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FEEDFORWARD OPTIMAL CONTROL FOR PRECISE DISPLACEMENT OF A RIGID BODY: MINIMAL ELECTRICAL ENERGY

ILYA IOSLOVICH, PER-OLOF GUTMAN, AND RAPHAEL LINKER

Abstract. The problem of optimal control for the precise movement of a rigid body with state and control constraints is considered. An important criterion for an optimal feedforward trajectory solution is the electrical energy consumption. This criterion is significant from the economical and technological points of view in the electronic industry and in industrial automation. The structure of the solutions is found and investigated for different cases. Algorithmic solutions are provided.

1. Introduction

An important criterion for an optimal feedforward trajectory solution is the electrical energy consumption, whose simple characterization is the copper loss, expressed as the quadratic function $J = \int u^2/2dt$ where u is the driving force, proportional to the electrical current in the motor. This criterion is significant for factory automation and industrial electronics. Note that motion control systems consume about 65% of the electricity used in industry, [9]. Another important reason to minimize electrical energy in industrial electronics is to reduce the temperature around the wafer and other electronic hardware. The theoretical description of the solution for the generic case (case 2) was presented in [4]. The solution for an alternative objective, kinetic energy, was presented in [5], and the solution for the minimal time criterion without friction was presented in [3], and with friction, respectively, in [1].

The generic solution is a solution where all the state constraints (velocity, upper and lower driving force) are active at some points or intervals in time. It consists of 7 time intervals, namely three with increasing velocity, one with constant velocity, and three with decreasing velocity. During the intervals no. 3 and 5 the control jerk (derivative of the driving force) is decreasing, on the intervals 1 and 7 the jerk is increasing, and on the intervals $2, 4$, and 6 jerk is zero. The plot of the velocity for this case with all constraints occasionally active for the minimal electric energy criterion is compared with the plot for the minimal time in Fig. 1.

If some of constraints are non-active, then the corresponding intervals disappear, and there are other cases that are considered below.

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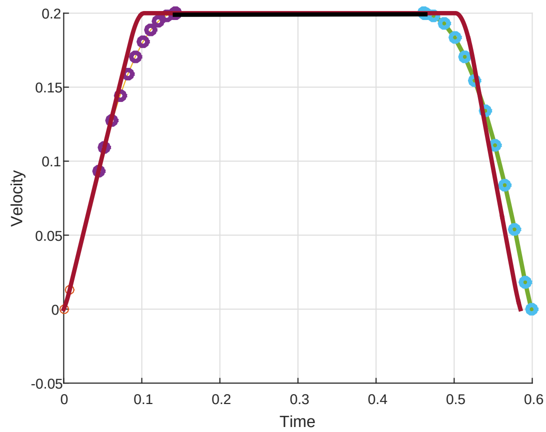

FIGURE 1. Velocities for minimal electrical energy and time optimal (red)

The numerical values of the parameters used for testing the algorithms, and generating the figures, are found in Table 1. The Table 2 summarizes definition of cases

TABLE 1. Parameter values for optimal profile generation for position control of the experimental X-Y stage

Parameter	Value	units	
$\bar{j}$	1625	N/s	jerk
$\bar{u}$	32.5	N	driving force max
$\bar{v}$	0.2	m/s	velocity max
m	13	kg	moving mass of the body
k	4	N	Coulomb friction coefficient
c	5.4	N·s/m	viscous friction coefficient
x_0	0	m	initial position
x_f	0.1	m	final position
k_1	20	N	stiction threshold = initial value of the force
u_f	-20	N	final value of the force
t_f	0.6	s	final time

and sub-cases.

1.1. **Fixing the final time.** In general the electrical energy consumed will be less if the fixed final time is larger. However the possible final time has a lower bound, and can be postulated to have an upper bound. The lower bound is the minimal time found from the solution of the corresponding minimal time problem. The upper bound is set to the time which has a solution with a marginal structure, i.e. some of the intervals become zero or equivalently some of the constraints cease to be active.

TABLE 2. Definition of cases and sub-cases. 1- active constraint, 0 inactive constraint, $\bar{v}$ = upper velocity, $\bar{u}$ = upper force constraint, $-\bar{u}$ = lower force constraint. Note that the lower velocity constraint is zero and is active at the initial and final time points.

v	$\bar{u}$	$-\bar{u}$	case/subcase	Number of intervals
0	0	0	1.2	2
0	0	0	1.3	3
1	1	1	2	7
1	0	0	3.1	4
1	0	0	3.2	5
0	1	1	4	6
0	1	0	5.1	3
0	1	0	5.2	4
1	1	0	6.1	5
1	1	0	6.2	6

2. Statement of the problem

The dynamic equations have the form:

$$m\frac{dv}{dt} = u(t) - k \cdot sign\,(v(t)) - c \cdot v(t) \tag{2.1}$$

$$\frac{dx}{dt} = v(t) \tag{2.2}$$

$$\frac{du}{dt} = j(t) \tag{2.3}$$

Here m [kg] is the mass of the plant, $x(t)$ [m] is the position, $u(t)$ [N] is the driving force proportional to the electrical current in the motor, $v(t)$ [m/sec] is the velocity, $j(t)$ [N/sec] is the driving force time derivative (called jerk for brevity), k [N] is the Coulomb friction, and c [N·sec/m] is the viscous friction coefficient. Note that we consider only non-negative velocities, so the function *sign* will be omitted below. $x(t)$, $v(t)$ and $u(t)$ are the state variables, and $j(t)$ is the control variable.

The following constraints are taken into account:

$$0 \leq v(t) \leq \bar{v}\ [m/sec] \tag{2.4}$$

$$-\bar{u} \leq u(t) \leq \bar{u}\ [N] \tag{2.5}$$

$$-\bar{j} \leq j(t) \leq \bar{j}\ [N/sec] \tag{2.6}$$

where $\bar{v}$, $\bar{u}$ and $\bar{j}$ are constants.

We consider in this section only the generic case (case 2) when all the state constraints are active at some interval(s) on the optimal trajectory. However this case is the most important because other cases can be obtained from this solution when parts of the trajectory vanish corresponding to constraints that cease to be active.

The minimal electrical energy problem can be formulated as follows. Starting from the initial conditions

$$x(0) = x_0,\ v(0) = 0,\ u(0) = u_0, \tag{2.7}$$

reach the required final point with final conditions

$$x(t_f) = x_f,\ v(t_f) = 0,\ u(t_f) = u_f \tag{2.8}$$

within the given time t_f [sec] which exceeds the corresponding minimal time value, and minimize the cost function

$$J = \int_0^{t_f} \frac{u^2}{2} dt \to \min . \tag{2.9}$$

Here u_f is a force whose absolute value does not exceed the minimal threshold k_1 (Table 1) of the break-away force (stiction). The value u_0 is equal to this threshold, and we shall show below that $u_f = -k_1$ is obtained in the solution. We also assume that $k \leq k_1$ such that when $u > k_1$ it holds that $dv/dt > 0$ when $v = 0$.

