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ON THE GENERALIZED SEQUENTIAL NORMAL COMPACTNESS IN VARIATIONAL ANALYSIS

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Dedicated to Boris S. Mordukhovich on the occasion of his 70th birthday.

Abstract. We study the generalized sequential normal compactness conditions in variational analysis and establish their complete calculus in Asplund spaces; we also extend the calculus of Mordukhovich generalized differential constructions utilizing these conditions.

1. Introduction and preliminaries

As variants/generalizations of Lipschitz properties of mappings and sets, sequential normal compactness conditions play an essential role in the development of Mordukhovich's generalized differentiation theory in variational analysis, which is a very active and fruitful field of mathematics in the past few decades; we refer the readers to [2,3] as well as [1,10] and references therein for extensive expositions of the field and its applications.

In this paper we study the generalized sequential normal compactness conditions introduced in [11] and provide further results following [12] in Asplund spaces, which are spaces on which every continuous convex function is generically Fréchet differentiable. The set of Asplund spaces is a very broad class of Banach spaces containing reflexive spaces, Fréchet smooth spaces, etc. We shall develop complete calculus rules of the generalized sequential normal compactness conditions for sets and mappings under various operations, and extend the calculus of Mordukhovich generalized differential constructions employing these conditions. The readers can find more discussion of these conditions and results in this direction in general Banach spaces in [14].

Let X be a Banach space with its dual space X^*. For a sequence $\{x_k^*\}_{k=0}^{\infty} \subset X^*$ and a functional $x^* \in X^*$, by $x_k^* \to x^*$ $(k \to \infty)$ we mean that the sequence converges to x^* in the norm topology of X^* when k approaches infinity (we often omit writing $k \to \infty$ when no confusion arises), and by $x_k^* \xrightarrow{w^*} x^*$ we mean that the sequence converges to x^* in the weak-star topology of X^*. Given a nonempty

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subset Ω of X with $\bar{x} \in \Omega$ and a scaler $\varepsilon \geq 0$, the set $\widehat{N}_\varepsilon(\bar{x};\Omega)$ of ε-normals of Ω at $\bar{x}$ is defined as

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq \varepsilon\Big\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ with $x \in \Omega$. When $\bar{x}$ is an isolated point, $\widehat{N}_\varepsilon(\bar{x};\Omega)$ is defined as X^*. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ is called the Fréchet normal cone.

In the sequel, we often consider products of Banach spaces in the form

$$X = \prod_{j \in J_X} X_j, \quad Y = \prod_{j \in J_Y} Y_j, \quad Z = \prod_{j \in J_Z} Z_j, \tag{1.1}$$

where $m, n, p \in \mathbb{N}$, $J_X = \{1, \cdots, m\}$, $J_Y = \{1, \cdots, n\}$, $J_Z = \{1, \cdots, p\}$, and X_j, Y_j, Z_j are Banach spaces, respectively. As the main object of the paper, the generalized sequential normal compactness (GSNC) can be defined as below:

Definition 1.1. Let $J_1, J_2 \subset J_X$, $\Omega \subset X$ be a nonempty set with $\bar{x} \in \Omega$. We say that Ω is generalized sequentially normally compact (GSNC) at $\bar{x} \in \Omega$ with respect to $\{X_j \mid j \in J_1\}$ (or J_1 for simplicity) through $\{X_j \mid j \in J_2\}$ (or J_2 for simplicity) if for all sequence $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* = (x_{1k}^*, \ldots, x_{mk}^*) \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$, the following holds:

$$\begin{bmatrix} x_{jk}^* \xrightarrow{w^*} 0 \ (j \notin J_2) \\ x_{jk}^* \to 0 \ (j \in J_2) \end{bmatrix} \Longrightarrow \big[x_{jk}^* \to 0 \ (j \in J_1)\big]. \tag{1.2}$$

When X is an Asplund space and the set Ω is closed around $\bar{x}$, we can equivalently replace the ε-normal cones in Definition 1.1 by the Fréchet normal cones. If $J_1 = J_X$, $J_2 = \emptyset$ in Definition 1.1, then the GSNC reduces to the sequential normal compactness (SNC) of Ω at $\bar{x}$; if $J_2 = J_X \backslash J_1$, then the GSNC reduces to the partial sequential normal compactness (PSNC) of Ω at $\bar{x}$ with respect to J_1; if $J_2 = \emptyset$, then the GSNC reduces to the strong partial sequential normal compactness (strong PSNC, or SPSNC) of Ω at $\bar{x}$ with respect to J_1. In this way the GSNC condition unifies the existing SNC/PSNC/strong PSNC conditions. We refer the readers to [4, 6, 7, 12–14] or [2] and references therein for more discussions of these compactness conditions and related notions both in Banach spaces and in Asplund spaces.

For the subset $J_1 \subset J_X$ as in Definition 1.1, and nonempty subsets Ω_1, Ω_2 of X, recall that $\{\Omega_1, \Omega_2\}$ is said to satisfy the mixed qualification condition at $\bar{x} \in \Omega_1 \cap \Omega_2$ with respect to $\{X_j \mid j \in J_1\}$ (or J_1 for simplicity) if for any $\varepsilon_k \downarrow 0$, $u_k \xrightarrow{\Omega_1} \bar{x}$, $v_k \xrightarrow{\Omega_2} \bar{x}$, $u_k^* = (u_{jk}^*)_{j \in J_X} \in \widehat{N}_{\varepsilon_k}(u_k;\Omega_1)$, $v_k^* = (v_{jk}^*)_{j \in J_X} \in \widehat{N}_{\varepsilon_k}(v_k;\Omega_2)$ with $(u_k^*, v_k^*) \xrightarrow{w^*} (u^*, v^*)$ $(k \to \infty)$, one has

$$\big[u_{jk}^* + v_{jk}^* \xrightarrow{w^*} 0 \ (j \in J \backslash J_1), \quad u_{jk}^* + v_{jk}^* \to 0 \ (j \in J_1)\big] \Rightarrow u^* = v^* = 0.$$

The mixed qualification condition was introduced in [6] for the development of the calculus of sequential normal compactness conditions. When $J_1 = J_X$, the mixed qualification condition reduces to the limiting qualification condition introduced in [5]. Clearly both qualification conditions are implied by the following more restrictive normal qualification condition corresponding to the case $J_1 = \emptyset$:

$$N(\bar{x};\Omega_1) \cap \big[-N(\bar{x};\Omega_2)\big] = \{0\}.$$

See [5, 6] or [2] for further discussions.

We proceed to recall more notions to be used in the paper. By taking the Painlevé-Kuratowski outer limit of ε-normal cones, we obtain the Mordukhovich normal cone $N(\bar{x};\Omega)$:

$$N(\bar{x};\Omega) := \big\{x^* \in X^* \mid \exists \varepsilon_k \downarrow 0, x_k^* \overset{w^*}{\to} x^*, x_k \overset{\Omega}{\to} \bar{x} \text{ with } x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)\big\}.$$

Let $F\colon X \rightrightarrows Y$ be a set-valued mapping/multifunction. By $\operatorname{gph} F$ we mean the graph of F, and by $\ker F$ we mean the kernel of F defined by

$$\ker F := \{x \in X \mid 0 \in F(x)\}.$$

TheMordukhovich normal coderivative $D_N^* F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is defined as

$$\begin{aligned} D_N^* F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \mid \exists \varepsilon_k \downarrow 0, x_k^* \overset{w^*}{\to} x^*, y_k^* \overset{w^*}{\to} x^*, (x_k, y_k) \overset{\operatorname{gph} F}{\to} (\bar{x},\bar{y}) \\ \text{with } (x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k,y_k);\operatorname{gph} F\big)\big\} \qquad \forall y^* \in Y^*. \end{aligned}$$

If the weak-star-convergence $y_k^* \overset{w^*}{\to} y^*$ in the above definition of $D_N^* F(\bar{x},\bar{y})$ is replaced by the norm-convergence $y_k^* \to y^*$, then we have the Mordukhovich mixed coderivative $D_M^* F(\bar{x},\bar{y})$. If $x_k^* \overset{w^*}{\to} x^*$ in the above definition of $D_N^* F(\bar{x},\bar{y})$ is replaced by $x_k^* \to x^*$, then we have the Mordukhovich reversed mixed coderivative $\widetilde{D}_M^* F(\bar{x},\bar{y})$.

