow}(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \\ - \operatorname{Flow}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \big\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))}. \tag{4.16}$$

Heat equation. We apply the embedding, cf. lemma 2.8,

$$W_0^{1,\frac{q}{2}}(0, T; W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0, T; W^{1,v(p)}(\Omega)) \hookrightarrow\hookrightarrow C([0, T]; L^p(\Omega)),$$

and the maximal parabolic regularity result assumption 1.6 (3) to the difference quotient of (4.10) and (0.5), and subtract (4.5). We also use $\left[\mathcal{D}_s \mathcal{G}^{\theta}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \delta\theta\right](0) = 0$ and observe the following chain of inequalities

$$\left\| \mathcal{D}_s \mathcal{G}^{\theta}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r)(t) - \delta\theta(t) \right\|_{L^p(\Omega)}$$

$$\le C \left\| \mathcal{D}_s \mathcal{G}^\theta(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \delta\theta \right\|_{W_0^{1,\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega)) \cap L^{\frac{q}{2}}(0,t;W^{1,v(p)}(\Omega))}$$

$$\le C \left\| \frac{f^s - f}{s} - \delta f \right\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))}, \tag{4.17}$$

where $f, f^s, \delta f \in L^{\frac{q}{2}}(0,T;W_\diamond^{-1,v(p)}(\Omega))$ are defined as the right hand sides of the corresponding heat equations (4.10), (0.5) and (4.5). It remains to bound the right hand side of (4.17) in a suitable way to exploit Gronwall's lemma.

We continue by adding zero terms as in the steps before and use the embedding $L^{\frac{p}{2}}(\Omega) \hookrightarrow W_\diamond^{-1,v(p)}(\Omega)$, cf. remark 1.5. For brevity, we define in the same way as we defined $\hat{\boldsymbol{\tau}}$ above the mapping $\hat{\boldsymbol{\sigma}} := \boldsymbol{\sigma}(\boldsymbol{u} + s\,\boldsymbol{\delta u}, \boldsymbol{p} + s\,\boldsymbol{\delta p}, \theta + s\,\delta\theta)$ and similarly for the backstress $\hat{\boldsymbol{\chi}}$. We estimate

$$\left\| \frac{f^s - f}{s} - \delta f \right\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))} \le \gamma\,(B_1 + B_2 + B_3) + B_4 + \ldots + B_9,$$

where the individual terms are given by

$$\begin{aligned}
B_1 &:= s^{-1} \left\| \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_2 &:= s^{-1} \left\| \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) : \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_3 &:= \left\| \mathcal{D}_s \operatorname{Heat}_1(\dot{\boldsymbol{u}}; \dot{\boldsymbol{\delta u}}) - \operatorname{Heat}_1{}'(\dot{\boldsymbol{u}}; \dot{\boldsymbol{\delta u}}) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_4 &:= s^{-1} \left\| (\boldsymbol{\sigma}^s + \boldsymbol{\chi}^s) : \dot{\boldsymbol{p}}^s - (\boldsymbol{\sigma}^s + \boldsymbol{\chi}^s) : (\dot{\boldsymbol{p}} + s\,\dot{\boldsymbol{\delta p}}) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_5 &:= s^{-1} \left\| (\boldsymbol{\sigma}^s + \boldsymbol{\chi}^s) : (\dot{\boldsymbol{p}} + s\,\dot{\boldsymbol{\delta p}}) - (\hat{\boldsymbol{\sigma}} + \hat{\boldsymbol{\chi}}) : (\dot{\boldsymbol{p}} + s\,\dot{\boldsymbol{\delta p}}) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_6 &:= \left\| \mathcal{D}_s \operatorname{Heat}_2(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) - \operatorname{Heat}_2{}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_7 &:= s^{-1} \left\| \theta^s \boldsymbol{t}'(\theta^s) : \left[\mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^s) - \dot{\boldsymbol{p}}^s) - \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) \right] - (\dot{\boldsymbol{p}} + s\,\dot{\boldsymbol{\delta p}})) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_8 &:= s^{-1} \Big\| \theta^s \boldsymbol{t}'(\theta^s) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) - (\dot{\boldsymbol{p}} + s\,\dot{\boldsymbol{\delta p}})) \\
&\qquad - (\theta + s\,\delta\theta)\, \boldsymbol{t}'(\theta + s\,\delta\theta) : \mathbb{C}(\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}} + s\,\dot{\boldsymbol{\delta u}}) - (\dot{\boldsymbol{p}} + s\,\dot{\boldsymbol{\delta p}})) \Big\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))} \\
B_9 &:= \left\| \mathcal{D}_s \operatorname{Heat}_3(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) - \operatorname{Heat}_3{}'(\boldsymbol{u}, \boldsymbol{p}, \theta; \boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta) \right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}.
\end{aligned}$$