3. Optimization and analysis

According to the Pontryagin Maximum principle (PMP), [8], the *Hamiltonian* must be formed and then maximized with respect to the control variable. The state constraints with corresponding Lagrange multipliers should be subtracted from the Hamiltonian, to yield the *augmented Hamiltonian*, see e.g. [7], [2]. Here the augmented Hamiltonian has the form

$$\begin{aligned} H &= p_v \frac{(u - k - cv)}{m} + p_x v + p_u j - \\ &- \bar{\lambda}_v (v - \bar{v}) - \underline{\lambda}_v (-v) - \\ &- \bar{\lambda}_u (u - \bar{u}) - \\ &- \underline{\lambda}_u (-u + \bar{u}) - u^2/2. \end{aligned} \tag{3.1}$$

The non-negative Lagrange multipliers $\bar{\lambda}_v, \underline{\lambda}_v, \bar{\lambda}_u, \underline{\lambda}_u$ can be non-zero only when the corresponding state constraint is an equality, i. e. when the constraint is active and the trajectory follows the constraint. Upper (lower) bars in the multiplier symbol denote multipliers related to the upper (lower) bounds.

The costate equations, [8], are

$$\begin{aligned} \tfrac{dp_x}{dt} &= -\frac{\partial H}{\partial x} = 0, \\ \tfrac{dp_v}{dt} &= -\frac{\partial H}{\partial v} = -p_x + c p_v/m + \\ &+ \bar{\lambda}_v - \underline{\lambda}_v, \\ \tfrac{dp_u}{dt} &= -\frac{\partial H}{\partial u} = -p_v/m + u + \bar{\lambda}_u - \underline{\lambda}_u. \end{aligned} \tag{3.2}$$

It follows that p_x is constant.

The condition of maximizing the augmented *Hamiltonian* (3.1) with respect to the control gives

$$j = \bar{j} \cdot \mathrm{sign}(p_u),\ \text{for } p_u \neq 0; j \in [-\bar{j}, \bar{j}] \text{ for } p_u = 0. \tag{3.3}$$

When $p_u = 0$, $dp_u/dt = 0$ there is a singular arc, where the control j cannot be determined by the maximization of the augmented Hamiltonian. The most important step is to make an assumption on the structure of the optimal solution.

Our initial assumption, that turns out to be correct (sic!), is as follows. This structure is based on the qualitative analysis of the state and costate equations.

4. Structure of optimal solution - generic case 2

(1) Interval 1. On the time interval $[0, t_1]$ we have $p_x > 0,\ p_v > 0,\ p_u > 0,\ j = \bar{j}$, u increases to $u = \bar{u}$, x and v increase, $dp_u/dt < 0$, $dp_v/dt < 0$, costates p_v and p_u decrease.

(2) Interval 2. At the time moment $t = t_1$ we have $u = \bar{u},\ p_u = 0,\ p_v/m > \bar{u},\ dp_v/dt < 0$. Then we have $\bar{\lambda}_u > 0,\ dp_u/dt = 0$. On the interval $[t_1, t_{21}]$ we have $p_u = 0$, and, accordingly, $j = 0$. Costate p_v decreases, x and v increase.

(3) Interval 3. At the moment $t = t_{21}$ we have $p_v/m = \bar{u}$, and then on the interval $[t_{21}, t_2]$ we have $p_v/m = u,\ j = (dp_v/dt)/m < 0$, and v decreases. $p_u = 0,\ dp_u/dt = 0,\ \bar{\lambda}_u = 0$, and u decreases along the new singular arc until the moment $t = t_2$ when we get $u = u_s,\ v = \bar{v},\ dv/dt = 0,\ p_v = u_s$. Here we use the notation $u_s = k + c \cdot v$.

(4) Interval 4. On the interval $[t_2, t_3]$ we have $dp_v/dt = 0,\ \bar{\lambda}_v > 0,\ dp_u/dt = 0,\ p_u = 0,\ p_v/m = u_s,\ u = u_s,\ j = 0$. The velocity v follows its upper bound.

(5) Interval 5. On the interval $[t_3, t_{31}]$ we have $\bar{\lambda}_v = 0,\ dp_v/dt < 0,\ j = dp_v/dt/m < 0,\ dp_u/dt = 0,\ p_u = 0,\ u = p_v/m$. The trajectory again follows a singular arc. Here v, p_v and u are decreasing.

(6) Interval 6, On the interval $[t_{31}, t_4]$ we have a point $t = t_{31}$ where $u = -\bar{u}$. Then
$$dp_u/dt = -p_v/m - \bar{u} - \underline{\lambda}_u = 0,$$
$\underline{\lambda}_u > 0,\ p_u = 0,\ j = 0,\ dp_v/dt < 0$, p_v and v are decreasing.

(7) Interval 7. At the point $t = t_4$ the singular arc stops, and on the interval $[t_4, t_f]$ we have $p_u > 0,\ j = \bar{j},\ \underline{\lambda}_u = 0,\ dp_u/dt = -p_v/m + u > 0, dv/dt < 0, du/dt > 0$. The variables x, u, v attain their fixed final values at the fixed time $t = t_f$. Note that when j at $t = t_4$ switches to $j = \bar{j}$ we have $d^2p_u/dt^2 > 0$ and thus p_u from $p_u = 0$ increases to be $p_u > 0$.

For the singular arcs on the intervals $[t_{21}, t_2]$ and $[t_3, t_{31}]$ it should be noted that the augmented Hamiltonian has the form

$$H = H_0 + jH_1, \tag{4.1}$$

with $H_1 = p_u$. The second order necessary condition by Kelley, for a singular arc, [6], is

$$(-1)^q \frac{\partial}{\partial j} \frac{d^{2q} H_1}{dt^{2q}} \leq 0. \tag{4.2}$$

The parameter q is a so-called *degree of singularity*, a value of q in this formula should be increased until the control variable will appear implicitly in the resulting expression for $d^{2q}H_1/dt^{2q}$. We see here that $q = 1$ and in the Kelley condition we have
$$-[\partial\{d^2p_u/dt^2\}/\partial j] = -[\partial\{-dp_v/dt + j\}/\partial j] = -1 \leq 0.$$
Thus the Kelley condition is satisfied.