Letting $\varphi\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\} = (-\infty,\infty]$, we denote its epigraph by $\operatorname{epi}\varphi$ and assume that $\varphi(\bar{x}) < \infty$ for some $\bar{x} \in X$. φ is said lower semicontinuous (l.s.c.) around $\bar{x}$ if $\operatorname{epi}\varphi$ is closed around $(\bar{x},\varphi(\bar{x}))$. Let $E_\varphi\colon X \rightrightarrows \mathbb{R}$ be a set-valued mapping specified by the relation $\operatorname{gph} E_\varphi = \operatorname{epi}\varphi$; then the Mordukhovich subdifferential $\partial\varphi(\bar{x})$ and singular subdifferential $\partial^\infty\varphi(\bar{x})$ of φ at $\bar{x}$ are respectively defined as

$$\partial\varphi(\bar{x}) := D_N^* E_\varphi(\bar{x},\varphi(\bar{x}))(1), \quad \partial^\infty\varphi(\bar{x}) := D_N^* E_\varphi(\bar{x},\varphi(\bar{x}))(0).$$

The Mordukhovich normal cone, coderivatives, and subdifferentials are fundamental notions in variational analysis and its applications; we refer the readers to [2, 3] and references therein for their history, calculus rules, and applications as well as related topics. Readers can find more development on these notions in [8, 9].

For a multifunction $F\colon X \rightrightarrows Y$ with $\bar{y} \in F(\bar{x})$, we say that F is inner semicontinuous at $(\bar{x},\bar{y})$ if for any sequence $x_k \to \bar{x}$ with $F(x_k) \neq \emptyset$, there is a sequence $y_k \in F(x_k)$ such that $y_k \to \bar{y}$ $(k \to \infty)$. We say that F is inner semicompact at $\bar{x}$ if for any sequence $x_k \to \bar{x}$ with $F(x_k) \neq \emptyset$, there is a sequence $y_k \in F(x_k)$ that contains a convergent subsequence. We refer the readers to [2] and references therein for more information about these two notions, and to [8, 9] for recent development. In the following two sections, when these two conditions are involved, we only present the results in the case of inner semicompactness, and omit the corresponding formulations for the case of inner semicontinuity for simplicity; see Remark 3.9 for more details.

In the rest of the paper, we develop calculus of the GSNC conditions in Asplund spaces in section 2, and then establish extended generalized differentiation results utilizing the GSNC conditions in section 3.

2. GSNC in Asplund spaces

We study preservations of the GSNC properties of sets/mappings under various set/mapping operations in Asplund spaces in this section. The obtained calculus rules extend/unify the corresponding results in [4,6]. First we recall the intersection rule for the GSNC of intersections of sets from [11,12] extending Theorem 3.3 in [6]:

Theorem 2.1. *Let X as in (1.1) be a product of Asplund spaces and $\Omega_1, \Omega_2 \subset X$ be locally closed around $\bar{x} \in \Omega_1 \cap \Omega_2$. Let $J_{i1}, J_{i2} \subset J_X$ with $J_{i1} \cap J_{i2} = \emptyset$ $(i = 1, 2)$ and $J_{11} \cup J_{21} = J_X$. Suppose that the following assumptions hold:*

(i) *Ω_i is GSNC at $\bar{x}$ with respect to J_{i1} through J_{i2} $(i = 1, 2)$;*
(ii) *Either Ω_1 is strongly PSNC at $\bar{x}$ with respect to J_{22}, or Ω_2 is strongly PSNC at $\bar{x}$ with respect to J_{12};*
(iii) *The mixed qualification condition with respect to $J_{12} \cup J_{22}$ holds for $\{\Omega_1, \Omega_2\}$ at $\bar{x}$.*

Then $\Omega_1 \cap \Omega_2$ is GSNC at $\bar{x}$ with respect to $J_{11} \cap J_{21}$ through $J_{12} \cup J_{22}$.

As demonstrated in the remaining part of the section, the intersection rule above implies many useful calculus results for the GSNC of sets or graphs of mappings. We first point out two simple cases. Let $F_i\colon X \rightrightarrows Y$, $\varphi_i\colon X \to \overline{\mathbb{R}}$ $(i = 1, 2)$ and consider the mappings $F_1 \cap F_2\colon X \rightrightarrows Y$, $\max\{\varphi_1, \varphi_2\}\colon X \to \overline{\mathbb{R}}$ defined by

$$(F_1 \cap F_2)(x) = F_1(x) \cap F_2(x), \quad \max\{\varphi_1, \varphi_2\}(x) = \max\{\varphi_1(x), \varphi_2(x)\} \quad \forall x \in X. \tag{2.1}$$

Then we can directly apply Theorem 2.1 to obtain the corresponding GSNC results for $\operatorname{gph}(F_1 \cap F_2)$ and $\operatorname{epi}\max\{\varphi_2, \varphi_2\}$ due to the relations

$$\operatorname{gph}(F_1 \cap F_2) = \operatorname{gph} F_1 \cap \operatorname{gph} F_2, \quad \operatorname{epi}\max\{\varphi_1, \varphi_2\} = \operatorname{epi}\varphi_1 \cap \operatorname{epi}\varphi_2.$$

For simplicity we omit the details (cf. Proposition 4.6 in [6]).

For a multifunction $F\colon X \rightrightarrows Y$ and a set $\Theta \subset Y$, the inverse image $F^{-1}(\Theta)$ of Θ under F is defined as

$$F^{-1}(\Theta) = \{x \in X \mid F(x) \cap \Theta \neq \emptyset\}.$$

The following result extends Theorem 3.8 in [6] regarding the normal compactness of inverse images.

Theorem 2.2. *Let $F\colon X \rightrightarrows Y$ be a multifunction with its graph $\operatorname{gph} F$ closed, $\Theta \subset Y$ be a closed subset with $\bar{x} \in F^{-1}(\Theta)$, where X, Y as in (1.1) are products of Asplund spaces, and let $J_{Xi} \subset J_X$, $J_{Yi} \subset J_Y$ $(i = 1, 2)$ with $J_{X1} \cap J_{X2} = \emptyset$, $J_{Y1} \cap J_{Y2} = \emptyset$. Assume that the mapping $S\colon X \rightrightarrows Y$ defined by*

$$S(x) = F(x) \cap \Theta \quad \forall x \in X$$

is inner semicompact at $\bar{x}$, and that for every $\bar{y} \in S(\bar{x})$ the following hold:

(i) *$\operatorname{gph} F$ is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X1}\} \cup \{Y_j \mid j \in J_{Y1}\}$ through $\{X_j \mid j \in J_{X2}\} \cup \{Y_j \mid j \in J_{Y2}\}$;*

(ii) *Θ is PSNC at $\bar{y}$ with respect to $\{Y_j \mid j \in J_Y \backslash J_{Y1}\}$, and is strongly PSNC at this point with respect to $\{Y_j \mid j \in J_{Y2}\}$;*

(iii) *F and Θ satisfy the qualification condition*

$$N(\bar{y};\Theta) \cap \ker D_N^* F(\bar{x},\bar{y}) = \{0\}. \tag{2.2}$$

Then the inverse image $F^{-1}(\Theta)$ is GSNC at $\bar{x}$ with respect to $\{X_j \mid j \in J_{X1}\}$ through $\{X_j \mid j \in J_{X2}\}$.