Note that the set $\{(\boldsymbol{u}^s, \boldsymbol{p}^s, \boldsymbol{\sigma}^s, \boldsymbol{\chi}^s, \theta^s)\}_{s\in[0,1]}$ is bounded independently of s since the solution operator for (0.1)–(0.5) is bounded according to lemma 2.7. Therefore we can estimate the individual terms (compare also to the proof of proposition 3.1) easily exploiting the Lipschitz properties of $\theta \mapsto \boldsymbol{t}(\theta)$ and $\theta \mapsto \theta\,\boldsymbol{t}'(\theta)$, see assumption 1.1, and obtain

$$\begin{aligned}
&\left\| \frac{f^s(t) - f(t)}{s} - \delta f(t) \right\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))} \\
&\quad \le C \left\| \mathcal{D}_s \mathcal{G}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}, \delta\theta \big) \right\|_{W^{1,q}(0,t;\boldsymbol{W}_{\mathfrak{D}}^{1,p}(\Omega)) \times W^{1,q}(0,t;\boldsymbol{L}^p(\Omega)) \times L^q(0,t;L^p(\Omega))}
\end{aligned} \tag{4.18}$$

$$+ \gamma \left\|\mathcal{D}_s \operatorname{Heat}_1(\boldsymbol{u};\boldsymbol{\delta u}) - \operatorname{Heat}_1{}'(\boldsymbol{u};\boldsymbol{\delta u})\right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}$$
$$+ \left\|\mathcal{D}_s \operatorname{Heat}_2(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) - \operatorname{Heat}_2{}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}$$
$$+ \left\|\mathcal{D}_s \operatorname{Heat}_3(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) - \operatorname{Heat}_3{}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}.$$

Recall that the constants C given above depend on $(\boldsymbol{\ell}, r)$ and on $(\boldsymbol{\delta\ell}, \delta r)$ but they are independent of s. Together with (4.15), (4.16) and the maximal parabolic regularity property (4.17), inequality (4.18) results in

$$\begin{aligned}
&\left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r)(t) - \theta(t)\right\|_{L^p(\Omega)} \le \left\|\frac{f^s(t)-f(t)}{s} - \delta f(t)\right\|_{L^{\frac{q}{2}}(0,t;W_\diamond^{-1,v(p)}(\Omega))}\\
&\quad\le C(\gamma,\epsilon,T)\ \left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|_{L^q(0,t;L^p(\Omega))}\\
&\quad+ C(\gamma,\epsilon,T)\ \left\|\mathcal{D}_s\boldsymbol{t}(\theta;\delta\theta) - \boldsymbol{t}'(\theta)\,\delta\theta\right\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))}\\
&\quad+ C(\epsilon,\gamma,T)\ \Big\|\mathcal{D}_s \operatorname{Flow}(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\\
&\qquad\qquad\qquad - \operatorname{Flow}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\Big\|_{L^q(0,t;\boldsymbol{L}^p(\Omega))}\\
&\quad+ \gamma \left\|\mathcal{D}_s \operatorname{Heat}_1(\boldsymbol{u};\boldsymbol{\delta u}) - \operatorname{Heat}_1{}'(\boldsymbol{u};\boldsymbol{\delta u})\right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}\\
&\quad+ \left\|\mathcal{D}_s \operatorname{Heat}_2(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) - \operatorname{Heat}_2{}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}\\
&\quad+ \left\|\mathcal{D}_s \operatorname{Heat}_3(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta) - \operatorname{Heat}_3{}'(\boldsymbol{u},\boldsymbol{p},\theta;\boldsymbol{\delta u},\boldsymbol{\delta p},\delta\theta)\right\|_{L^{\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))}\\
&\quad=: C\ \left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|_{L^q(0,t;L^p(\Omega))} + D_s(t)
\end{aligned}$$

and we obtain by the convexity of $z \mapsto z^q$ for $z \ge 0$ the inequality

$$\left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r)(t) - \theta(t)\right\|^q_{L^p(\Omega)} \le C\int_0^t \left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|^q_{L^p(\Omega)}\,\mathrm{d}\xi + C\,D_s(t)^q.$$

Now we can again use the Gronwall lemma to get

$$\left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|_{L^\infty(0,T;L^p(\Omega))} \le C(T)\,D_s(T). \tag{4.19}$$

