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SEMICIRCULAR-LIKE AND SEMICIRCULAR ELEMENTS INDUCED BY p-ADIC ANALYTIC DYNAMICAL SYSTEMS

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Abstract. In this paper, we study semicircular-like elements, and semicircular elements induced by semigroup dynamical systems of p-adic number fields $\mathbb{Q}_p$ over a unital C^*-probability space (A, ψ), for all primes p.

1. Introduction

Remark that, for studying geometry with "very small" distance, the p-adic-analytic structures have been widely used in quantum physics (e.g., see [31]; also see [5, 9, 10, 11] and [30]). In this paper, we are interested in certain quantum observables dictated by geometry with very small distance (which are self-adjoint operators in ceratin Banach $*$-algebras), equipped with their quantum-statistical data followed by the *semicircular*(*-like*) *law*(*s*). In noncommutative statistical operator-algebraic structures, the semicircular law plays a key role in theories by the *central limit theorem* (e.g., see [35], also see [1, 2, 18, 23, 32] and [34]); i,e., the semicircular law (as the free distributions of semicircular elements in free-probability-theory point of view) is understood as the noncommutative version of the *Gaussian distribution* (or, the *normal distribution* in classical-statistics point of view).

The *weighted-semicircular elements* (whose free distributions are semicircular-like laws), and the corresponding *semicircular elements* (whose free distributions are the semicircular law) induced from the p-adic analysis on the p-adic number fields $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, have been studied in our earlier works [8, 9, 11] and [12], where $\mathcal{P}$ is the set of all *primes* in the set $\mathbb{N}$ of all *natural numbers*.

But, here, we generalize the main constructions-and-results of [8, 9] and [12], and universalize the theories of them under *dynamical system*. The main purpose of this paper is to construct-and-study *weighted-semicircular elements* and *semicircular elements* induced by certain *semigroup dynamical systems* of the *σ-algebras* $\sigma(\mathbb{Q}_p)$ of the *p-adic number fields* $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, by regarding $\sigma(\mathbb{Q}_p)$ as a *semigroup*.

$$\sigma(\mathbb{Q}_p) = (\sigma_p(\mathbb{Q}_p), \cap),$$

where $\cap$ is the usual *set-intersection*.

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Let S be an arbitrary *semigroup*, and assume that there exists a well-defined *injective map*,

$$\pi \quad \to Hom(A),$$

where A is a *topological* $*$*-algebra* (C^**-algebra*, or W^**-algebra*, or *Banach* $*$*-algebra*, etc.) such that:

$$\pi(s) \in Hom(A),$$

for all $s \in S$, and

$$\pi(s_1 s_2) = \pi(s_1)\pi(s_2) \text{ on } A,$$

for all s_1, $s_2 \in S$, where $s_1 s_2$ is a semigroup-product of s_1 and s_2 in S, and $\pi(s_1)\pi(s_2)$ means the product (or composition) of $\pi(s_1)$ and $\pi(s_2)$ in $Hom(A)$, and where

$$Hom(A) = \left\{ \beta : A \to A \,\middle|\, \begin{array}{c} \beta \text{ is a} \\ *\text{-homomorphism} \\ \text{on } A \end{array} \right\}.$$

Then one can obtain the corresponding *semigroup dynamical system*,

$$(A, S, \pi).$$

For such a dynamical system (A, S, π), one can construct the corresponding *crossed product topological* $*$*-algebra*,

$$\mathfrak{A}_S = A \times_\pi S,$$

generated by A and $\pi(S)$, satisfying the π*-relation*:

$$(a_1, s_1)(a_2, s_2) = (a_1 \pi(s_1)(a_2), s_1 s_2),$$

and

$$(a_1, s_1)^* = (\pi(a_1^*), s_1),$$

for all a_1, $a_2 \in A$, and s_1, $s_2 \in S$.

Here, for a fixed *unital* C^**-probability space* (A, ψ), we construct a suitable semigroup dynamical system,

$$(A \otimes_{\mathbb{C}} M_p, \sigma(\mathbb{Q}_p), \theta^p),$$

and corresponding *crossed product* C^**-algebra* $\mathfrak{A}_p$,

$$\mathfrak{A}_p = (A \otimes_{\mathbb{C}} M_p) \times_{\theta^p} \sigma(\mathbb{Q}_p),$$

where M_p is the C^*-algebra of measurable functions on $\mathbb{Q}_p$.

We study how $\mathfrak{A}_p$ generate *weighted-semicircular elements* and *semicircular elements*. In particular, our weighted-semicircular elements will imply both free-distributional data on (A, ψ), and p-adic-analytic information on $\mathbb{Q}_p$, for $p \in \mathcal{P}$.

Our main results illustrate close connections among *number theory*, *operator theory*, *operator algebra theory*, *representation theory*, *theory of dynamical systems*, via *free probability theory*.

1.1. **Remark: Non-Traditional & Traditional Approaches.** Note that the (usual, or traditional) free probability theory provides noncommutative operator-algebraic version of *measure theory* and *statistics* (e.g., [10, 12, 23] through [29], and [31] through [35]). But the $*$-*algebra* $\mathcal{M}_p$ and corresponding C^*-*algebra* M_p in the sense of [8, 11] and [12] are "commutative," (Also, see Sections 3, 4 and 5 below), and hence, they have commutative *functional analysis* (determined up to suitable *linear functionals* on them). But, we applied free-probability-theoretic "methods," "tools," and "concepts" to study such *analysis* on these algebras, non-traditionally because we were only interested in operators having their distributional data, the semicircular(-like) law(s) there. Remark that, under such "non-traditional" senses, free probability theory well-covers commutative operator-algebraic analysis, however, *freeness* on commutative structures is trivial (which is not interesting in free-probabilistic, operator-algebra point of view), while the operators equipped with the semicircular(like) law(s) are nicely characterized-and-explained by free-probabilistic settings and language. In fact, in [8] and [11], by taking *free product* on our commutative structures, we considered "traditional" free probability on noncommutative free product algebras, and realized that our (weighted-)semicircular elements (of [12]) are well-fit for traditional free probability theory.

In this paper, with help of (non-traditional) free-probability-theoretic approaches of [8, 9, 11] and [12], we directly work on "traditional" free-probability-theoretic structures determined by *dynamical systems*.

1.2. **Preview and Motivation.** Relations between number theory and operator algebra theory have been studied in various different approaches (e.g., [5, 13, 14, 15, 18, 19, 20, 21, 22, 30] and [31]). Especially, we cannot help emphasizing close connections between *primes* and *operators*. For example, in [10], we studied operator theory on number-theoretic *Hecke algebras* $\mathcal{H}\left(GL_2(\mathbb{Q}_p)\right)$, where $GL_2(X)$ mean the *general linear groups* consisting of all invertible (2×2)-matrices over X, via *representation theory* and (traditional) free probability theory.

In [12], the author and Jorgensen constructed weighted-semicircular elements, and corresponding semicircular elements in a certain *Banach* $*$-*algebra* $\mathfrak{LS}_p$ induced from the $*$-*algebra* $\mathcal{M}_p$ consisting of all *measurable functions* on a p-adic number fields $\mathbb{Q}_p$, for $p \in \mathcal{P}$. For any fixed prime p, one can obtain $|\mathbb{Z}|$-many weighted-semicircular elements $Q_{p,j}$ in certain $|\mathbb{Z}|$-many non-traditional Banach $*$-probability spaces $\mathfrak{LS}_p(j)$, for all $j \in \mathbb{Z}$. By doing suitable scalar-multiplication on $Q_{p,j}$'s, we obtained corresponding semicircular elements $\Theta_{p,j}$'s in $\mathfrak{LS}_p(j)$, for all $j \in \mathbb{Z}$.

In [8], the author constructed the *free product Banach* $*$-*probability space* $\mathfrak{LS}$ of the system $\{\mathfrak{LS}_p(j)\}_{p\in\mathcal{P},j\in\mathbb{Z}}$, over both primes and integers (which is a traditional free-probability-theoretic structure), and studied weighted-semicircular elements $Q_{p,j}$'s, and semicircular elements $\Theta_{p,j}$'s in $\mathfrak{LS}$, as *free generators* of $\mathfrak{LS}$. The free distributions of *free reduced words* in $Q_{p,j}$'s, or $\Theta_{p,j}$'s were considered there. The main results of [8] demonstrate that the non-traditionally obtained semicircular(-like) elements of [12] are indeed semicircular(-like) elements traditionally.

As an application of [8], in [11], we studied *free stochastic calculus* on $\mathfrak{LS}$ for the *free stochastic motions* (or *free stochastic processes*) determined by the weighted-semicircular laws, and the semicircular law of [8].

In this paper, we will extend the foundations of [8, 11] and [12], and generalize the results of them under dynamical systems in the traditional free-probability-theoretic senses.

1.3. **Overview.** In Sections 2, we briefly introduce backgrounds of our works: free probability theory, and p-adic analysis. Our (non-traditional) free-probabilistic models on $\mathcal{M}_p$ is established and considered in Sections 3. And then, we construct suitable Hilbert-space representations of $\mathcal{M}_p$, preserving the free-distributional data of Section 3 implying number-theoretic information in Section 4. Under representation, the C^*-algebras M_p and corresponding (non-traditional) C^*-probabilistic structures are studied in Section 5.

In Section 6, from non-traditional C^*-probabilistic structures of Section 5, we construct (traditional) C^*-probability spaces $(\mathfrak{A}_p, \psi_{p,j})$ induced by *semigroup-dynamical systems*

$$((A \otimes_{\mathbb{C}} M_p), \sigma(p), \theta^p)$$

of the semigroup

$$\sigma(p) = (\sigma(\mathbb{Q}_p), \cap),$$

the σ-algebra $\sigma(\mathbb{Q}_p)$ with the usual set-intersection $\cap$, acting on a unital C^*-probability space (A, ψ), where

$$\mathfrak{A}_p = (A \otimes_{\mathbb{C}} M_p) \times_{\theta^p} \sigma(p)$$

are the *crossed product C^*-algebras* of the semigroup-dynamical systems. We study free probability on $\mathfrak{A}_p$.

In Sections 7 and 8, the weighted-semicircular elements, implying free distributions from (A, ψ), and p-adic analysis from M_p, are established-and-studied. Under suitable additional conditions, semicircular elements are naturally obtained from our weighted-semicircular elements.

In Sections 9, we universalize the weighted-semicircularity and the corresponding semicircularity of Section 8 under *free product.*

The free-distributional data of *free reduced words* in our weighted-semicircular elements, and those in semicircular elements, are computed in Section 10.

2. Preliminaries

In this section, we briefly introduce backgrounds of our proceeding works. For more about pure number-theoretic motivations, and backgrounds, e.g., see [16] and [17].