Proof. Take any sequence $\varepsilon_k \downarrow 0$, $x_k \to \bar{x}$ with $x_k \in F^{-1}(\Theta)$, and $x_k^* = (x_{jk}^*)_{j \in J_X} \in \widehat{N}_{\varepsilon_k}\big(x_k; F^{-1}(\Theta)\big)$ with $x_{jk}^* \to 0$ $(j \in J_{X2})$ and $x_{jk}^* \overset{w^*}{\to} 0$ $(j \in J_X \backslash J_{X2})$. It suffices to show that $x_{jk}^* \to 0$ $(j \in J_{X1})$ along a subsequence under the assumptions made. According to the choice of x_k, $F(x_k) \cap \Theta \neq \emptyset$. Then by the inner semicompactness of the mapping S at $\bar{x}$, there is a sequence $y_k \in F(x_k) \cap \Theta$ that contains a convergent subsequence. We may assume, without loss of generality, that $y_k \to \bar{y}$. Then $\bar{y} \in F(\bar{x}) \cap \Theta$ due to the closedness of $\operatorname{gph} F$ and Θ. Let

$$\Omega_1 := \operatorname{gph} F, \quad \Omega_2 := X \times \Theta.$$

Then both Ω_1 and Ω_2 are closed and $(\bar{x},\bar{y}) \in \Omega_1 \cap \Omega_2$. One can verify by the definition of ε-normal cones that

$$(x_k^*, 0) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \Omega_1 \cap \Omega_2\big), \quad k \in \mathbb{N}. \tag{2.3}$$

We see that Ω_2 is PSNC at $(\bar{x},\bar{y})$ with respect to $\{X_j \mid j \in J_X\} \cup \{Y_j \mid j \in J_Y \backslash Y_1\}$, is strongly PSNC at this point with respect to $\{X_j \mid j \in J_{X2}\} \cup \{Y_j \mid j \in J_{Y2}\}$, and the qualification condition (2.2) implies the qualification for $\{\Omega_1, \Omega_2\}$ in Theorem 2.1. Taking into account assumption (i) on the GSNC property of $\operatorname{gph} F$, we can apply Theorem 2.1 and obtain $x_{jk}^* \to 0$ $(j \in J_{X1})$ by (2.3), which completes the proof. $\square$

Theorem 2.2 reduces to the two cases of Theorem 3.8 in [6] in the following two special situations: (i) $J_{X1} = J_X$, $J_{X2} = \emptyset$, $J_{Y1} = \emptyset$, $J_{Y2} = J_Y$; (ii) $J_{X1} = J_X$, $J_{X2} = \emptyset$, $J_{Y1} = J_Y$, $J_{Y2} = \emptyset$. When $Y = \mathbb{R}$ in Theorem 2.2, we obtain the GSNC properties of level sets of scalar functions below, which extends Corollary 3.9 in [6].

Theorem 2.3. *Let $\varphi\colon X \to \overline{\mathbb{R}}$ with $\varphi(\bar{x}) = 0$, where X as in (1.1) is a product of Asplund spaces, and let $J_{X1}, J_{X2} \subset J_X$ with $J_{X1} \cap J_{X2} = \emptyset$. Then the following assertions hold:*

(i) *If φ is l.s.c. around $\bar{x}$, $\operatorname{epi}\varphi$ is GSNC at $(\bar{x}, 0)$ with respect to $\{X_j \mid j \in J_{X1}\}$ through $\{X_j \mid j \in J_{X2}\}$, and the qualification condition*

$$0 \notin \partial\varphi(\bar{x}) \tag{2.4}$$

holds, then the set $\{x \in X \mid \varphi(x) \leq 0\}$ is GSNC at $\bar{x}$ with respect to $\{X_j \mid j \in J_{X1}\}$ through $\{X_j \in J_{X2}\}$.

(ii) *If φ is continuous around $\bar{x}$, $\operatorname{gph}\varphi$ is GSNC at $(\bar{x}, 0)$ with respect to $\{X_j \mid j \in J_{X1}\}$ through $\{X_j \in J_{X2}\}$, and the qualification condition*

$$0 \notin \partial\varphi(\bar{x}) \cup \partial(-\varphi)(\bar{x}) \tag{2.5}$$

holds, then the set $\{x \in X \mid \varphi(x) = 0\}$ is GSNC at $\bar{x}$ with respect to $\{X_j \mid j \in J_{X1}\}$ through $\{X_j \in J_{X2}\}$.

Proof. Case (i) corresponds to Theorem 2.2 with $F = E_\varphi$, $Y = \mathbb{R}$, and $J_{Y1} = J_Y$, $J_{Y2} = \emptyset$; case (ii) corresponds to Theorem 2.2 with $F = \varphi$, and $J_{Y1} = J_Y$, $J_{Y2} = \emptyset$. □

Combining Theorem 2.1 and Theorem 2.3, we can develop results concerning GSNC properties of the constraint sets

$$\{x \in X \mid \varphi_i(x) \le 0\,(1 \le i \le q), \varphi_i(x) = 0\,(q+1 \le i \le q+r)\}$$

for scalar functions $\varphi_i \colon X \to \overline{\mathbb{R}}$ with $\varphi_i(\bar{x}) = 0$ $(1 \le i \le q+r)$, where $p, r \in \mathbb{N}$. We leave the formulations of these results to the readers (cf. Theorem 3.10 in [6]).

We proceed to study preservations of GSNC properties of mappings under additions.

Theorem 2.4. *Let $F_i \colon X \rightrightarrows Y$ $(i = 1, 2)$ be multifunctions with closed graphs and $\bar{y} \in (F_1 + F_2)(\bar{x})$, where X, Y as in (1.1) are products of Asplund spaces, and let $J_{Xi1}, J_{Xi2} \subset J_X$, $J_{Yi1}, J_{Yi2} \subset J_Y$ with $J_{X11} \cup J_{X21} = J_X$, $J_{Xi1} \cap J_{Xi2} = J_{Yi1} \cap J_{Yi2} = \emptyset$ $(i = 1, 2)$. Assume that the mapping $S \colon X \times Y \rightrightarrows Y \times Y$ defined by*

$$S(x, y) = \{(y_1, y_2) \in Y \times Y | y_1 \in F_1(x), y_2 \in F_2(x), y_1 + y_2 = y\} \quad \forall (x, y) \in X \times Y$$

is inner semicompact at $(\bar{x}, \bar{y})$, and that for every $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ the following hold:

(i) $\operatorname{gph} F_i$ *is GSNC at $(\bar{x}, \bar{y}_i)$ with respect to $\{X_j \mid j \in J_{Xi1}\} \cup \{Y_j \mid j \in J_{Yi1}\}$ through $\{X_j \mid j \in J_{Xi2}\} \cup \{Y_j \mid j \in J_{Yi2}\}$ $(i = 1, 2)$;*

(ii) *Either* $\operatorname{gph} F_1$ *is strongly PSNC at $(\bar{x}, \bar{y}_1)$ with respect to $\{X_j \mid j \in J_{X22}\}$, or* $\operatorname{gph} F_2$ *is strongly PSNC at $(\bar{x}, \bar{y}_2)$ with respect to $\{X_j \mid j \in J_{X12}\}$;*

(iii) *F_1, F_2 satisfy the qualification condition*

$$D^*_N F_1(\bar{x}, \bar{y}_1)(0) \cap \big[-D^*_N F_2(\bar{x}, \bar{y}_2)(0)\big] = \{0\}. \tag{2.6}$$

Then $\operatorname{gph}(F_1 + F_2)$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X11} \cap J_{X21}\} \cup \{Y_j \mid j \in J_{Y11} \cup J_{Y21}\}$ through $\{X_j \mid j \in J_{X12} \cup J_{X22}\} \cup \{Y_j \mid j \in J_{Y12} \cup J_{Y22}\}$.*

Proof. It is sufficient to show that for any sequence $\varepsilon_k \downarrow 0$, $(x_k, y_k) \to (\bar{x}, \bar{y})$ with $y_k \in (F_1 + F_2)(x_k)$, and $x^*_k = (x^*_{jk})_{j \in J_X}$, $y^*_k = (y^*_{jk})_{j \in J_Y}$ with

$$(x^*_k, y^*_k) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_k); \operatorname{gph}(F_1 + F_2)\big) \tag{2.7}$$

and $x^*_{jk} \to 0$ $(j \in J_{X12} \cup J_{X22})$, $x^*_{jk} \xrightarrow{w^*} 0$ $(j \notin J_{X12} \cup J_{X22})$, $y^*_{jk} \to 0$ $(j \in J_{Y12} \cup J_{Y22})$, $y^*_{jk} \xrightarrow{w^*} 0$ $(j \notin J_{Y12} \cup J_{Y22})$, one has

$$x^*_{jk} \to 0\,(j \in J_{X11} \cap J_{X21}, \quad y^*_{jk} \to 0\,(j \in J_{Y11} \cup J_{Y21}) \tag{2.8}$$

along some subsequence. By the choice of (x_k, y_k), $S(x_k, y_k) \ne \emptyset$. Since the mapping S is inner semicompact at $(\bar{x}, \bar{y})$, there exists a sequence $(y_{1k}, y_{2k}) \in S(x_k, y_k)$ that contains a convergent subsequence. We may assume, without loss of generality, that $(y_{1k}, y_{2k}) \to (\bar{y}_1, \bar{y}_2)$. It follows that $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ because $\operatorname{gph} F_1$, $\operatorname{gph} F_2$ are closed. Construct the sets $\Omega_1, \Omega_2 \subset X \times Y \times Y$ such that

$$\Omega_i := \{(x, u_1, u_2) \in X \times Y \times Y \mid u_i \in F_i(x)\}\,(i = 1, 2).$$