In addition we obtain from inequality (4.17) and (4.18)

$$\begin{aligned}
&\left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|_{W_0^{1,\frac{q}{2}}(0,t;L^{\frac{p}{2}}(\Omega))\cap L^{\frac{q}{2}}(0,t;W^{1,v(p)}(\Omega))}\\
&\quad\le C\ \left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|_{L^q(0,t;L^p(\Omega))} + D_s(t).
\end{aligned}\tag{4.20}$$

Putting everything together. It remains to take the limit for $s \to 0$. We start with (4.19) and obtain

$$\lim_{s\downarrow 0}\left\|\mathcal{D}_s\mathcal{G}^\theta(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r) - \delta\theta\right\|_{L^\infty(0,T;L^p(\Omega))} \le \lim_{s\downarrow 0} C(T)\,D_s(T) = 0$$

using lemma 4.4. Therefore we infer with (4.20) that

$$\lim_{s\downarrow 0} \left\| \mathcal{D}_s \mathcal{G}^\theta(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \delta\theta \right\|_{W_0^{1,\frac{q}{2}}(0,T;L^{\frac{p}{2}}(\Omega))\cap L^{\frac{q}{2}}(0,T;W^{1,v(p)}(\Omega))} = 0.$$

Furthermore we end up using (4.15) and (4.16) with

$$\lim_{s\downarrow 0} \left\| \mathcal{D}_s \mathcal{G}^{\boldsymbol{u},\boldsymbol{p}}(\boldsymbol{\ell}, r; \boldsymbol{\delta\ell}, \delta r) - \big(\boldsymbol{\delta u}, \boldsymbol{\delta p}\big) \right\|_{W^{1,q}(0,T;\boldsymbol{W}^{1,p}_{\mathfrak{D}}(\Omega))\times W^{1,q}(0,T;\boldsymbol{L}^p(\Omega))} = 0.$$

This shows the assertion. □

Having theorem 4.2 at hand we immediately obtain the following corollary.

Corollary 4.6 (Hadamard differentiability of the solution map)**.** *Under the assumptions of theorem 2.6, the solution map defined in* (2.5) *is Hadamard differentiable and its Hadamard derivative coincides with the directional derivative given in theorem 4.2.*

Proof. Since the solution map $\mathcal{G}$ is directionally differentiable by theorem 4.2 and locally Lipschitz continuous by proposition 3.1, we can apply lemma 2.9, which shows the assertion. □

Remark 4.7 (Consequences of differentiability results)**.**

(1) It is straightforward to obtain from the representation of the directional derivative of $\mathcal{G}$ as the solution of the system (4.1)–(4.5) a sufficient condition for Gâteaux differentiability of the solution map since the only nondifferentiable term is the second term on the right hand side of the linearized viscoplastic flow rule (4.3). This observation shows that if the control $(\boldsymbol{\ell}, r) \in L^q(0,T;\boldsymbol{W}^{-1,p}_{\mathfrak{D}}(\Omega)) \times L^{\frac{q}{2}}(0,T;W^{-1,v(p)}_{\diamond}(\Omega))$ fulfills

$$\lambda\Big(\big\{(t,\boldsymbol{x}) \in [0,T]\times\Omega : \phi(\boldsymbol{u},\boldsymbol{p},\theta)(t,\boldsymbol{x}) = 0 \text{ and } \exists(\boldsymbol{\delta\ell},\delta r) \text{ such that } \phi'(\boldsymbol{u},\boldsymbol{p},\theta;\mathcal{G}'(\boldsymbol{\ell},r;\boldsymbol{\delta\ell},\delta r))(t,\boldsymbol{x}) < 0\big\}\Big) = 0 \tag{4.21}$$

then the solution map $\mathcal{G}$ is Gâteaux differentiable in $(\boldsymbol{\ell}, r)$. Here λ represents the Lebesgue measure, $(\boldsymbol{u},\boldsymbol{p},\theta) := \mathcal{G}(\boldsymbol{\ell}, r)$, $\phi(\boldsymbol{u},\boldsymbol{p},\theta) := \tilde{\sigma}(\theta) - \big|[\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{p},\theta) + \boldsymbol{\chi}(\boldsymbol{u},\boldsymbol{p},\theta)]^D\big|$ and $\boldsymbol{\sigma}(\cdot)$ and $\boldsymbol{\chi}(\cdot)$ are defined by the algebraic equations (0.1) and (0.2), respectively.

We remark that it is an open question whether (4.21) is also a necessary condition for Gâteaux differentiability.

(2) In the context of an optimization problem where the thermoplastic system appears as a constraint, we can exploit the Hadamard differentiability of the solution map in order to derive first order necessary optimality conditions in primal form using standard techniques including the notion of the tangent cone to the set of feasible controls.

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