It is important however to note that along the singular arc the condition $-\bar{j} \leq dp_v/dt$ must be satisfied on the intervals $[t_{21}, t_2]$ and $[t_3, t_{31}]$ in order to keep the equality $j = dp_v/dt$. The opposite case means that the case 2 is not feasible for current initial and final data.

We see that similarly to the minimal time solution, [1], there are 7 different time intervals, but on the intervals 3 and 5 there are singular arcs with intermediate negative values of jerk control j.

4.1. **Final value of the driving force.** The final value of the driving force u_f must be such that the movement at the time $t = t_f$ when $v(t_f) = 0$ will switch from the slip mode to the stick mode. It means that $-k_1 \leq u_f \leq k_1$. thus potentially there are three options:
1. u_f is not fixed and $-k_1 \leq u_f \leq k_1$,
2. $u_f = k1$,
3. $u_f = -k_1$.
If the value u_f is not fixed (option 1) we shall have the transversality condition $p_u(t_f) = 0$. This is unfeasible if the interval 7 exists because at the interval 7 we have $p_u > 0$, $dp_u/dt > 0$.
However if the final interval 7 is missing and interval 6 is also missing (means $u > -\bar{u}$, then the option 1 is feasible and will correspond to subcases $x.1$. Note that in these subcases we should have at the end the transversality condition $p_u = 0$.
In case of the option 2 we have $u(t_f) = k_1$ and thus $dv/dt > 0$, because $v(t_f) = 0$, $k_1 > k$. Thus there is a small interval $[t, t_f]$ where $v(t) < 0$. This contradicts our assumption that we always have $v(t) \geq 0$.
We can formulate a *Theorem* (proved above):
The option 1 is possibly feasible only if the intervals $6, 7$ are missing.

5. Description of algorithms for different cases and subcases

We are using the original problem-oriented algorithms that are less computationally expensive then the shooting method. The programs for different cases and subcases have the main script that invokes the function to minimize residuals. Usually the matlab optimization function *fminsearch* is used. The names of scripts and functions for each case-subcase are shown in Table 3.

6. Case 1, subcases 1.2 and 1.3

6.1. **Subcase 1.2.** In subcase 1.2 we have no active constraints for state variables v, u and thus intervals $2, 4, 6$ vanished. At the interval 1 costate p_u is positive and decreased to zero at the end point of this interval denoted as t_1. From there the singular solution with $p_u = 0$ holds. Intervals $3, 4, 5$ merged and the intervals $2, 6, 7$ vanished. Starting from the moment t_1 the control j is negative with intermediate value. At the singular solution we have

$$\frac{dp_u}{dt} = -\frac{p_v}{m} + u = 0, \ \frac{p_v}{m} = u; \ \frac{dp_v}{dt} = -p_x + c * p_v/m. \tag{6.1}$$

TABLE 3. Names of scripts and functions for all cases and sub-cases

Case/subcase	Script	Function
1.2	mm_electro_12	mm_el_12
1.3	mm_electro_13	mm_el_13
2	mm_electro_2	mm_el_2
3.1	mm_electro_31	mm_el_31
3.2	mm_electro_32	mm_el_32
4	mm_electro_4	mm_el_4
5.1	mm_electro_51	mm_el_51
5.2	mm_electro_52	mm_el_52
6.1	mm_electro_61	mm_el_61
6.2	mm_electro_62	mm_el_62

From there it follows that

$$\frac{du}{dt} = (-p_x + c * u)/m \tag{6.2}$$

We use 3 inputs for the function mm_el_12: p_x, t_1, tfa, where tfa should be equal to the fixed final time value (denoted as tff. From $t = 0$ to $t = t_1$ the estate equations are integrated with $j = \bar{j}$ and we obtain $u(t_1) = u_1,\ x(t_1) = x_1, v(t_1) = v_1$. Then we integrate state equations from $t = t_1$ to $t = tfa$ using equation (6.2) for the singular arc and get final values $x(tfa) = x_2,\ v(tfa) = v_2,\ u(tfa) = u_2$. The residuals are $(tfa - tff)^2,\ (x_f - x2)^2,\ v_2^2$. They should be zero as result of optimization. If we obtain $u_2 > -k_1$ and state constraints are satisfied then this solution is optimal because it has no state constraints and less final constraints, otherwise we should try case 1.2. The value of the objective is shown in variable $u2tl$. The plot of the force u for $tff = 0.6,\ x_f = 0.1$ is shown in Fig. 2. and the plot for the corresponding velocity is shown in Fig.3

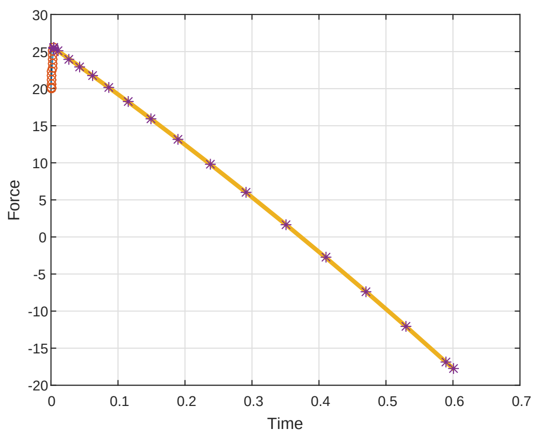

FIGURE 2. Driving force u for the case 1.2 with $tff = 0.6,\ xf = 0.1$

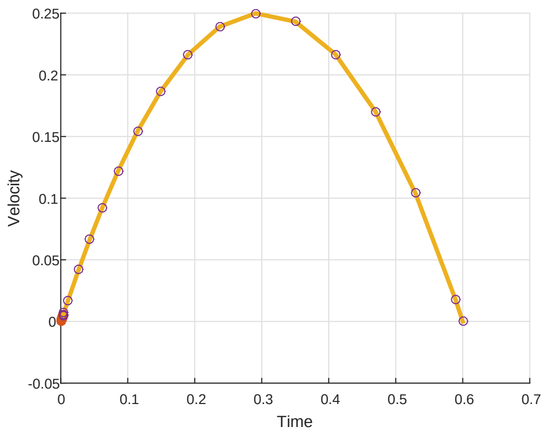

FIGURE 3. Velocity v for the case 1.2 with $tff = 0.6,\ xf = 0.1$

6.2. **Subcase 1.3.** In the subcase 1.3 we have additional final time interval with $j = \bar{j}$ when the driving force u must increase to the final value $u(tff) = -k_1$ in order to prevent the slide movement in the opposite direction after the stop. This subcase slightly differs from the subcase 1.2 and the function mm_el_13 has an additional $3rd$ input $t4$. This is a length of the final time interval. On the final time interval state equations are integrated with control $j = \bar{j}$ and the final values are denoted as $x3,\ v3,\ u3$. The minimized residuals are $(tff - tfa - t4)^2,\ (xf - x3)^2,\ -min(0, v2),\ v3^2, (u3 + k_1)^2$. Altogether there are 3 inputs: $px, tfa, t4$ and 3 time intervals: 1 with $j = \bar{j}$, 2 with singular arc and $j < 0$ and 3 with $j = \bar{j}$. The plot of the driving force u for $tff = 0.46,\ x_f = 0.1$ is shown in Fig. 4. and the plot for the corresponding velocity is shown in Fig.5

