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NECESSARY CONDITIONS FOR SOLUTIONS OF SET OPTIMIZATION PROBLEMS WITH RESPECT TO VARIABLE DOMINATION STRUCTURES

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Abstract. In this paper, we introduce several set relations equipped with a variable domination structure, which is not necessarily given by a cone-valued map. We also derive relationships between the solution concepts for set optimization problems given by set approach and vector approach. The main part of this work is devoted to investigate necessary optimality conditions for solutions of set optimization problems based on set approach. These necessary optimality conditions are derived in terms of Mordukhovich's coderivative for optimal solutions of set-valued problems with respect to various set relations.

1. Introduction

Set optimization is a growing field of optimization theory with many applications in bilevel programming, economics, mathematical finance, health care and uncertain optimization (see [2, 4, 14] and references therein). There are three approaches to defining solution concepts in set optimization, namely the vector approach, the set approach and the lattice approach (see [14, Section 2.6]). The so-called vector approach is an interesting concept from the mathematical point of view and many authors have investigated it. Optimality conditions for set-valued optimization based on the Bouligand derivative and generalized differentiation objects by Mordukhovich are presented in the literature (see [6] and references therein).

In this paper, we consider set optimization problems with respect to (w.r.t.) variable domination structures. These optimization problems are very important and interesting from the mathematical as well as from the practical perspective because they have many applications in production theory, radio therapy treatment and welfare economics (see, for instance, [3, 7, 9, 14, 15, 16, 17, 19, 20]).
We study set-valued optimization problems with a set-valued objective map $F : X \rightrightarrows Y$, where X and Y are Banach spaces. The solution concepts are defined using a domination map $\mathcal{K} : Y \rightrightarrows Y$ or a domination map $\mathcal{Q} : X \rightrightarrows Y$. This leads to the following set-valued optimization problems

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$$(P_{\mathcal{K}}) \qquad \mathcal{K} - \operatorname*{Min}_{x \in X} F(x)$$

and

$$(P_{\mathcal{Q}}) \qquad \mathcal{Q} - \operatorname*{Min}_{x \in X} F(x).$$

It is important to mention that the solution concepts based on the vector approach in [10, 11] are given by a domination map $\mathcal{K} : Y \rightrightarrows Y$, where Y is the image space of the set-valued objective map $F : X \rightrightarrows Y$, whereas the domination map $\mathcal{Q} : X \rightrightarrows Y$ in [7, 19, 20] is acting between the same spaces like the objective map F. In addition, [19, 20] have followed the set approach to derive characterizations for many kinds of solutions for problem $(P_{\mathcal{K}})$ by using suitable scalarizing functionals.

In the literature, necessary optimality conditions for solutions of set optimization problems based on vector approach are derived (see [3, 7] and references therein). Although the solution concept based on vector approach is of mathematical interest, it cannot be often used in practice. A solution $\bar{x}$ given by the vector approach depends on only one certain special element $\bar{y}$ of the image set $F(\overline{x})$ and other elements of $F(\overline{x})$ are ignored, see Definition 3.8 in this paper. In other words, this definition does not care how elements in the set $F(\bar{x}) \setminus \{\bar{y}\}$ perform. The aim of our paper is to derive necessary optimality conditions for solutions of set optimization problems $(P_{\mathcal{K}})$ based on set approach using corresponding results for solutions of set optimization problems $(P_{\mathcal{Q}})$ defined by the vector approach (see [7]). For this reason we show relationships between the solution concepts given by the vector approach and by the set approach.

Recently, these relationships are investigated in [18] for the case that $\mathcal{K}(\cdot)$ and $\mathcal{Q}(\cdot)$ are fixed-cone valued mappings. Eichfelder and Pilecka [11, 12] have derived some of these relationships in which the vector approach is equipped with a domination structure acting onto the image space of the objective function. Furthermore, Eichfelder and Pilecka [12, Theorem 5.1] have presented necessary optimality conditions working on primal spaces where derivative concepts for set-valued maps are used. Moreover, necessary optimality conditions for solutions of set-valued optimization problems are shown by Dempe and Pilecka in [4, 5].

Our paper is organized as follows: The following section prepares the notions which will be used in the sequel. In the subsequent section, we recall six binary relations to compare two sets and some properties of them. These relations are equipped with a domination structure not necessarily given by a cone-valued map. This section is also concerned with the set approach and the vector approach to define solutions of a set-valued problem. In the subsequent part, we derive some results concerning relationships between these solution concepts based on two given approaches. The final section deals with finding optimality conditions for solutions of the set-valued problem based on the set approach. By means of the results given in the previous sections and the optimality conditions for solutions of set-valued problems with variable domination structure given in [7], we derive necessary conditions for solutions w.r.t. lower set relations ($\preccurlyeq_l^{\mathcal{K}}, \preccurlyeq_{cl}^{\mathcal{K}}$, and $\preccurlyeq_{pl}^{\mathcal{K}}$). Furthermore, we obtain necessary optimality conditions for solutions w.r.t. the upper set relations ($\preccurlyeq_u^{\mathcal{K}}$, $\preccurlyeq_{cu}^{\mathcal{K}}$) by means of coderivatives in Asplund spaces.

2. Preliminaries

Let X, Y be Banach spaces over the real field $\mathbb{R}$. We denote the open ball with center $x \in X$ and radius $\varepsilon > 0$ by $B_X(x, \varepsilon)$, and by S_{Y^*} the unit sphere of Y^*. $\mathcal{P}(Y)$ denotes the set of all nonempty subsets of Y. X^* is the topological dual space of X and w^* is the weak star topology on X^*. For $x \in X$, we denote by $\mathcal{V}(x)$ the system of the neighborhoods of x. Let $C \subset X$, we denote by $\operatorname{cl} C$ and $\operatorname{int} C$ the closure and interior of C, respectively. C is a cone if for each $c \in C$, $\lambda \geq 0$ it holds that $\lambda\, c \in C$. C is pointed if $C \cap (-C) = \{0\}$. In addition, C is proper if $C \neq X$ and $C \neq \{0\}$. For a cone C, we set $C^+ := \{y^* \in Y^* \mid \forall y \in C : \ y^*(y) \geq 0\}$ for the positive dual cone of C.
Let $A \subseteq X$ be given. We denote the smallest cone containing A by $\operatorname{cone}(A)$, that is,

$$\operatorname{cone}(A) := \bigcup_{t \geq 0} tA.$$

Let $A \subseteq X$ be given, $A \neq \emptyset$. The algebraic interior of A is denoted by $\operatorname{core}(A)$ and given as follows

$$\operatorname{core}(A) := \{a \in A \mid \forall\, x \in X, \ \exists\, \delta > 0, \forall\, \lambda \in [0, \delta] : \ a + \lambda x \in A\}.$$

It is stated in [27], if A is convex, then the following assertion holds true

$$\operatorname{core}(A) := \{a \in A | \ \operatorname{cone}(A - a) = X\}.$$

Let $\mathcal{K} : Y \rightrightarrows Y$ and $F, \ \mathcal{Q} : X \rightrightarrows Y$ be set-valued maps. As usual, we denote by $\operatorname{Gr} F$ and $\operatorname{Dom} F$ the graph and the domain of F, respectively. They are defined as follows

$$\operatorname{Dom} F := \{x \in X | \ F(x) \neq \emptyset\},$$

and

$$\operatorname{Gr} F := \{(x, y) \in X \times Y | \ y \in F(x)\}.$$

If $S \subseteq X$, we denote the image of S under F by $F(S) := \bigcup_{x \in S} F(x)$ and the inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $(y, x) \in \operatorname{Gr} F^{-1}$ if $(x, y) \in \operatorname{Gr} F$.

In this paper, we are dealing with the dual approach and derive necessary optimality conditions by means of Mordukhovich's coderivative (see the book [22] by Mordukhovich for more details). We now introduce the mains objects we use in the sequel.

Definition 2.1. Let Ω be a nonempty subset of a normed space X and let $\bar{x}, \ x \in \Omega$, $\varepsilon \geq 0$.

(i) The set of $\varepsilon-$ normals to Ω at x is defined by

$$\hat{N}_\varepsilon(\Omega, x) := \left\{ x^* \in X^* | \ \limsup_{u \xrightarrow{\Omega} x} \frac{x^*(u - x)}{\|u - x\|} \leq \varepsilon \right\}, \tag{2.1}$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ and $u \in \Omega$. If $\varepsilon = 0$, we call elements of (2.1) Fréchet normals and their collection, denoted by $\hat{N}(\Omega, x)$, is the Fréchet normal cone to Ω at x.

(ii) The basic (or limiting, or Mordukhovich) normal cone to Ω at $\bar{x}$ is defined as

$$N(\Omega,\bar{x}) := \{x^* \in X^* |\ \exists \varepsilon_n \downarrow 0,\ x_n \xrightarrow{\Omega} \bar{x},\ x_n^* \xrightarrow{w^*} x^*,\ x_n^* \in \hat{N}_{\varepsilon_n}(\Omega, x_n)\ (n \in \mathbb{N})\}.$$

Remark 2.2. If X is an Asplund space (i.e., a Banach space where every convex continuous function is generically Fréchet differentiable, see [22] for more details), and Ω is closed around $\bar{x}$ (i.e., there is a neighborhood V of $\bar{x}$ such that $\Omega \cap \operatorname{cl} V$ is closed), the formula for the basic normal cone looks as follows:

$$N(\Omega,\bar{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{\Omega} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(\Omega, x_n)\ (n \in \mathbb{N})\}.$$

Correspondingly, two concepts of coderivatives for set-valued maps are introduced in the following definition.

Definition 2.3. Let $F : X \rightrightarrows Y$ be a set-valued map and $(\bar{x},\bar{y}) \in \operatorname{Gr} F$.

(i) The Fréchet coderivative of F at $(\bar{x},\bar{y})$ is the set-valued map $\hat{D}^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$\hat{D}^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* |\ (x^*, -y^*) \in \hat{N}(\operatorname{Gr} F, (\bar{x},\bar{y})\}.$$

(ii) The normal coderivative of F at $(\bar{x},\bar{y})$ is the set-valued map $D^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* |\ (x^*, -y^*) \in N(\operatorname{Gr} F, (\bar{x},\bar{y})\}.$$

A main tool for the proof of the necessary optimality conditions in terms of Mordukhovich's coderivative is the incompatibility between openness and optimality. Here, we introduce the corresponding notations.

Definition 2.4. $F : X \rightrightarrows Y$ is said to be open at linear rate $L > 0$ (or $L-$open) around $(\bar{x},\bar{y}) \in \operatorname{Gr} F$ if there exist two neighborhoods $U \in \mathcal{V}(\bar{x}), V \in \mathcal{V}(\bar{y})$ and a positive number $\varepsilon > 0$ such that, for every $(x,y) \in \operatorname{Gr} F \cap (U \times V)$ and every $\rho \in (0,\varepsilon)$,

$$B_Y(y,\rho L) \subseteq F(B_X(x,\rho)).$$

F is said to be open at linear rate $L > 0$ or ($L-$open) at $(\overline{x},\overline{y})$ if there exists a positive number $\varepsilon > 0$ such that, for every $\rho \in (0,\varepsilon)$,

$$B_Y(\overline{y},\rho L) \subseteq F(B_X(\overline{x},\rho)).$$

It is necessary to mention that the openness at linear rate is a stronger property than the openness. We say that F is open at $(\bar{x},\bar{y}) \in \operatorname{Gr} F$ if the image through F of every neighborhood of $\bar{x}$ is a neighborhood of $\bar{y}$.

Now, we are concerned with some compactness requirements related to a subset of an Asplund space and a set-valued map.

