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NECESSARY AND SUFFICIENT TURNPIKE CONDITIONS

ALEXANDER J. ZASLAVSKI

Abstract. In this paper we survey our results on the turnpike property for some classes of variational and optimal control problems. To have this property means that approximate solutions of the problems are determined mainly by objective functions and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. We discuss necessary and sufficient conditions for turnpike properties of approximate solutions for variational problems and discrete-time optimal control problems.

1. Introduction

In this work we survey our results on the structure of approximate solutions of variational and optimal control problems which deals with the turnpike property of optimal control problems. To have this property means that the approximate solutions of the problems are determined mainly by objective functions and are essentially independent of the choice of intervals and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [25]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics [1, 6, 9, 11, 16, 19, 24, 32, 34] and optimal control problems [6, 8, 17, 22, 23, 27, 32, 34, 35]. The related infinite horizon problems are analyzed in [3–5, 10, 12, 15, 32, 34, 35]. Our paper has the following structure. In Section 2 we study turnpike properties for discrete-time unconstrained optimal control problems acting on a compact metric space. Turnpike properties of approximate solutions of variational problems are discussed in Sections 3 and 4. Section 5 contains the analysis of discrete-time optimal control problems with constraints.

2. Discrete-time unconstrained problems

In this section we analyze the structure of solutions of the optimization problems

$$(P) \qquad \sum_{i=m_1}^{m_2-1} v_i(z_i, z_{i+1}) \to \min, \ \{z_i\}_{i=m_1}^{m_2} \subset X \text{ and } z_{m_1} = x, \ z_{m_2} = y,$$

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where $v_i : X \times X \to R^1$, $i = 0, \pm 1, \pm 2, \dots$ is a continuous function defined on a metric space X and $x, y \in X$. The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework (see [6, 32, 34] and the references mentioned there). The results of this section were obtained in [30].

Let $\mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$ be the set of all integers, (X, ρ) be a compact metric space and let $v_i : X \times X \to R^1$, $i = 0, \pm 1, \pm 2, \dots$ be a sequence of continuous functions such that

$$\sup\{|v_i(x, y)| : \ x, y \in X, \ i \in \mathbf{Z}\} < \infty$$

and which satisfy the following assumption:

(A) For each $\epsilon > 0$ there exists $\delta > 0$ such that if $i \in \mathbf{Z}$ and if

$$x_1, x_2, y_1, y_2 \in X$$

satisfy $\rho(x_j, y_j) \le \delta$, $j = 1, 2$, then $|v_i(x_1, x_2) - v_i(y_1, y_2)| \le \epsilon$.

For each $y, z \in X$ and each pair of integers $n_1, n_2 > n_1$ set

$$\sigma(n_1, n_2, y, z) = \inf \left\{ \sum_{i=n_1}^{n_2 - 1} v_i(x_i, x_{i+1}) : \ \{x_i\}_{i=n_1}^{n_2} \subset X, \ x_{n_1} = y, \ x_{n_2} = z \right\},$$

$$\sigma(n_1, n_2) = \inf \left\{ \sum_{i=n_1}^{n_2 - 1} v_i(x_i, x_{i+1}) : \ \{x_i\}_{i=n_1}^{n_2} \subset X \right\}.$$

Choose a positive number d_0 such that

$$|v_i(x, y)| \le d_0, \ x, y \in X, \ i \in \mathbf{Z}.$$

A sequence $\{y_i\}_{i=-\infty}^{\infty} \subset X$ is called good if there is $c > 0$ such that for each pair of integers $m_1, m_2 > m_1$,

$$\sum_{i=m_1}^{m_2 - 1} v_i(y_i, y_{i+1}) \le \sigma(m_1, m_2, y_{m_1}, y_{m_2}) + c.$$

We say that the sequence $\{v_i\}_{i=-\infty}^{\infty}$ has the turnpike property (TP) if there exists a sequence $\{\widehat{x}_i\}_{i=-\infty}^{\infty} \subset X$ which satisfies the following condition:

For each $\epsilon > 0$ there are $\delta > 0$ and a natural number N such that for each pair of integers $T_1, T_2 \ge T_1 + 2N$ and each sequence $\{y_i\}_{i=T_1}^{T_2} \subset X$ which satisfies

$$\sum_{i=T_1}^{T_2 - 1} v_i(y_i, y_{i+1}) \le \sigma(T_1, T_2, y_{T_1}, y_{T_2}) + \delta$$

there are integers $\tau_1 \in \{T_1, \dots, T_1 + N\}, \tau_2 \in \{T_2 - N, \dots, T_2\}$ such that:

(i) $\rho(y_i, \widehat{x}_i) \le \epsilon, \ i = \tau_1, \dots, \tau_2$;

(ii) if $\rho(y_{T_1}, \widehat{x}_{T_1}) \le \delta$, then $\tau_1 = T_1$ and if $\rho(y_{T_2}, \widehat{x}_{T_2}) \le \delta$, then $\tau_2 = T_2$.