2.1. **Free Probability.** Readers can review fundamental combinatorial-and-analytic *free probability* from [26] and [35]. *Free probability* is understood as the noncommutative operator-algebraic version of classical *measure theory* and *statistics.* The classical *independence* is replaced to be the *freeness* by replacing *measures* on sets to *linear functionals* on algebras. It has various applications not only in pure mathematics (e.g., [23, 24, 25, 27, 28, 29, 32, 33] and [34]), but also in related fields ([6, 4] through [12]).

Here, we will use *combinatorial free probability theory* of Speicher (e.g., [26]). Especially, in the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed. Also, the *free product of noncommutative (free) probability spaces* is considered without detailed introduction.

2.2. **Calculus on $\mathbb{Q}_p$.** For more about *p-adic analysis* and *Adelic analysis*, see [30] and [31]. Let $p \in \mathcal{P}$ be a prime, and let $\mathbb{Q}$ be the set of all *rational numbers.* Define a *non-Archimedean norm* $|.|_p$ on $\mathbb{Q}$ by

$$|x|_p = \left|p^k \frac{a}{b}\right|_p = \frac{1}{p^k},$$

whenever $x = p^k \frac{a}{b}$, where k, $a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$. We call $|.|_p$, the *p-norm on* $\mathbb{Q}$, for all $p \in \mathcal{P}$. For instance,

$$\left|\frac{8}{3}\right|_2 = \left|2^3 \cdot \frac{1}{3}\right|_2 = \frac{1}{2^3} = \frac{1}{8},$$

and

$$\left|\frac{8}{3}\right|_3 = \left|3^{-1} \cdot 8\right|_3 = \frac{1}{3^{-1}} = 3,$$

and

$$\left|\frac{8}{3}\right|_q = \frac{1}{q^0} = 1, whenever q \in \mathcal{P} \setminus \{2, 3\}.$$

The *p-adic number fields* $\mathbb{Q}_p$ are the maximal p-norm closures in $\mathbb{Q}$. So, under norm topology, the set $\mathbb{Q}_p$ forms a *Banach space* (e.g., [31]).

All elements x of $\mathbb{Q}_p$ are expressed by

$$x = \sum_{k=-N}^{\infty} x_k p^k, \text{ with } x_k \in \{0, 1, ..., p-1\},$$

for $N \in \mathbb{N}$, decomposed by

$$x = \sum_{l=-N}^{-1} x_l p^l + \sum_{k=0}^{\infty} x_k p^k.$$

If $x = \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$, then we call x, a *p-adic integer.* Remark that, $x \in \mathbb{Q}_p$ is a p-adic integer, if and only if $|x|_p \leq 1$. So, by collecting all p-adic integers in $\mathbb{Q}_p$, one can define the *unit disk* $\mathbb{Z}_p$ *of* $\mathbb{Q}_p$,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \, |x|_p \leq 1\}.$$

Under the *p-adic addition* and the *p-adic multiplication* in the sense of [31], this Banach space $\mathbb{Q}_p$ forms a *ring* algebraically, i.e., $\mathbb{Q}_p$ is a *Banach ring.*

Also, one can understand this *Banach ring* $\mathbb{Q}_p$ as a *measure space*,

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

where $\sigma(\mathbb{Q}_p)$ is the *σ-algebra of* $\mathbb{Q}_p$, consisting of all *μ_p-measurable subsets*, where μ_p is a left-and-right additive invariant *Haar measure* on $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1.$$

If we define

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{2.1}$$

for all $k \in \mathbb{Z}$, then these μ_p-measurable subsets U_k's of (2.1) satisfy

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\cup} U_k,$$

and

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \text{ for all } x \in \mathbb{Q}_p, \tag{2.2}$$

and

$$\cdots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \cdots.$$

i.e., the family $\{U_k\}_{k \in \mathbb{Z}}$ of (2.1) forms a *basis* of the Banach topology for $\mathbb{Q}_p$ (e.g., [31]).

Define now subsets $\partial_k \in \sigma(\mathbb{Q}_p)$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \; in \mathbb{Z}. \tag{2.3}$$

We call such μ_p-measurable subsets ∂_k, the *k-th boundaries* (*of* U_k) in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. By (2.2) and (2.3), one obtains that

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\sqcup} \partial_k,$$

and

$$\mu_p(\partial_k) = \mu_p(U_k) - \mu_p(U_{k+1}) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{2.4}$$

and

$$\partial_{k_1} \cap \partial_{k_2} = \delta_{k_1,k_2} \partial_{k_1},$$

for all k, k_1, $k_2 \in \mathbb{Z}$, where $\sqcup$ means the *disjoint union*.

Now, let $\mathcal{M}_p$ be the (pure-algebraic) *algebra*,

$$\mathcal{M}_p = \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}\right], \tag{2.5}$$

where χ_S are the usual *characteristic functions* of μ_p-measurable subsets S of $\mathbb{Q}_p$.

So, $f \in \mathcal{M}_p$, if and only if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S, \text{ with } t_S \in \mathbb{C}, \tag{2.5$'$}$$

where $\sum$ is the *finite sum*.

Remark that the algebra $\mathcal{M}_p$ of (2.5) forms a $*$-*algebra over* $\mathbb{C}$, with its well-defined *adjoint*,

$$\left(\sum_{S \in \sigma(G_p)} t_S \chi_S\right)^* \overset{def}{=} \sum_{S \in \sigma(G_p)} \overline{t_S} \chi_S,$$

where $t_S \in \mathbb{C}$ with their *conjugates* $\overline{t_S}$ in $\mathbb{C}$.

Let $f \in \mathcal{M}_p$ be in the sense of (2.5)$'$. Then one can define the *p-adic integral of* f by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S). \tag{2.6}$$

Note that, by (2.4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a unique subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} S \cap \partial_j \neq \varnothing\}, \tag{2.7}$$

satisfying

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \int_{\mathbb{Q}_p} \sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} d\mu_p$$

$$\text{(by (2.6))} \qquad = \sum_{j \in \Lambda_S} \mu_p \left(S \cap \partial_j\right)$$

$$\leq \sum_{j \in \Lambda_S} \mu_p \left(\partial_j\right) = \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{2.8}$$

by (2.4), for the subset Λ_S of $\mathbb{Z}$ of (2.7).

Proposition 2.1. *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that*

$$0 \leq r_j \leq 1 in \mathbb{R}, \text{ forall } j \in \Lambda_S, \text{ and} \tag{2.9}$$

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Proof. The existence of r_j and the p-adic integration in (2.9) is guaranteed by (2.6), (2.7) and (2.8). □

3. Analysis on $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and $\mathbb{Q}_p$, the corresponding p-adic number field, and let $\mathcal{M}_p$ be the $*$-algebra (2.5). In this section, we establish a suitable (non-traditional) free-probabilistic model on $\mathcal{M}_p$ to study analysis on $\mathcal{M}_p$.

Let U_k and ∂_k be in the sense of (2.1), respectively, (2.3) in $\mathbb{Q}_p$, i.e.,

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \text{and } \partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{3.1}$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by the p-adic integration (2.6),

$$\varphi_p\left(f\right) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p. \tag{3.2}$$

Then, by (3.2), one naturally obtain that

$$\varphi_p\left(\chi_{U_j}\right) = \frac{1}{p^j}, \text{ and } \varphi_p\left(\chi_{\partial_j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

for all $j \in \mathbb{Z}$, by (2.2), respectively, (2.4).

Definition 3.1. The pair $(\mathcal{M}_p, \varphi_p)$ is called the (non-traditional) p-adic free probability space, for $p \in \mathcal{P}$, where φ_p is the linear functional (3.2) on $\mathcal{M}_p$.

Let U_k be in the sense of (3.1) in $\mathbb{Q}_p$, and $\chi_{U_k} \in \mathcal{M}_p$, for all $k \in \mathbb{Z}$. Then

$$\chi_{U_{k_1}} \chi_{U_{k_2}} = \chi_{U_{k_1} \cap U_{k_2}} = \chi_{U_{\max\{k_1, k_2\}}},$$

by (2.2), where $\max\{k_1, k_2\}$ means the *maximum* in $\{k_1, k_2\}$. Thus, one can verify that

$$\varphi_p\left(\chi_{U_{k_1}}\chi_{U_{k_2}}\right) = \frac{1}{p^{\max\{k_1,k_2\}}}. \tag{3.3}$$

Inductive to (3.3), we obtain the following result.

Proposition 3.2. *Let* $(j_1, ..., j_N) \in \mathbb{Z}^N$, *for* $N \in \mathbb{N}$. *Then*

$$\prod_{l=1}^{N}\chi_{U_{j_l}} = \chi_{U_{\max\{j_1,\dots,j_N\}}} \text{ in } \mathcal{M}_p, \text{ and} \tag{3.4}$$
$$\varphi_p\left(\prod_{l=1}^{N}\chi_{U_{j_l}}\right) = \frac{1}{p^{\max\{j_1,\dots,j_N\}}}.$$

Proof. The proof of (3.4) is done by induction on (3.3). □

Now, let ∂_k be the k-th boundary of (3.1) in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for k_1, k_2 $\in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}}\chi_{\partial_{k_2}} = \chi_{\partial_{k_1}\cap\partial_{k_2}} = \delta_{k_1,k_2}\chi_{\partial_{k_1}}, \text{ and} \tag{3.5}$$
$$\varphi_p\left(\chi_{\partial_{k_1}}\chi_{\partial_{k_2}}\right) = \delta_{k_1,k_2}\left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}}\right),$$

where δ is the *Kronecker delta.*

Proposition 3.3. *Let* $(j_1, ..., j_N) \in \mathbb{Z}^N$, *for* $N \in \mathbb{N}$. *Then*

$$\prod_{l=1}^{N}\chi_{\partial_{j_l}} = \delta_{(j_1,\dots,j_N)}\chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p, \text{ and} \tag{3.6}$$
$$\varphi_p\left(\prod_{l=1}^{N}\chi_{\partial_{j_l}}\right) = \delta_{(j_1,\dots,j_N)}\left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}}\right),$$

where

$$\delta_{(j_1,\dots,j_N)} = \left(\prod_{l=1}^{N-1}\delta_{j_l,j_{l+1}}\right)\left(\delta_{j_N,j_1}\right).$$

Proof. The proof of (3.6) is done by induction on (3.5). □

Thus, one can get that, for any $S \in \sigma(\mathbb{Q}_p)$,

$$\varphi_p(\chi_S) = \sum_{j\in\Lambda_S} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{3.7}$$

by (3.6), where $0 \le r_j \le 1$ are in the sense of (2.9), for all $j \in \mathbb{Z}$.