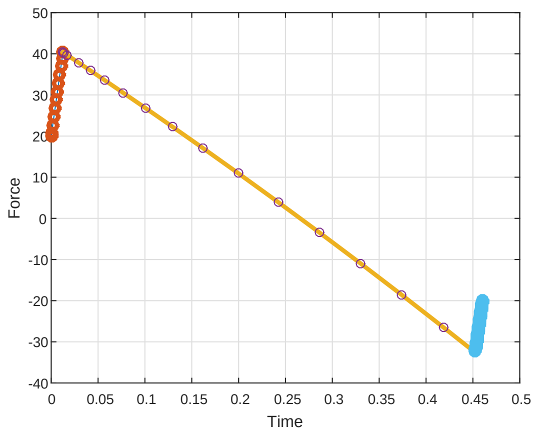

FIGURE 4. Driving force u for the case 1.3 with $tff = 0.46,\ xf = 0.1$

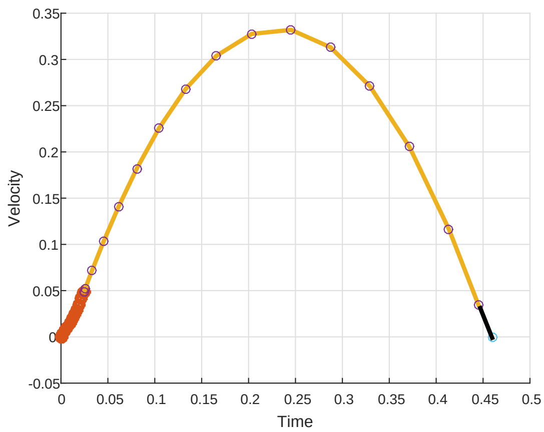

FIGURE 5. Velocity v for the case 1.3 with $tff = 0.46$, $xf = 0.1$

The value of the objective significantly depends of the fixed final time. The objective J for $tff = 0.42$ is equal to $J = 146.07\ N^2 s$, for $T = 0.46$ is equal 111.56, for $tff = 0.5$ we have $J = 88.03\ N^2 s$, for $tff = 0.55$ we have $J = 67.92\ N^2 s$ and for $tff = 0.6$ we have $J = 54.22\ N^2 s$. The plot for these values is shown in Fig.6

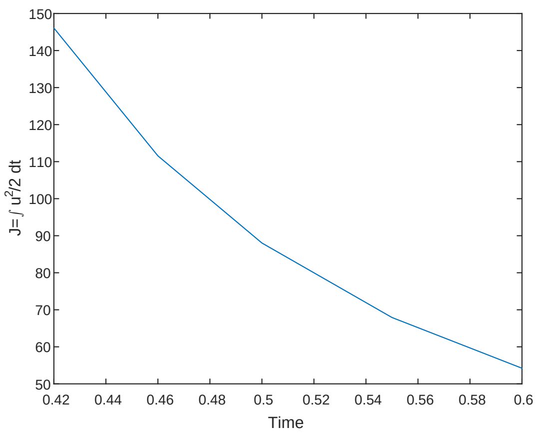

FIGURE 6. Objective $\int u^2/2dt$ vs final time, $xf = 0.1$ m

7. CASE 2

In this generic case we have 7 time intervals and singular arcs of different nature appeared on intervals 2 and 6. At the interval 2 we have $j = 0$ and $u = \bar{u}$, and at the interval 6 we have $j = 0$ and $u = -\bar{u}$, and also on the interval 4 where $j = 0$ and $v = \bar{v}$, $u = u_s$. Another type of the singular arcs appeared on intervals 3 and 5 where j has a negative intermediate value according to u that satisfies equation (6.2). The only unknown input value is parameter p_x that has to be found by minimization

of residuals in the function *mm_el2*. The residuals are: $(tfa - tff)^2$ where tfa is a calculated time of the process to the end point; $-min(0, t210)$ where $t210$ is a time length of the interval 2; $-min(0, t22)$ where $t22$ is a time length of the interval 3;$-min(0, t23)$ where $t23$ is a time length of the interval 4; $-min(0, t33)$ where $t33$ is a time length of the interval 5; $-min(0, t310)$ where $t310$ is a time length of the interval 6; $-min(0, v21)$ where $v21$ is a calculated value of the velocity at the beginning of the interval 3; $-min(0, v31)$ where $v31$ is a calculated value of the velocity at the end of the interval 5.
Note that for the case 2 we have final condition $u_f = -k1$ (option 3) because at the interval 7 we have $u = -\bar{u} < k1$, thus option 1 is unfeasible.
Our procedure uses the known structure of the solution.

Recall that p_x is constant. We know that $u(0) = u_0 = k1$, and $u(t_1) = \bar{u}$, thus we can easily find time point $t = t_1$ from the equation

$$t_1 = \frac{\bar{u} - u_0}{\bar{j}}.$$

Then the values $x(t_1), v(t_1)$ can be found from the state equations with $u = u_0 + t \cdot \bar{j}$. At the moment $t = t_{21}$ at the beginning of the interval 3 we have $p_v/m = \bar{u}$, and at the moment $t = t_2$ we have $p_v/m = u_s$, look at the description above. Note that the costate variable p_v is monotonously decreasing here, so we can use it as an independent variable for integration of the state equation for v backwards from $v = \bar{v}$ to $v(t_{21}$. The time interval $t22 = t_2 - t_{21}$ can be found by integration of the equation dt/dp_v, and thus also the value $v(t_{21})$ can be determined. Now the time interval $t210 = t_{21} - t_1$ can be found by integration of the state equation dv/dt with $u = \bar{u}$. Thus we have found t_{21}, t_2 and all the state variables at these points.