The sequence $\{\widehat{x}_i\}_{i=-\infty}^{\infty} \subset X$ is called the turnpike of $\{v_i\}_{i=-\infty}^{\infty}$.

The turnpike property is very important for applications. Suppose that our sequence of cost functions $\{v_i\}_{i=-\infty}^{\infty}$ has the turnpike property and we know a finite number of "approximate" solutions of the problem (P). Then we know the turnpike $\{\widehat{x}_i\}_{i=-\infty}^{\infty}$, or at least its approximation, and the constant N (see the definition of (TP)) which is an estimate for the time period required to reach the turnpike. This information can be useful if we need to find an "approximate" solution of the problem (P) with a new time interval $[m_1, m_2]$ and the new values $x, y \in X$ at the end points m_1 and m_2. Namely instead of solving this new problem on the "large" interval $[m_1, m_2]$ we can find an "approximate" solution of the problem (P) on the "small" interval $[m_1, m_1 + N]$ with the values $x, \widehat{x}_{m_1+N}$ at the end points and an "approximate" solution of the problem (P) on the "small" interval $[m_2 - N, m_2]$ with the values $\widehat{x}_{m_2-N}, y$ at the end points. Then the concatenation of the first solution, the sequence $\{\widehat{x}_i\}_{m_1+N}^{m_2-N}$ and the second solution is an "approximate" solution of the problem (P) on the interval $[m_1, m_2]$ with the values x, y at the end points. Sometimes as an "approximate" solution of the problem (P) we can choose any sequence $\{x_i\}_{i=m_1}^{m_2}$ satisfying

$$x_{m_1} = x, \ x_{m_2} = y \text{ and } x_i = \widehat{x}_i \text{ for all } i = m_1 + N, \dots, m_2 - N.$$

Assume that $\{\widehat{x}_i\}_{i=-\infty}^{\infty} \subset X$. How to verify if the sequence of cost functions $\{v_i\}_{i=-\infty}^{\infty}$ has (TP) and $\{\widehat{x}_i\}_{i=-\infty}^{\infty}$ is its turnpike? In [30] we introduced three properties (P1), (P2) and (P3) and showed that $\{v_i\}_{i=-\infty}^{\infty}$ has (TP) if and only if $\{v_i\}_{i=-\infty}^{\infty}$ possesses the properties (P1), (P2) and (P3). The property (P1) means that all good sequences have the same asymptotic behavior. Property (P2) means that for each pair of integers $m_1, m_2 > m_1$ the sequence $\{\widehat{x}_i\}_{i=m_1}^{m_2}$ is a unique solution of problem (P) with $x = \widehat{x}_{m_1}$, $y = \widehat{x}_{m_2}$ and that if a sequence $\{y_i\}_{i=-\infty}^{\infty} \subset X$ is a solution of problem (P) for each pair of integers $m_1, m_2 > m_1$ with $x = y_{m_1}$, $y = y_{m_2}$, then $y_i = \widehat{x}_i$ for all integers i. Property (P3) means that if a sequence $\{y_i\}_{i=m_1}^{m_2} \subset X$ is an approximate solution of problem (P) and $m_2 - m_1$ is large enough, then there is $j \in [m_1, m_2]$ such that y_j is close to $\widehat{x}_j$.

The next theorem is the main result of [30].

Theorem 2.1. *Let $\{\widehat{x}_i\}_{i=-\infty}^{\infty} \subset X$. Then the sequence $\{v_i\}_{i=-\infty}^{\infty}$ has the turnpike property and $\{\widehat{x}_i\}_{i=-\infty}^{\infty}$ is its turnpike if and only if the following properties hold:*

(P1) *If $\{y_i\}_{i=-\infty}^{\infty} \subset X$ is good, then*

$$\lim_{i \to \infty} \rho(y_i, \widehat{x}_i) = 0, \ \lim_{i \to -\infty} \rho(y_i, \widehat{x}_i) = 0;$$

(P2) *For each pair of integers $m_1, m_2 > m_1$*

$$\sum_{i=m_1}^{m_2-1} v_i(\widehat{x}_i, \widehat{x}_{i+1}) = \sigma(m_1, m_2, \widehat{x}_{m_1}, \widehat{x}_{m_2})$$

and if a sequence $\{y_i\}_{i=-\infty}^{\infty} \subset X$ satisfies

$$\sum_{i=m_1}^{m_2-1} v_i(y_i, y_{i+1}) = \sigma(m_1, m_2, y_{m_1}, y_{m_2})$$

for each pair of integers $m_1, m_2 > m_1$, then $y_i = \widehat{x}_i$, $i \in \mathbf{Z}$;

(P3) *For each* $\epsilon > 0$ *there exist* $\delta > 0$ *and a natural number* L *such that for each integer* m *and each sequence* $\{y_i\}_{i=m}^{m+L} \subset X$ *which satisfies*

$$\sum_{i=m}^{m+L-1} v_i(y_i, y_{i+1}) \le \sigma(m, m+L, y_m, y_{m+L}) + \delta$$

there is $j \in \{m, \dots, m+L\}$ *for which* $\rho(y_j, \widehat{x}_j) \le \epsilon$.