Definition 2.5. Let X, Y be Asplund spaces, Ω be a subset of Y and $\mathcal{Q} : X \rightrightarrows Y$ be a set-valued map. Let $(\bar{x},\bar{y}) \in \operatorname{Gr} \mathcal{Q}$ and $\bar{u} \in \Omega$ be given.

(i) Ω is said to be sequentially normally compact (SNC) at $\bar{u}$, if

$$[y_n \xrightarrow{\Omega} \bar{u},\ y_n^* \xrightarrow{w^*} 0,\ y_n^* \in \hat{N}(\Omega, y_n)] \quad \Longrightarrow \quad y_n^* \to 0.$$

(ii) $\mathcal{Q}$ is said to be partially sequentially normally compact (PSNC) at $(\bar{x}, \bar{y})$, if

$$(x_n, y_n) \xrightarrow{\operatorname{Gr} \mathcal{Q}} (\bar{x}, \bar{y}),\ x_n^* \xrightarrow{w^*} 0,\ y_n^* \to 0,\ (x_n^*, y_n^*) \in \hat{N}(\operatorname{Gr} \mathcal{Q}, (x_n, y_n))$$

implies $x_n^* \to 0$.

Observe that, if C is a proper, closed, convex cone and $\operatorname{int} C \neq \emptyset$ then C is (SNC) at 0, see [7] for more details.

In order to show necessary optimality conditions in terms of Mordukhovich's coderivative, we need certain assumptions concerning the alliedness of sets (for more details, see [24, 25] and the references therein).

Definition 2.6. (Allied sets) Let $S_1, S_2, ..., S_k$ be closed subsets of a normed vector space Z, $\bar{z} \in \bigcap_{i=1}^k S_i$. One says that they are allied at $\bar{z}$ whenever $(z_{in}) \subset S_i$, $(z_{in}) \to \bar{z}$, $z_{in}^* \in \hat{N}(S_i, z_{in})$, the relation $\sum_{i=1}^k z_{in}^* \to 0$ implies $(z_{in}^*) \to 0$ for all $i = 1, \dots, k$.

Notice that Definition 2.6 is equivalent to the definition of η-regularity introduced and characterized in [21, Definition 7, Proposition 10]: The sets $S_1, S_2, \dots, S_k$ are η-regular at $\bar{z} \in S_1 \cap \dots \cap S_k$ if there exist $\gamma,\ \delta > 0$ such that

$$\|\sum_{i=1}^k z_i^*\| \geq \gamma \sum_{i=1}^k \|z_i^*\|,$$

for every $z_i \in B(\bar{z}, \delta) \cap S_i,\ z_i^* \in \hat{N}(S_i, x_i),\ i = 1, \dots, k$. In addition, these notions also imply the metric inequality of $(S_1, \dots, S_k)$ at $\bar{z}$ [8, Theorem 4.1], which is used as a main tool to establish chain rules for the limiting Fréchet subdifferentials [23].

In order to derive optimality conditions for set optimization problems based on set approach, we will concern the alliedness property of two sets given by

$$\begin{aligned} & C_1 := \{(x, y, k) : (x, y) \in \operatorname{Gr} F,\ k \in Y\}, \\ \text{and} \quad & C_2 := \{(x, y, k) : (x, k) \in \operatorname{Gr} \mathcal{Q},\ y \in Y\}. \end{aligned}$$

Definition 2.7 (Lower semicontinuous mapping [1, Definition 1.4.2])**.** The set-valued map $F : X \rightrightarrows Y$ is called lower semicontinuous (lsc, for short) at $\bar{x} \in \operatorname{Dom} F$ if for any $y \in F(\bar{x})$ and for any sequence $(x_n) \to \bar{x}$ there exists a sequence $(y_n) \to y$ with $y_n \in F(x_n)$. F is said to be lower semicontinuous if it is lower semicontinuous at any point $\bar{x} \in \operatorname{Dom} F$.

Remark 2.8. As shown in [1, Pages 40 and 42], if $\operatorname{Dom} F$ is closed then F is lsc if and only if the core of any closed subset is closed. We also have that F is lsc at $x \in \operatorname{Dom} F$ if and only if $F(x) \subseteq \liminf_{x' \to x} F(x')$.

3. Solution onsepts for set-valued optimization problems

3.1. **SET RELATIONS.** In this part, we will introduce different solution concepts for set-valued optimization problems. For doing this, we recall some concepts from the theory of ordered sets that will be used.

Definition 3.1. Let $A, B, C \in \mathcal{P}(Y)$ and a binary relation $\preccurlyeq$ be given. $\preccurlyeq$ is said to be

(i) reflexive, if $A \preccurlyeq A$.
(ii) transitive, if $A \preccurlyeq B$, $B \preccurlyeq C$ implies $A \preccurlyeq C$.
(iii) symmetric, if $A \preccurlyeq B$ implies $B \preccurlyeq A$.
(iv) antisymmetric, if $A \preccurlyeq B$, $B \preccurlyeq A$ implies $A = B$.

In the following, we recall the definition of several set relations given in [11]. Notice that we do not require the domination structure $\mathcal{K} : Y \rightrightarrows Y$ to be a cone-valued map.

Definition 3.2. Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. We define the binary relations on $\mathcal{P}(Y)$ w.r.t. $\mathcal{K}$ as follows:

(i) The variable generalized lower less relation ($\preccurlyeq_l^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_l^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)).$$

(ii) The variable generalized upper less relation ($\preccurlyeq_u^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_u^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \subseteq \bigcup_{b \in B} (b - \mathcal{K}(b)).$$

(iii) The variable generalized certainly lower less relation ($\preccurlyeq_{cl}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{cl}^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \subseteq \bigcap_{a \in A} (a + \mathcal{K}(a)).$$

(iv) The variable generalized certainly upper less relation ($\preccurlyeq_{cu}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{cu}^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \subseteq \bigcap_{b \in B} (b - \mathcal{K}(b)).$$

(v) The variable generalized possibly lower less relation ($\preccurlyeq_{pl}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{pl}^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \cap \bigcup_{a \in A} (a + \mathcal{K}(a)) \neq \emptyset.$$

(vi) The variable generalized possibly upper less relation ($\preccurlyeq_{pu}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{pu}^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \cap \bigcup_{b \in B} (b - \mathcal{K}(b)) \neq \emptyset.$$

We derive the following proposition by directly using Definition 3.2.

Proposition 3.3. *Let $A, B \in \mathcal{P}(Y)$ and consider the relations (i)-(vi) given by Definition 3.2 . The following assertions hold true.*

(i) $A \preccurlyeq_u^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \preccurlyeq_l^{-\mathcal{K}} A.$
(ii) $A \preccurlyeq_{cu}^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \preccurlyeq_{cl}^{-\mathcal{K}} A.$

(iii) $A \preccurlyeq_{pu}^{\mathcal{K}} B \iff B \preccurlyeq_{pl}^{-\mathcal{K}} A.$
(iv) $A \preccurlyeq_{cl}^{\mathcal{K}} B \implies A \preccurlyeq_{l}^{\mathcal{K}} B \implies A \preccurlyeq_{pl}^{\mathcal{K}} B.$
(v) $A \preccurlyeq_{cu}^{\mathcal{K}} B \implies A \preccurlyeq_{u}^{\mathcal{K}} B \implies A \preccurlyeq_{pu}^{\mathcal{K}} B.$

Remark 3.4. Each of the above relations has its own meaning in practical problems. For example, in uncertain optimization problems $\preccurlyeq_l^{\mathcal{K}}$ is used by a decision maker who is interested in minimizing the best case, and when the worst case is concerned, he will choose the relation $\preccurlyeq_u^{\mathcal{K}}$.

In order to derive some properties of the relations given in Definition 3.2, some of the following properties of the domination structure given by a set-valued map $\mathcal{K} : Y \rightrightarrows Y$ will be used:

$$\forall y \in Y : \ 0 \in \mathcal{K}(y); \tag{3.1}$$

$$\forall y \in Y : \ \mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y); \tag{3.2}$$

$$\forall y \in Y, d \in \mathcal{K}(y) : \ \mathcal{K}(y + d) \subseteq \mathcal{K}(y); \tag{3.3}$$

$$\forall y \in Y, d \in \mathcal{K}(y) : \ \mathcal{K}(y - d) \subseteq \mathcal{K}(y); \tag{3.4}$$

$$\forall y \in Y : \ \mathcal{K}(y) \bigcap (-\mathcal{K}(y)) = \{0\}. \tag{3.5}$$

Obviously, if $\mathcal{K}(y)$ is a convex, pointed cone in Y for all $y \in Y$, then $\mathcal{K}$ satisfies the properties (3.1), (3.2) and (3.5).
The relations given in Definition 3.2 satisfy the following properties.

Proposition 3.5. *The following statements hold true:*

(i) *If $\mathcal{K}$ satisfies property (3.1), then the binary relations $\preccurlyeq_l^{\mathcal{K}}$ and $\preccurlyeq_u^{\mathcal{K}}$ are reflexive.*
(ii) *If $\mathcal{K}$ satisfies properties (3.2) and (3.3), then the relations $\preccurlyeq_l^{\mathcal{K}}$ and $\preccurlyeq_{cl}^{\mathcal{K}}$ are transitive.*
(iii) *If $\mathcal{K}$ satisfies properties (3.2) and (3.4), then the relations $\preccurlyeq_u^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}}$ are transitive.*
(iv) *If $\mathcal{K}$ satisfies $\mathcal{K}(Y) \cap (-\mathcal{K}(Y)) = \{0\}$, then the relations $\preccurlyeq_{cl}^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}}$ are antisymmetric.*

The proof of Proposition 3.5 is similar to that one in [11, Lemma 2.1] for a cone-valued map $\mathcal{K}$.
From now on, we denote by $\preccurlyeq_t^{\mathcal{K}}$ one of the relations $(i) - (vi)$ given in Definition 3.2, $t \in \{l, u, cl, cu, pl, pu\}$. If $A, B \in \mathcal{P}(Y)$ such that $A \preccurlyeq_t^{\mathcal{K}} B$ and $B \preccurlyeq_t^{\mathcal{K}} A$, we will write $A \sim B$. Let $\mathcal{A}$ be a family of sets in $\mathcal{P}(Y)$. We recall some minimality notions of $\mathcal{A}$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$, which are used in the next sections.

Definition 3.6. Let $\mathcal{A}$ be a family of nonempty subsets of Y, $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map and $t \in \{l, u, cl, cu, pl, pu\}$.

(a) A set $\bar{A} \in \mathcal{A}$ is called a minimal element of $\mathcal{A}$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$ if

$$A \in \mathcal{A}, A \preccurlyeq_t^{\mathcal{K}} \bar{A} \implies \bar{A} \preccurlyeq_t^{\mathcal{K}} A.$$

(b) A set $\bar{A} \in \mathcal{A}$ is called a strongly minimal element of $\mathcal{A}$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$ if

$$\forall\ A \in \mathcal{A} \setminus \{\bar{A}\} : \ \bar{A} \preccurlyeq_t^{\mathcal{K}} \mathcal{A}.$$

(c) A set $\bar{A} \in \mathcal{A}$ is called a strictly minimal element of $\mathcal{A}$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$ if

$$A \in \mathcal{A}, A \preccurlyeq_t^{\mathcal{K}} \bar{A} \quad \Longrightarrow \quad \bar{A} = A.$$

We denote respectively by $\mathrm{Min}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$, $\mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$ and $\mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$ the sets of all minimal, strongly minimal and strictly minimal elements of $\mathcal{A}$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$. The index "$Y$" in Min_Y, SoMin_Y, SiMin_Y is used to mark that we consider concepts of minimality in the image space Y. In Definition 3.9, we introduce additionally corresponding concepts in the pre-image space X.

Remark 3.7. Let $t \in \{l, u, cl, cu, pl, pu\}$. We note the following properties by taking into account the definition of the sets $\mathrm{Min}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$, $\mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$ and $\mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$.