Clearly these two sets are closed, $(\bar{x}, \bar{y}_1, \bar{y}_2) \in \Omega_1 \cap \Omega_2$, and one can verify directly by the definition of ε-normal cones that (2.7) implies

$$(x_k^*, y_k^*, y_k^*) \in \widehat{N}_{\varepsilon_k}\big((x_k, y_{1k}, y_{2k}); \Omega_1 \cap \Omega_2\big), \quad k \in \mathbb{N}. \tag{2.9}$$

By the structures of Ω_1, Ω_2 and the GSNC assumptions on F_1, F_2, it follows that

(i) Ω_1 is GSNC at $(\bar{x}, \bar{y}_1, \bar{y}_2)$ with respect to $\{X_j \mid j \in J_{X11}\}$, $\{Y_j \mid j \in J_{Y11}\}$ in the first Y-component in $X \times Y \times Y$, and the second Y-component in $X \times Y \times Y$ through $\{X_j \mid j \in J_{X12}\}$ and $\{Y_j \mid j \in J_{Y12}\}$ in the first Y-component in $X \times Y \times Y$;

(ii) Ω_2 is GSNC at $(\bar{x}, \bar{y}_1, \bar{y}_2)$ with respect to $\{X_j \mid j \in J_{X21}\}$, the first Y-component in $X \times Y \times Y$, and $\{Y_j \mid j \in J_{Y21}\}$ in the second Y-component in $X \times Y \times Y$ through $\{X_j \mid j \in J_{X22}\}$ and $\{Y_j \mid j \in J_{Y22}\}$ in the second Y-component in $X \times Y \times Y$;

(iii) Either Ω_1 is strongly PSNC at $(\bar{x}, \bar{y}_1, \bar{y}_2)$ with respect to $\{X_j \mid j \in J_{X22}\}$ and $\{Y_j \mid j \in J_{Y22}\}$ in the second Y-component in $X \times Y \times Y$, or Ω_2 is strongly PSNC at $(\bar{x}, \bar{y}_1, \bar{y}_2)$ with respect to $\{X_j \mid j \in J_{X12}\}$ and $\{Y_j \mid j \in J_{Y12}\}$ in the first Y-component in $X \times Y \times Y$;

(iv) the qualification condition (2.11) implies the qualification condition (iii) in Theorem 2.1.

Therefore we can apply Theorem 2.1 and obtain from (2.9) that (2.8) holds. The proof is complete. □

When $J_{X11} = J_{X21} = J_X$, $J_{X12} = J_{X22} = \emptyset$, $J_{Y11} = J_{Y21} = J_Y$, $J_{Y12} = J_{Y22} = \emptyset$, Theorem 2.4 reduces to Theorem 4.4 in [6]. Note that the qualification condition (2.6) corresponds to the normal qualification condition of $\operatorname{gph} F_i$, which can be replaced by a more delicate condition involving the mixed qualification in terms of normal cones of $\operatorname{gph} F_i$ $(i = 1, 2)$. In the case $J_{Y11} = J_{Y21} = \emptyset$, $J_{Y12} = J_{Y22} = J_Y$, we can improve (2.6) to (2.10) using the mixed coderivatives of F_i $(i = 1, 2)$ in the following result.

Theorem 2.5. *Let $F_i \colon X \rightrightarrows Y$ $(i = 1, 2)$ be multifunctions with closed graphs and $\bar{y} \in (F_1 + F_2)(\bar{x})$, where X, Y as in (1.1) are products of Asplund spaces, and let $J_{Xi1}, J_{Xi2} \subset J_X$ with $J_{X11} \cup J_{X21} = J_X$, $J_{Xi1} \cap J_{Xi2} = \emptyset$ $(i = 1, 2)$. Assume that the mapping $S \colon X \times Y \rightrightarrows Y \times Y$ defined in Theorem 2.4 is inner semicompact at $(\bar{x}, \bar{y})$, and that for every $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ the following hold:*

(i) $\operatorname{gph} F_i$ *is GSNC at $(\bar{x}, \bar{y}_i)$ with respect to $\{X_j \mid j \in J_{Xi1}\}$ through $\{X_j \mid j \in J_{Xi2}\} \cup \{Y_j \mid j \in J_Y\}$ $(i = 1, 2)$;*

(ii) *Either* $\operatorname{gph} F_1$ *is strongly PSNC at $(\bar{x}, \bar{y}_1)$ with respect to $\{X_j \mid j \in J_{X22}\}$, or* $\operatorname{gph} F_2$ *is strongly PSNC at $(\bar{x}, \bar{y}_2)$ with respect to $\{X_j \mid j \in J_{X12}\}$;*

(iii) *F_1, F_2 satisfy the qualification condition*

$$D_M^* F_1(\bar{x}, \bar{y}_1)(0) \cap \big[-D_M^* F_2(\bar{x}, \bar{y}_2)(0)\big] = \{0\}. \tag{2.10}$$

Then $\operatorname{gph}(F_1 + F_2)$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X11} \cap J_{X21}\}$ through $\{X_j \mid j \in J_{X12} \cup J_{X22}\} \cup \{Y_j \mid j \in J_Y\}$.*

Proof. The proof is similar to the proof of Theorem 2.4 (for the case $J_{Y11} = J_{Y21} = \emptyset$, $J_{Y12} = J_{Y22} = J_Y$) except we can directly check that the qualification condition

(2.10) implies the qualification condition (iii) in Theorem 2.1 for the set system $\{\Omega_1, \Omega_2\}$ defined in the proof of Theorem 2.4 at $(\bar{x}, \bar{y}_1, \bar{y}_2)$.. $\square$

We single out a special case ($J_{X12} = J_{X22} = \emptyset$) of Theorem 2.5 which gives a natural extension of Theorem 4.1 in [6] (corresponding to the case $J_{X1} = J_{X2} = J_X$ below):

Theorem 2.6. *Let $F_i \colon X \rightrightarrows Y$ $(i = 1, 2)$ be multifunctions with closed graphs and $\bar{y} \in (F_1 + F_2)(\bar{x})$, where X, Y as in (1.1) are products of Asplund spaces, and let $J_{X1}, J_{X2} \subset J_X$ with $J_{X1} \cup J_{X2} = J_X$. Assume that the mapping $S \colon X \times Y \rightrightarrows Y \times Y$ defined in Theorem 2.4 is inner semicompact at $(\bar{x}, \bar{y})$, and that for every $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ the following hold:*

(i) gph F_i *is GSNC at $(\bar{x}, \bar{y}_i)$ with respect to $\{X_j \mid j \in J_{Xi}\}$ through $\{Y_j \mid j \in J_Y\}$ $(i = 1, 2)$;*

(ii) *F_1, F_2 satisfy the qualification condition*

$$D^*_M F_1(\bar{x}, \bar{y}_1)(0) \cap \big[- D^*_M F_2(\bar{x}, \bar{y}_2)(0) \big] = \{0\}. \tag{2.11}$$

Then gph$(F_1 + F_2)$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X1} \cap J_{X2}\}$ through $\{Y_j \mid j \in J_Y\}$.*

In the case of l.s.c. scalar functions, we have the following corollary of Theorem 2.4 or Theorem 2.5 by setting $F_i = E_{\varphi_i}$ $(i = 1, 2)$.