Next we should make similar calculations starting from the point $t = t_f$ backwards. The value of time interval $t_f - t_4$ is determined from the equation

$$t_f - t_4 = \frac{\bar{u} + u_f}{\bar{j}}.$$

Then we can find $v(t_4), (x_f - x_4)$ by integrating the state equations backwards with

$$u(t) = u_f - (t_f - t)\bar{j}$$

from the point $v(t_f) = 0,\ x(t_f) = x_f$.

At the point $t = t_{31}$ we have that $p_v/m = -\bar{u}$ and at the point $t = t_3$ we have $p_v/m = u_s$. Thus we can use p_v as an independent variable and integrate dt/dp_v to find the value of the time interval $t_{31} - t_3$. We can integrate dv/dp_v on the same interval with starting point $v(t_3) = \bar{v}$ to find $v(t_{31})$, and then $x(t_{31}) - x(t_3)$.

Now we know $x_3 - x_2$ and we can find the value of the interval $t_3 - t_2 = (x_3 - x_2)/\bar{v}$. We computed value of the unknown p_x by using the Matlab procedure *fminsearch* to minimize a weighted sum of residuals.
For different integrations there are used functions $dtpv1, dtpv1a, dtvp3, test_t2, t_65$ and also functions $testV12, testX12, testV65, testX65$ that were generated by the Matlab Symbolic Toolbox.
The plot of the driving force u for $tff = 0.59,\ x_f = 0.1$ is shown in Fig. 7, and the plot for the corresponding velocity is shown in Fig.8.

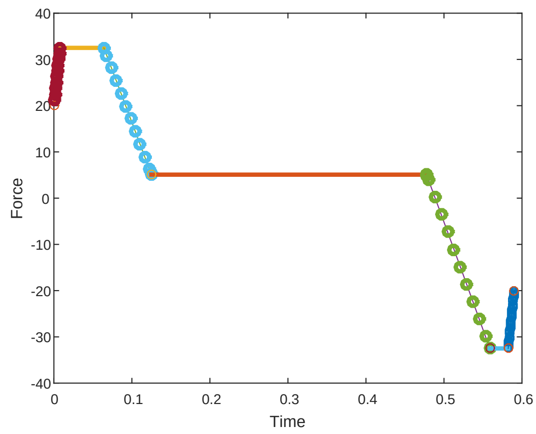

FIGURE 7. Driving force u for the case 2 with $tff = 0.59,\ xf = 0.1$

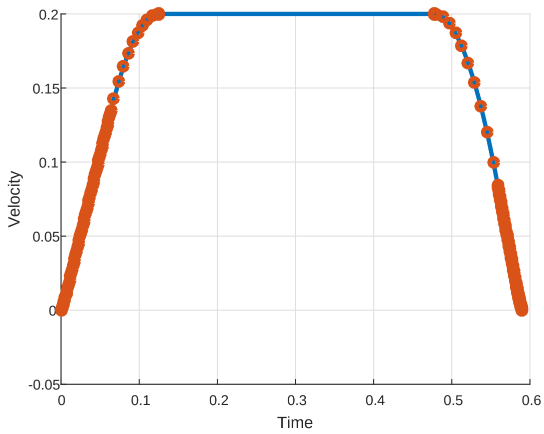

FIGURE 8. Velocity v for the case 2 with $tff = 0.59,\ xf = 0.1$

8. CASE 3, SUBCASES 3.1 AND 3.2

In case 3 we have the only active state constraint $v \le \bar{v}$. The intervals 2 and 6 related to the upper and lower bounds of u are missing. In subcase 3.1 also the interval 7 with $j = \bar{j}$ is missing.

8.1. **Subcase 3.1.** In this subcase we have 4 intervals: $1, 3, 4, 5$. For the interval 1 we have $j = \bar{j}$, for the interval 4 we have $v = \bar{v}$, $j = 0$, and for the intervals $3, 5$ we have $j < 0$ and the singular arc were the equation 6.2) holds. The function mm_el_31 is used to minimize the residuals. There are 3 input values: $p_x, t1, t310$ where p_x is a constant costate for x, $t1$ is a time length of the interval 1, and $t310$ is a time length of the interval 5. The values of the increments of a distance for intervals $1, 3, 5$ is calculated and then compared with xf. The difference is a distance length of the interval 4. More detailed description is as follows. First the state equation

are integrated on the interval 1 with $j = \bar{j}$ during time $t1$. Thus values $x1, v1, u1$ are found. On the interval 3 the costate p_v is changed from $u1 * m$ to the value $us * m$. Thus we have used p_v as the independent variable and integrate dt/dp_v in order to find $t22$ - the time length of the interval 3. Next we integrate state equations on the known length of the interval 3 using equation (6.2) and get values of $x2, v2$. Note that we must have $v2 = \bar{v}$. Note that in the programs we denote $vm = \bar{v}$. In this program (and also for the case 3.2) we denote $v21 = v2 - vm$ and this value should be zero. The value $v21^2$ is one of the residuals. In the program $\bar{v}$ is denoted as vm. On the interval 5 we integrate state equations with equation (6.2) and get $v3, u3, x3$ where $x3$ is an increment of the distance during interval 5. After that we can calculate $x23$ for the interval 4. The difference $v21 = vm - v2$ must be zero. he total time of the process is calculated and denoted as tfa. The residuals to be minimized are $(tfa - tff)^2; -min(0, t23), v3^2, v21^2$.
The plots for the driving force and velocity for $tff = 0.65$ and $xf = 0.1$ for subcase 3.1 are shown in Fig. 9 and Fig. 10 respectively.

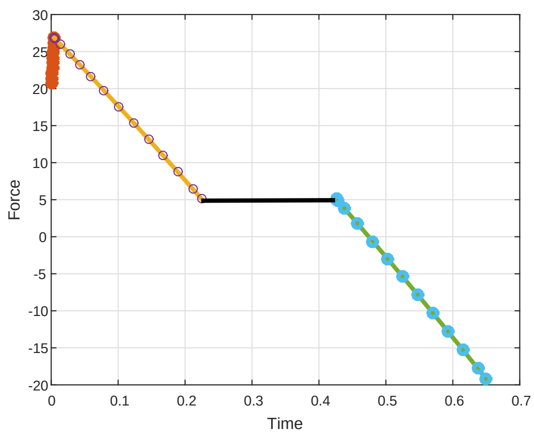

FIGURE 9. Driving force u for the case 3.1 with $tff = 0.65$, $xf = 0.1$

8.2. **Subcase 3.2.** For the subcase 3.2 we have 5 intervals, namely $1, 3, 4, 5, 7$. The optimization function $mmel\_32$ has 4 inputs, namely $p_x, t1, t310, t4$. We have to satisfy additional end condition $u4 + k11 = 0$. Here additional input $t4$ is a time length of the interval 7 with $j = \bar{j}$. We know all the initial values for the interval 7 and thus can integrate state equations with known control $j = \bar{j}$ without difficulty. Values $v4, u4$ are the correspondent values of the state variables at the end of the interval 7. The residuals to be minimized are $(tfa - tff)^2; -min(0, t23), -min(0, v3), v21^2, v4^2, (u4 - k11)^2$.
The plots for the driving force and velocity for $tff = 0.57$ and $xf = 0.1$ for subcase 3.2 are shown in Fig. 11 and Fig. 12 respectively.