(i) It is clear that if $\preccurlyeq_t^{\mathcal{K}}$ is transitive and $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$, then for all B such that $B \sim \bar{A}$, it holds that $B \in \mathrm{Min}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$.

(ii) Obviously, we have the inclusions $\mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}}) \subseteq \mathrm{Min}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$ and $\mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}}) \subseteq \mathrm{Min}_Y(\mathcal{A}, \preccurlyeq_t^{\mathcal{K}})$.

(iii) We have that

$$\begin{aligned}
\bar{A} \in \mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{cl}^{\mathcal{K}}) &\Longleftrightarrow \forall\, A \in \mathcal{A} \setminus \{\bar{A}\} : \ \bar{A} \preccurlyeq_{cl}^{\mathcal{K}} A \\
&\overset{\text{Proposition 3.3}}{\Longrightarrow} \forall\, A \in \mathcal{A} \setminus \{\bar{A}\} : \ \bar{A} \preccurlyeq_{l}^{\mathcal{K}} A \\
&\Longrightarrow \bar{A} \in \mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{l}^{\mathcal{K}}).
\end{aligned}$$

Therefore, we get that $\mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{l}^{\mathcal{K}})$. Similarly, we can obtain that $\mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{l}^{\mathcal{K}}) \subseteq \mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{pl}^{\mathcal{K}})$.
In addition, it yields from Definition 3.6(c) that

$$\begin{aligned}
\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{pl}^{\mathcal{K}}) &\Longleftrightarrow \forall\, A \neq \bar{A} : \ A \not\preccurlyeq_{pl}^{\mathcal{K}} \bar{A} \\
&\overset{\text{Proposition 3.3}}{\Longrightarrow} \forall\, A \neq \bar{A} : \ A \not\preccurlyeq_{l}^{\mathcal{K}} \bar{A} \\
&\Longleftrightarrow \bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{l}^{\mathcal{K}}).
\end{aligned}$$

Thus, $\mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{pl}^{\mathcal{K}}) \subseteq \mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{l}^{\mathcal{K}})$.
Similarly, we have that $\mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{l}^{\mathcal{K}}) \subseteq \mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{cl}^{\mathcal{K}})$.
For the relations $\preccurlyeq_{u}^{\mathcal{K}}$, $\preccurlyeq_{cu}^{\mathcal{K}}$, and $\preccurlyeq_{pu}^{\mathcal{K}}$ the following assertions also hold true:

$$\mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{cu}^{\mathcal{K}}) \subseteq \mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{u}^{\mathcal{K}}) \subseteq \mathrm{SoMin}_Y(\mathcal{A}, \preccurlyeq_{pu}^{\mathcal{K}})$$

and

$$\mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{pu}^{\mathcal{K}}) \subseteq \mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{u}^{\mathcal{K}}) \subseteq \mathrm{SiMin}_Y(\mathcal{A}, \preccurlyeq_{cu}^{\mathcal{K}}).$$

3.2. SOLUTION CONCEPTS BASED ON VECTOR APPROACH AND SET APPROACH FOR SET OPTIMIZATION PROBLEMS.

First, we consider the problem $(P_{\mathcal{Q}})$, where the domination map $\mathcal{Q} : X \rightrightarrows Y$ is acting between the same spaces as the set-valued objective map $F : X \rightrightarrows Y$. Furthermore, we assume that for all $x \in X : F(x) \neq \emptyset$. We define solutions of $(P_{\mathcal{Q}})$ by using the vector approach as follows.

Definition 3.8. Let $F : X \rightrightarrows Y$ and $\mathcal{Q} : X \rightrightarrows Y$ be two given set-valued maps, and $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{y} \in F(\bar{x})$.

(i) $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is called a nondominated solution of the problem $(P_{\mathcal{Q}})$ w.r.t. $\mathcal{Q}$ if

$$\bar{y} \notin \bigcup_{x \in X} \big(F(x) + (\mathcal{Q}(x) \setminus \{0\})\big).$$

The set of all nondominated solutions of $(P_{\mathcal{Q}})$ w.r.t. $\mathcal{Q}$ is denoted by $\mathrm{ND}(F(X), \mathcal{Q})$.

(ii) $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is called a minimal solution of the problem $(P_{\mathcal{Q}})$ w.r.t. $\mathcal{Q}$ if

$$\bar{y} \notin F(X) + (\mathcal{Q}(\bar{x}) \setminus \{0\}).$$

We denote the set of all minimal solutions of $(P_{\mathcal{Q}})$ w.r.t. $\mathcal{Q}$ by $\mathrm{Min}(F(X), \mathcal{Q})$.

Obviously, if $(\bar{x}, \bar{y}) \in \mathrm{Min}(F(X), \mathcal{Q})$, then it is also a nondominated solution of $(P_{\mathcal{Q}})$ w.r.t. $\widetilde{\mathcal{Q}} : X \rightrightarrows Y$ defined by

$$\forall x \in X : \ \widetilde{\mathcal{Q}}(x) \equiv \mathcal{Q}(\bar{x}).$$

Now, we follow the set approach to define solutions of $(P_{\mathcal{K}})$, $\mathcal{K} : Y \rightrightarrows Y$, w.r.t. the relation $\preccurlyeq_t^{\mathcal{K}}$, where $t \in \{u, l, cu, cl, pu, pl\}$. The solution concepts in the following definition are given in the pre-image space X, whereas the solution concepts in Definition 3.6 are formulated in the image space Y.

Definition 3.9 ([11, Definition 5.2])**.** Let $F : X \rightrightarrows Y$, $\mathcal{K} : Y \rightrightarrows Y$ be two given set-valued maps and $t \in \{u, l, cu, cl, pu, pl\}$.

(a) A point $\bar{x} \in X$ is called a minimal solution of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$ if

$$x \in X, F(x) \preccurlyeq_t^{\mathcal{K}} F(\bar{x}) \quad \Longrightarrow \quad F(\bar{x}) \preccurlyeq_t^{\mathcal{K}} F(x).$$

We denote by $\mathrm{Min}(F(X), \preccurlyeq_t^{\mathcal{K}})$ the set of all minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$.

(b) A point $\bar{x} \in X$ is called a strongly minimal solution of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$ if

$$\forall\, x \in X \setminus \{\bar{x}\} : \ F(\bar{x}) \preccurlyeq_t^{\mathcal{K}} F(x).$$

We denote by $\mathrm{SoMin}(F(X), \preccurlyeq_t^{\mathcal{K}})$ the set of all strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$.

(c) A point $\bar{x} \in X$ is called a strictly minimal solution of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$ if

$$x \in X, \ F(x) \preccurlyeq_t^{\mathcal{K}} F(\bar{x}) \text{ or } F(x) = F(\bar{x}) \quad \Longrightarrow \quad \bar{x} = x.$$

We denote by $\mathrm{SiMin}(F(X), \preccurlyeq_t^{\mathcal{K}})$ the set of all strictly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_t^{\mathcal{K}}$.

Remark 3.10. Let $t \in \{l, u, cl, cu, pl, pu\}$.

(i) Observe that if $\preccurlyeq_t^{\mathcal{K}}$ is transitive and $\bar{x} \in \mathrm{Min}(F(X), \preccurlyeq_t^{\mathcal{K}})$, then it also holds true for all $x \in X$ satisfying $F(x) \sim F(\bar{x})$.
If for all $x \neq x'$, $F(x) \neq F(x')$ holds true, then the Definition 3.9(b) and (c) are equivalent to $F(\bar{x}) \in \mathrm{SoMin}_Y(F(X), \preccurlyeq_t^{\mathcal{K}}$) and $F(\bar{x}) \in \mathrm{SiMin}_Y(F(X), \preccurlyeq_t^{\mathcal{K}}$), respectively.
Let $[F(\bar{x})] := \{F(x) \in F(X) |\ F(x) \sim F(\bar{x})\}$. Then, it holds that

$$\bar{x} \in \mathrm{SiMin}(F(X), \preccurlyeq_t^{\mathcal{K}}) \quad \Longrightarrow \quad [F(\bar{x})] = \{F(\bar{x})\}.$$

(ii) Definition 3.9 implies that $\mathrm{SiMin}(F(X), \preccurlyeq_t^{\mathcal{K}})$ and $\mathrm{SoMin}(F(X), \preccurlyeq_t^{\mathcal{K}})$ are subsets of $\mathrm{Min}(F(X), \preccurlyeq_t^{\mathcal{K}})$. Furthermore, by using the same lines as in Remark 3.7(iii) the following relationships for the lower relations $\preccurlyeq_l^{\mathcal{K}}$, $\preccurlyeq_{cl}^{\mathcal{K}}$, and $\preccurlyeq_{pl}^{\mathcal{K}}$ hold true:

$$\mathrm{SoMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preccurlyeq_l^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preccurlyeq_{pl}^{\mathcal{K}})$$

and

$$\mathrm{SiMin}(F(X), \preccurlyeq_{pl}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preccurlyeq_l^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}}).$$

Similarly, we have the following relationships for the upper relations $\preccurlyeq_u^{\mathcal{K}}$, $\preccurlyeq_{cu}^{\mathcal{K}}$ and $\preccurlyeq_{pu}^{\mathcal{K}}$:

$$\mathrm{SoMin}(F(X), \preccurlyeq_{cu}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preccurlyeq_u^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preccurlyeq_{pu}^{\mathcal{K}})$$

and

$$\mathrm{SiMin}(F(X), \preccurlyeq_{pu}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preccurlyeq_u^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preccurlyeq_{cu}^{\mathcal{K}}).$$

4. Relationships between solution concepts based on set approach and vector approach for set optimization problems

In the following, we derive the relationships between solution concepts of $(P_{\mathcal{K}})$, $\mathcal{K} : Y \rightrightarrows Y$, and $(P_{\mathcal{Q}})$, $\mathcal{Q} : X \rightrightarrows Y$ given in Definition 3.8 and Definition 3.9. The following theorems will use two set-valued maps $\hat{\mathcal{Q}} : X \rightrightarrows Y$ and $\hat{\mathcal{Q}}' : X \rightrightarrows Y$ respectively determined by:

$$\forall x \in X: \ \hat{\mathcal{Q}}(x) := \bigcap_{y \in F(x)} \mathcal{K}(y), \tag{4.1}$$

and

$$\forall x \in X: \ \hat{\mathcal{Q}}'(x) := \bigcup_{y \in F(x)} \mathcal{K}(y). \tag{4.2}$$

Let us recall a result by Eichfelder and Pilecka [12] about the relationships between strictly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the variable generalized possibly lower less relation $(\preccurlyeq_{pl}^{\mathcal{K}})$ introduced in Definition 3.2, (v), and nondominated solutions (see Definition 3.8, (i)) of the set-valued optimization problem $(P_{\mathcal{Q}})$ w.r.t. $\hat{\mathcal{Q}}$.

Theorem 4.1 ([12, Lemma 5.1])**.** *Consider problem $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_{pl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$ which satisfies that $\mathcal{K}(y)$ is a proper, convex cone for all $y \in Y$, and let some vector $\bar{x} \in \mathrm{SiMin}(F(X), \preccurlyeq_{pl}^{\mathcal{K}})$ be given. Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ be given by (4.1). If there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \notin F(\bar{x}) + \hat{\mathcal{Q}}(\bar{x}) \setminus \{0\}$, then $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$.*

Now, our intention is to derive a corresponding result for strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the variable generalized lower less relation $\preccurlyeq_l^{\mathcal{K}}$, introduced in Definition 3.2, (i), and nondominated solutions of the set-valued optimization problem introduced in Definition 3.8, (i).