It should be mentioned that (P1)-(P3) easily follow from the turnpike property. However it is very nontrivial to show that (P1)-(P3) are sufficient for this property.

3. Turnpike property for variational problems

In this section, which is based on [29], we discuss the structure of approximate solutions of variational problems with continuous integrands $f : [0, \infty) \times R^n \times R^n \to R^1$ which belong to a complete metric space of functions. We do not impose any convexity assumption. The main result of this section, obtained in [29] deals with the turnpike property of variational problems.

We consider the variational problems

$$(P) \qquad \int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min,\ z(T_1) = x,\ z(T_2) = y,$$

$$z :\ [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \ge 0$, $T_2 > T_1$, $x, y \in R^n$ and $f : [0, \infty) \times R^n \times R^n \to R^1$ belongs to a space of integrands described below.

It is well known that the solutions of the problems (P) exist for integrands f which satisfy two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex [26]. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs, although there are some interesting theorems without convexity [7, 20, 21]. For integrands f which do not satisfy the convexity assumption the existence of solutions of the problems (P) is not guaranteed and in this situation we consider δ-*approximate solutions.*

Let $T_1 \ge 0$, $T_2 > T_1$, $x, y \in R^n$, $f : [0, \infty) \times R^n \times R^n \to R^1$ be an integrand and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to R^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \le \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

for each a.c. function $z :\ [T_1, T_2] \to R^n$ satisfying $z(T_1) = x,\ z(T_2) = y$.

The main result of [29] deals with the turnpike property of the variational problems (P). As usual, to have this property means, roughly speaking, that the approximate solutions of the problems (P) are determined mainly by the integrand and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints.

In the classical turnpike theory, it was assumed that a cost function (integrand) is convex. The convexity of the cost function played a crucial role there. In [29] we get rid of convexity of integrands and establish necessary and sufficient conditions for the turnpike property for a space of nonconvex integrands $\mathcal{M}$ described below.

Let us now define the space of integrands. Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. Denote by $\mathcal{M}$ the set of all continuous functions $f : [0,\infty) \times R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) the function f is bounded on $[0,\infty) \times E$ for any bounded set $E \subset R^n \times R^n$;

A(ii) $f(t,x,u) \ge \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(t,x,u) \in [0,\infty) \times R^n \times R^n$;

A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(t,x_1,u) - f(t,x_2,u)| \le \epsilon \max\{f(t,x_1,u), f(t,x_2,u)\}$$

for each $t \in [0,\infty)$ and each $u, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M,\ i = 1,2,\ |u| \ge \Gamma, \quad |x_1 - x_2| \le \delta;$$

A (iv) for each $M, \epsilon > 0$ there exists $\delta > 0$ such that $|f(t,x_1,u_1) - f(t,x_2,u_2)| \le \epsilon$ for each $t \in [0,\infty)$ and each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i|, |u_i| \le M,\ i = 1,2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta.$$

It is easy to show that an integrand $f = f(t,x,u) \in C^1([0,\infty) \times R^n \times R^n)$ belongs to $\mathcal{M}$ if f satisfies assumption A(ii), and if $\sup\{|f(t,0,0)| :\ t \in [0,\infty)\} < \infty$ and also there exists an increasing function $\psi_0 :\ [0,\infty) \to [0,\infty)$ such that

$$\sup\{|\partial f/\partial x(t,x,u)|,\ |\partial f/\partial u(t,x,u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $t \in [0,\infty)$ and each $x, u \in R^n$.

For the set $\mathcal{M}$ we consider the uniformity which is determined by the following base:

$$\begin{gathered} E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{M} \times \mathcal{M} : |f(t,x,u) - g(t,x,u)| \le \epsilon \\ \text{for each } t \in [0,\infty) \text{ and each } x, u \in R^n \text{ satisfying } |x|, |u| \le N \\ \text{and } (|f(t,x,u)| + 1)(|g(t,x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \\ \text{for each } t \in [0,\infty) \text{ and each } x, u \in R^n \text{ satisfying } |x| \le N\}, \end{gathered}$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$.

It is not difficult to show that the space $\mathcal{M}$ with this uniformity is metrizable (by a metric ρ_w). It is known (see [29]) that the metric space $(\mathcal{M}, \rho_w)$ is complete. The metric ρ_w induces in $\mathcal{M}$ a topology.

We consider functionals of the form

$$I^f(T_1,T_2,x) = \int_{T_1}^{T_2} f(t,x(t),x'(t))dt$$

where $f \in \mathcal{M}$, $0 \le T_1 < T_2 < +\infty$ and $x :\ [T_1,T_2] \to R^n$ is an a.c. function.