Theorem 2.7. *Let $\varphi_i \colon X \to \overline{\mathbb{R}}$ be l.s.c. functions around $\bar{x} \in X$ with $\varphi(\bar{x}) < \infty$ $(i = 1, 2)$, where X as in (1.1) is a product of Asplund spaces, and let $J_{Xi1}, J_{Xi2} \subset J_X$ with $J_{X11} \cup J_{X21} = J_X$, $J_{Xi1} \cap J_{Xi2} = \emptyset$ $(i = 1, 2)$. Assume that*

(i) epi φ_i *is GSNC at $(\bar{x}, \varphi_i(\bar{x}))$ with respect to $\{X_j \mid j \in J_{Xi1}\}$ through $\{X_j \mid j \in J_{Xi2}\}$ $(i = 1, 2)$;*

(ii) *either* epi φ_1 *is strongly PSNC at $(\bar{x}, \varphi_1(\bar{x}))$ with respect to $\{X_j \mid j \in J_{X22}\}$, or* epi φ_2 *is strongly PSNC at $(\bar{x}, \varphi_2(\bar{x}))$ with respect to $\{X_j \mid j \in J_{X12}\}$;*

(iii) *φ_1, φ_2 satisfy the qualification condition*

$$\partial^\infty \varphi_1(\bar{x}) \cap \big[- \partial^\infty \varphi_2(\bar{x}) \big] = \{0\}. \tag{2.12}$$

Then epi$(\varphi_1 + \varphi_2)$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X11} \cap J_{X21}\}$ through $\{X_j \mid j \in J_{X12} \cup J_{X22}\}$.*

When $J_{X12} = J_{X22} = \emptyset$, Theorem 2.7 reduces to the following result extending Corollary 4.3 in [6].

Theorem 2.8. *Let $\varphi_i \colon X \to \overline{\mathbb{R}}$ be l.s.c. functions around $\bar{x} \in X$ with $\varphi(\bar{x}) < \infty$ $(i = 1, 2)$, where X as in (1.1) is a product of Asplund spaces, and let $J_{X1}, J_{X2} \subset J_X$ with $J_{X1} \cup J_{X2} = J_X$. Assume that*

(i) epi φ_i *is strongly PSNC at $(\bar{x}, \varphi_i(\bar{x}))$ with respect to $\{X_j \mid j \in J_{Xi}\}$ $(i = 1, 2)$;*

(ii) *φ_1, φ_2 satisfy the qualification condition*

$$\partial^\infty \varphi_1(\bar{x}) \cap \big[- \partial^\infty \varphi_2(\bar{x}) \big] = \{0\}. \tag{2.13}$$

Then epi$(\varphi_1 + \varphi_2)$ *is strongly PSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X1} \cap J_{X2}\}$.*

In the case of continuous scalar functions, we have the following corollary of Theorem 2.4 or Theorem 2.5 in the case $F_i = \varphi_i$ $(i = 1, 2)$.

Theorem 2.9. *Let $\varphi_i \colon X \to \overline{\mathbb{R}}$ be continuous functions around $\bar{x} \in X$ $(i = 1, 2)$, where X as in (1.1) is a product of Asplund spaces, and let $J_{Xi1}, J_{Xi2} \subset J_X$ with $J_{X11} \cup J_{X21} = J_X$, $J_{Xi1} \cap J_{Xi2} = \emptyset$ $(i = 1, 2)$. Assume that*

(i) $\operatorname{gph} \varphi_i$ *is GSNC at $(\bar{x}, \varphi_i(\bar{x}))$ with respect to $\{X_j \mid j \in J_{Xi1}\}$ through $\{X_j \mid j \in J_{Xi2}\}$ $(i = 1, 2)$;*
(ii) *either* $\operatorname{gph} \varphi_1$ *is strongly PSNC at $(\bar{x}, \varphi_1(\bar{x}))$ with respect to $\{X_j \mid j \in J_{X22}\}$, or* $\operatorname{gph} \varphi_2$ *is strongly PSNC at $(\bar{x}, \varphi_2(\bar{x}))$ with respect to $\{X_j \mid j \in J_{X12}\}$;*
(iii) *φ_1, φ_2 satisfy the qualification condition*

$$\big[\partial^\infty \varphi_1(\bar{x}) \cup \partial^\infty(-\varphi_1)(\bar{x})\big] \cap \big[-\big(\partial^\infty \varphi_2(\bar{x}) \cup \partial^\infty(-\varphi_2)(\bar{x})\big)\big] = \{0\}. \tag{2.14}$$

Then $\operatorname{gph}(\alpha_1\varphi_1 + \alpha_2\varphi_2)$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X11} \cap J_{X21}\}$ through $\{X_j \mid j \in J_{X12} \cup J_{X22}\}$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$.*

When $J_{X12} = J_{X22} = \emptyset$, Theorem 2.9 reduces to the following result extending Corollary 4.5 in [6].

Theorem 2.10. *Let $\varphi_i \colon X \to \overline{\mathbb{R}}$ be continuous functions around $\bar{x} \in X (i = 1, 2)$, where X as in (1.1) is a product of Asplund spaces, and $J_{X1}, J_{X2} \subset J_X$ with $J_{X1} \cup J_{X2} = J_X$. Assume that*

(i) $\operatorname{gph} \varphi_i$ *is strongly PSNC at $(\bar{x}, \varphi_i(\bar{x}))$ with respect to $\{X_j \mid j \in J_{Xi}\}$ $(i = 1, 2)$;*
(ii) *φ_1, φ_2 satisfy the qualification condition*

$$\big[\partial^\infty \varphi_1(\bar{x}) \cup \partial^\infty(-\varphi_1)(\bar{x})\big] \cap \big[-\big(\partial^\infty \varphi_2(\bar{x}) \cup \partial^\infty(-\varphi_2)(\bar{x})\big)\big] = \{0\}. \tag{2.15}$$

Then $\operatorname{gph}(\alpha_1\varphi_1 + \alpha_2\varphi_2)$ *is strongly PSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X1} \cap J_{X2}\}$ for any $\alpha_1, \alpha_2 \in \mathbb{R}$.*

Next we establish the GSNC calculus for compositions of mappings.

Theorem 2.11. *Let $G \colon X \rightrightarrows Y$ and $F \colon Y \rightrightarrows Z$ be multifunctions with closed graphs and $\bar{z} \in (F \circ G)(\bar{x})$, where X, Y, Z as in (1.1) are products of Asplund spaces, and let $J_{Xi} \subset J_X$, $J_{Yi1}, J_{Yi2} \subset J_Y$, $J_{Zi} \subset J_Z$ $(i = 1, 2)$ with $J_{X_1} \cap J_{X_2} = J_{Y11} \cap J_{Y12} = J_{Y21} \cap J_{Y22} = J_{Z1} \cap J_{Z2} = \emptyset$, $J_{Y11} \cup J_{Y21} = J_Y$. Assume that the mapping $S \colon X \times Z \rightrightarrows Y$ defined by*

$$S(x, z) = G(x) \cap F^{-1}(z) \quad \forall (x, z) \in X \times Z$$

is inner semicompact at $(\bar{x}, \bar{z})$, and that for every $\bar{y} \in S(\bar{x}, \bar{z})$ the following hold:

(i) $\operatorname{gph} G$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{X_j \mid j \in J_{X1}\} \cup \{Y_j \mid j \in J_{Y11}\}$ through $\{X_j \mid j \in J_{X2}\} \cup \{Y_j \mid j \in J_{Y12}\}$;*
(ii) $\operatorname{gph} F$ *is GSNC at $(\bar{y}, \bar{z})$ with respect to $\{Y_j \mid j \in J_{Y21}\} \cup \{Z_j \mid j \in J_{Z1}\}$ through $\{Y_j \mid j \in J_{Y22}\} \cup \{Z_j \mid j \in J_{Z2}\}$;*
(iii) *Either* $\operatorname{gph} G$ *is strongly PSNC at $(\bar{x}, \bar{y})$ with respect to $\{Y_j \mid j \in J_{Y22}\}$, or* $\operatorname{gph} F$ *is strongly PSNC at $(\bar{y}, \bar{z})$ with respect to $\{Y_j \mid j \in J_{Y12}\}$.*
(iv) *F, G satisfy the qualification condition*

$$D^*_N F(\bar{y}, \bar{z})(0) \cap \ker D^*_N G(\bar{x}, \bar{y}) = \{0\}. \tag{2.16}$$

Then $\operatorname{gph}(F\circ G)$ *is GSNC at* $(\bar x,\bar z)$ *with respect to* $\{X_j \mid j\in J_{X1}\}\cup\{Z_j\mid j\in J_{Z1}\}$ *through* $\{X_j\mid j\in J_{X2}\}\cup\{Z_j\mid j\in J_{Z2}\}$.