Theorem 4.2. *Consider problem ($P_{\mathcal{K}}$) w.r.t.* $\preccurlyeq_l^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, *and let some vector* $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_l^{\mathcal{K}})$ *be given. Suppose that there is* $\bar{y} \in F(\bar{x})$ *such that*

$$\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y). \tag{4.3}$$

Furthermore, assume that $\mathcal{K} : Y \rightrightarrows Y$ *satisfies properties (3.1)-(3.3), and (3.5). Let* $\hat{\mathcal{Q}} : X \rightrightarrows Y$ *be given by (4.1). Then,* $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$.

Proof. Since $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_l^{\mathcal{K}})$, it holds that

$$\forall x \in X \setminus \{\bar{x}\} : \ F(\bar{x}) \preccurlyeq_l^{\mathcal{K}} F(x).$$

Furthermore, it holds that

$$F(\bar{x}) \preccurlyeq_l^{\mathcal{K}} F(\bar{x}),$$

since $0 \in \mathcal{K}(y)$ for all $y \in Y$. Thus, we obtain

$$\forall x \in X : \ F(\bar{x}) \preccurlyeq_l^{\mathcal{K}} F(x),$$

which is equivalent to

$$\forall x \in X : F(x) \subseteq \bigcup_{y \in F(\bar{x})} (y + \mathcal{K}(y)). \tag{4.4}$$

Suppose by contradiction that $(\bar{x}, \bar{y}) \notin \mathrm{ND}(F(X), \hat{\mathcal{Q}})$. This means that

$$\begin{aligned} & \exists x \in X : \ \bar{y} \in F(x) + \hat{\mathcal{Q}}(x) \setminus \{0\} \\ \iff & \exists x \in X, \ \exists y \in F(x) \setminus \{\bar{y}\} : \ \bar{y} \in y + \hat{\mathcal{Q}}(x) \setminus \{0\} \\ \iff & \exists x \in X, \ \exists y \in F(x) \setminus \{\bar{y}\} : \ \bar{y} \in y + \hat{\mathcal{Q}}(x) \subseteq y + \mathcal{K}(y). \end{aligned} \tag{4.5}$$

From (4.4), taking into account that $\mathcal{K}$ satisfies (3.3), we have that

$$\exists \hat{y} \in F(\bar{x}) : \ y \in \hat{y} + \mathcal{K}(\hat{y}) \quad \Longrightarrow \quad \mathcal{K}(y) \subseteq \mathcal{K}(\hat{y}). \tag{4.6}$$

Therefore, we can conclude

$$\begin{aligned} \bar{y} \in y + \mathcal{K}(y) & \subseteq \hat{y} + \mathcal{K}(\hat{y}) + \mathcal{K}(y) \\ & \subseteq \hat{y} + \mathcal{K}(\hat{y}) + \mathcal{K}(\hat{y}) \subseteq \hat{y} + \mathcal{K}(\hat{y}) \\ & \Longrightarrow \quad \bar{y} \in y + \mathcal{K}(y) \subseteq \hat{y} + \mathcal{K}(\hat{y}). \end{aligned} \tag{4.7}$$

Taking into account (4.3), we obtain that $\bar{y} = \hat{y}$. By (4.7), we get

$$y + \mathcal{K}(y) \subseteq \hat{y} + \mathcal{K}(\hat{y}) = \bar{y} + \mathcal{K}(\bar{y}).$$

Furthermore,

$$\bar{y} \in y + \mathcal{K}(y) \quad \Longrightarrow \quad \mathcal{K}(\bar{y}) \subseteq \mathcal{K}(y). \tag{4.8}$$

Since (4.6) and (4.8), it holds that $\mathcal{K}(y) \subseteq \mathcal{K}(\hat{y}) = \mathcal{K}(\bar{y}) \subseteq \mathcal{K}(y)$.
Thus, $\mathcal{K}(\bar{y}) = \mathcal{K}(y)$. Taking into account (4.6), (4.8) and $\bar{y} = \hat{y}$, we get that

$$y - \bar{y} \in \mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})) = \{0\}.$$

This means that $y = \bar{y}$. This is a contradiction to $y \in F(x) \setminus \{\bar{y}\}$. Therefore, $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$. □

In the following theorem, we show a result about the relationships between strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the variable generalized certainly lower less relation ($\preccurlyeq_{cl}^{\mathcal{K}}$) introduced in Definition 3.2, (iii), and nondominated solutions of the set-valued optimization problem introduced in Definition 3.8, (i). It is shown in Remark 3.10(ii) that $\mathrm{SoMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preccurlyeq_{l}^{\mathcal{K}})$, i.e., if $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}})$ then it holds that $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{l}^{\mathcal{K}})$.
Therefore, we present the following corollary without proof since it can be proved by using the same arguments as in that one of Theorem 4.2.

Corollary 4.3. *Consider problem $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_{cl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$ and $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}})$. Suppose that there is $\bar{y} \in F(\bar{x})$ satisfying condition* (4.3)*. Furthermore, assume that $\mathcal{K} : Y \rightrightarrows Y$ satisfies properties (3.1)-(3.3), and (3.5). Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ be given by (4.1). Then, $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$.*

On the other hand, under some assumptions we can derive a stronger result as follows.

Theorem 4.4. *Consider problem $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_{cl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, where $\mathcal{K}$ satisfies properties (3.2) and (3.3), and $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}})$. Assume that there exists $\bar{y} \in F(\bar{x})$ satisfying $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$. Let $\hat{\mathcal{Q}}' : X \rightrightarrows Y$ given by* (4.2) *such that*

$$\forall x \in X : \hat{\mathcal{Q}}'(x) \bigcap (-\hat{\mathcal{Q}}'(x)) = \{0\}.$$

Then, $(\bar{x}, \bar{y}) \in \mathrm{Min}(F(X), \hat{\mathcal{Q}}')$.

Proof. Since $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cl}^{\mathcal{K}})$, it holds that

$$\begin{aligned} & \forall\, x \in X \setminus \{\bar{x}\} : \; F(\bar{x}) \preccurlyeq_{cl}^{\mathcal{K}} F(x) \\ \Longleftrightarrow \quad & \forall\, x \in X \setminus \{\bar{x}\} : \; F(x) \subseteq \bigcap_{y \in F(\bar{x})} (y + \mathcal{K}(y)) \\ \Longleftrightarrow \quad & \forall\, x \in X \setminus \{\bar{x}\}, \; \forall\, y \in F(\bar{x}) : \; F(x) \subseteq y + \mathcal{K}(y). \end{aligned} \tag{4.9}$$

From the assumption $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$ and taking into account that $\mathcal{K}$ satisfies (3.3), it holds that

$$\forall\, y \in F(\bar{x}) : \; y \in \bar{y} + \mathcal{K}(\bar{y}) \quad \Longrightarrow \quad \mathcal{K}(y) \subseteq \mathcal{K}(\bar{y}).$$

This implies

$$y + \mathcal{K}(y) \subseteq \bar{y} + \mathcal{K}(\bar{y}) + \mathcal{K}(\bar{y}) \subseteq \bar{y} + \mathcal{K}(\bar{y}),$$

since $\mathcal{K}$ satisfies (3.2). Taking into account (4.9), we get

$$\forall x \in X \setminus \{\bar{x}\}, \; \forall\, y \in F(\bar{x}) : F(x) \subseteq y + \mathcal{K}(y) \subseteq \bar{y} + \mathcal{K}(\bar{y}).$$

In addition, because of $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$, we obtain

$$F(X) \subseteq \bar{y} + \mathcal{K}(\bar{y}) \subseteq \bar{y} + \hat{\mathcal{Q}}'(\bar{x}).$$

Now, we claim that $(\bar{x}, \bar{y}) \in \mathrm{Min}(\mathrm{F(X)}, \hat{\mathcal{Q}}')$. Indeed, suppose that there is $x \in X$ satisfying

$$\begin{aligned} & \bar{y} \in F(x) + \hat{\mathcal{Q}}'(\bar{x}) \setminus \{0\} \\ \Longrightarrow \quad & \exists\, y \in F(x), \; t \in \hat{\mathcal{Q}}'(\bar{x}) \setminus \{0\} : \; \bar{y} = y + t. \end{aligned} \tag{4.10}$$

Since $F(X) \subseteq \bar{y} + \hat{\mathcal{Q}}'(\bar{x})$ and $y \in F(X)$, we have that:

$$\exists\, t' \in \hat{\mathcal{Q}}'(\bar{x}) : \; y = \bar{y} + t'. \tag{4.11}$$

From (4.10) and (4.11), we get that

$$t = -t' \in \hat{\mathcal{Q}}'(\bar{x}) \setminus \{0\} \bigcap (-\hat{\mathcal{Q}}'(\bar{x})) = \emptyset, \text{ a contradiction.}$$

The proof is complete. □

We illustrate the three theorems above by Figure 1.

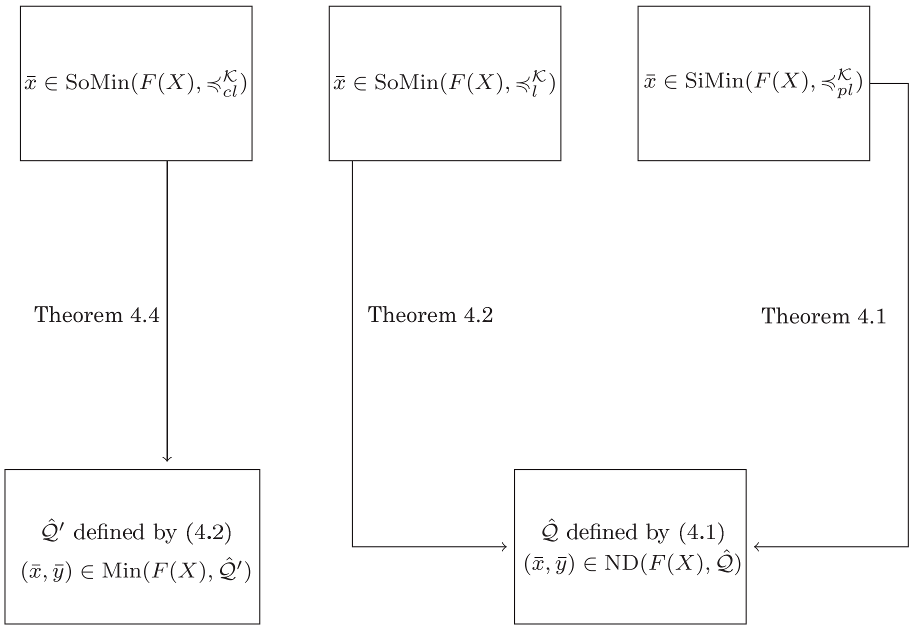

FIGURE 1. Illustration of Theorem 4.1, Theorem 4.2 and Theorem 4.4

In order to derive some relationships between the solution concepts from Definition 3.9 and Definition 3.8 in the converse direction, it is necessary to introduce the following concepts of domination property of a family of sets $\mathcal{A} \subseteq \mathcal{P}(Y)$. These concepts are introduced in [11] in order to study relationships between optimal solutions according to the set approach and those ones according to the vector approach, where the domination structure in the latter one acts onto the output space of the objective function F.

Definition 4.5. Let a family $\mathcal{A} \subseteq \mathcal{P}(Y)$ of nonempty sets and a relation $\preccurlyeq^{\mathcal{K}}_t$ ($t \in \{u, l, cu, cl, pu, pl\}$) be given. We say that,

(i) $\mathcal{A}$ has the weak domination property w.r.t. $\preccurlyeq^{\mathcal{K}}_t$ if for each set $A \in \mathcal{A}$ there exists a family of sets $\Gamma^A_{\mathcal{A}} \subseteq \mathcal{A}$ such that $\Gamma^A_{\mathcal{A}} \subseteq \mathrm{Min}_Y(\mathcal{A}, \preccurlyeq^{\mathcal{K}}_t)$ and
$$\bigcup \{B : B \in \Gamma^A_{\mathcal{A}}\} \preccurlyeq^{\mathcal{K}}_t A.$$

(ii) $\bar{\mathcal{A}} \subseteq \mathcal{A}$ has the domination property w.r.t. $\preccurlyeq^{\mathcal{K}}_t$ if $\mathrm{Min}_Y(\bar{\mathcal{A}}, \preccurlyeq^{\mathcal{K}}_t) \neq \emptyset$ and for each set $A \in \mathcal{A}$ there exists a set $B \in \mathrm{Min}_Y(\bar{\mathcal{A}}, \preccurlyeq^{\mathcal{K}}_t)$ such that $B \preccurlyeq^{\mathcal{K}}_t A$.