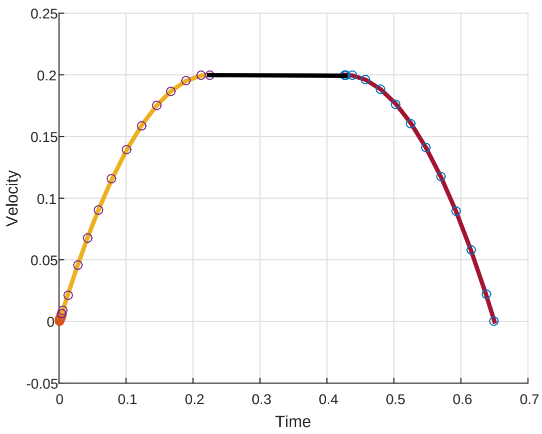

FIGURE 10. Velocity v for the case 3.1 with $tff = 0.65,\ xf = 0.1$

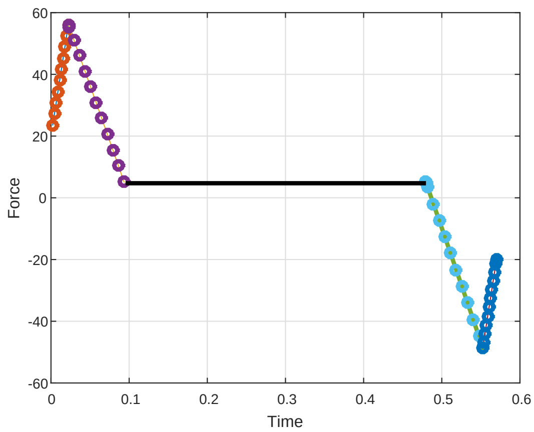

FIGURE 11. Driving force u for the case 3.2 with $tff = 0.57,\ xf = 0.1$

9. Case 4

The case 4 is rather similar to the case 2 but here the upper constraint for velocity is not active and thus the interval 4 is missing. Intervals 3 and 5 we consider separately though they are merged and the control j is continuous at the point of their junction where v has the maximum value. The algorithm is the same as for the case 2 but the maximal value of the velocity is not known and is added as an additional input to the function mm_el_4. Similarly to the parameter px it should be found in the process of minimization of residuals. Note that the value $t23$ is not a residual because the interval 4 is missing. The plots for the driving force and velocity for $tff = 0.42$ and $xf = 0.1$ for case 4 are shown in Fig. 13 and Fig. 14 respectively.

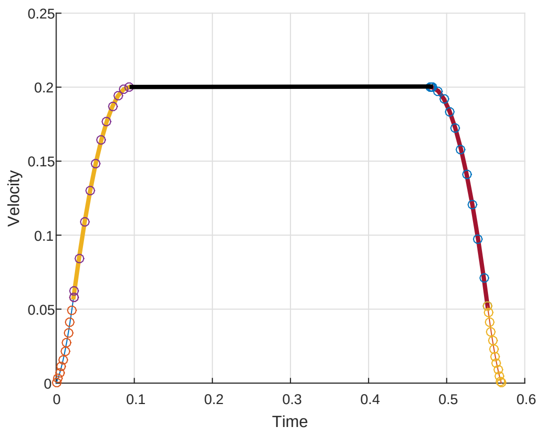

FIGURE 12. Velocity v for the case 3.2 with $tff = 0.57,\ xf = 0.1$

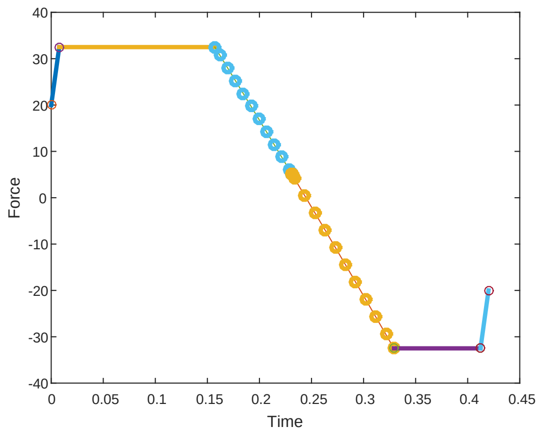

FIGURE 13. Driving force u for the case 4 with $tff = 0.42,\ xf = 0.1$

10. CASE 5, SUBCASES 5.1 AND 5.2

10.1. **Subcase 5.1.** In the subcase 5.1 we have only active state constraint for the upper limit of the driving force u and the final interval 7 with $j = \bar{j}$ is also missing. Accordingly the intervals $4, 6$ are missing and the interval 5 is merged with the interval 3. The maximal value of the velocity vm is unknown and is added to the inputs of the function $mm\_el\_ - 51$. We have 3 inputs: $px, vm, t310$, where $t310$ is a time length of the interval 5 and is added for the technical reasons. We have used the Matlab optimization function $fmincon$ for constrained optimization with nominal minimized function $mm\_el\_5\_f$ which is empty. The weighted equality constraints are the final velocity is zero, $v3 = 0$, the final distance is equal to xf, $xf - x2 - x3 = 0$, and the final time is reached, $tf = t3,\ t3 = t2 + t310$. The inequality constraint is $t210 > 0$ for the non-negativity of the time length of the

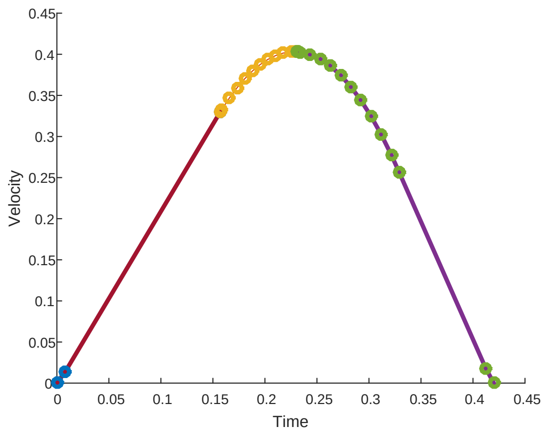

FIGURE 14. Velocity v for the case 4 with $tff = 0.42,\ xf = 0.1$

interval 2.
The plots for the driving force and velocity for $tff = 0.518$ and $xf = 0.1$ for subcase 5.1 are shown in Fig. 15 and Fig. 16 respectively.