Also, if $S_1,\, S_2 \in \sigma(\mathbb{Q}_p)$, then

$$\begin{aligned}\chi_{S_1}\chi_{S_2} &= \left(\sum_{k\in\Lambda_{S_1}} \chi_{S_1\cap\partial_k}\right)\left(\sum_{j\in\Lambda_{S_2}} \chi_{S_2\cap\partial_j}\right) \\ &= \sum_{(k,j)\in\Lambda_{S_1}\times\Lambda_{S_2}} \left(\chi_{S_1\cap\partial_k}\chi_{S_2\cap\partial_j}\right) \\ &= \sum_{(k,j)\in\Lambda_{S_1}\times\Lambda_{S_2}} \delta_{k,j}\chi_{(S_1\cap S_2)\cap\partial_j} \\ &= \sum_{j\in\Lambda_{S_1,S_2}} \chi_{(S_1\cap S_2)\cap\partial_j},\end{aligned} \tag{3.8}$$

where

$$\Lambda_{S_1,S_2} = \Lambda_{S_1}\cap\Lambda_{S_2}.$$

By (3.7) and (3.8), one can get that there exist $w_j \in \mathbb{R}$, such that

$$0 \le w_j \le 1, \text{ for all } j \in \Lambda_{S_1,S_2}, \text{ and} \tag{3.9}$$

$$\varphi_p\left(\chi_{S_1}\chi_{S_2}\right) = \sum_{j\in\Lambda_{S_1,S_2}} w_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for all $S_1,\, S_2 \in \sigma(\mathbb{Q}_p)$. In (3.9), definitely, if Λ_{S_1,S_2} is empty, then $\varphi_p\left(\chi_{S_1}\chi_{S_2}\right) = 0$.

Proposition 3.4. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Let*

$$\Lambda_{S_1,\ldots,S_N} = \bigcap_{l=1}^{N}\Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} are in the sense of (2.7)*, for $l = 1, \ldots, N$. Then there exist $r_j \in \mathbb{R}$, such that*

$$0 \le r_j \le 1 \text{ in } \mathbb{R}, \text{ for } j \in \Lambda_{S_1,\ldots,S_N}, \text{ and} \tag{3.10}$$

$$\varphi_p\left(\prod_{l=1}^{N}\chi_{S_l}\right) = \sum_{j\in\Lambda_{S_1,\ldots,S_N}} r_j\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right).$$

Proof. The proof of (3.10) is done by induction on (3.9). □

The above free-moment formula (3.10) provides a universal tool to compute the distributional data of elements in our p-adic free probability space $(\mathcal{M}_p, \varphi_p)$.

4. Representations of $(\mathcal{M}_p, \varphi_p)$

Fix a prime p in $\mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the p-adic free probability space. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2-*space* of $\mathbb{Q}_p$,

$$H_p \overset{def}{=} L^2\left(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p\right) = L^2\left(\mathbb{Q}_p\right), \tag{4.1}$$

over $\mathbb{C}$, consisting of all *square-integrable μ_p-measurable functions* on $\mathbb{Q}_p$. Then this L^2-space is a well-defined *Hilbert space* equipped with its *inner product* $\langle\ ,\ \rangle_2$,

$$\langle h_1, h_2\rangle_2 \overset{def}{=} \int_{\mathbb{Q}_p} h_1 h_2^* d\mu_p, \tag{4.2}$$

for all h_1, $h_2 \in H_p$.

The L^2-space H_p of (4.1) is the $\|.\|_2$-*norm* completion in $\mathcal{M}_p$, where

$$\|f\|_2 \overset{def}{=} \sqrt{\langle f, f \rangle_2}, \text{ for all } f \in H_p,$$

where $\langle\ ,\ \rangle_2$ is the inner product (4.2) on H_p.

Definition 4.1. We call the Hilbert space H_p of (4.1), the p-adic Hilbert space.

By the definition (4.1) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an *algebra-action* α^p,

$$\alpha^p(f)(h) = fh, \text{ for all } h \in H_p, \tag{4.3}$$

for all $f \in \mathcal{M}_p$. i.e., for any $f \in \mathcal{M}_p$, the image $\alpha^p(f)$ is a *multiplication operator on* H_p *with its symbol* f contained in the *operator algebra* $B(H_p)$ of all bounded linear operators on H_p.

Notation. Denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{M}_p$. Also, for convenience, denote $\alpha_{\chi_S}^p$ simply by α_S^p, for all $S \in \sigma(\mathbb{Q}_p)$. For instance,

$$\alpha_{U_k}^p = \alpha_{\chi_{U_k}}^p = \alpha^p(\chi_{U_k}),$$

and

$$\alpha_{\partial_k}^p = \alpha_{\chi_{\partial_k}}^p = \alpha^p(\chi_{\partial_k}),$$

for all $k \in \mathbb{Z}$, where U_k and ∂_k are in the sense of (3.1), for all $k \in \mathbb{Z}$.

By the definition (4.3), the linear morphism α^p is a well-determined $*$-algebra-action of $\mathcal{M}_p$ acting on H_p, equivalently, it is a well-defined $*$-homomorphism from $\mathcal{M}_p$ into $B(H_p)$; Indeed, it is not difficult to check that

$$\alpha_{f_1 f_2}^p = \alpha_{f_1}^p \alpha_{f_2}^p \tag{4.4}$$

for all f_1, $f_2 \in \mathcal{M}_p$; and

$$\left(\alpha_f^p\right)^* = \alpha_{f^*} \tag{4.5}$$

for all $f \in \mathcal{M}_p$.

Proposition 4.2. *The linear morphism α^p of (4.3) is a well-defined $*$-algebra-action of $\mathcal{M}_p$ acting on H_p. Equivalently, the pair (H_p, α^p) is a well-determined Hilbert-space representation of $\mathcal{M}_p$.*

Proof. By (4.4) and (4.5), the morphism α^p of (4.3) is a bounded $*$-homomorphism from $\mathcal{M}_p$ to $B(H_p)$. So, the pair (H_p, α^p) is a Hilbert-space representation of $\mathcal{M}_p$. □

Definition 4.3. The Hilbert-space representation (H_p, α^p) is said to be the p-adic representation of $\mathcal{M}_p$.

Depending on the p-adic representation (H_p, α^p) of $\mathcal{M}_p$, one can construct the C^*-*algebra* M_p in the operator algebra $B(H_p)$.

Definition 4.4. Let M_p be the operator-norm closure of $\mathcal{M}_p$ in the operator algebra $B(H_p)$, i.e.,

$$M_p \overset{def}{=} \overline{\alpha^p(\mathcal{M}_p)} = \overline{\mathbb{C}\left[\alpha_f^p : f \in \mathcal{M}_p\right]} \text{ in } B(H_p), \tag{4.6}$$

where $\overline{X}$ mean the operator-norm closures of subsets X of $B(H_p)$. Then this C^*-subalgebra M_p of $B(H_p)$ is called the p-adic C^*-algebra of $(\mathcal{M}_p, \varphi_p)$.

5. Free-probabilistic models on M_p

Throughout this section, let's fix a prime $p \in \mathcal{P}$, and let $(\mathcal{M}_p, \varphi_p)$ be the p-adic free probability space. And let M_p be the p-adic C^*-algebra (4.6) determined by the p-adic representation $(\mathcal{M}_p, \varphi_p)$. We here consider suitable (non-traditional) free-probabilistic models on M_p.

Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi_j^p(a) \overset{def}{=} \left\langle \alpha_a^p(\chi_{\partial_j}), \chi_{\partial_j} \right\rangle_2, \text{ for all } a \in M_p, \tag{5.1}$$

for all $j \in \mathbb{Z}$, where $\langle\ ,\ \rangle_2$ is the inner product (4.2) on the p-adic Hilbert space H_p of (4.1), and where $\{\partial_j\}_{j\in\mathbb{Z}}$ are the boundaries of $\mathbb{Q}_p$.

First, remark that, if $a \in M_p$, then

$$a = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S \chi_S \text{ in } M_p,$$

where $\sum$ is finite or infinite (limit of finite) sum(s) under C^*-topology of M_p.

Definition 5.1. Let $j \in \mathbb{Z}$, and let φ_j^p be the linear functional (5.1) on the p-adic C^*-algebra M_p. Then the pair $\left(M_p, \varphi_j^p\right)$ is said to be the (non-traditional) j-th p-adic C^*-probability space.

So, one can get the system

$$\{(M_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of the j-th p-adic C^*-probability spaces.

Now, fix $j \in \mathbb{Z}$, and take the corresponding j-th p-adic C^*-probability space $\left(M_p, \varphi_j^p\right)$. For $S \in \sigma(\mathbb{Q}_p)$, and a generating operator $\alpha_S^p \in M_p$, one has that

$$\begin{aligned}
\varphi_j^p(\alpha_S^p) &= \left\langle \alpha_S^p(\chi_{\partial_j}), \chi_{\partial_j} \right\rangle_2 \\
&= \left\langle \chi_{S\cap\partial_j}, \chi_{\partial_j} \right\rangle_2 \\
&= \int_{\mathbb{Q}_p} \chi_{S\cap\partial_j} d\mu_p \\
&= \mu_p(S \cap \partial_j) \\
&= r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),
\end{aligned} \tag{5.2}$$

for some $0 \le r_S \le 1$ in $\mathbb{R}$.

Proposition 5.2. *Let* $S \in \sigma(\mathbb{Q}_p)$, *and* $\alpha_S^p = \alpha_{\chi_S}^p \in \left(M_p, \varphi_j^p\right)$, *for a fixed* $j \in \mathbb{Z}$. *Then there exists* $r_S \in \mathbb{R}$, *such that*

$$0 \leq r_S \leq 1 \text{ in } \mathbb{R}, \tag{5.3}$$
$$\textit{and}$$
$$\varphi_j^p\left(\left(\alpha_S^p\right)^n\right) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \text{ for all } n \in \mathbb{N}.$$

Proof. Remark that the generating operator α_S^p is a projection in M_p, in the sense that:

$$\left(\alpha_S^p\right)^* = \alpha_S^p = \left(\alpha_S^p\right)^2, in M_p.$$

So,

$$\left(\alpha_S^p\right)^n = \alpha_S^p, \text{ for all } n \in \mathbb{N}.$$

Thus, for any $n \in \mathbb{N}$, we have

$$\varphi_j^p\left(\left(\alpha_S^p\right)^n\right) = \varphi_j^p(\alpha_S^p) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

for some $0 \leq r_S \leq 1$ in $\mathbb{R}$, by (5.2). □

The above formula (5.3) characterizes the distributions of the projections α_S^p in the j-th p-adic C^*-probability space $\left(M_p, \varphi_j^p\right)$, for $j \in \mathbb{Z}$.