Proof. It suffices to show that for any sequence $\varepsilon_k\downarrow 0$, $(x_k,z_k)\to(\bar x,\bar z)$ with $(x_k,z_k)\in\operatorname{gph}(F\circ G)$, and $x_k^*=(x_{jk}^*)_{j\in J_X}$, $z_k^*=(z_{jk}^*)_{j\in J_Z}$ with

$$(x_k^*,z_k^*)\in\widehat N_{\varepsilon_k}\big((x_k,z_k);\operatorname{gph}(F\circ G)\big) \tag{2.17}$$

and $x_{jk}^*\to 0$ $(j\in J_{X2})$, $z_{jk}^*\to 0$ $(j\in J_{Z2})$, $x_{jk}^*\xrightarrow{w^*}0$ $(j\notin J_{X2})$, $z_{jk}^*\xrightarrow{w^*}0$ $(j\notin J_{Z2})$, one has

$$x_{jk}^*\to 0\,(j\in J_{X1}),\quad z_{jk}^*\to 0\,(j\in J_{Z1}) \tag{2.18}$$

along some subsequence. By the choice of (x_k,z_k), $S(x_k,z_k)\neq\emptyset$. According to the semicompactness assumption on S, there is a sequence $y_k\in S(x_k,z_k)$ containing a convergent subsequence. Without loss of generality, we assume $y_k\to\bar y$. Because $\operatorname{gph}G$, $\operatorname{gph}F$ are closed, it follows that $\bar y\in S(\bar x,\bar z)$. Consider closed sets $\Omega_1,\Omega_2\subset X\times Y\times Z$ defined by

$$\Omega_1:=\operatorname{gph}G\times Z,\quad \Omega_2:=X\times\operatorname{gph}F;$$

then $(\bar x,\bar y,\bar z)\in\Omega_1\cap\Omega_2$ and it can be verified by (2.17) and the definition of ε-normal cones that

$$(x_k^*,0,z_k^*)\in\widehat N_{\varepsilon_k}\big((x_k,y_k,z_k);\Omega_1\cap\Omega_2\big),\quad k\in\mathbb N. \tag{2.19}$$

By the structures of Ω_1 and Ω_2, Ω_1 is GSNC with respect to $\{X_j\mid j\in J_{X1}\}\cup\{Y_j\mid j\in J_{Y11}\}\cup\{Z_j\mid j\in J_Z\}$ through $\{X_j\mid j\in J_{X2}\}\cup\{Y_j\mid j\in J_{Y12}\}$ at $(\bar x,\bar y,\bar z)$, Ω_2 is GSNC with respect to $\{X_j\mid j\in J_X\}\cup\{Y_j\mid j\in J_{Y21}\}\cup\{Z_j\mid j\in J_{Z1}\}$ through $\{Y_j\mid j\in J_{Y22}\}\cup\{Z_j\mid j\in J_{Z2}\}$ at $(\bar x,\bar y,\bar z)$, and either Ω_1 is strongly PSNC at $(\bar x,\bar y,\bar z)$ with respect to $\{Y_j\mid j\in J_{Y22}\}\cup\{Z_j\mid j\in J_{Z2}\}$, or Ω_2 is strongly PSNC at $(\bar x,\bar y,\bar z)$ with respect to $\{X_j\mid j\in J_{X2}\}\cup\{Y_j\mid j\in Y_{12}\}$; also (2.16) implies the qualification condition (iii) in Theorem 2.1. Applying Theorem 2.1 to the set system $\{\Omega_1,\Omega_2\}$ at $(\bar x,\bar y,\bar z)$, we obtain (2.18) by (2.19) and completes the proof. ☐

When $J_{X1}=J_X$, $J_{Y12}=J_{Y21}=J_Y$, $J_{Z1}=J_Z$, and $J_{X2}=J_{Y11}=J_{Y22}=J_{Z2}=\emptyset$, Theorem 2.11 reduces to the first case of Theorem 5.4 in [6]; when $J_{X1}=J_X$, $J_{Y11}=J_{Y22}=J_Y$, $J_{Z1}=J_Z$, and $J_{X2}=J_{Y12}=J_{Y21}=J_{Z2}=\emptyset$, Theorem 2.11 reduces to the second case of Theorem 5.4 in [6]; in this way, the two cases of the latter theorem is unified. When $J_{Z1}=\emptyset$, $J_{Z2}=J_Z$, we can improve the qualification (2.16) in terms of the mixed coderivative of F as below. The proof is similar except that we need to directly check that (2.20) implies the qualification condition (iii) in Theorem 2.1 for the set system $\{\Omega_1,\Omega_2\}$ defined in the proof of Theorem 2.11 at $(\bar x,\bar y,\bar z)$.

Theorem 2.12. *Let* $G\colon X\rightrightarrows Y$ *and* $F\colon Y\rightrightarrows Z$ *be multifunctions with closed graphs and* $\bar z\in(F\circ G)(\bar x)$, *where* X, Y, Z *as in (1.1) are products of Asplund spaces, and let* $J_{Xi}\subset J_X$, $J_{Yi1},J_{Yi2}\subset J_Y$ $(i=1,2)$ *with* $J_{X_1}\cap J_{X_2}=J_{Y11}\cap J_{Y12}=J_{Y21}\cap J_{Y22}=\emptyset$, $J_{Y11}\cup J_{Y21}=J_Y$. *Assume that the mapping* $S\colon X\times Z\rightrightarrows Y$ *defined in Theorem 2.11 is inner semicompact at* $(\bar x,\bar z)$, *and that for every* $\bar y\in S(\bar x,\bar z)$ *the following hold:*

(i) $\operatorname{gph}G$ *is GSNC at* $(\bar x,\bar y)$ *with respect to* $\{X_j\mid j\in J_{X1}\}\cup\{Y_j\mid j\in J_{Y11}\}$ *through* $\{X_j\mid j\in J_{X2}\}\cup\{Y_j\mid j\in J_{Y12}\}$;

(ii) $\operatorname{gph} F$ *is GSNC at* $(\bar{y}, \bar{z})$ *with respect to* $\{Y_j \mid j \in J_{Y21}\}$ *through* $\{Y_j \mid j \in J_{Y22}\} \cup \{Z_j \mid j \in J_Z\}$;

(iii) *Either* $\operatorname{gph} G$ *is strongly PSNC at* $(\bar{x}, \bar{y})$ *with respect to* $\{Y_j \mid j \in J_{Y22}\}$, *or* $\operatorname{gph} F$ *is strongly PSNC at* $(\bar{y}, \bar{z})$ *with respect to* $\{Y_j \mid j \in J_{Y12}\}$.

(iv) $\{F, G\}$ *satisfies the qualification condition*

$$D_M^* F(\bar{y}, \bar{z})(0) \cap \ker D_N^* G(\bar{x}, \bar{y}) = \{0\}. \tag{2.20}$$

Then $\operatorname{gph}(F \circ G)$ *is GSNC at* $(\bar{x}, \bar{z})$ *with respect to* $\{X_j \mid j \in J_{X1}\}$ *through* $\{X_j \mid j \in J_{X2}\} \cup \{Z_j \mid j \in J_Z\}$.

When $J_{X1} = J_X$, $J_{Y12} = J_{Y21} = J_Y$, and $J_{X2} = J_{Y11} = J_{Y22} = \emptyset$, Theorem 2.12 reduces to the first case of Theorem 5.1 in [6]; when $J_{X1} = J_X$, $J_{Y11} = J_{Y22} = J_Y$, and $J_{X2} = J_{Y12} = J_{Y21} = \emptyset$, Theorem 2.12 reduces to the second case of Theorem 5.1 in [6].

To conclude this section, we provide a corollary of Theorem 2.12 (or Theorem 2.11) when F is a scalar function and G is single-valued. This result extends Corollary 5.3 in [6].