For $f \in \mathcal{M}$, $y, z \in R^n$ and numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ we set

$$\begin{gathered} U^f(T_1,T_2,y,z) = \inf\{I^f(T_1,T_2,x) :\ x :\ [T_1,T_2] \to R^n \\ \text{is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\}. \end{gathered}$$

It is easy to see that $-\infty < U^f(T_1,T_2,y,z) < +\infty$ for each $f \in \mathcal{M}$, each $y, z \in R^n$ and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{M}$. A locally absolutely continuous (a.c.) function $x : [0,\infty) \to R^n$ is called an (f)-*good function* [32, 34, 35] if for any a.c function $y : [0,\infty) \to R^n$ there is a number M_y such that

$$I^f(0,T,y) \geq M_y + I^f(0,T,x) \text{ for each } T \in (0,\infty).$$

The following result was proved in [29].

Proposition 3.1. *Let $f \in \mathcal{M}$ and let $x : [0,\infty) \to R^n$ be a bounded a.c. function. Then the function x is (f)-good if and only if there is $M > 0$ such that*

$$I^f(0,T,x) \leq U^f(0,T,x(0),x(T)) + M \textit{ for any } T > 0.$$

Let us now give the precise definition of the turnpike property.

Assume that $f \in \mathcal{M}$. We say that f has the turnpike property, or breifly (TP), if there exists a bounded continuous function $X_f : [0,\infty) \to R^n$ which satisfies the following condition:

For each $K, \epsilon > 0$ there exist constants $\delta, L > 0$ such that for each $x, y \in R^n$ satisfying $|x|, |y| \leq K$, each $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$v(T_1) = x,\ v(T_2) = y,\ I^f(T_1,T_2,v) \leq U^f(T_1,T_2,x,y) + \delta$$

the inequality $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + L, T_2 - L]$.

The function X_f is called the turnpike of f.

Assume that $f \in \mathcal{M}$ and $X : [0,\infty) \to R^n$ is a bounded continuous function. How to verify if the integrand f has (TP) and X is its turnpike? In [29] we introduced two properties (P1) and (P2) and show that f has (TP) if and only if f possesses the properties (P1) and (P2). The property (P2) means that all (f)-good functions have the same asymptotic behavior while the property (P1) means that if an a.c. function $v : [0,T] \to R^n$ is an approximate solution and T is large enough, then there is $\tau \in [0,T]$ such that $v(\tau)$ is close to $X(\tau)$.

The next theorem is the main result [29].

Theorem 3.2. *Let $f \in \mathcal{M}$ and $X_f : [0,\infty) \to R^n$ be a bounded absolutely continuous function. Then f has the turnpike property with X_f being the turnpike if and only if the following two properties hold:*

(P1) *For each $K, \epsilon > 0$ there exist $\gamma, l > 0$ such that for each $T \geq 0$ and each a.c. function $w : [T, T+l] \to R^n$ which satisfies*

$$|w(T)|, |w(T+l)| \leq K,\ I^f(T,T+l,w) \leq U^f(T,T+l,w(T),w(T+l)) + \gamma$$

there is $\tau \in [T, T+l]$ for which $|X_f(\tau) - v(\tau)| \leq \epsilon$.

(P2) *For each (f)-good function $v : [0,\infty) \to R^n$,*

$$|v(t) - X_f(t)| \to 0 \textit{ as } t \to \infty.$$

In [29] we proved the following theorem which is an extension of Theorem 3.2.

Theorem 3.3. *Let $f \in \mathcal{M}$, $X_f : [0,\infty) \to R^n$ be an (f)-good function. Assume that the properties (P1), (P2) hold. Then for each $K, \epsilon > 0$ there exist $\delta, L > 0$ and* *a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{M}$ *such that for each* $g \in \mathcal{U}$*, each* $T_1 \geq 0$*,* $T_2 \geq T_1 + 2L$ *and each a.c. function* $v : [T_1, T_2] \to R^n$ *which satisfies*

$$|v(T_1)|, |v(T_2)| \leq K, \ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta$$

the inequality $|v(t) - X_f(t)| \leq \epsilon$ *holds for all* $t \in [T_1 + L, T_2 - L]$*.*

4. Strong turnpike property for variational problems

In this section we use the notation, definitions and assumptions introduced in Section 3 and discuss the results of [31].

The next result was proved in [31].