6. Semigroup dynamical systems induced by $\mathbb{Q}_p$

In this section, we study dynamical systems induced by the σ-algebra $\sigma(\mathbb{Q}_p)$ of a p-adic number field $\mathbb{Q}_p$, for $p \in \mathcal{P}$. Throughout this section, let's fix a *unital* C^**-probability space* (A, ψ) of a noncommutative C^*-algebra A in an *operator algebra* $B(H)$ (consisting of all bounded operators on H), where H is a Hilbert space where A acts, and a bounded linear functional ψ on A, satisfying

$$\psi(1_A) = 1,$$

where 1_A is the *unit* (or the *multiplication-identity*) of A.

Let M_p be our p-adic C^*-algebra (4.6) in the operator algebra $B(H_p)$, where H_p is the p-adic Hilbert space of Section 4.

Define now a new C^*-algebra $\mathcal{A}_p$ by the *tensor product* C^**-algebra* $A \otimes_{\mathbb{C}} M_p$ in the operator algebra $B(H \otimes H_p)$, where $\otimes_{\mathbb{C}}$ is the tensor product of C^*-algebras, and $\otimes$ is the tensor product of Hilbert spaces, i.e.,

$$\mathcal{A}_p \overset{def}{=} A \otimes_{\mathbb{C}} M_p \subset B(\mathcal{H}_p), \tag{6.1}$$

where $\mathcal{H}_p = H \otimes H_p$.

Now, understand $\sigma(\mathbb{Q}_p)$ as a *semigroup* $(\sigma(\mathbb{Q}_p), \cap)$, where $\cap$ is the usual *set-intersection.* We denote this semigroup simply by $\sigma(p)$, i.e.,

$$\sigma(p) \overset{denote}{=} (\sigma(\mathbb{Q}_p), \cap). \tag{6.2}$$

Indeed, such an algebraic structure $\sigma(p)$ of (6.2) is a well-defined semigroup because $\cap$ is not only well-defined, but also associative on $\sigma(\mathbb{Q}_p)$.

Now, note that there exists an *injective map* $\pi^p : \sigma(p) \to B(\mathcal{H}_p)$ assigning $S \in \sigma(p)$ to an operator,

$$\pi^p(S) \overset{def}{=} 1_A \otimes \alpha_S^p \in \mathcal{A}_p, \tag{6.3}$$

in the C^*-algebra $\mathcal{A}_p$ of (6.1), where α^p is the representation (4.3), satisfying (4.4) and (4.5) on the p-adic Hilbert space H_p. Remark that, since we assumed that the given C^*-probability space (A, ψ) is unital, the unit $1_A \in A$ is well-determined, and hence, the morphism π^p of (6.3) is well-defined, injectively.

Then this morphism π^p of (6.3) satisfies that:

$$\begin{gathered} \pi^p(S_1 \cap S_2) = \pi^p(S_1)\pi^p(S_2) \text{ in } \mathcal{A}_p, \\ and \\ \pi^p(S)^* = \pi^p(S) \text{ in } mathcal A_p, \end{gathered} \tag{6.4}$$

for all S, S_1, $S_2 \in \sigma(p)$.

Since the map π^p of (6.3) is well-determined satisfying (6.4), one can define a *semigroup-action* θ^p *of the semigroup* $\sigma(p)$ of (6.2) acting on the C^*-algebra $\mathcal{A}_p$ by

$$\begin{aligned} (\theta^p(S))\left(a \otimes \alpha_Y^p\right) &= \pi^p(S)^*(a \otimes \alpha_Y^p)\pi^p(S) \\ &= a \otimes \alpha_S^p \alpha_Y^p \alpha_S^p = a \otimes \alpha_{Y \cap S}^p, \end{aligned} \tag{6.5}$$

for all $a \in (A, \psi)$, and S, $Y \in \sigma(p)$, in $\mathcal{A}_p$.

Notation. For convenience, we denote the images $\theta^p(S)$ by θ_S^p, $\forall S \in \sigma(p)$.

Let θ^p be the morphism (6.5). Then

$$\begin{aligned} \theta^p_{S_1 \cap S_2}(a \otimes \alpha_Y) &= a \otimes \alpha^p_{Y \cap S_1 \cap S_2} \\ \text{(by (6.5))} \qquad &= a \otimes \alpha^p_{Y \cap S_2 \cap S_1} = \left(a \otimes \alpha^p_{Y \cap S_2}\right)\left(1_A \otimes \alpha^p_{S_1}\right) \\ \text{(by (6.4))} \qquad &= \left(\theta^p_{S_1}(a)\right)\left(1_A \otimes \alpha^p_Y \alpha^p_{S_2}\right) = \theta^p_{S_1}\left(\theta^p_{S_2}(a \otimes \alpha^p_Y)\right) \\ &= \left(\theta^p_{S_1}\theta^p_{S_2}\right)(a \otimes \alpha^p_Y), \end{aligned}$$

for all $a \in (A, \psi)$, and S_1, $S_2 \in \sigma(p)$, i.e.,

$$\theta^p_{S_1 \cap S_2} = \theta^p_{S_1}\theta^p_{S_2} \text{ on } A, \ \forall S_1, S_2 \in \sigma(p). \tag{6.6}$$

Also, one can get that

$$\begin{aligned} \text{by (6.5)} \qquad \theta^p_S\left((a \otimes \alpha^p_Y)^*\right) &= \theta^p_S\left(a^* \otimes \alpha^p_Y\right) = a^* \otimes \alpha^p_{Y \cap S} \\ &= \left(a \otimes \alpha^p_{Y \cap S}\right)^* = \left(\theta^p_S\left(a \otimes \alpha^p_Y\right)\right)^*, \end{aligned}$$

for all $a \in A$, and Y, $S \in \sigma(p)$, i.e.,

$$\theta^p_S\left(T^*\right) = \left(\theta^p_S(T)\right)^*, \tag{6.7}$$

for all $T \in \mathcal{A}_p$.

By (6.6) and (6.7), indeed, the morphism θ^p of (6.5) forms a semigroup-action of $\sigma(p)$ acting on A in $\mathcal{A}_p$, because

$$\theta^p_S \in Hom(\mathcal{A}_p), \text{ for all } S \in \sigma(p), \tag{6.8}$$

where

$$Hom(\mathcal{A}_p) = \left\{ \theta \,\middle|\, \begin{array}{c} \theta \text{ is a bounded} \\ *\text{-homomorphism on } \mathcal{A}_p \end{array} \right\}.$$

Now, construct the mathematical triple,

$$(\mathcal{A}_p, \sigma(p), \theta^p), \text{ for } p \in \mathcal{P}. \tag{6.9}$$

Proposition 6.1. *Let $(\mathcal{A}_p, \sigma(p), \theta^p)$ be a triple (6.9). Then it forms a semigroup C^*-dynamical system of $\sigma(p)$ acting on $\mathcal{A}_p$.*

Proof. It suffices to show that the morphism θ^p of (6.5) is a well-defined semigroup-action of $\sigma(p)$ acting on $\mathcal{A}_p$. But, it is shown by (6.8) with help of (6.6) and (6.7). □

The above proposition shows that there exists a well-determined semigroup dynamical system (6.9) from the p-adic number field $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, whenever we fix a suitable C^*-probability space (A, ψ).

Definition 6.2. Let $(\mathcal{A}_p, \sigma(p), \theta^p)$ be a semigroup dynamical system (6.9) for $p \in \mathcal{P}$. We call it a p-adic (semigroup-)A-dynamical system of $\sigma(p)$ (on $\mathcal{A}_p$), where (A, ψ) is a fixed unital C^*-probability space.

For a p-adic A-dynamical system $(\mathcal{A}_p, \sigma(p), \theta^p)$ of (6.9), one can define the corresponding *crossed product C^*-algebra* $\mathfrak{A}_p$ by a ($*$-isomorphic) C^*-(sub)algebra of $B(\mathcal{H}_p \otimes H_p)$ generated by A and $\theta^p(\sigma(p))$,

$$\mathfrak{A}_p = \mathcal{A}_p \times_{\theta^p} \sigma(p), \tag{6.10}$$

consisting of limits of linear combinations of

$$\left(a \otimes \sigma_Y^p, S\right) \in \mathfrak{A}_p,$$

with $a \in (A, \psi)$ for all $Y, S \in \sigma(p)$.

Then, on this crossed product C^*-algebra $\mathfrak{A}_p$ of (6.10) follow so-called the θ^p-*relation*:

$$\left(a_1 \otimes \alpha_{Y_1}^p, S_1\right)\left(a_2 \otimes \alpha_{Y_2}^p, S_2\right) = \left((a_1 \otimes \alpha_{Y_1}^p)\theta_{S_1}^p(a_2 \otimes \alpha_{Y_2}^p), S_1 \cap S_2\right), \text{ and} \tag{6.11}$$
$$\left(a \otimes \alpha_Y^p, S\right)^* = \left(\theta_S^p\left((a \otimes \alpha_Y^p)^*\right), S\right),$$

for all $a_1, a_2, a \in (A, \psi)$, and $Y_1, Y_2, Y, S_1, S_2, S \in \sigma(p)$.

By (6.5), the above θ^p-relation (6.11) can be re-written by

$$\begin{aligned} \left(a_1 \otimes \alpha_{Y_1}^p, S_1\right)\left(a_2 \otimes \alpha_{Y_2}^p, S_2\right) &= \left((a_1 \otimes \alpha_{Y_1}^p)(a_2 \otimes \alpha_{Y_2 \cap S_1}^p), S_1 \cap S_2\right) \\ &= \left(a_1 a_2 \otimes \alpha_{Y_1 \cap Y_2 \cap S_1}^p, S_1 \cap S_2\right), \text{ and} \\ \left(a \otimes \alpha_Y^p, S\right)^* &= \left(a^* \otimes \alpha_{Y \cap S}^p, S\right), \end{aligned} \tag{6.11$'$}$$

for all $a_1, a_2 \in (A, \psi)$, and $Y_1, Y_2, Y, S_1, S_2, S \in \sigma(p)$.

So, without loss of generality, if we mention about the θ^p -relation (6.11) on the crossed product C^*-algebra $\mathfrak{A}_p$ of (6.10), we directly regard it as (6.11)$'$ from below.

Definition 6.3. Let $\mathfrak{A}_p = \mathcal{A}_p \times_{\theta^p} \sigma(p) = (A \otimes_{\mathbb{C}} M_p) \times_{\theta^p} \sigma(p)$ be the crossed product C^*-algebra (6.10) generated by a p-adic A-dynamical system $(\mathcal{A}_p, \sigma(p), \theta^p)$ of (6.9), satisfying the θ^p-relation (6.11), or (6.11)$'$, where $(A,\ \psi)$ be a fixed unital C^*-probability space. Then we call $\mathfrak{A}_p$, the p-adic A-dynamical (C^*-)algebra of $\sigma(p)$, for $p \in \mathcal{P}$.