Theorem 2.13. *Let* $g\colon X \to Y$ *be a continuous function around* $\bar{x} \in X$, *and* $\varphi\colon Y \to \overline{\mathbb{R}}$ *be an l.s.c. function around* $\bar{y} := g(\bar{x})$, *where* X, Y *as in (1.1) are products of Asplund spaces, and let* $J_{Xi} \subset J_X$, $J_{Yi1}, J_{Yi2} \subset J_Y$ $(i = 1, 2)$ *with* $J_{X_1} \cap J_{X_2} = J_{Y11} \cap J_{Y12} = J_{Y21} \cap J_{Y22} = \emptyset$, $J_{Y11} \cup J_{Y21} = J_Y$. *Assume that the following hold:*

(i) $\operatorname{gph} g$ *is GSNC at* $(\bar{x}, \bar{y})$ *with respect to* $\{X_j \mid j \in J_{X1}\} \cup \{Y_j \mid j \in J_{Y11}\}$ *through* $\{X_j \mid j \in J_{X2}\} \cup \{Y_j \mid j \in J_{Y12}\}$;

(ii) $\operatorname{epi} \varphi$ *is GSNC at* $(\bar{y}, \varphi(\bar{y}))$ *with respect to* $\{Y_j \mid j \in J_{Y21}\}$ *through* $\{Y_j \mid j \in J_{Y22}\}$;

(iii) *Either* $\operatorname{gph} g$ *is strongly PSNC at* $(\bar{x}, \bar{y})$ *with respect to* $\{Y_j \mid j \in J_{Y22}\}$, *or* $\operatorname{epi} \varphi$ *is strongly PSNC at* $(\bar{y}, \bar{z})$ *with respect to* $\{Y_j \mid j \in J_{Y12}\}$.

(iv) g, φ *satisfy the qualification condition*

$$\partial^\infty \varphi(\bar{y}) \cap \ker D_N^* g(\bar{x}, \bar{y}) = \{0\}. \tag{2.21}$$

Then $\operatorname{epi}(\varphi \circ g)$ *is GSNC at* $(\bar{x}, \bar{z})$ *with respect to* $\{X_j \mid j \in J_{X1}\}$ *through* $\{X_j \mid j \in J_{X2}\}$.

Proof. The theorem reduces to the case $G = g$, $F = E_\varphi$, and $Z = \mathbb{R}$ of Theorem 2.12. □

3. GSNC and generalized differentiation

The generalized sequential normal compactness opens the door to extended generalized differential calculus involving Mordukhovich constructions for sets and set-valued mappings. In fact, an improved exact extremal principle was established in [12], where it also contains the extended intersection rule as below.

Theorem 3.1. *Let* X *as in (1.1) be a product of Asplund spaces, nonempty sets* $\Omega_1, \Omega_2 \subset X$ *be locally closed around* $\bar{x} \in \Omega_1 \cap \Omega_2$, *and* J_i $(i = 1, 2, 3, 4)$ *form a partition of* J_X. *Suppose that the following assumptions hold:*

(i) Ω_1 *is GSNC at* $\bar{x}$ *with respect to* $J_2 \cup J_3$ *through* J_1*, and* Ω_2 *is GSNC at* $\bar{x}$ *with respect to* $J_1 \cup J_4$ *through* J_2*;*

(ii) *either* Ω_1 *is strongly PSNC at* $\bar{x}$ *with respect to* J_2*, or* Ω_2 *is strongly PSNC at* $\bar{x}$ *with respect to* J_1*;*

(iii) *The limiting qualification condition holds for* $\{\Omega_1, \Omega_2\}$ *at* $\bar{x}$*.*

Then one has the inclusion

$$N(\bar{x}; \Omega_1 \cap \Omega_2) \subset N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2). \tag{3.1}$$

In the remaining part of the section, we establish calculus rules, mostly based on Theorem 3.1, involving Mordukhovich generalized differential constructions of sets, set-valued mappings, and scalar functions. These results can be justified using similar schemes as for the cases of GSNC calculus in section 2, and we omit these proofs for simplicity (cf. the proofs of corresponding results in [5]). The obtained calculus extend the corresponding cases of the general results in [5] involving a topology τ. Actually results in this section can be extended to this general case with the topology τ naturally, and we omit the details. First we present the rule for inverse images under set-valued mappings.

Theorem 3.2. *Let* $F\colon X \rightrightarrows Y$ *be a multifunction with a closed graph,* $\Theta \subset Y$ *be a closed subset with* $\bar{x} \in F^{-1}(\Theta)$*, where* X*,* Y *as in (1.1) are products of Asplund spaces, and let* $J_{Yi} \subset J_Y$ *(*$i = 1, 2$*) with*$J_{Y1} \cap J_{Y2} = \emptyset$*. Assume that the mapping* $S\colon X \rightrightarrows Y$ *defined in Theorem 2.2 is inner semicompact at* $\bar{x}$*, and that for every* $\bar{y} \in S(\bar{x})$ *the following hold:*

(i) $\operatorname{gph} F$ *is GSNC at* $(\bar{x}, \bar{y})$ *with respect to* $\{Y_j \mid j \in J_{Y1}\}$ *through* $\{X_j \in J_X\} \cup \{Y_j \mid j \in J_{Y2}\}$*;*

(ii) Θ *is PSNC at* $\bar{y}$ *with respect to* $\{Y_j \mid j \in J_Y \backslash J_{Y1}\}$*, and is strongly PSNC at this point with respect to* $\{Y_j \mid j \in J_{Y2}\}$*;*

(iii) F *and* Θ *satisfy the qualification condition*

$$N(\bar{y}; \Theta) \cap \ker \widetilde{D}^*_M F(\bar{x}, \bar{y}) = \{0\}. \tag{3.2}$$

Then

$$N\big(\bar{x}; F^{-1}(\Theta)\big) \subset \bigcup_{\bar{y} \in F(\bar{x}) \cap \Theta, y^* \in N(\bar{y}; \Theta)} D^*_N F(\bar{x}, \bar{y})(y^*). \tag{3.3}$$

Next we provide the rule for the coderivatives of sums of set-valued mappings. Note that the case for mixed coderivative (i.e., $D^* = D^*_M$) can not be derived from Theorem 3.1; it can be proved directly using the set system $\{\Omega_1, \Omega_2\}$ defined in the proof of Theorem 2.4 following the scheme of the proof of Theorem 3.1 in [12].

Theorem 3.3. *Let* $F_i\colon X \rightrightarrows Y$ *(*$i = 1, 2$*) be multifunctions with closed graphs and* $\bar{y} \in (F_1 + F_2)(\bar{x})$*, where* X*,* Y *as in (1.1) are products of Asplund spaces, and let* $J_{Xi} \subset J_X$ *(*$i = 1, 2, 3, 4$*) form a partition of* J_X*. Assume that the mapping* $S\colon X \times Y \rightrightarrows Y \times Y$ *defined in Theorem 2.4 is inner semicompact at* $(\bar{x}, \bar{y})$*, and that for every* $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ *the following hold:*

(i) $\operatorname{gph} F_1$ *is GSNC at* $(\bar{x}, \bar{y}_1)$ *with respect to* $\{X_j \mid j \in J_{X2} \cup J_{X3}\}$ *through* $\{X_j \mid j \in J_{X1}\} \cup \{Y_j \mid j \in J_Y\}$*;*

(ii) $\operatorname{gph} F_2$ *is GSNC at* $(\bar{x}, \bar{y}_2)$ *with respect to* $\{X_j \mid j \in J_{X1} \cup J_{X4}\}$ *through* $\{X_j \mid j \in J_{X2}\} \cup \{Y_j \mid j \in J_Y\}$*;*

(iii) *Either* $\operatorname{gph} F_1$ *is strongly PSNC at* $(\bar{x}, \bar{y}_1)$ *with respect to* $\{X_j \mid j \in J_{X2}\}$*, or* $\operatorname{gph} F_2$ *is strongly PSNC at* $(\bar{x}, \bar{y}_2)$ *with respect to* $\{X_j \mid j \in J_{X1}\}$*;*

(iv) F_1*,* F_2 *satisfy the qualification condition*

$$D_M^* F_1(\bar{x}, \bar{y}_1)(0) \cap \big[-D_M^* F_2(\bar{x}, \bar{y}_2)(0)\big] = \{0\}. \tag{3.4}$$

Then for all $y^* \in Y^*$*, and for* $D^* = D_N^*$ *or* D_M^**,*

$$D^*(F_1 + F_2)(\bar{x}, \bar{y})(y^*) \subset \bigcup_{(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})} \big[D^* F_1(\bar{x}, \bar{y}_1)(y^*) + D^* F_2(\bar{x}, \bar{y}_2)(y^*)\big]. \tag{3.5}$$

When $J_{X1} = J_{X2} = \emptyset$, Theorem 3.3 reduces to the following result which does not have the strong PSNC assumptions and the assumptions on F_1 and F_2 are symmetric.