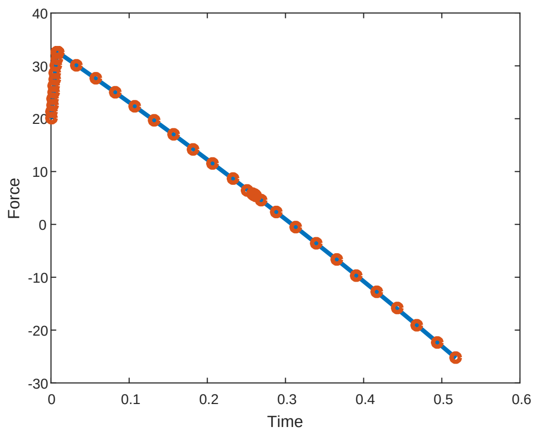

FIGURE 15. Driving force u for the case 5.1 with $tff = 0.518,\ xf = 0.1$

10.2. **Subcase 5.2.** The case 5.2 comparing to subcase 5.1 has an additional final time interval 7 with time length $t4$. The inputs to the function mm_el_52 are the same as for the function mm_el_51. The value $t4$ is determined as $tff - t3$. During the time $t4$ we have $j = \bar{j}$. The function mm_el_51 used Matlab optimization procedure $fminsearch$ to minimize residuals. The residuals are $v4^2, (xf - x4 - x3 - x2)^2, -min(0, t210), (tff - t3 - t4)^2, (k11 + u4)^2$. Here the value $k11$ is the program notation for the Karnopp threshold $k1$ and the last residual is needed to prevent the movement in the opposite direction at the final point.
The plots for the driving force and velocity for $tff = 0.48$ and $xf = 0.1$ for subcase

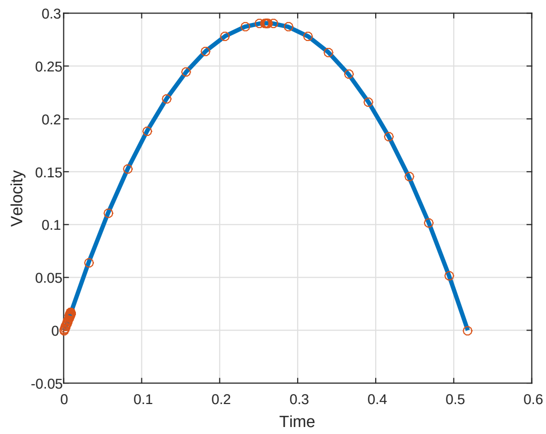

FIGURE 16. Velocity v for the case 5.1 with $tff = 0.518,\ xf = 0.1$

5.2 are shown in Fig. 17 and Fig. 18 respectively.

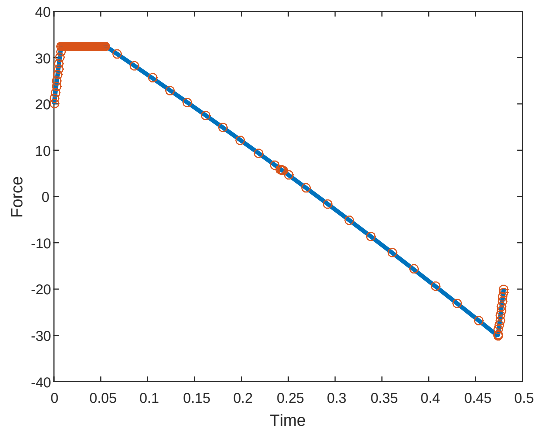

FIGURE 17. Driving force u for the case 5.2 with $tff = 0.48,\ xf = 0.1$

11. CASE 6, SUBCASES 6.1 AND 6.2

11.1. **Subcase** 6.1**.** In subcase 6.1 we have intervals $1, 2, 3, 4, 5$ of non-zero length and intervals $6, 7$ missing. The matlab function mm_el_61 has inputs $px, t310$. The values related to the intervals $1, 2, 3$ have to be found similarly as for the case 2, and the values related to the interval 5 have to be found by integration from the end of the interval 4 with the use of equation (6.2) with initial values $v = vm; u = us$. Thus the increment of the distance on the interval 5, namely $x3$, has to be found. Then the interval $x23$ related to the interval 4 have to be found as $x23 = xf - x2 - x3$, and the value $t23 = x23/vm$. The residuals are $v3^2$ - final value of the velocity;

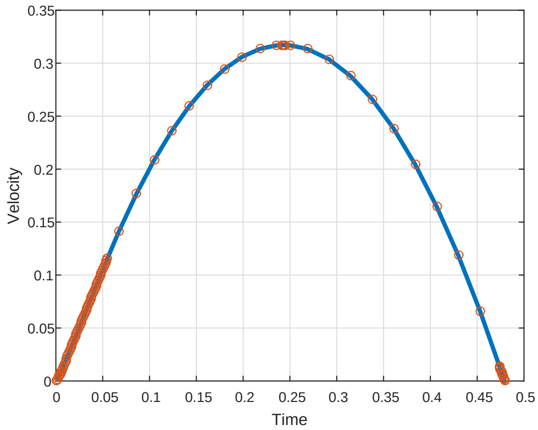

FIGURE 18. Velocity v for the case 5.2 with $tff = 0.48,\ xf = 0.1$

$-min(0, t210)$ - time length of the interval 2, $-min(0, x23)$ - length of the increment of the distance on the interval 4, and $(tff - t3)^2$ - the satisfaction of the equality constraint for the fixed final time. Here we have $t3 = t2 + t310 + t23$. The plots for the driving force and velocity for $tff = 0.58$ and $xf = 0.1$ for subcase 6.1 are shown in Fig. 19 and Fig. 20 respectively.