Observe that Definition 4.5(i) is weaker than Definition 4.5(ii) which is first introduced in [13, Definition 4.9] for constant ordering cones.

In the next theorem, we discuss the relationships between nondominated solutions of $(P_{\mathcal{Q}})$ (see Definition 3.8) and minimal solutions of a set-valued optimization problem where the solution concept is governed by the variable generalized lower less relation $\preccurlyeq_l^{\mathcal{K}}$ (see Definition 3.2, (i)) with respect to a certain set-valued map $\mathcal{K}$ under the assumption that the set $\mathcal{F}(X) := \{F(x)|\ x \in X\}$ has the weak domination property w.r.t. $\preccurlyeq_l^{\mathcal{K}}$.

Theorem 4.6. *Consider problem $(P_{\mathcal{Q}})$, $\mathcal{Q} : X \rightrightarrows Y$, and $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$. Let $\mathcal{K} : Y \rightrightarrows Y$ be given by*

$$\mathcal{K}(y) := \begin{cases} \bigcap\limits_{x \in X : y \in F(x)} \mathcal{Q}(x) & \text{if } y \in F(X), \\ \{0\} & \text{if } y \notin F(X). \end{cases} \tag{4.12}$$

Suppose that $\mathcal{F}(X)$ has the weak domination property w.r.t. $\preccurlyeq_l^{\mathcal{K}}$. Then, there exists a minimal solution $x' \in \mathrm{Min}(F(X), \preccurlyeq_l^{\mathcal{K}})$ provided that $\bar{y} \in F(x')$.

Proof. Since $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$, it holds that

$$\bar{y} \notin \bigcup_{x \in X} (F(x) + (\mathcal{Q}(x) \setminus \{0\})). \tag{4.13}$$

Taking into account the weak domination property of $\mathcal{F}(X)$, it holds that there is a family of sets $\tilde{\mathcal{F}}(X) \subseteq \mathcal{F}(X)$ such that

$$\tilde{\mathcal{F}}(X) \subseteq \mathrm{Min}_Y(\mathcal{F}(X), \preccurlyeq_l^{\mathcal{K}}),$$

and

$$\bigcup \{B : B \in \tilde{\mathcal{F}}(X)\} \preccurlyeq_l^{\mathcal{K}} F(\bar{x}).$$

We suppose that there exists $\bar{S} \subseteq X$ such that for all $x \in \bar{S} :\ F(x) \in \tilde{\mathcal{F}}(X)$, *i.e.*, $x \in \mathrm{Min}(F(X), \preccurlyeq_l^{\mathcal{K}})$. Let $F(\bar{S}) = \bigcup \{B :\ B \in \tilde{\mathcal{F}}(X)\}$. Thus, $F(\bar{S}) \preccurlyeq_l^{\mathcal{K}} F(\bar{x})$. This is equivalent to

$$F(\bar{x}) \subseteq \bigcup_{y \in F(\bar{S})} (y + \mathcal{K}(y)).$$

Taking into account $\bar{y} \in F(\bar{x})$, there is $x' \in \bar{S} \subseteq X,\ y \in F(x')$ such that

$$\bar{y} \in y + \mathcal{K}(y) = y + \bigcap_{\{x \in X:\ y \in F(x)\}} \mathcal{Q}(x) \subseteq y + \mathcal{Q}(x'). \tag{4.14}$$

(4.13) and (4.14) imply that $y = \bar{y} \in F(x')$.
In addition, $F(x') \in \mathrm{Min}_Y(F(X), \preccurlyeq_l^{\mathcal{K}})$, i.e., $x' \in \mathrm{Min}(F(X), \preccurlyeq_l^{\mathcal{K}})$, and the proof is complete. □

Furthermore, we explain the relationships between nondominated solutions of $(P_{\mathcal{Q}})$ (see Definition 3.8) and minimal solutions of a set-valued optimization problem where the solution concept is governed by the variable generalized certainly lower less relation $\preccurlyeq_{cl}^{\mathcal{K}}$ (see Definition 3.2, (iii)) with respect to a certain set-valued map $\mathcal{K}$ under the assumption that $\mathcal{F}(X)$ has the weak domination property w.r.t. $\preccurlyeq_{cl}^{\mathcal{K}}$.

Theorem 4.7. *Assume that* $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$, $\mathcal{Q} : X \rightrightarrows Y$. *Suppose that* $\mathcal{F}(X)$ *has the weak domination property w.r.t.* $\preccurlyeq_{cl}^{\mathcal{K}}$ *with* $\mathcal{K} : Y \rightrightarrows Y$ *given by (4.12). Then, there is* $\bar{S} \subseteq X$ *such that for all* $x \in \bar{S} : F(x) = \{\bar{y}\}$ *and* $x \in \mathrm{Min}(F(X), \preccurlyeq_{cl}^{\mathcal{K}})$.

Proof. By using the same arguments as in Theorem 4.6, it holds that there is $\bar{S} \subseteq X$ such that $\bar{S} \subseteq \mathrm{Min}(F(X), \preccurlyeq_{cl}^{\mathcal{K}})$ and $F(\bar{S}) \preccurlyeq_{cl}^{\mathcal{K}} F(\bar{x})$. Taking into account the definition of $\preccurlyeq_{cl}^{\mathcal{K}}$, we get

$$F(\bar{x}) \subseteq \bigcap_{y \in F(\bar{S})} (y + \mathcal{K}(y)).$$

This yields

$$\forall x \in \bar{S}, \forall\, y \in F(x) : \bar{y} \in y + \mathcal{K}(y) \subseteq y + \mathcal{Q}(x).$$

Taking into account $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$, it holds that $\bar{y} = y$. This conclusion holds true for all $y \in F(x)$, such that we obtain for all $x \in \bar{S} : F(x) = \{\bar{y}\}$. □

We finish this section by the following figure illustrating Theorem 4.6 and Theorem 4.7.

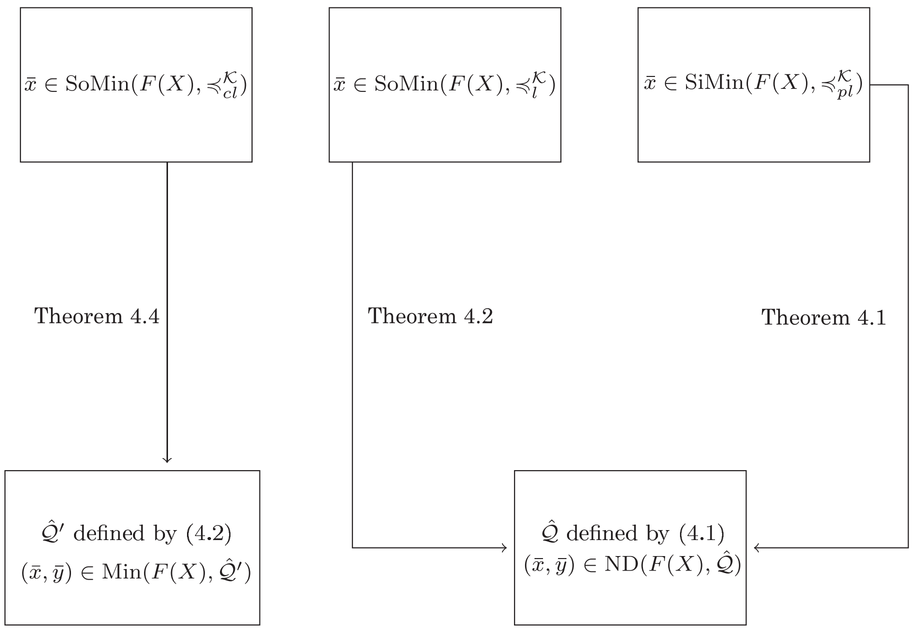

FIGURE 2. Illustration of Theorem 4.6 and Theorem 4.7

5. OPTIMALITY CONDITIONS

This section is devoted to deriving necessary optimality conditions for strongly and strictly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq_{t}^{\mathcal{K}}$, where $t \in \{l, pl, cl, u, cu\}$. The so-called vector approach for handling set optimization problems is very suitable for deriving these necessary optimality conditions due to relationships between between solutions of the vector and set approach. In this section, we will formulate such optimality conditions based on the derived results from Section 4. Less general results, for a fixed domination structure, can be found in [18] under the following optimality notion, where the spaces X and Y and the map F are defined as in this paper, and for $A, B \in \mathcal{P}(Y)$ and K being a proper, closed, convex cone in Y, we define

$$A \preccurlyeq_t^{\operatorname{int} K} B \quad :\Longleftrightarrow \quad A \preccurlyeq_t^{\mathcal{K}} B \text{ with } \mathcal{K} :\equiv \operatorname{int} K$$

for $t \in \{l, pl\}$.

Definition 5.1 (Local Minimal Points w.r.t. $\preccurlyeq_t^K$ ($\preccurlyeq_t^{\operatorname{int} K}$))**.** Let K be a closed, convex, proper and pointed cone in Y.

(i) $\overline{x} \in X$ is a local minimal point w.r.t. $\preccurlyeq_t^K$ if there is a neighborhood U of $\overline{x}$ such that there does not exist $x \in U \setminus \{\overline{x}\}$ such that $F(x) \preccurlyeq_t^K F(\overline{x})$.

(ii) Let $\operatorname{int} K \neq \emptyset$. $\overline{x} \in X$ is a local minimal point w.r.t. $\preccurlyeq_t^{\operatorname{int} K}$ if there is a neighborhood U of $\overline{x}$ such that there does not exist $x \in U \setminus \{\overline{x}\}$ such that $F(x) \preccurlyeq_t^{\operatorname{int} K} F(\overline{x})$.

Note that for all necessary optimality conditions for a local minimal point w.r.t. $\preccurlyeq_t^K$ and $\preccurlyeq_t^{\operatorname{int} K}$ (with $t \in \{l, pl\}$) in Section 4 of [18], the following assumption needs to be added: For a given $\overline{x} \in X$, assume that for all $\overline{y} \in F(\overline{x})$, $(F(\overline{x}) - \overline{y}) \cap (-K) \subseteq \{0\}$, or $(F(\overline{x}) - \overline{y}) \cap (-\operatorname{int} K) = \emptyset$, respectively. We demonstrate this with the following result.

Theorem 5.2 ([18, Theorem 4.10])**.** *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a closed-graph multifunction, K be a proper, closed, convex cone in Y and $\overline{x} \in X$ be a local minimal point w.r.t. $\preccurlyeq_{pl}^K$. Suppose that K is (SNC) at 0. Assume further that for all $\overline{y} \in F(\overline{x})$, it holds that $(F(\overline{x}) - \overline{y}) \cap (-K) \subseteq \{0\}$. Then, there exists some $\overline{y} \in F(\overline{x})$ and some $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*F(\overline{x}, \overline{y})(y^*).$$

We discuss the usefulness of the assumption $(F(\overline{x}) - \overline{y}) \cap (-K) \subseteq \{0\}$, or $(F(\overline{x}) - \overline{y}) \cap (-\operatorname{int} K) = \emptyset$, respectively, for all $\overline{y} \in F(\overline{x})$, in the remark below.