Proposition 4.1. *Let* $f \in \mathcal{M}$ *and let for each* $(t, x) \in [0, \infty) \times R^n$ *the function* $f(t, x, \cdot) : R^n \to R^1$ *be convex. Then for each* $z \in R^n$ *there is a bounded* (f)*-good function* $Z : [0, \infty) \to R^n$ *such that* $Z(0) = z$ *and that for each* $T > 0$*,*

$$I^f(0, T, Z) = U^f(0, T, Z(0), Z(T)).$$

Let $f \in \mathcal{M}$. We say that f has the strong turnpike property, or briefly (STP), if there exists a bounded a.c. function $X_f : [0, \infty) \to R^n$ which satisfies the following condition:

For each $K, \epsilon > 0$ there exist constants $\delta, L > 0$ such that for each $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_1)|, |v(T_2)| \leq K, \ I^f(T_1, T_2, v) \leq U^f(T_1, T_2, v(T_1), v(T_2)) + \delta$$

(i) there are $\tau_1 \in [T_1, T_1 + L]$ and $\tau_2 \in [T_2 - L, T_2]$ for which

$$|v(t) - X_f(t)| \leq \epsilon, \ t \in [\tau_1, \tau_2];$$

(ii) if $|v(T_1) - X_f(T_1)| \leq \delta$, then $\tau_1 = T_1$ and if $|v(T_2) - X_f(T_2)| \leq \delta$, then $\tau_2 = T_2$.

The function X_f is called the turnpike of f.

If in the definition above condition (ii) is not assumed, then we say that the integrand f has the turnpike property which was discussed in Section 3.

Let $f \in \mathcal{M}$. We say that an a.c. function $x : [0, \infty) \to R^n$ is (f)-overtaking optimal if for each a.c. function $y : [0, \infty) \to R^n$ satisfying $y(0) = x(0)$,

$$\limsup_{T \to \infty} [I^f(0, T, x) - I^f(0, T, y)] \leq 0.$$

Assume that $f \in \mathcal{M}$ and $X : [0, \infty) \to R^n$ is a bounded a.c. function. How to verify if the integrand f has (STP) and X is its turnpike? In [31] we introduced three properties (P1), (P2) and (P3) and showed that f has (STP) if and only if f possesses properties (P1), (P2) and (P3). Property (P1) means that all (f)-good functions have the same asymptotic behavior while property (P2) means that X is a unique (f)-overtaking optimal function whose value at zero is $X(0)$. Property (P3) means that if an a.c. function $v : [0, T] \to R^n$ is an approximate solution and T is large enough, then there is $\tau \in [0, T]$ such that $v(\tau)$ is close to $X(\tau)$.

The next theorem is the main result of [31].

Theorem 4.2. *Let* $f \in \mathcal{M}$*, for each* $(t, x) \in [0, \infty) \times R^n$ *the function* $f(t, x, \cdot) : R^n \to R^1$ *be convex and let* $X_f : [0, \infty) \to R^n$ *be a bounded a.c. function. Then* f *has the strong turnpike property with* X_f *being the turnpike if and only if the following three properties hold:*

(P1) *For each pair of (f)-good functions $v_1, v_2 : [0,\infty) \to R^n$,*

$$|v_1(t) - v_2(t)| \to 0 \text{ as } t \to \infty.$$

(P2) *X_f is an (f)-overtaking optimal function and if an (f)-overtaking optimal function $v : [0,\infty) \to R^n$ satisfies $v(0) = X_f(0)$, then $v = X_f$.*

(P3) *For each $K, \epsilon > 0$ there exist $\gamma, l > 0$ such that for each $T \geq 0$ and each a.c. function $w : [T, T+l] \to R^n$ which satisfies*

$$|w(T)|, |w(T+l)| \leq K,\ I^f(T, T+l, w) \leq U^f(T, T+l, w(T), w(T+l)) + \gamma$$

there is $\tau \in [T, T+l]$ for which $|X_f(\tau) - v(\tau)| \leq \epsilon$.

In [31] we obtained the following theorem which is an extension of Theorem 4.2.

Theorem 4.3. *Let $f \in \mathcal{M}$, for each $(t,x) \in [0,\infty) \times R^n$ the function $f(t,x,\cdot) : R^n \to R^1$ is convex and let $X_f : [0,\infty) \to R^n$ be a bounded a.c. function. Assume that properties (P1), (P2) and (P3) from Theorem 4.2 hold.*

Then for each $K, \epsilon > 0$ there exist $\delta, L > 0$ and a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the following property holds:

For each $g \in \mathcal{U}$, each $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_1)|, |v(T_2)| \leq K,\ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta$$

there exist $\tau_1 \in [T_1, T_1 + L]$, $\tau_2 \in [T_2 - L, T_2]$ such that

$$|v(t) - X_f(t)| \leq \epsilon,\ t \in [\tau_1, \tau_2].$$

Moreover, if $|v(T_1) - X_f(T_1)| \leq \delta$, then $\tau_1 = T_1$, and if $|v(T_2) - X_f(T_2)| \leq \delta$, then $T_2 = \tau_2$,

5. Discrete-time problems with constraints

In this section we discuss the structure of approximate solutions of nonautonomous discrete-time optimal control systems arising in economic dynamics which are determined by sequences of lower semicontinuous objective functions. The results of this section we obtained in [33].