Define now a "*conditional*" *tensor product C^*-algebra*

$$\mathfrak{A}_p^0 = \mathcal{A}_p \otimes_{\theta^p} M_p = (A \otimes_{\mathbb{C}} M_p) \otimes_{\theta^p} M_p, \tag{6.12}$$

by the C^*-subalgebra the (usual) tensor product C^*-algebra $\mathcal{A}_p \otimes_{\mathbb{C}} M_p$ of the C^*-algebra $\mathcal{A}_p$ of (6.1) and the p-adic C^*-algebra M_p of (4.6), satisfying the θ^p-*condition*:

$$\left((a_1 \otimes \alpha_{Y_1}^p) \otimes \alpha_{S_1}^p\right)\left((a_2 \otimes \alpha_{Y_2}^p) \otimes \alpha_{S_2}^p\right) = \left((a_1 \otimes \alpha_{Y_1}^p)\theta_{S_1}^p(a_2 \otimes \alpha_{Y_2}^p)\right) \otimes \alpha_{S_1}^p \alpha_{S_2}^p,$$

$$\text{and} \tag{6.13}$$

$$\left((a \otimes \alpha_Y^p) \otimes \alpha_S^p\right)^* = \left(\theta_S^p\left((a \otimes \alpha_Y^p)^*\right)\right) \otimes \left(\alpha_S^p\right)^*,$$

equivalently,

$$\left((a_1 \otimes \alpha_{Y_1}^p) \otimes \alpha_{S_1}^p\right)\left((a_2 \otimes \alpha_{Y_2}^p) \otimes \alpha_{S_2}^p\right) = \left(a_1 a_2 \otimes \alpha_{Y_1 \cap Y_2 \cap S_1}^p\right) \otimes \alpha_{S_1 \cap S_2}^p,$$

$$\textit{and} \tag{6.13$'$}$$

$$\left((a \otimes \alpha_Y^p) \otimes \alpha_S^p\right)^* = \left(a^* \otimes \alpha_{Y \cap S}^p\right) \otimes \alpha_S^p,$$

for all $a_1,\ a_2,\ a \in (A,\ \psi)$, and $Y_1,\ Y_2,\ Y,\ S_1,\ S_2,\ S \in \sigma(p)$.

Then we obtain the following isomorphism theorem of our p-adic A-dynamical algebra $\mathfrak{A}_p$ of (6.10).

Theorem 6.4. *Let $\mathfrak{A}_p = A \times_{\theta^p} \sigma(p)$ be the p-adic A-dynamical algebra (6.10) of a p-adic A-dynamical system $(A,\ \sigma(p), \theta^p)$ of (6.9), and let $\mathfrak{A}_p^0 = A \otimes_{\theta^p} M_p$ be the conditional tensor product C^*-algebra (6.12) with its θ^p-condition (6.13), or (6.13)$'$. Then*

$$\mathfrak{A}_p \overset{*\text{-iso}}{=} \mathfrak{A}_p^0 \text{ in } B\left(\mathcal{H}_p \otimes H_p\right), \tag{6.14}$$

where "$\overset{*\text{-iso}}{=}$" means "being $*$-isomorphic."

Proof. First of all, note that the p-adic C^*-algebra M_p of (4.6) is generated by $\sigma(\mathbb{Q}_p)$ $= \sigma(p)$.

Let $\mathfrak{A}_p$ and $\mathfrak{A}_p^0$ be in the sense of (6.10), respectively, (6.12), as C^*-subalgebras of $B\left(\mathcal{H}_p \otimes H_p\right)$. Remark that $\mathfrak{A}_p$ satisfies the θ^p-relation (6.11), and $\mathfrak{A}_p^0$ satisfies the θ^p-conditional relation (6.13).

Define now a morphism $\Phi : \mathfrak{A}_p \to \mathfrak{A}_p^0$ by a linear transformation satisfying

$$\Phi\left((a \otimes \alpha_Y^p), S\right) = (a \otimes \alpha_Y^p) \otimes \alpha_S^p \text{ in } \mathfrak{A}_p^0, \tag{6.15}$$

for all $a \in (A,\ \psi)$ and $Y,\ S \in \sigma(p)$.

By the definition (6.15) of Φ, it is bounded and bijective under linearity because of the generator-preserving property.

This bijective bounded linear transformation Φ of (6.15) satisfies that

by $(6.11)'$
$$\Phi\left((a_1 \otimes \alpha^p_{Y_1}, S_1)(a_2 \otimes \alpha^p_{Y_2}, S_2)\right) = \Phi\left(a_1 a_2 \otimes \alpha^p_{Y_1 \cap Y_2 \cap S_1}, S_1 \cap S_2\right)$$

by (6.15)
$$= \left(a_1 a_2 \otimes \alpha^p_{Y_1 \cap Y_2 \cap S_1}\right) \otimes \left(\alpha^p_{S_1 \cap S_2}\right)$$

by $(6.13)'$
$$= \left((a_1 \otimes \alpha^p_{Y_1}) \otimes \alpha^p_{S_1}\right)\left((a_2 \otimes \alpha^p_{Y_2}) \otimes \alpha^p_{S_2}\right)$$
$$= \Phi\left((a_1 \otimes \alpha^p_{Y_1}), S_1\right)\Phi\left((a_2 \otimes \alpha^p_{Y_2}, S_2\right),$$

in $\mathfrak{A}^0_p$ by (6.15), for all $a_l \in (A, \psi)$, and Y_l, $S_l \in \sigma(p)$, for all $l = 1, 2$. Thus, under linearity, one obtains that:

$$\Phi(T_1 T_2) = \Phi(T_1)\Phi(T_2) \text{ in } \mathfrak{A}^0_p, \tag{6.16}$$

for all T_1, $T_2 \in \mathfrak{A}_p$.

Also, one has that
$$\Phi\left((a \otimes \alpha^p_Y, S)^*\right) = \Phi\left(((a^* \otimes \alpha^p_{Y \cap S}), S)\right)$$

by $(6.11)'$
$$= \left(a^* \otimes \alpha^p_{Y \cap S}\right) \otimes \alpha^p_S$$

by (6.15)
$$= \left(a \otimes \alpha^p_{Y \cap S}\right)^* \otimes \left(\alpha^p_S\right)^*$$

by $(6.13)'$
$$= \left(\Phi\left((a \otimes \alpha^p_Y), S\right)\right)^*,$$

in $\mathfrak{A}^0_p$, for all $a \in (A, \psi)$ and Y, $S \in \sigma(p)$. So, one has

$$\Phi(T^*) = (\Phi(T))^* \text{ in } \mathfrak{A}^0_p, \text{ for all } T \in \mathfrak{A}_p. \tag{6.17}$$

Therefore, by (6.16) and (6.17), the bijective bounded linear transformation Φ of (6.15) is a $*$-homomorphism from $\mathfrak{A}_p$ to $\mathfrak{A}^0_p$, equivalently, it is a $*$-isomorphism from $\mathfrak{A}_p$ onto $\mathfrak{A}^0_p$. Thus, two C^*-algebras $\mathfrak{A}_p$ and $\mathfrak{A}^0_p$ are $*$-isomorphic. $\square$

By the structure theorem (6.14), our p-adic A-dynamical algebra $\mathfrak{A}_p$ of (6.10) and the conditional tensor product C^*-algebra $\mathfrak{A}^0_p$ of (6.13) are same as C^*-algebras. So, in the following text, we use $\mathfrak{A}_p$ and $\mathfrak{A}^0_p$ alternatively, case-by-case.

Notation and Assumption. From below, we use the notation $\mathfrak{A}_p$ for both $\mathfrak{A}_p$ and $\mathfrak{A}^0_p$, and we call $\mathfrak{A}^o_p$, the *p-adic A-dynamical algebra* of $(\mathcal{A}_p, \sigma(p), \theta^p)$, too. Similarly, we use the terms
$$\left(a \otimes \alpha^p_Y, S\right) \text{ and } (a \otimes \alpha^p_Y) \otimes \alpha^p_S,$$
as a same object in the A-dynamical p-adic C^*-algebra $\mathfrak{A}_p = \mathfrak{A}^o_p$, case-by-case, for all $a \in (A, \psi)$ and Y, $S \in \sigma(p)$.

Also, if there are no confusions, we sometimes denote
$$\theta^p_S(a \otimes \alpha^p_Y) \text{ simply by } (a \otimes \alpha^p_Y)^S,$$
for all $a \in (A, \psi)$, and Y, $S \in \sigma(p)$.

Before proceeding, let's consider the following basic computations. Observe that, for any $N \in \mathbb{N} \setminus \{1\}$, if $\beta_l = a_l \otimes \alpha^p_{Y_l} \in \mathcal{A}_p$, for $l = 1, ..., N$, then

$$\prod_{l=1}^{N} (\beta_l, S_l) = \left(\left(\beta_1 \beta_2^{S_1} \beta_3^{S_1 \cap S_2} \ldots \beta_N^{S_1 \cap \cdots \cap S_{N-1}}\right), S_1 \cap ... \cap S_N\right), \tag{6.18}$$

by the induction on (6.11) (or that on (6.13)), for all $a_l \in (A, \psi)$, and Y_l, $S_l \in \sigma(p)$, for $l = 1, ..., N$.

Lemma 6.5. *Let* $(a_l \otimes \alpha^p_{Y_l}, S_l)$ *be generating operators of the* p*-adic* A*-dynamical algebra* $\mathfrak{A}_p$, *with* $a_l \in (A, \psi)$, *and* Y_l, $S_l \in \sigma(p)$, *for all* $l = 1, ..., N$, *for* $N \in \mathbb{N} \setminus \{1\}$. *Then*

$$\underset{l=1}{\overset{N}{\Pi}} \left(a_l \otimes \alpha^p_{Y_l}, S_l\right) = \left(\left(\underset{l=1}{\overset{N}{\Pi}} a_l\right) \otimes \left(\alpha^p_{\left(\underset{l=1}{\overset{N}{\cap}} Y_l\right) \cap \left(\underset{l=1}{\overset{N-1}{\cap}} S_l\right)}\right), \underset{l=1}{\overset{N}{\cap}} S_l\right) \tag{6.19}$$

Proof. The proof of (6.19) is done by (6.18), and by (6.11)′, or (6.13)′.