Remark 5.3. The relation $(F(\overline{x}) - \overline{y}) \cap (-K) \subseteq \{0\}$, or $(F(\overline{x}) - \overline{y}) \cap (-\operatorname{int} K) = \emptyset$, respectively, for all $\overline{y} \in F(\overline{x})$, is useful if the set-valued objective map F at the point $\overline{x}$ is given as Pareto frontier of a vector optimization problem. Moreover, it is possible to generalize our results to the case of so-called *minmax-set relations* (see Jahn, Ha [13]), that compare sets based on their minimal or maximal elements in terms of the vector approach. In this case, the above assumption does not impose any additional restriction.

Now, we extend our discussion to set optimization problems equipped with a variable domination structure. Before recalling a theorem about necessary optimality conditions for nondominated solutions of set-valued optimization problems given in [7], it is necessary to discuss an assumption concerning the domination structure $\mathcal{Q} : X \rightrightarrows Y$ by which the original problem $(P_{\mathcal{Q}})$ is equipped. That is

$$\forall x \in X : \ \mathcal{Q}(x) \text{ is a closed, convex, pointed, proper cone.}$$

Note that when $\mathcal{K}(y)$ is not necessarily a cone-valued map for all $y \in Y$, this requirement can be fulfilled for the mappings $\hat{\mathcal{Q}}$ and $\hat{\mathcal{Q}}'$ given respectively by (4.1) and (4.2). This will be illustrated by the following examples.

Example 5.4. Let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be given as

$$\forall (x_1, x_2) \in \mathbb{R}^2 : F(x_1, x_2) := \{(d_1, d_2) |\ 0 \le d_1 \le |x_1|,\ 0 \le d_2 \le |x_2|\}$$

and $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be determined by:

$$\forall (d_1, d_2) \in \mathbb{R}^2 :\ \mathcal{K}(d_1, d_2) := \begin{cases} \{(y_1, y_2) | y_2 \ge \frac{d_2}{d_1} y_1\} \cup \{(d_1, 0)\} & \text{if } d_1 \ne 0, \\ \{(y_1, y_2) | y_1 \le 0, y_2 \ge 0\} & \text{if } d_1 = 0. \end{cases}$$

Then, for all $(x_1, x_2) \in \mathbb{R}^2$ it holds that

$$\hat{\mathcal{Q}}(x_1, x_2) = \bigcap_{(d_1, d_2) \in F(x)} \mathcal{K}(d_1, d_2) = \{(y_1,\ y_2) |\ y_1 \le 0,\ y_2 \ge 0\}.$$

It is obvious that for all $(x_1, x_2) \in \mathbb{R}^2$ we have that $\hat{\mathcal{Q}}(x_1, x_2)$ is a closed, convex, pointed cone. However, $\mathcal{K}(y_1, y_2)$ is not a cone for all $(y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}$. See Figure 3 for the illustration of this example, where the image spaces of F, $\mathcal{K}$ and $\hat{\mathcal{Q}}$ are combined.

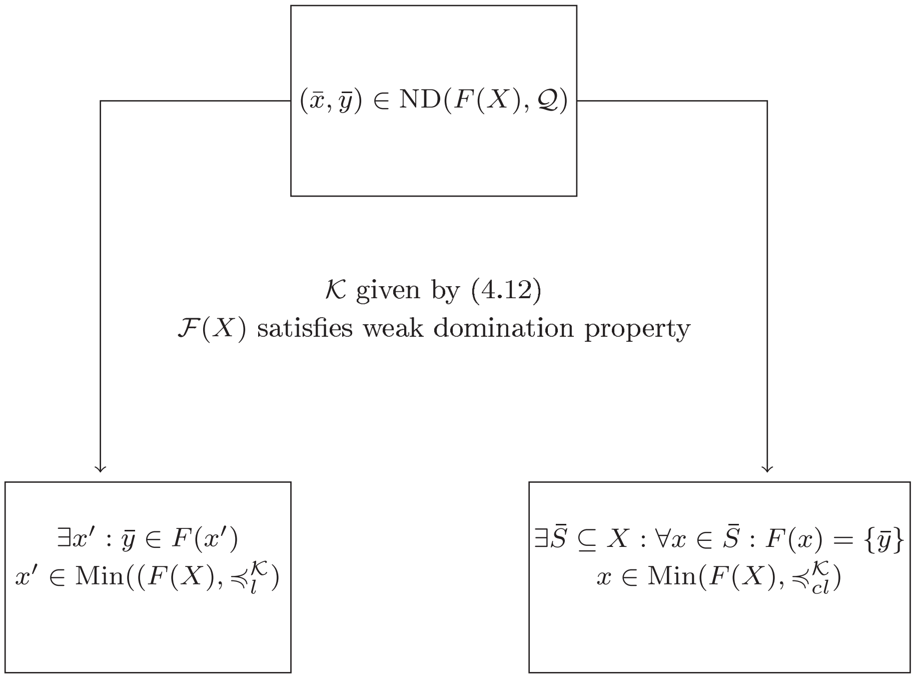

FIGURE 3. Illustration of Example 5.4

Example 5.5. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as

$$\forall x \in \mathbb{R} : F(x) = \{(d_1, d_2) \in \mathbb{R}^2 |\ d_2 = |x| d_1\}$$

and $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be determined by

$$\forall (d_1, d_2) \in \mathbb{R}^2 : \ \mathcal{K}(d_1, d_2) := \begin{cases} \{(y_1, y_2) |\ |d_2| \leq y_2 \leq \frac{|d_2|}{|d_1|} y_1\}, & \text{if } d_1 \neq 0, \\ \{(y_1, y_2) |\ y_1 \geq 0,\ y_2 = 0\}, & \text{if } d_1 = 0. \end{cases}$$

Then, it holds that:

$$\forall x \in \mathbb{R} : \hat{\mathcal{Q}}'(x) = \bigcup_{(d_1,d_2) \in F(x)} \mathcal{K}(d_1, d_2) = \{(y_1, y_2) |\ 0 \leq y_2 \leq |x| y_1\}.$$

This is a closed, convex and pointed cone. However, $\mathcal{K}(d_1, d_2)$ is not a cone for all $(d_1, d_2) \in F(x) \setminus \{0\}$. For an illustration, see Figure 4, where the image spaces of F, $\mathcal{K}$ and $\hat{\mathcal{Q}}'$ are combined.

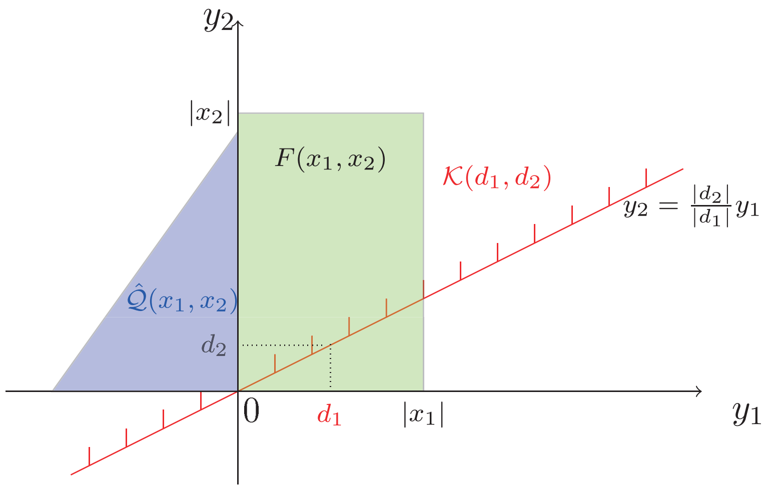

FIGURE 4. Illustration of Example 5.5

Thus, from now on, we consider the vector approach to define the solution of $(P_{\mathcal{Q}})$ with the assumption that for all $x \in X$, $\mathcal{Q}(x)$ ($\mathcal{Q}(x) := \hat{\mathcal{Q}}(x)$ or $\mathcal{Q}(x) := \hat{\mathcal{Q}}'(x)$) is a closed, convex, pointed, proper cone in Y.

In the next theorem, we recall a necessary condition for nondominated solutions of $(P_{\mathcal{Q}})$ (see Definition 3.8) given by Durea, Strugariu and Tammer in [7, Theorem 4.10]. The main idea in the proof of the necessary condition in [7, Theorem 4.10] is the incompatibility between openness and optimality (previously developed in [6]). It is interesting to mention that for the proof of [7, Theorem 4.10] Ekeland's Variational Principle is involved by the application of sufficient conditions in terms of coderivatives for the openness of the composition of multifunctions in Durea, Huynh, Nguyen and Strugariu [8, Theorem 4.2]). A corresponding result is shown by Bao and Mordukhovich in [3, Theorem 3.8] where the authors derived the result using the Extremal Principle as the main tool.

Theorem 5.6 ([7, Theorem 4.10]). *Let X, Y be Asplund spaces, F, $\mathcal{Q} : X \rightrightarrows Y$ be two set-valued maps such that for all $x \in X$, $F(x) \neq \emptyset$. Consider the set-valued optimization problem $(P_{\mathcal{Q}})$ and $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$. Furthermore, assume that*

$\operatorname{Gr} F$ and $\operatorname{Gr}\mathcal{Q}$ are closed around $(\bar{x}, \bar{y})$ and $(\bar{x}, 0)$, respectively. In addition, suppose that the following assumptions hold:

(a) *The sets $C_1 = \{(x, y, k) : (x, y) \in \operatorname{Gr} F,\ k \in Y\}$ and $C_2 = \{(x, y, k) : (x, k) \in \operatorname{Gr}\mathcal{Q},\ y \in Y\}$ are allied at $(\bar{x}, \bar{y}, 0)$;*

(b) *there is a neighborhood U of $\bar{x}$ such that $\bigcap_{x \in U} \mathcal{Q}(x) \neq \{0\}$;*

(c) *$\mathcal{Q}$ is lower semicontinuous at $\bar{x}$;*

(d) *F^{-1} is (PSNC) at $(\bar{y}, \bar{x})$ or $\mathcal{Q}^{-1}$ is (PSNC) at $(0, \bar{x})$.*

Then, there exists $y^* \in \mathcal{Q}(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\mathcal{Q}(\bar{x}, 0)(y^*).$$

Now, we are ready to show necessary optimality conditions for minimal solutions of $(P_{\mathcal{K}})$ based on the set approach.

Necessary optimality conditions for strictly optimal solutions of a set-valued optimization problem working on primal spaces (where the Bouligand derivative of a set-valued map is used) are presented by Eichfelder and Pilecka in [11, Theorem 5.1].

In the following theorems, we derive necessary optimality conditions using generalized differentiation objects lying in the dual spaces, i.e., we will use Mordukhovich's coderivative (see Definition 2.3).

First, we show a necessary optimality condition for strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the variable generalized lower less relation $\preccurlyeq_l^{\mathcal{K}}$, introduced in Definition 3.2, (i) using Theorem 5.6.

Theorem 5.7. *Consider problem $(P_{\mathcal{K}})$ w.r.t. the variable generalized lower less relation $\preccurlyeq_l^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \operatorname{SoMin}(F(X), \preccurlyeq_l^{\mathcal{K}})$. Assume that there is an element $\bar{y} \in F(\bar{x})$ satisfying*

$$\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y).$$

Furthermore, suppose that $\mathcal{K} : Y \rightrightarrows Y$ satisfies properties (3.1)-(3.3), and (3.5). Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ given by (4.1) and assume that the assumptions in Theorem 5.6 hold true for the two multifunctions $F, \hat{\mathcal{Q}}$. Then, there exists $y^* \in \hat{\mathcal{Q}}(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

Proof. Consider $\bar{x} \in \operatorname{SoMin}(F(X), \preccurlyeq_l^{\mathcal{K}})$. Then, from Theorem 4.2, it holds that for the element $\bar{y} \in F(\bar{x})$ satisfying $\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y)$ that $(\bar{x}, \bar{y}) \in \operatorname{ND}(F(X), \hat{\mathcal{Q}})$. Since all assumptions in Theorem 5.6 hold true for F and $\hat{\mathcal{Q}}$, it holds that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

The proof is complete. □

The following result is an assertion about a necessary optimality condition for strictly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the variable generalized possibly lower less relation $\preccurlyeq_{pl}^{\mathcal{K}}$ introduced in Definition 3.2, (v).