For each nonempty set Y denote by $\mathcal{B}(Y)$ the set of all bounded functions $f : Y \to R^1$ and for each $f \in \mathcal{B}(Y)$ set

$$\|f\| = \sup\{|f(y)| :\ y \in Y\}.$$

For each nonempty compact metric space Y denote by $C(Y)$ the set of all continuous functions $f : Y \to R^1$.

Let (X, ρ) be a compact metric space with the metric ρ. The set $X \times X$ is equipped with the metric ρ_1 defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2),\ (x_1, x_2),\ (y_1, y_2) \in X \times X.$$

For each integer $t \geq 0$ let Ω_t be a nonempty closed subset of the metric space $X \times X$.

Let $T \geq 0$ be an integer. A sequence $\{x_t\}_{t=T}^{\infty} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers $t \geq T$.

Let T_1, T_2 be integers such that $0 \le T_1 < T_2$. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a program if $(x_t, x_{t+1}) \in \Omega_t$ for all integers t satisfying $T_1 \le t < T_2$.

We assume that there exists a program $\{x_t\}_{t=0}^{\infty}$. Denote by $\mathcal{M}$ the set of all sequences of functions $\{f_t\}_{t=0}^{\infty}$ such that for each integer $t \ge 0$

$$f_t \in \mathcal{B}(\Omega_t)$$

and that

$$\sup\{\|f_t\| : \ t = 0, 1, \dots\} < \infty.$$

For each pair of sequences $\{f_t\}_{t=0}^{\infty}$, $\{g_t\}_{t=0}^{\infty} \in \mathcal{M}$ set

$$d(\{f_t\}_{t=0}^{\infty}, \{g_t\}_{t=0}^{\infty}) = \sup\{\|f_t - g_t\| : \ t = 0, 1, \dots\}.$$

It is easy to see that $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a metric on $\mathcal{M}$ and that the metric space $(\mathcal{M}, d)$ is complete.

Let $\{f_t\}_{t=0}^{\infty} \in \mathcal{M}$. We consider the following optimization problems

(P_{T_1,T_2}) $$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \ \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program,}$$

$(P_{T_1,T_2}^{(y)})$ $$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \ \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y,$$

$(P_{T_1,T_2}^{(y,z)})$
$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \to \min \ \text{ s. t. } \{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y, x_{T_2} = z,$$

where $y, z \in X$ and integers T_1, T_2 satisfy $0 \le T_1 < T_2$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e. g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [13], the study of the discrete Frenkel-Kontorova model related to dislocations in one-dimensional crystals [2, 28] and the analysis of a long slender bar of a polymeric material under tension in [14, 18]. Similar optimization problems are also considered in mathematical economics.

For each $y, z \in X$ and each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ set

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) = \inf\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) : \ \{x_t\}_{t=T_1}^{T_2} \text{ is a program}\},$$

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y) = \inf\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) :$$
$$\{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y\},$$

$$U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y, z) = \inf\{\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) :$$

$$\{x_t\}_{t=T_1}^{T_2} \text{ is a program and } x_{T_1} = y,\ x_{T_2} = z\}.$$

Here we assume that the infimum over empty set is ∞.

We are interested in the structure of approximate solutions of problems (P_{T_1,T_2}), $P^{(y)}_{T_1,T_2}$ and $P^{(y,z)}_{T_1,T_2}$ which are defined as follows.

Let $M \geq 0$, $y, z \in X$ and let integers T_1, T_2 satisfy $0 \leq T_1 < T_2$. A program $\{x_t\}_{t=T_1}^{T_2}$ is called an (M)-approximate solution of problem $P^{(y,z)}_{T_1,T_2}$ if

$$x_{T_1} = y,\ x_{T_2} = z \text{ and } \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y, z) + M.$$

It is called an (M)-approximate solution of problem $P^{(y)}_{T_1,T_2}$ if

$$x_{T_1} = y \text{ and } \sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_t\}_{t=0}^{\infty}, T_1, T_2, y) + M.$$

The program $\{x_t\}_{t=T_1}^{T_2}$ is called an (M)-approximate solution of problem P_{T_1,T_2} if

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) + M.$$

A program $\{x_t\}_{t=0}^{\infty}$ is called an (M)-approximate solution of the corresponding infinite horizon problem if for each pair of integers $T_1 \geq 0$, $T_2 > T_1$,

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) + M.$$

Denote by $\mathcal{M}_{reg}$ the set of all sequences of functions $\{f_i\}_{i=0}^{\infty} \in \mathcal{M}$ for which there exist a program $\{x_t^f\}_{t=0}^{\infty}$ and constants $c_f > 0$, $\gamma_f > 0$ such that the following conditions hold:

(C1) the function f_t is lower semicontinuous for all integers $t \geq 0$;

(C2) for each pair of integers $T_1 \geq 0$, $T_2 > T_1$,

$$\sum_{t=T_1}^{T_2-1} f_t(x_t^f, x_{t+1}^f) \leq U(\{f_t\}_{t=0}^{\infty}, T_1, T_2) + c_f;$$