Indeed, by (6.18), one can get that, if $\beta_l = a_l \otimes \alpha^p_{Y_l} \in \mathcal{A}_p$, and hence, if $(\beta_l, S_l) \in \mathfrak{A}_p$, for $l = 1, ..., N$, then

$$\begin{aligned}
&\underset{l=1}{\overset{N}{\Pi}} (\beta_l, S_l) \\
&= \left(\beta_1 \beta_2^{S_1} \beta_3^{S_1 \cap S_2} ... \beta_N^{S_1 \cap ... \cap S_{N-1}}, S_1 \cap ... \cap S_N\right) \\
&= \left((a_1 a_2 ... a_N) \otimes \alpha^p_{Y_1} \alpha^p_{Y_2 \cap S_1} \alpha^p_{Y_3 \cap S_1 \cap S_2} ... \alpha^p_{Y_{N-1} \cap S_1 \cap S_2 \cap \cdots \cap S_{N-1}}, S_1 \cap \cdots \cap S_N\right) \\
&= \left((a_1 a_2 ... a_N) \otimes \alpha^p_{Y_1 \cap (Y_2 \cap S_1) \cap \cdots \cap (Y_{N-1} \cap S_1 \cap S_2 \cap \cdots \cap S_{N-1})}, S_1 \cap \cdots \cap S_N\right) \\
&= \left((a_1 a_2 ... a_N) \otimes \alpha^p_{(Y_1 \cap Y_2 \cap \cdots \cap Y_N) \cap (S_1 \cap S_2 \cap \cdots \cap S_{N-1})}, S_1 \cap ... \cap S_N\right).
\end{aligned}$$

Therefore, the formula (6.19) holds. □

7. On C^*-Subalgebras $\mathfrak{S}_p$ of M_p

As we have seen in Section 6, if we have a fixed unital C^*-probability space (A, ψ), one can construct the p-adic A-dynamical C^*-algebra

$$\mathfrak{A}_p = \mathcal{A}_p \times_{\theta^p} \sigma(p) = \mathcal{A}_p \otimes_{\theta^p} M_p$$

of (6.10) with its θ^p-relation (6.11), induced by a p-adic dynamical system,

$$(\mathcal{A}_p, \sigma(p), \theta^p),$$

where $\sigma(p)$ is the semigroup $(\sigma(\mathbb{Q}_p), \cap)$, and $\mathcal{A}_p = A \otimes_{\mathbb{C}} M_p$, for $p \in \mathcal{P}$.

In this section, for any fixed $p \in \mathcal{P}$, we define a certain C^*-subalgebra $\mathfrak{S}_p$ of our p-adic C^*-algebra M_p of (4.6), and construct the corresponding C^*-subalgebra

$$\mathfrak{S}^A_p = A \otimes_{\theta^p} \mathfrak{S}_p \text{ of } \mathfrak{A}_p.$$

And consider (traditional) free probability on $\mathfrak{S}^A_p$ determined by sectionized linear functionals

$$\{\psi_{p,j} : j \in \mathbb{Z}\}.$$

7.1. C^***-Subalgebras** $\mathfrak{S}_p$ **of** M_p**.** Throughout this section, let's fix $p \in \mathcal{P}$, and let M_p be the p-adic C^*-algebra (4.6). Take operators

$$P_{p,j} = \alpha^p_{\partial_j} \in M_p, \tag{7.1}$$

for all $j \in \mathbb{Z}$, for $p \in \mathcal{P}$, where ∂_j are the j-th boundaries (2.3) of $\mathbb{Q}_p$ contained in the semigroup $\sigma(p)$.

As we have seen, these operators $P_{p,j}$ of (8.1) are *projections* on the p-adic Hilbert space H_p in M_p, i.e.,

$$(7.1)' \qquad P^*_{p,j} = P_{p,j} = P^2_{p,j}, \text{ moreover, } P_{p,j_1}P_{p,j_2} = \delta_{j_1,j_2}P_{p,j_1},$$

for all $p \in \mathcal{P}$, and j, j_1, $j_2 \in \mathbb{Z}$. We now restrict our interests to these projections $P_{p,j}$ of (7.1).

Definition 7.1. Let $\mathfrak{S}_p$ be the C^*-subalgebra

$$\mathfrak{S}_p = C^*\left(\{P_{p,j}\}_{j\in\mathbb{Z}}\right) = \overline{\mathbb{C}\left[\{P_{p,j}\}_{j\in\mathbb{Z}}\right]} \text{ of } M_p, \tag{7.2}$$

where $P_{p,j}$ are projections (7.1), for all $j \in \mathbb{Z}$. We call this C^*-subalgebra $\mathfrak{S}_p$, the p-adic projection (C^*-sub)algebra of the p-adic C^*-algebra M_p.

Let $\mathfrak{S}_p$ be the p-adic projection algebra of M_p. Then it satisfies the following structure theorem.

Proposition 7.2. *Let* $\mathfrak{S}_p$ *be the p-adic projection algebra* (7.2)*. Then*

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\oplus} \left(\mathbb{C} \cdot P_{p,j}\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \tag{7.3}$$

in M_p.

Proof. The isomorphism theorem (7.3) is proven by $(7.1)'$ and (7.2). □

By the structure theorem (7.3) of $\mathfrak{S}_p$, one can realize that $\mathfrak{S}_p$ acts like a diagonal subalgebra inside M_p. By the definition (7.2) of $\mathfrak{S}_p$, one can get the (non-traditional) C^*-probability spaces,

$$\left(\mathfrak{S}_p, \varphi^p_j\right), \text{ for } p \in \mathcal{P} \text{ and } j \in \mathbb{Z}. \tag{7.4}$$

as a free-probabilistic sub-structures of the p-adic C^*-probability spaces (M_p, φ^p_j)'s, where the linear functionals φ^p_j in (7.4) mean the restrictions $\varphi^p_j \mid_{\mathfrak{S}_p}$ on $\mathfrak{S}_p$ of the linear functionals φ^p_j of (5.1) on M_p, for all $j \in \mathbb{Z}$.

Notation. For convenience, we denote $(\mathfrak{S}_p, \varphi^p_j)$ of (7.4) by $\mathfrak{S}_{p,j}$, for all $p \in \mathcal{P}$, and $j \in \mathbb{Z}$.

Proposition 7.3. *Let* $\mathfrak{S}_{p,j}$ *be in the sense of* (7.4) *for* $j \in \mathbb{Z}$*, and let* $P_{p,k}$ *be the generating projections* (7.1) *of* $\mathfrak{S}_{p,j}$*, for all* $k \in \mathbb{N}$*. Then*

$$\varphi^p_j\left((P_{p,k})^n\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \forall n \in \mathbb{N}, \tag{7.5}$$

for all $k \in \mathbb{Z}$.

Proof. The formula (7.5) is proven by (5.3), (5.4) and (7.1). □

Let ϕ be the *Euler totient function*, i.e., it is an *arithmetic function* $\phi : \mathbb{N} \to \mathbb{C}$ defined by

$$\phi(n) = \left| \left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{c} 1 \leq k \leq n \text{ in } \mathbb{N}, \\ \text{and } \gcd(k, n) = 1 \end{array} \right\} \right|, \tag{7.6}$$

for all $n \in \mathbb{N}$, where $|Y|$ mean the *cardinalities of sets* Y, and $\gcd(,)$ is the *greatest common divisor*. It is well-known that

$$\phi(p) = p - 1 = p\left(1 - \frac{1}{p}\right), \text{ for all } p \in \mathcal{P}.$$

So, the above formula (7.5) can be re-formulated to be

$$\varphi_j^p\left(P_{p,k}\right) = \delta_{j,k}\left(\frac{\phi(p)}{p^{j+1}}\right), \text{ for all } p \in \mathcal{P}, j \in \mathbb{Z}. \tag{7.7}$$

7.2. **On Conditional Tensor Product C^*-Algebras $A \otimes_{\theta^p} \mathfrak{S}_p$ in $\mathfrak{A}_p$.** Let (A, ψ) be a fixed unital C^*-probability space, and let $\mathfrak{S}_p$ be the p-adic projection subalgebra of M_p. Define now the *conditional tensor product C^*-algebra* $\mathfrak{S}_p^A$ by

$$\mathfrak{S}_p^A \overset{def}{=} \mathcal{A}_p \otimes_{\theta^p} \mathfrak{S}_p = \left(A \otimes_{\mathbb{C}} M_p\right) \otimes_{\theta^p} \mathfrak{S}_p \tag{7.8}$$

as a C^*-subalgebra of our p-adic A-dynamical algebra $\mathcal{A}_p \otimes_{\theta^p} M_p = \mathfrak{A}_p$ under the θ^p-relation (6.13) (equivalent to (6.11)), where θ^p is the semigroup-action (6.5) of $\sigma(p)$. Such a C^*-algebra $\mathfrak{S}_p^A$ of (7.8) is well-defined as a C^*-subalgebra of $\mathfrak{A}_p$ since the tensor-factor $\mathfrak{S}_p$ is a well-defined C^*-subalgebra of M_p, for $p \in \mathcal{P}$.

By (7.8), all elements of $\mathfrak{S}_p^A$ are generated by the operators formed by

$$\left(a \otimes \alpha_Y^p\right) \otimes P_{p,k},$$

for all $a \in (A, \psi)$, $Y \in \sigma(p)$, and $k \in \mathbb{Z}$.

Define a linear morphism

$$\Psi_p : \mathfrak{S}_p^A = \mathcal{A}_p \otimes_{\theta^p} \mathfrak{S}_p \to \mathcal{A}_p \tag{7.9}$$

by the linear transformation satisfying

$$\Psi_p\left(\left(a \otimes \alpha_Y^p\right) \otimes P_{p,j}\right) = a \otimes \alpha_Y^p P_{p,j} = a \otimes \alpha_{Y \cap \partial_j^p}^p,$$

for all $a \in (A, \psi)$, $Y \in \sigma(p)$, for all $j \in \mathbb{Z}$.

Then, by the definition (7.9) of Ψ_p, it is a well-defined surjective bounded linear transformation from $\mathfrak{S}_p^A$ onto $\mathcal{A}_p$, by (2.3) and (7.9).

It is not difficult to check that: the morphisms $g_j^p : \mathcal{A}_p \to \mathbb{C}$, defined by linear morphisms satisfying

$$g_j^p\left(a \otimes \alpha_Y^p\right) = \varphi_j^p\left(\psi(a)\alpha_Y^p\right) \tag{7.10}$$

are well-defined linear functionals on $\mathcal{A}_p$, for all $a \in (A, \psi)$, $Y \in \sigma(p)$, for all $j \in \mathbb{Z}$, where φ_j^p are in the sense of (7.4).