Theorem 5.8. *Consider problem $(P_{\mathcal{K}})$ w.r.t. the variable generalized possibly lower less relation $\preccurlyeq_{pl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, which satisfies that $\mathcal{K}(y)$ is a proper, convex cone* *for all $y \in Y$, and $\bar{x} \in \operatorname{SiMin}(F(X), \preccurlyeq^{\mathcal{K}}_{pl})$. Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ be determined by (4.1). Suppose that there is $\bar{y} \in F(\bar{x})$ satisfying $\bar{y} \notin F(\bar{x}) + \hat{\mathcal{Q}}(\bar{x}) \setminus \{0\}$. Assume that the two multifunctions $F, \hat{\mathcal{Q}}$ satisfy the assumptions given in Theorem 5.6. Then, there exists $y^* \in \hat{\mathcal{Q}}(\bar{x})^+ \setminus \{0\}$ such that*

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

Proof. We follow the line of the proof of Theorem 5.7. □

Now, we show a necessary optimality condition for strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the variable generalized certainly lower less relation $\preccurlyeq^{\mathcal{K}}_{cl}$, introduced in Definition 3.2, (iii).
The following theorem is a consequence of Theorem 5.7. It is proved by directly applying Collorary 4.3 and Theorem 5.6 and therefore, the proof is skipped.

Theorem 5.9. *Consider problem $(P_{\mathcal{K}})$ w.r.t. the variable generalized certainly lower less relation $\preccurlyeq^{\mathcal{K}}_{cl}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \operatorname{SoMin}(F(X), \preccurlyeq^{\mathcal{K}}_{cl})$. Assume that there is an element $\bar{y} \in F(\bar{x})$ satisfying*

$$\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y).$$

Furthermore, suppose that $\mathcal{K} : Y \rightrightarrows Y$ satisfies properties (3.1)-(3.3), and (3.5). Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ given by (4.1) and assume that the assumptions in Theorem 5.6 hold true for the two multifunctions $F, \hat{\mathcal{Q}}$. Then, there exists $y^* \in \hat{\mathcal{Q}}(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

In addition, we obtain a stronger necessary optimality condition for strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preccurlyeq^{\mathcal{K}}_{cl}$ again using Theorem 5.6 as follows:

Theorem 5.10. *Consider problem $(P_{\mathcal{K}})$ w.r.t. the variable generalized certainly lower less relation $\preccurlyeq^{\mathcal{K}}_{cl}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \operatorname{SoMin}(F(X), \preccurlyeq^{\mathcal{K}}_{cl})$, where $\mathcal{K} : Y \rightrightarrows Y$ satisfies properties (3.2) and (3.3). Suppose that there exists $\bar{y} \in F(\bar{x})$ satisfying $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$. Let $\hat{\mathcal{Q}}' : X \rightrightarrows Y$ be given by (4.2) such that*

$$\forall x \in X : \ \hat{\mathcal{Q}}'(x) \bigcap (-\hat{\mathcal{Q}}'(x)) = \{0\}.$$

Assume in addition that the following assumptions are fulfilled:

(i) *$\hat{\mathcal{Q}}'(\bar{x}) \neq \{0\}$ and $\hat{\mathcal{Q}}'(\bar{x})$is closed ;*
(ii) *F^{-1} is (PSNC) at $(\bar{y}, \bar{x})$ or $\hat{\mathcal{Q}}'^{-1}$ is (PSNC) at $(0, \bar{x})$.*

Then, there exists $y^* \in \hat{\mathcal{Q}}'(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*).$$

Proof. By Theorem 4.4, it holds that $(\bar{x}, \bar{y}) \in \operatorname{Min}(F(X), \hat{\mathcal{Q}}')$. Thus, $(\bar{x}, \bar{y}) \in \operatorname{ND}(F(X), \hat{\mathcal{Q}}'_*)$, where $\hat{\mathcal{Q}}'_* : X \rightrightarrows Y$ is defined by

$$\forall x \in X : \hat{\mathcal{Q}}'_*(x) := \hat{\mathcal{Q}}'(\bar{x}).$$

Now, we will prove that F and $\hat{\mathcal{Q}}'_*$ satisfy all assumptions supposed in Theorem 5.6.

$$\begin{aligned}\hat{D}^*\hat{\mathcal{Q}}'_*(u,k)(k^*) &= \{x^* \in X^* | (x^*, -k^*) \in \hat{N}(\operatorname{Gr}\hat{\mathcal{Q}}'_*, (u,k))\} \\ &= \{x^* \in X^* | (x^*, -k^*) \in \hat{N}(X \times \hat{\mathcal{Q}}'(\bar{x}), (u,k)\} \\ &= \{x^* \in X^* | (x^*, -k^*) \in \hat{N}(X,u) \times \hat{N}(\hat{\mathcal{Q}}'(\bar{x}), k)\} \\ &= \{0\}. \end{aligned} \tag{5.1}$$

The last equation is obtained by using [26, Proposition 6.41] and $\hat{N}(X,u) = \{0\}$. Therefore, the alliedness property of $(C_1,\ C_2)$ trivially holds (see [7]).
We have for all neighborhood U of $\bar{x}$ that $\bigcap_{x\in U} \hat{\mathcal{Q}}'_*(x) = \hat{\mathcal{Q}}'(\bar{x}) \neq \{0\}$, i.e., assumption (b) in Theorem 5.6 is fulfilled.
Since $\hat{\mathcal{Q}}'_*(\bar{x}) = \liminf_{x'\to\bar{x}} \hat{\mathcal{Q}}'_*(x') = \hat{\mathcal{Q}}'(\bar{x})$, it holds that $\hat{\mathcal{Q}}'_*$ is lsc at $\bar{x}$ (Remark 2.8).
Now, we apply Theorem 5.6 and get that there exists $y^* \in \hat{\mathcal{Q}}'_*(\bar{x})^+ \setminus \{0\} = \hat{\mathcal{Q}}'(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x},\bar{y})(y^*) + D^*\hat{\mathcal{Q}}'_*(\bar{x},0)(y^*).$$

By using the same lines to obtain (5.1), it also holds that $D^*\hat{\mathcal{Q}}'_*(\bar{x},0)(y^*) = \{0\}$ and thus we get the desired conclusion as follows:

$$\exists\ y^* \in \hat{\mathcal{Q}}'(\bar{x})^+ \setminus \{0\} \text{ such that } 0 \in D^*F(\bar{x},\bar{y})(y^*).$$

□

As shown in the previous parts, necessary optimality conditions are derived by means of the relationships between the minimal solutions of problem $(P_{\mathcal{K}})$ w.r.t. the lower set less relations ($\preccurlyeq^{\mathcal{K}}_l, \preccurlyeq^{\mathcal{K}}_{cl}$ and $\preccurlyeq^{\mathcal{K}}_{pl}$) and solution concepts of problem $(P_{\mathcal{Q}})$. According results concerning the upper set less relations ($\preccurlyeq^{\mathcal{K}}_u, \preccurlyeq^{\mathcal{K}}_{cu}$ and $\preccurlyeq^{\mathcal{K}}_{pu}$) do not hold true in general, since these relations are related to the 'worst cases'. On the other hand, the definitions of nondominated and minimal solutions of $(P_{\mathcal{Q}})$ are given similarly when we concern to the 'best' cases. Therefore, in the sequel, we use another approach to obtain necessary optimality conditions for solutions of problem $(P_{\mathcal{K}})$ w.r.t. the upper set less relations ($\preccurlyeq^{\mathcal{K}}_u$ and $\preccurlyeq^{\mathcal{K}}_{cu}$). Now, we briefly illustrate our method to derive these necessary optimality conditions. It is necessary to mention that in [7] the authors derive optimality conditions for solutions of problem $(P_{\mathcal{Q}})$ by means of the sufficient conditions for the openness of a sum valued-mappings, that is $F + \mathcal{Q}$. However, we are concerning the set-valued problem $(P_{\mathcal{K}})$, where $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$, i.e., F and $\mathcal{K}$ have different pre-image spaces. Futhermore, looking at the definitions of the relations $\preccurlyeq^{\mathcal{K}}_u$ and $\preccurlyeq^{\mathcal{K}}_{cu}$, we can see that they are related to composition of multifunctions F and $\mathcal{K}$. For that reason, we study the sufficient conditions for the openness of a composition of multifunctions contributed from our objective function F and the variable domination structure $\mathcal{K}$. These sufficient conditions in terms of Mordukhovich's coderivative are recently given by Durea, Huynh, Nguyen and Strugariu in [8].
Let $F_1 : X \rightrightarrows Y_1$, $F_2 :\ X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings where X, Y_1, Y_2, Z are Asplund spaces. Consider the following composition multifunctions

$H : X \rightrightarrows Z$ defined as

$$H(x) := \bigcup_{\substack{y_2 \in F_2(x) \\ y_1 \in F_1(x)}} G(y_1, y_2). \tag{5.2}$$

The next theorem gives sufficient conditions in terms of coderivatives for the openness of the composition of set-valued mappings (see [8, Theorem 4.2]), where Ekeland's Variational Principle is the main tool in the proof.

Theorem 5.11. [8, Theorem 4.2] *Let X, Y_1, Y_2, Z be Asplund spaces. Suppose that $F_1 : X \rightrightarrows Y_1$, $F_2 :\ X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ are closed-graph multifunctions and $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $\bar{z} \in G(\bar{y}_1, \bar{y}_2)$, $(\bar{y}_1, \bar{y}_2) \in F_1(\bar{x}) \times F_2(\bar{x})$. Assume that the following sets are allied at $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$*

$$\begin{aligned} \hat{C}_1 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z :\ y_1 \in F_1(x)\}, \\ \hat{C}_2 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z :\ y_2 \in F_2(x)\}, \\ \hat{C}_3 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z :\ z \in G(y_1, y_2)\}. \end{aligned} \tag{5.3}$$

Suppose that there exists $c > 0$ such that

$$c < \liminf_{\substack{(t_1,t_2,w) \xrightarrow{\mathrm{Gr}\, G} (\bar{y}_1,\bar{y}_2,\bar{z}),\delta\downarrow 0 \\ (u_1,v_1) \xrightarrow{\mathrm{Gr}\, F_1} (\bar{x},\bar{y}_1),\ (u_2,v_2) \xrightarrow{\mathrm{Gr}\, F_2} (\bar{x},\bar{y}_2)}} \left\{ \|x_1^* + x_2^*\| : \begin{cases} x_1^* \in \hat{D}^* F_1(u_1, v_1)(t_1^*) \\ x_2^* \in \hat{D}^* F_2(u_2, v_2)(t_2^*) \\ (z_1^* + t_1^*, z_2^* + t_2^*) \in \hat{D}^* G(t_1, t_2, w)(w^*) \\ \|w^*\| = 1,\ \|z_1^*\| < \delta,\ \|z_2^*\| < \delta \end{cases} \right\}. \tag{5.4}$$

Then, for every $L \in (0, c)$, H (given by (5.2)) is L-open at $(\bar{x}, \bar{z})$.