(C3) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each integer $t \geq 0$ and each $(x, y) \in \Omega_t$ satisfying $\rho(x, x_t^f) \leq \delta$, $\rho(y, x_{t+1}^f) \leq \delta$ we have

$$|f_t(x_t^f, x_{t+1}^f) - f_t(x, y)| \leq \epsilon;$$

(C4) for each integer $t \geq 0$, each $(x_t, x_{t+1}) \in \Omega_t$ satisfying $\rho(x_t, x_t^f) \leq \gamma_f$ and each $(x'_{t+1}, x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x'_{t+2}, x^f_{t+2}) \leq \gamma_f$ there is $x \in X$ such that

$$(x_t, x) \in \Omega_t,\ (x, x'_{t+2}) \in \Omega_{t+1};$$

moreover, for each $\epsilon > 0$ there exists $\delta \in (0, \gamma_f)$ such that for each integer $t \geq 0$, each $(x_t, x_{t+1}) \in \Omega_t$ and each $(x'_{t+1}, x'_{t+2}) \in \Omega_{t+1}$ satisfying $\rho(x_t, x_t^f) \leq \delta$ and

$\rho(x'_{t+2}, x^f_{t+2}) \leq \delta$ there is $x \in X$ such that

$$(x_t, x) \in \Omega_t,\ (x, x'_{t+2}) \in \Omega_{t+1},\ \rho(x, x^f_{t+1}) \leq \epsilon.$$

Denote by $\bar{\mathcal{M}}_{reg}$ the closure of $\mathcal{M}_{reg}$ in $(\mathcal{M}, d)$. Denote by $\mathcal{M}_{c,reg}$ the set of all sequences $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$ such that $f_i \in C(\Omega_i)$ for all integers $i \geq 0$ and by $\bar{\mathcal{M}}_{c,reg}$ the closure of $\mathcal{M}_{c,reg}$ in $(\mathcal{M}, d)$.

We study the optimization problems stated above with the sequence of objective functions $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$. Our study is based on the relation between these finite horizon problems and the corresponding infinite horizon optimization problem determined by $\{f_i\}_{i=0}^\infty$. Note that the condition (C2) means that the program $\{x^f_t\}_{t=0}^\infty$ is an approximate solution of this infinite horizon problem.

Let $\{f_i\}_{i=0}^\infty \in \mathcal{M}_{reg}$, a program $\{x^f_i\}_{i=0}^\infty$, $c_f > 0$ and $\gamma_f > 0$ be such that (C1)-(C4) hold.

We begin with the following useful result.

Proposition 5.1. *Let $S \geq 0$ be an integer and $\{x_i\}_{i=S}^\infty$ be a program. Then either the sequence $\{\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x^f_i, x^f_{i+1})\}_{T=S+1}^\infty$ is bounded or*

$$\lim_{T\to\infty} [\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x^f_i, x^f_{i+1})] = \infty. \tag{2.1}$$

A program $\{x_t\}_{t=S}^\infty$, where $S \geq 0$ is an integer, is called $(\{f_i\}_{i=0}^\infty)$-good if the sequence

$$\{\sum_{i=S}^{T-1} f_i(x_i, x_{i+1}) - \sum_{i=S}^{T-1} f_i(x^f_i, x^f_{i+1})\}_{T=S+1}^\infty$$

is bounded.

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses an asymptotic turnpike property (or briefly (ATP)) with $\{x^f_i\}_{i=0}^\infty$ being the turnpike if for each integer $S \geq 0$ and each $(\{f_i\}_{i=0}^\infty)$-good program $\{x_i\}_{i=S}^\infty$,

$$\lim_{i\to\infty} \rho(x_i, x^f_i) = 0.$$

We say that the sequence $\{f_i\}_{i=0}^\infty$ possesses a turnpike property (or briefly (TP)) if for each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies

$$\sum_{i=T_1}^{T_2-1} f_i(x_i, x_{i+1}) \leq \min\{U(\{f_i\}_{i=0}^\infty, T_1, T_2, x_{T_1}, x_{T_2}) + \delta,$$
$$U(\{f_i\}_{i=0}^\infty, T_1, T_2) + M\},$$

the inequality $\rho(x_i, x^f_i) \leq \epsilon$ holds for all integers $i = T_1 + L, \dots, T_2 - L$.

The sequence $\{x^f_i\}_{i=0}^\infty$ is called the turnpike of $\{f_i\}_{i=0}^\infty$.

In [33] we obtained the following results.