By (7.10), one can get that

$$g_j^p\left(a \otimes \alpha_Y^p\right) = \varphi_j^p\left(\alpha_Y^p\right)\psi\left(a\right). \tag{7.10$'$}$$

Define now linear functionals $\psi_{p,j}$ on the C^*-algebra $\mathfrak{S}_p^A$ of (7.8) by bounded linear morphisms from $\mathfrak{S}_p^A$ to $\mathbb{C}$ satisfying,

$$\psi_{p,j} = g_j^p \circ \Psi_p \text{ on } \mathfrak{S}_p^A, \tag{7.11}$$

where $\left\{g_j^p\right\}_{j\in\mathbb{Z}}$ are in the sense of (7.10) (satisfying (7.10)$'$), and Ψ_p is in the sense of (7.9).

Let $P_{p,k} \in M_p$ and $P_{p,l} \in \mathfrak{S}_p$, for $k,\ l \in \mathbb{Z}$, and $a \in (A,\ \psi)$, and let

$$T_{a,k}^{p,l} = (a \otimes P_{p,k}) \otimes P_{p,l} \in \mathfrak{S}_p^A. \tag{7.12}$$

Then, by (7.11), for any $j \in \mathbb{Z}$, if $T_{a,k}^{p,l} \in \mathfrak{S}_p^A$ is in the sense of (7.12), then

$$\begin{aligned}
\psi_{p,j}\left(T_{a,k}^{p,l}\right) &= g_j^p\left(a \otimes P_{p,k}P_{p,l}\right) \\
&= \delta_{k,l}g_j^p\left(a \otimes P_{p,k}\right) \\
&= \delta_{k,l}\varphi_j^p\left(\psi(a)P_{p,k}\right) \\
\text{by } (7.10)' \qquad &= \delta_{k,l}\varphi_j^p\left(P_{p,k}\right)\psi(a) \\
&= \delta_{k,l}\delta_{j,k}\varphi_j^p\left(P_{p,j}\right)\psi(a) \\
&= \delta_{k,l}\delta_{k,j}\psi(a)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \\
&= \delta_{k,l}\delta_{k,j}\frac{\psi(a)\phi(p)}{p^{j+1}}, \qquad (7.13)
\end{aligned}$$

by (5.3).

Since these linear functionals $\psi_{p,j}$'s of (7.11) are well-defined linear functionals on $\mathfrak{S}_p^A$, the pairs

$$\mathfrak{S}_{p,j}^A \overset{denote}{=} \left(\mathfrak{S}_p^A, \psi_{p,j}\right), \tag{7.14}$$

form well-defined (traditional) C^*-probability spaces, for all $j \in \mathbb{Z}$ (for all $p \in \mathcal{P}$).

Definition 7.4. Let $\mathfrak{S}_p^A$ be in the sense of (7.8) for a fixed $p \in \mathcal{P}$, and let $\psi_{p,j}$ be linear functionals (7.11), for all $j \in \mathbb{Z}$. Then $\mathfrak{S}_p^A$ is called the A-dynamical p(-adic)-projection (C^*-)algebra. The C^*-probability spaces $\mathfrak{S}_{p,j}^A$ of (7.14) are said to be the A-dynamical j-th p(-adic)-projection (C^*-)probability spaces (induced by $\mathfrak{S}_p$), for all $j \in \mathbb{Z}$.

As we discussed above, one can find the following free-distributional data on the A-dynamical j-th p-projection probability space.

Theorem 7.5. *Let $T_{a,k}^{p,l} = (a \otimes P_{p,k}) \otimes P_{p,l}$ be a free random variable of the A-dynamical j-th p-projection probability space $\mathfrak{S}_{p,j}^A$ of* (7.14)*, where $a \in (A,\ \psi)$, $P_{p,k} \in M_p$, and $P_{p,l} \in \mathfrak{S}_p$, for $j,\ k,\ l \in \mathbb{Z}$. Then*

$$\begin{aligned}
\psi_{p,j}\left(\left(T_{a,k}^{p,l}\right)^n\right) &= \delta_{k,l}\delta_{k,j}\psi(a^n)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) \\
&= \delta_{k,l}\delta_{k,j}\left(\frac{\psi(a^n)\phi(p)}{p^{j+1}}\right),
\end{aligned} \tag{7.15}$$

for all $n \in \mathbb{N}$.

Proof. Let $T_{a,k}^{p,l}$ be a given free random variable (7.12) in $\mathfrak{S}_{p,j}^{A}$. Then

$$\left(T_{a,k}^{p,l}\right)^n = ((a \otimes P_{p,k}) \otimes P_{p,l})^n$$

by (6.19) and (7.9) $\qquad = \left(a^n \otimes P_{p,k}^n P_{p,l}^{n-1}\right) \otimes P_{p,l}^n$

since $\{P_{p,j}\}_{j\in\mathbb{Z}}$ are mutually orthogonal in $\mathfrak{S}_p$ $\qquad = \delta_{k,l}\left(a^n \otimes P_{p,k}\right) \otimes P_{p,k}$

(7.16) $\qquad = \delta_{k,l} T_{a^n,k}^{p,k}$,

where $T_{a^n,k}^{p,k}$ is in the sense of (7.12) in $\mathfrak{S}_p^A$.

Thus, one has that

(by (7.16))
$$\psi_{p,j}\left(\left(T_{a,k}^{p,l}\right)^n\right) = \delta_{k,l}\psi_{p,j}\left(T_{a^n,k}^{p,k}\right) = \delta_{k,l}\delta_{k,j}\varphi_j^p\left(P_{p,j}\right)\psi(a^n) = \delta_{k,l}\delta_{k,j}\psi(a^n)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),$$

by (7.13), for all $n \in \mathbb{N}$. Therefore, the free-distributional data (7.15) holds. ☐

Assume that a fixed free random variable a of (A, ψ) is self-adjoint in A, in the sense that: $a^* = a$ in A. Then the operator $T_{a,k}^{p,l}$ of (7.12) is self-adjoint in the A-dynamical p-projection algebra $\mathfrak{S}_p^A$, since

$$\left(T_{a,k}^{p,l}\right)^* = \left(a^* \otimes P_{p,k}^*\right) \otimes P_{p,l}^* = T_{a,k}^{p,l} \text{ in } frakS_p^A.$$

In such a case, the free distribution of $T_{a,k}^{p,l}$ is completely characterized by the free moments (7.15).

Corollary 7.6. *Let* $T_{a,j}^{p,j}$ *be a free random variable of* (7.12) *in the* A*-dynamical* p*-projection probability space* $\mathfrak{S}_{p,j}^A$*, for* $j \in \mathbb{Z}$*. Then*

$$\psi_{p,j}\left(\left(T_{a,j}^{p,j}\right)^n\right) = \psi(a^n)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) = \frac{\psi(a^n)\phi(p)}{p^{j+1}}, \tag{7.17}$$

for all $n \in \mathbb{N}$.

Proof. The proof of (7.17) is done by (7.16). ☐

If $a \in (A, \psi)$ is self-adjoint, then the free distribution of $T_{a,j}^{p,j}$ is characterized by the free moments in (7.17). As one can see in (7.17), such non-vanishing free moments are determined by both number-theoretic data $\phi(p)$, and the free distribution $\{\psi(a^n)\}_{n=1}^{\infty}$ of the self-adjoint element $a \in (A, \psi)$.

Corollary 7.7. *Let* $T_{1_A,k}^{p,l}$ *be a free random variable of the* j*-th* A*-dynamical* p*-projection probability space* $\mathfrak{S}_{p,j}^A$*, where* 1_A *is the unit of* (A, ψ)*. Then*

$$\psi_{p,j}\left(\left(T_{1_A,k}^{p,l}\right)^n\right) = \delta_{k,l}\delta_{k,j}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) = \frac{\delta_{k,l}\delta_{k,j}\phi(p)}{p^{j+1}}, \tag{7.18}$$

and hence,

$$\psi_{p,j}\left(\left(T_{1_A,j}^{p,j}\right)^n\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}} = \frac{\phi(p)}{p^{j+1}}, \tag{7.19}$$

for all $n \in \mathbb{N}$.

Proof. The free-distributional data (7.18) is obtained by (7.15), and the free moments (7.19) are gotten from (7.17). □

7.3. **Certain Banach $*$-Probability Spaces $\mathfrak{L}\mathfrak{S}_p^A(j)$ Induced by $\mathfrak{S}_{p,j}^A$.** In this section, we fix $p \in \mathcal{P}$, and $j \in \mathbb{Z}$, and a unital C^*-probability space (A, ψ), and let

$$\mathfrak{S}_{p,j}^A = \left(\mathfrak{S}_p^A, \psi_{p,j}\right)$$

be the j-th A-dynamical p-projection probability space (7.14). Also, we let

$$T_{a,k}^{p,l} \overset{denote}{=} (a \otimes P_{p,k}) \otimes P_{p,l}, \text{ with } P_{p,m} = \alpha_{\partial_m}^p, \tag{7.20}$$

be generating operators (7.12) in the A-dynamical p-projection algebra $\mathfrak{S}_p^A$, for all $a \in (A, \psi)$, and k, l, $m \in \mathbb{Z}$.

Define now linear morphisms c_p^A and a_p^A "acting on the C^*-algebra $\mathfrak{S}_p^A$," by the bounded linear transformations satisfying

$$c_p^A\left(T_{a,k}^{p,l}\right) = T_{a,k}^{l+1}, \text{ and } a_p^A\left(T_{a,k}^{p,l}\right) = T_{a,k}^{l+1}, \tag{7.21}$$

in $\mathfrak{S}_p^A$.

Note that such bounded linear transformations c_p^A and a_p^A of (7.21) are regarded as Banach-space operators by understanding the C^*-algebra $\mathfrak{S}_p^A$ as a Banach space (under the C^*-norm on $\mathfrak{S}_p^A$). i.e., they are Banach-space operators contained in the *operator space* $B\left(\mathfrak{S}_p^A\right)$ in the sense of [13], consisting of all bounded operators on the Banach space $\mathfrak{S}_p^A$.

Remark 7.8. In [8, 11] and [12], we introduced the operators c_p and a_p directly on the p-adic projection algebra $\mathfrak{S}_p$, defined by

$$c_p(P_{p,k}) = P_{p,k+1}, \text{ and } a_p(P_{p,k}) = P_{p,k-1},$$

for all $k \in \mathbb{Z}$. So, one may regard our operators c_p^A and a_p^A of (7.21) as

$$\left(1_A \otimes 1_{M_p}\right) \otimes c_p, \text{ respectively, } \left(1_A \otimes 1_{M_p}\right) \otimes a_p$$

on $A \otimes_{\theta^p} \mathfrak{S}_p = \mathfrak{S}_p^A$, where the unit 1_A is understood as the multiplication operator "on A" with its symbol 1_A, and 1_{M_p} is the identity $*$-isomorphism on M_p, satisfying $1_{M_p}(T) = T$, for all $T \in M_p$.