In order to apply the Theorem 5.11 to our problem $(P_{\mathcal{K}})$, it is necessary to determine appropriate set-valued maps as follows.
Let Y_1, Y_2 and Z be equal to the space Y, and suppose that the set-valued mappings $F_1,\ F_2 : X \rightrightarrows Y$ and $G : Y \times Y \rightrightarrows Y$ are respectively determined by

$$\begin{gathered} \forall x \in X :\ F_1(x) := F(x), \\ \forall\ x \in X :\ F_2(x) := \{0\}, \\ \forall\ (y_1, y_2) \in Y \times Y :\ G(y_1, y_2) := (I - \mathcal{K})(y_1) = y_1 - \mathcal{K}(y_1). \end{gathered}$$

Because G only depends on y_1, instead of studying G, we invest the following set-valued map

$$\hat{G} : Y \rightrightarrows Y$$

such that

$$\forall\ y \in Y :\ \hat{G}(y) := (I - \mathcal{K})(y) = y - \mathcal{K}(y).$$

Let $\hat{H} : X \rightrightarrows Y$ defined by

$$\hat{H}(x) := \bigcup_{y \in F(x)} \hat{G}(y) = \bigcup_{y \in F(x)} (y - \mathcal{K}(y)).$$

From the setting of $F_1, F_2, \hat{G}$, the allied property of $(\hat{C}_1, \hat{C}_2, \hat{C}_3)$ in (5.3) becomes the allied property of (E_1, E_2) given as

$$E_1 := \{(x, y, z) \in X \times Y \times Y :\ y \in F(x)\},$$

$$E_2 := \{(x, y, z) \in X \times Y \times Y : \ z \in \hat{G}(y)\}.$$

Proposition 5.12. *Consider problem ($P_{\mathcal{K}}$) w.r.t. the variable generalized upper less relation $\preccurlyeq_u^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$ satisfying* (3.1), *and* $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_u^{\mathcal{K}})$ *and* $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, F$. *Suppose that there is a neighborhood U of $\bar{x}$ such that $\hat{H}(U) - \bigcap_{x \in U} \hat{H}(x)$ is a proper cone. Then, $\hat{H}$ is not open at $(\bar{x}, \bar{y})$.*

Proof. Since for all $y \in Y$, $0 \in \mathcal{K}(y)$, we get that $F(\bar{x}) \preccurlyeq_u^{\mathcal{K}} F(\bar{x})$. Taking into account that $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_u^{\mathcal{K}})$, it holds that

$$\forall x \in X : \ F(\bar{x}) \preccurlyeq_u^{\mathcal{K}} F(x) \quad \Longleftrightarrow \quad F(\bar{x}) \subseteq \bigcup_{y \in F(x)} (y - \mathcal{K}(y)). \tag{5.5}$$

Let $\bar{y} \in F(\bar{x})$ be arbitrarily given. Then, (5.5) implies that

$$\forall x \in X : \ \bar{y} \in \bigcup_{y \in F(x)} (y - \mathcal{K}(y)) = \hat{H}(x).$$

Suppose, by contradiction, that $\hat{H}$ is open at $(\bar{x}, \bar{y})$. Then, for the given neighborhood U of $\bar{x}$, there is an open set $V (\bar{y} \in V)$ such that $V \subseteq \hat{H}(U)$, which is equivalent to

$$V \subseteq \bigcup_{y \in F(U)} (y - \mathcal{K}(y)).$$

Let us choose $y \in V$ arbitrarily. Then, there is $x \in U$ such that

$$y \in \bigcup_{y \in F(x)} (y - \mathcal{K}(y)) = \hat{H}(x).$$

Therefore,

$$\begin{aligned} y - \bar{y} \in \hat{H}(x) - \bar{y} &\subseteq \hat{H}(x) - \bigcap_{x \in U} \hat{H}(x) \\ &\subseteq \hat{H}(U) - \bigcap_{x \in U} \hat{H}(x). \end{aligned}$$

This implies

$$V - \bar{y} \subseteq \hat{H}(U) - \bigcap_{x \in U} \hat{H}(x).$$

Since the first set is absorbing and the second one is a cone, it follows that

$$Y \subseteq \hat{H}(U) - \bigcap_{x \in U} \hat{H}(x),$$

contradicting the fact that $\hat{H}(U) - \bigcap_{x \in U} \hat{H}(x)$ is proper.

□

Now, we show a necessary optimality condition for strongly minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the variable generalized upper less relation $\preccurlyeq_u^{\mathcal{K}}$, introduced in Definition 3.2, (ii).

Theorem 5.13. *Let X, Y be Asplund spaces. Consider problem ($P_{\mathcal{K}}$) w.r.t. the variable generalized upper less relation $\preccurlyeq_u^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_u^{\mathcal{K}})$. Suppose $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, F$, F and $\hat{G} := I - \mathcal{K}$ be closed graph multifunctions. Suppose in addition that the following assertions hold true:*

(i) $\forall y \in Y : \; 0 \in \mathcal{K}(y)$;

(ii) *there is a neighborhood U of $\bar{x}$ such that $\hat{H}(U) - \bigcap_{x \in U} \hat{H}(x)$ is a proper cone;*

(iii) $\{E_1, E_2\}$ *are allied at* $(\bar{x}, \bar{y}, \bar{y})$;

(iv) F^{-1} *is (PSNC) at* $(\bar{y}, \bar{x})$ *and* $\hat{G}^{-1}$ *is (PSNC) at* $(\bar{y}, \bar{y})$;

(v) $D^*\hat{G}(\bar{y}, \bar{y})(0) = \{0\}$.

Then, for all $\bar{y} \in F(\bar{x})$ there exist $w^* \in Y^* \setminus \{0\}$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(t^*).$$

Proof. We have from Proposition 5.12 that $\hat{H}$ is not open at $(\bar{x}, \bar{y})$, hence it is not linearly open at this point. Since the other conditions from Theorem 5.11 are satisfied, the condition (5.4) does not hold true. Consequently, there exist sequences $(u_n, v_n) \xrightarrow{\mathrm{Gr}\, F} (\bar{x}, \bar{y})$, $(t_n, w_n) \xrightarrow{\mathrm{Gr}\, \hat{G}} (\bar{y}, \bar{y})$, $(w_n^*) \subset S_{Y^*}$, $(x_n^*) \in X^*$, $z_n^* \to 0$ such that

$$\forall\, n : \; x_n^* \in \hat{D}^*F(u_n, v_n)(t_n^*), z_n^* + t_n^* \in \hat{D}^*\hat{G}(t_n, w_n)(w_n^*) \text{ and } \|x_n^*\| \to 0. \tag{5.6}$$

Now we prove that (t_n^*) is bounded. Suppose the contradiction and by $z_n^* \to 0$ we get that for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ sufficiently large such that

$$n < \|t_{kn}^*\| + \|z_{kn}^*\|. \tag{5.7}$$

For the reason of keeping the notation simple, we denote the subsequences (t_{kn}^*), (z_{kn}^*) by (t_n^*), (z_n^*), respectively. Because of the positive homogeneity of the Fréchet coderivatives, we have that

$$\frac{x_n^*}{n} \in \hat{D}^*F(u_n, v_n)(\frac{t_n^*}{n}),$$

and

$$\frac{1}{n}(z_n^* + t_n^*) \in \hat{D}^*\hat{G}(t_n, w_n)(\frac{w_n^*}{n}).$$

It yields

$$(\frac{x_n^*}{n}, \frac{-t_n^*}{n}) \in \hat{N}(\mathrm{Gr}\, F, (u_n, v_n)) \text{ and } (\frac{z_n^* + t_n^*}{n}, \frac{-w_n^*}{n}) \in \hat{N}(\mathrm{Gr}\, \hat{G}, (t_n, w_n)).$$

Thus,

$$(\frac{x_n^*}{n}, \frac{-t_n^*}{n}, 0) \in \hat{N}(E_1, (u_n, v_n, \bar{y})) \text{ and } (0, \frac{z_n^* + t_n^*}{n}, \frac{-w_n^*}{n}) \in \hat{N}(E_2, (\bar{x}, t_n, w_n)).$$

Since $w_n^* \in S_{Y^*}, z_n^* \to 0$ and $\|x_n^*\| \to 0$, it holds that

$$(\frac{x_n^*}{n}, \frac{-t_n^*}{n}, 0) + (0, \frac{z_n^* + t_n^*}{n}, \frac{-w_n^*}{n}) = (\frac{x_n^*}{n}, \frac{z_n^*}{n}, \frac{-w_n^*}{n}) \to 0.$$

Then, the alliedness of the sets (E_1, E_2) implies $\frac{1}{n}(z_n^* + t_n^*) \to 0$, which is impossible in virtue of relation (5.7).

Consequently, since Y is Asplund and (t_n^*) is bounded, we get that there is a subsequence of (t_n^*) which weak* converges to $t^* \in Y^*$. Also, since $(w_n^*) \subset S_{Y^*}$, (w_n^*)

contains a weak* convergent subsequence to an element w^*. For simplicity, we denote this subsequence also by (w_n^*).
We claim that $t^* = w^* = 0$ does not hold true. Indeed, suppose that $t^* = w^* = 0$, i.e., $t_n^* \xrightarrow{w^*} 0$ and $w_n^* \xrightarrow{w^*} 0$. Taking into account F^{-1} is (PSNC) at $(\bar{y}, \bar{x})$, and $x_n^* \to 0$, it holds that $t_n^* \to 0$. From $z_n^* \to 0$ we get $(t_n^* + z_n^*) \to 0$. In addition, $w_n^* \xrightarrow{w^*} 0$ and $\hat{G}^{-1}$ is (PSNC) at $(\bar{y}, \bar{y})$, it holds that $w_n^* \to 0$, which contradicts the fact that $(w_n^*) \subset S_{Y^*}$.
In addition, taking into account (5.6), it holds that there exist some $w^* \in Y^*$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ satisfying $0 \in D^*F(\bar{x}, \bar{y})(t^*)$.
It is obvious from the assumption $D^*\hat{G}(\bar{y}, \bar{y})(0) = \{0\}$ that if $w^* = 0$, then $t^* = 0$, a contradiction. Then, $w^* \neq 0$. The proof is complete. □

Now, we consider the variable generalized certainly upper less relation $\preccurlyeq_{cu}^{\mathcal{K}}$. From Remark 3.10 (ii), we know that $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cu}^{\mathcal{K}})$ implies $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{u}^{\mathcal{K}})$. Therefore, from Theorem 5.13, we have the following result.

Theorem 5.14. *Let X, Y be Asplund spaces. Consider problem $(P_{\mathcal{K}})$ w.r.t. the variable generalized certainly upper less relation $\preccurlyeq_{cu}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cu}^{\mathcal{K}})$. Assume $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, F$, F and $\hat{G} := I - \mathcal{K}$ be closed graph multifunctions. Suppose in addition that the assumptions (i)-(v) in Theorem 5.13 are fulfilled. Then, for all $\bar{y} \in F(\bar{x})$ there exist $w^* \in Y^* \setminus \{0\}$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ such that*

$$0 \in D^*F(\bar{x}, \bar{y})(t^*).$$

Proof. Since $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{cu}^{\mathcal{K}})$, we get $\bar{x} \in \mathrm{SoMin}(F(X), \preccurlyeq_{u}^{\mathcal{K}})$. Since all assumptions given in Theorem 5.13 hold true, we follow the same line of its proof to obtain that: for all $\bar{y} \in F(\bar{x})$ there exist $w^* \in Y^* \setminus \{0\}$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(t^*).$$

□

6. Conclusions

In this work, we investigate very general set relations w.r.t. a variable domination structure, which is not necessarily given by a cone-valued map. We consider two approaches to define solutions of a set optimization problem and investigate the relationships between these two approaches. The main result of this paper consists in deriving necessary optimality conditions in terms of Mordukhovich's coderivative for solutions of set-valued problems w.r.t. to various set relations. Our research leaves many possibilities for future investigations. The next step is to apply our results to real-world applications, for example taking into account uncertainties in economic, radiotherapy treatment and behavioral sciences.

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