Theorem 5.2. *The sequence $\{f_i\}_{i=0}^\infty$ possesses the turnpike property if and only if $\{f_i\}_{i=0}^\infty$ possesses (ATP) and the following property:*

(P) *For each $\epsilon > 0$ and each $M > 0$ there exist $\delta > 0$ and a natural number L such that for each integer $T \geq 0$ and each program $\{x_t\}_{t=T}^{T+L}$ which satisfies*

$$\sum_{i=T}^{T+L-1} f_i(x_i, x_{i+1})$$

$$\leq \min\{U(\{f_i\}_{i=0}^{\infty}, T, T+L, x_T, x_{T+L}) + \delta,\ U(\{f_i\}_{i=0}^{\infty}, T, T+L) + M\}$$

there is an integer $j \in \{T, \dots, T+L\}$ for which $\rho(x_j, x_j^f) \leq \epsilon$.

The property (P) means that if a natural number L is large enough and a program $\{x_t\}_{t=T}^{T+L}$ is an approximate solution of the corresponding finite horizon problem, then there is $j \in \{T, \dots, T+L\}$ such that x_j is close to x_j^f.

We denote by Card(A) the cardinality of the set A.

Theorem 5.3. *Assume that the sequence $\{f_i\}_{i=0}^{\infty}$ possesses (ATP) and the property (P), $\epsilon > 0$ and $M > 0$. Then there exists a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 > T_1 + L$ and each program $\{x_t\}_{t=T_1}^{T_2}$ which satisfies*

$$\sum_{t=T_1}^{T_2-1} f_t(x_t, x_{t+1}) \leq U(\{f_i\}_{i=0}^{\infty}, T_1, T_2) + M$$

the following inequality holds:

$$Card(\{t \in \{T_1, \dots, T_2\} :\ \rho(x_t, x_t^f) > \epsilon\}) \leq L.$$

Let $S \geq 0$ be an integer. A program $\{x_t\}_{t=S}^{\infty}$ is called $(\{f_i\}_{i=0}^{\infty})$-minimal if for each integer $T > S$,

$$\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) = U(\{f_i\}_{i=0}^{\infty}, S, T, x_S, x_T).$$

A program $\{x_t\}_{t=S}^{\infty}$ is called $(\{f_i\}_{i=0}^{\infty})$-overtaking optimal if for each program $\{x'_t\}_{t=S}^{\infty}$ satisfying $x_S = x'_S$,

$$\limsup_{T\to\infty}\Big(\sum_{t=S}^{T-1} f_t(x_t, x_{t+1}) - \sum_{t=S}^{T-1} f_t(x'_t, x'_{t+1})\Big) \leq 0.$$

Theorem 5.4. *Assume that the sequence $\{f_i\}_{i=0}^{\infty}$ possesses (ATP), $z \in X$, $S \geq 0$ is an integer and that there exists an $(\{f_i\}_{i=0}^{\infty})$-good program $\{x_t\}_{t=S}^{\infty}$ satisfying $x_S = z$. Then there exists an $(\{f_i\}_{i=0}^{\infty})$-overtaking optimal program $\{x_t^*\}_{t=S}^{\infty}$ satisfying $x_S^* = z$.*

Theorem 5.5. *Assume that the sequence $\{f_i\}_{i=0}^{\infty}$ possesses (ATP), $z \in X$, $S \geq 0$ is an integer and that there exists an $(\{f_i\}_{i=0}^{\infty})$-good program $\{\bar{x}_t\}_{t=S}^{\infty}$ satisfying $\bar{x}_S = z$. Let a program $\{x_t\}_{t=S}^{\infty}$ satisfy $x_S = z$. Then the following properties are equivalent.*

(i) *$\{x_t\}_{t=S}^{\infty}$ is an $(\{f_i\}_{i=0}^{\infty})$-overtaking optimal program;*

(ii) *the program $\{x_t\}_{t=S}^{\infty}$ is $(\{f_i\}_{i=0}^{\infty})$-minimal and $(\{f_i\}_{i=0}^{\infty})$-good;*

(iii) *the program $\{x_t\}_{t=S}^{\infty}$ is $(\{f_i\}_{i=0}^{\infty})$-minimal and satisfies*

$$\lim_{t\to\infty} \rho(x_t, x_t^f) = 0.$$

In [33] we also showed that $\{f_i\}_{i=0}^{\infty}$ is approximated by elements of $\mathcal{M}_{reg}$ possessing (TP).

For each $r \in (0,1)$ and all integers $i \geq 0$ set

$$f_i^{(r)}(x,y) = f_i(x,y) + r\rho(x, x_i^f),\ (x,y) \in \Omega_i.$$

Clearly, $\{f_i^{(r)}\}_{i=0}^{\infty} \in \mathcal{M}_{reg}$ for all $r \in (0,1)$ and

$$\lim_{r \to 0^+} d(\{f_i^{(r)}\}_{i=0}^{\infty}, \{f_i\}_{i=0}^{\infty}) = 0.$$

Proposition 5.6. *Let $r \in (0,1)$. Then the sequence $\{f_i^{(r)}\}_{i=0}^{\infty}$ possesses (TP) with $\{x_i^f\}_{i=0}^{\infty}$ being the turnpike.*

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A. J. Zaslavski
Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: `ajzasl@tx.technion.ac.il`