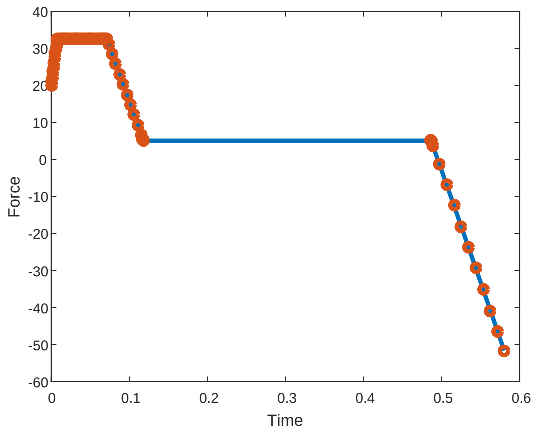

FIGURE 19. Driving force u for the case 6.1 with $tff = 0.58,\ xf = 0.1$

11.2. **Subcase** 6.2**.** The subcase 6.2 is very similar to the subcase 6.1 but it has additionally nonzero interval 7 with $j = \bar{j}$. Accordingly the function mm_el_62 has additional input $t4$. The residuals are $v4^2$ - the final value of the velocity, $-(min(0, t210)$ - the time length of the interval 2 must be non-negative, $(xf - x4 - x23 - x3 - x2)^2$ - the final distance xf must be achieved, $(tff - t3 - t4)^2$ - the final time must be equal to the fixed tff value, and $(k11 + u4)^2)$ - the final value of the driving force must be equal to the value of the Karnopp threshold with sign minus.

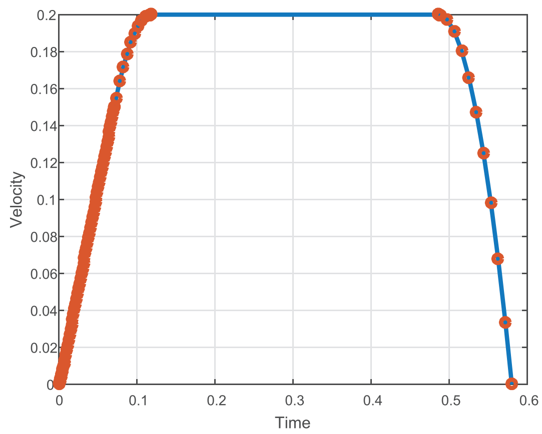

FIGURE 20. Velocity v for the case 6.1 with $tff = 0.58,\ xf = 0.1$

The plots for the driving force and velocity for $tff = 0.58$ and $xf = 0.1$ for subcase 6.2 are shown in Fig. 21 and Fig. 22 respectively.

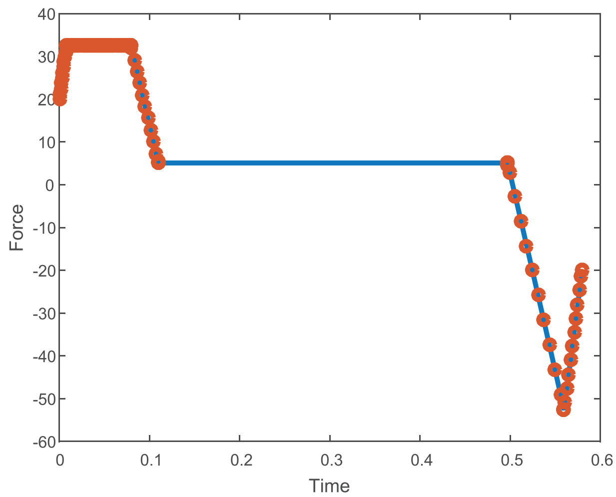

FIGURE 21. Driving force u for the case 6.2 with $tff = 0.58,\ xf = 0.1$

12. CONCLUSION

We have presented the optimal control solution for the precise rigid body displacement with Coulomb and viscous friction for all possible cases of the active and non-active constraints. The objective functional, namely minimal electrical energy consumed, was found to be sensitive to the final fixed time of the process. With combination of the minimal time solutions these results are useful for design of the suitable feedforward system. The original algorithms are less computationally expensive then the shooting method.

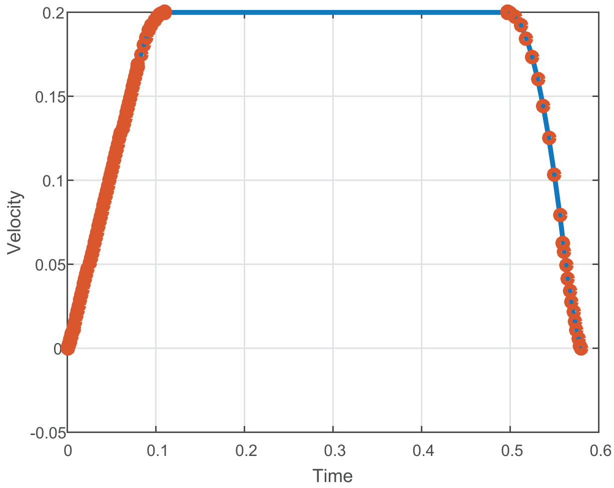

FIGURE 22. Velocity v for the case 6.2 with $tff = 0.58$, $xf = 0.1$

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References

[1] A. Berger and I. Ioslovich and P. O. Gutman, *Time optimal trajectory planning with feedforward and friction compensation*, in: American Control Conference (ACC 2015) proceedings, Chicago, IL, USA, 2015, pp. 4143–4148.

[2] R. Hartl, S. Sethi and R. Vickson, *A survey of the maximum principles for optimal control problems with state constraints*, SIAM Review **2** (1995), 181–218.

[3] I. Ioslovich and P. O. Gutman, *Time-optimal control of wafer stage positioning using simplified models*, Contemporary Mathematics **619** (2014), 99–107.

[4] I. Ioslovich,P. O. Gutman, R. Linker and S. Moshenberg, *Optimal rigid body precise displacement - minimization of electrical energy*, in: The 20th IFAC 2017 World Congress, Preprints, Toulouse, France, 2017, pp. 776–780.

[5] I. Ioslovich,P. O. Gutman and S. Moshenberg, *On energy-optimal and time-optimal precise displacement of rigid body with friction*, J. Optim. Theory Appl. **172** (2017), 466–480.

[6] J. Kelley, R. E. Kopp and H. G. Moyer, *Singular extremals*, in: Topics in optimization, G. Leitmann (ed), Academic press, NY, 1967, pp. 63–101.

[7] M. M. Khrustalev, *Necessary and sufficient dynamic programming conditions for optimal control problem with state constraints*, Lecture Notes in Control and Information Sciences, H.-J. Sebastian and K. Tammer (eds), Springer Berlin Heidelberg, 1990, pp. 311–320.

[8] L.S. Pontryagin and V. G. Boltyansky and R. Gamkrelidze and E. F. Mischenko, *Mathematical Theory of Optimal Processes*, Wiley-Interscience, 1962.

[9] E. Worrell, L. Bernstein, J. Roy and J. Harnisch, *Industrial energy and climate change mitigation*, Energy Efficiency **2** (2009), 109–123.

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