Theorem 3.4. *Let* $F_i \colon X \rightrightarrows Y$ $(i = 1, 2)$ *be multifunctions with closed graphs and* $\bar{y} \in (F_1 + F_2)(\bar{x})$*, where* X*,* Y *as in (1.1) are products of Asplund spaces, and let* $J_{Xi} \subset J_X$ $(i = 1, 2)$ *with* $J_{X1} \cup J_{X2} = J_X$*,* $J_{X1} \cap J_{X2} = \emptyset$*. Assume that the mapping* $S \colon X \times Y \rightrightarrows Y \times Y$ *defined in Theorem 2.4 is inner semicompact at* $(\bar{x}, \bar{y})$*, and that for every* $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$ *the following hold:*

(i) $\operatorname{gph} F_i$ *is GSNC at* $(\bar{x}, \bar{y}_i)$ *with respect to* $\{X_j \mid j \in J_{Xi}\}$ *through* $\{Y_j \mid j \in J_Y\}$ $(i = 1, 2)$*;*

(ii) F_1*,* F_2 *satisfy the qualification condition*

$$D_M^* F_1(\bar{x}, \bar{y}_1)(0) \cap \big[-D_M^* F_2(\bar{x}, \bar{y}_2)(0)\big] = \{0\}. \tag{3.6}$$

Then for all $y^* \in Y^*$*, and for* $D^* = D_N^*$ *or* D_M^**,*

$$D^*(F_1 + F_2)(\bar{x}, \bar{y})(y^*) \subset \bigcup_{(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})} \big[D^* F_1(\bar{x}, \bar{y}_1)(y^*) + D^* F_2(\bar{x}, \bar{y}_2)(y^*)\big]. \tag{3.7}$$

When $F_i = E_{\varphi_i}$ for scalar function φ_i $(i = 1, 2)$, Theorem 3.3 reduces to the following subdifferential sum rule.

Theorem 3.5. *Let* $\varphi_i \colon X \to \overline{\mathbb{R}}$ $(i = 1, 2)$ *be l.s.c. functions around* $\bar{x} \in X$*, where* X *as in (1.1) is a product of Asplund spaces, and let* $J_{Xi} \subset J_X$ $(i = 1, 2, 3, 4)$ *form a partition of* J_X*. Assume that the following hold:*

(i) $\operatorname{epi} \varphi_1$ *is GSNC at* $(\bar{x}, \varphi_1(\bar{x}))$ *with respect to* $\{X_j \mid j \in J_{X2} \cup J_{X3}\}$ *through* $\{X_j \mid j \in J_{X1}\}$*;*

(ii) $\operatorname{epi} \varphi_2$ *is GSNC at* $(\bar{x}, \varphi_2(\bar{x}))$ *with respect to* $\{X_j \mid j \in J_{X1} \cup J_{X4}\}$ *through* $\{X_j \mid j \in J_{X2}\}$*;*

(iii) *Either* $\operatorname{epi} \varphi_1$ *is strongly PSNC at* $(\bar{x}, \varphi_1(\bar{x}))$ *with respect to* $\{X_j \mid j \in J_{X2}\}$*, or* $\operatorname{epi} \varphi_2$ *is strongly PSNC at* $(\bar{x}, \varphi_2(\bar{x}))$ *with respect to* $\{X_j \mid j \in J_{X1}\}$*;*

(iv) φ_1*,* φ_2 *satisfy the qualification condition*

$$\partial^\infty \varphi_1(\bar{x}) \cap \big[-\partial^\infty \varphi_2(\bar{x})\big] = \{0\}. \tag{3.8}$$

Then

$$\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial \varphi_1(\bar{x}) + \partial \varphi_2(\bar{x}), \tag{3.9}$$

$$\partial^\infty(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial^\infty \varphi_1(\bar{x}) + \partial^\infty \varphi_2(\bar{x}), \tag{3.10}$$

When $J_{X1} = J_{X2} = \emptyset$ in Theorem 3.5, we have the following corollary.

Theorem 3.6. *Let $\varphi_i\colon X \to \overline{\mathbb{R}}$ $(i = 1, 2)$ be lower semicontinuous and around $\bar{x} \in X$, where X as in (1.1) is a product of Asplund spaces and $J_{Xi} \subset J_X$ $(i = 1, 2)$ with $J_{X1} \cup J_{X2} = J_X$, $J_{X1} \cap J_{X2} = \emptyset$. Assume that the following hold:*

(i) $\operatorname{epi}\varphi_i$ *is strongly PSNC at $(\bar{x}, \varphi_i(\bar{x}))$ with respect to $\{X_j \mid j \in J_{Xi}\}$ $(i = 1, 2)$;*

(ii) *φ_1, φ_2 satisfy the qualification condition*

$$\partial^\infty\varphi_1(\bar{x}) \cap \big[-\partial^\infty\varphi_2(\bar{x})\big] = \{0\}. \tag{3.11}$$

Then

$$\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x}), \tag{3.12}$$

$$\partial^\infty(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial^\infty\varphi_1(\bar{x}) + \partial^\infty\varphi_2(\bar{x}), \tag{3.13}$$

To conclude the paper, we provide the chain rule for coderivatives of compositions of set-valued mappings. The case $D^* = D^*_N$ can be derived from Theorem 3.1, while the case $D^* = D^*_M$ can be proved directly using the set system $\{\Omega_1, \Omega_2\}$ defined in the proof of Theorem 2.11 following the scheme of the proof of Theorem 3.1 in [12].

Theorem 3.7. *Let $G\colon X \rightrightarrows Y$ and $F\colon Y \rightrightarrows Z$ with closed graphs and $\bar{z} \in (F \circ G)(\bar{x})$, where X, Y, Z as in (1.1) are products of Asplund spaces and $J_{Yi} \subset J_Y$ $(i = 1, 2)$ with $J_{Y1} \cap J_{Y2} = \emptyset$, $J_{Y1} \cup J_{Y2} = J_Y$. Assume that $S\colon X \times Z \rightrightarrows Y$ defined in Theorem 2.11 is inner semicompact at $(\bar{x}, \bar{z})$, and that for every $\bar{y} \in G(\bar{x}) \cap F^{-1}(\bar{z})$ the following hold:*

(i) $\operatorname{gph} G$ *is GSNC at $(\bar{x}, \bar{y})$ with respect to $\{Y_j \mid j \in J_{Y1}\}$ through $\{X_j \mid j \in J_X\}$;*

(ii) $\operatorname{gph} F$ *is GSNC at $(\bar{y}, \bar{z})$ with respect to $\{Y_j \mid j \in J_{Y2}\}$ through $\{Z_j \mid j \in J_Z\}$;*

(iii) *$\{F, G\}$ satisfies the qualification condition*

$$D^*_M F(\bar{y}, \bar{z})(0) \cap \big[-D^*_M G^{-1}(\bar{y}, \bar{x})(0)\big] = \{0\}. \tag{3.14}$$

Then for all $z^* \in Z^*$ and for $D^* = D^*_N$ or D^*_M,

$$D^*(F \circ G)(\bar{x}, \bar{z})(z^*) \subset \bigcup_{\bar{y} \in S(\bar{x}, \bar{z})} \big[D^*_N G(\bar{x}, \bar{y}) \circ D^* F(\bar{y}, \bar{z})(z^*)\big]. \tag{3.15}$$

Remark 3.8. By Theorem 3.1 and the relations in (2.1), it is possible to derive formulas for coderivatives of intersections of mappings, and subdifferentials of maxima of scalar functions; we omit the details (cf. Proposition 3.20 and Theorem 3.46 in [2]).

Remark 3.9. For those results in section 2 and 3 involving the inner semicompactness, we can also establish the corresponding versions involving the inner semicontinuity. They are similar with similar proofs, and we omit these results for simplicity.

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