Definition 7.9. Let c_p^A and a_p^A be the Banach-space operators (7.21) acting "on $\mathfrak{S}_p^A$." Then we call them, the p-adic A-creation, respectively, the p-adic A-annihilation on $\mathfrak{S}_p^A$. Define now a new operator l_p^A by

$$l_p^A = c_p^A + a_p^A \text{ on } frakS_p^A. \tag{7.22}$$

This Banach-space operator $l_p^A \in B\left(\mathfrak{S}_p^A\right)$ is called the p-adic A-radial operator on $\mathfrak{S}_p^A$.

Consider that: if $T_{a,k}^{p,l}$ are in the sense of (7.20) in $\mathfrak{S}_p^A$, then

$$c_p^A a_p^A \left(T_{a,k}^{p,l}\right) = c_p^A \left(T_{a,k}^{p,l-1}\right) = T_{a,k}^{p,l},$$

and

$$a_p^A c_p^A \left(T_{a,k}^{p,l}\right) = a_p^A \left(T_{a,k}^{p,l+1}\right) = T_{a,k}^{p,l},$$

and hence, for any $S \in \sigma(\mathbb{Q}_p)$, if $T_{a,S}^{p,l} = (a \otimes \alpha_S^p) \otimes P_{p,l} \in \mathfrak{S}_p^A$, then

$$c_p^A a_p^A \left(T_{a,S}^{p,l}\right) = T_{a,S}^{p,l} = a_p^A c_p^A \left(T_{a,S}^{p,l}\right).$$

Therefore,

$$c_p^A a_p^A = 1_{\mathfrak{S}_p^A} = a_p^A c_p^A, \tag{7.23}$$

where

$$1_{\mathfrak{S}_p^A} = \left(1_A \otimes 1_{M_p}\right) \otimes 1_{\mathfrak{S}_p} \in B\left(\mathfrak{S}_p^A\right),$$

where $1_{\mathfrak{S}_p}$ is the identity $*$-isomorphism on $\mathfrak{S}_p$, satisfying

$$1_{\mathfrak{S}_p}(P_{p,k}) = P_{p,k}, \text{ for all } k \in \mathbb{Z}.$$

Proposition 7.10. *Let c_p^A and a_p^A be the p-adic A-creation, respectively, the p-adic A-annihilation of* (7.21) *in $B\left(\mathfrak{S}_p^A\right)$. Then*

$$\left(c_p^A\right)^{n_1} \left(a_p^A\right)^{n_2} = \left(a_p^A\right)^{n_2} \left(c_p^A\right)^{n_1},$$

(7.24) and

$$\begin{aligned}\left(c_p^A a_p^A\right)^n &= \left(c_p^A\right)^n \left(a_p^A\right)^n = 1_{\mathfrak{S}_p^A} \\ &= \left(a_p^A\right)^n \left(c_p^A\right)^n = \left(a_p^A c_p^A\right)^n,\end{aligned}$$

for all n_1, n_2, $n \in \mathbb{N}$.

Proof. The proof of (7.24) is done by (7.23). □

By (7.24), one can realize that

$$\left(l_p^A\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(c_p^A\right)^k \left(a_p^A\right)^{n-k}, \tag{7.25}$$

on $\mathfrak{S}_p^A$, for all $n \in \mathbb{N}$, with axiomatization:

$$\left(c_p^A\right)^0 = 1_{\mathfrak{S}_p^A} = \left(a_p^A\right)^0,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for all } k \leq n \in \mathbb{N}_0$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Define a *closed subspace* $\mathfrak{L}_p^A$ of the operator space $B\left(\mathfrak{S}_p^A\right)$ by

$$\mathfrak{L}_p^A = \overline{\mathbb{C}[\{l_p^A\}]}^{\|.\|}, \tag{7.26}$$

where $\overline{Y}^{\|.\|}$ mean the operator-norm $\|.\|$-closures of subsets Y of $B\left(\mathfrak{S}_p^A\right)$, where

$$\|X\| = \sup\left\{ \|X(T)\|_{\mathfrak{S}_p^A} \left| \begin{array}{c} T \in \mathfrak{S}_p^A, \text{ such that} \\ \|T\|_{\mathfrak{S}_p^A} = 1 \end{array} \right. \right\},$$

where $\|.\|_{\mathfrak{S}_p^A}$ is the C^*-*norm* on the A-dynamical p-projection algebra $\mathfrak{S}_p^A$.

By the definition (7.26), this closed subspace forms an algebra embedded in the topological vector space $B\left(\mathfrak{S}_p^A\right)$, furthermore, if we define an operation

$$\left(\sum_{n=0}^{\infty} t_n \left(l_p^A\right)^n\right)^* = \sum_{n=0}^{\infty} \overline{t_n} \left(l_p^A\right)^n,$$

on $\mathfrak{L}_p^A$, where $\overline{t_n}$ are the conjugates of t_n in $\mathbb{C}$, then it forms a $*$-algebra over $\mathbb{C}$. i.e., all elements of $\mathfrak{L}_p^A$ are *adjointable* in $B\left(\mathfrak{S}_p^A\right)$ in the sense of [13]. In conclusion, the topological structure $\mathfrak{L}_p^A$ of (7.26) forms a *Banach $*$-algebra* in the operator space $B\left(\mathfrak{S}_p^A\right)$.

Now, define a (usual) *tensor product Banach $*$-algebra* $\mathfrak{L}\mathfrak{S}_p^A$ by

$$\mathfrak{L}\mathfrak{S}_p^A \overset{def}{=} \mathfrak{L}_p^A \otimes_{\mathbb{C}} \mathfrak{S}_p^A. \tag{7.27}$$

i.e.,

$$\mathfrak{L}\mathfrak{S}_p^A = frakL_p^A \otimes_{\mathbb{C}} \left(\left(A \otimes_{\mathbb{C}} M_p\right) \otimes_{\theta^p} \mathfrak{S}_p\right).$$

In the definition (7.27) of $\mathfrak{L}\mathfrak{S}_p^A$, note that the first tensor product $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras, and the second tensor product $\otimes_{\mathbb{C}}$ is the tensor product of C^*-algebras, while $\otimes_{\theta^p}$ is the conditional tensor product (6.12) with θ^p-relations (6.13) (equivalent to (6.11)). Since $\mathfrak{L}_p^A$ is a Banach $*$-algebra, and $\mathfrak{S}_p^A$ is a C^*-algebra, this tensor product $*$-algebra $\mathfrak{L}\mathfrak{S}_p^A$ of (7.27) is a well-defined Banach $*$-algebra under product topology.

Definition 7.11. The tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p^A$ of (7.27) is called the A-dynamical p(-adic-)radial-projection (Banach-$*$-)algebra for $p \in \mathcal{P}$.

Let $\mathfrak{L}\mathfrak{S}_p^A$ be the A-dynamical p-radial-projection algebra for a fixed prime p. Define a linear morphism

$$E_{p,j}^A \mathfrak{L}\mathfrak{S}_p^A \to \mathfrak{S}_p^A$$

by the linear transformation satisfying that

$$E_{p,j}^A\left(\left(l_p^A\right)^n \otimes T_{a,k}^{p,l}\right) = \begin{cases} \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1} \left(l_p^A\right)^n \left(T_{a,k}^l\right) & \text{if } l = j \\ & \\ 0_{\mathfrak{S}_p^A} & \text{otherwise,} \end{cases} \tag{7.28}$$

for all generating operators

$$\left(l_p^A\right)^n \otimes T_{a,k}^{p,l}, \forall n \in \mathbb{N}, a \in (A, \psi),$$

of $\mathfrak{L}\mathfrak{S}_p^A$, where $T_{a,k}^l$ are in the sense of (7.20), for all $k \in \mathbb{Z}$.

Indeed, one can get that

$$\left(l_p^A \otimes T_{a,k}^{p,l}\right)^n = \left(l_p^A\right)^n \otimes \left(\delta_{k,l} T_{a^n,k}^{p,k}\right),$$

by (7.16), and hence, all non-zero generating operators have their forms,

$$\left(l_p^A\right)^n \otimes T_{x,k}^{p,l}, \text{ for some } x \in (A, \psi).$$

So, the definition (7.28) is meaningful under linearity. Indeed, the linear morphism $E_{p,j}^A$ of (7.28) is a well-defined bounded linear transformation from $\mathfrak{L}\mathfrak{S}_p^A$ onto $\mathfrak{S}_p^A$, by (7.3), (7.8), (7.26) and (7.27).

Now, define linear functionals $\tau_{p,j}^A$ on the A-tensor p-adic radial-projection algebra $\mathfrak{L}\mathfrak{S}_p$ of $\mathfrak{S}_p$ by

$$\tau_{p,j}^A = \left(\frac{1}{\phi(p)}\psi_{p,j}\right) \circ E_{p,j}^A \text{ on } \mathfrak{L}\mathfrak{S}_p^A, \tag{7.29}$$

where $\psi_{p,j}$ are the linear functionals (7.11) on $\mathfrak{S}_p^A$, and $E_{p,j}^A$ is the bounded linear morphism (7.28), for all $j \in \mathbb{Z}$.

Thus, for a fixed prime p, the pairs

$$\mathfrak{L}\mathfrak{S}_p^A(j) \overset{denote}{=} \left(\mathfrak{L}\mathfrak{S}_p^A, \tau_{p,j}^A\right) \tag{7.30}$$

form well-defined Banach $*$-probability spaces for all $j \in \mathbb{Z}$.

Definition 7.12. Let $\mathfrak{L}\mathfrak{S}_p^A$ be the A-dynamical p-radial-projection algebra (7.27) for $p \in \mathcal{P}$, and let $\{\tau_{p,j}^A\}_{j\in\mathbb{Z}}$ be the linear functionals (7.29) on $\mathfrak{L}\mathfrak{S}_p^A$. Then the corresponding Banach $*$-probability spaces $\mathfrak{L}\mathfrak{S}_p^A(j)$ of (7.30) are called the j-th filtered A-dynamical p-(radial-projection Banach $*$-)probability spaces, for all $j \in \mathbb{Z}$.

Let $X_{a,k}^{p,l}$ be the generating operators of $\mathfrak{L}\mathfrak{S}_p^A$ formed by

$$X_{a,k}^{p,l} = l_p^A \otimes T_{a,k}^{p,l} \in \mathfrak{L}\mathfrak{S}_p^A, \tag{7.31}$$

for all $a \in (A,\ \psi)$, and $k,\ l \in \mathbb{Z}$, where $T_{a,k}^{p,l} \in \mathfrak{S}_p^A$ are in the sense of (7.20).

If $X_{a,k}^{p,l}$ are the generating operators (7.31) of $\mathfrak{L}\mathfrak{S}_p^A$, then

$$\begin{aligned} \left(X_{a,k}^{p,l}\right)^n &= \left(l_p^A \otimes T_{a,k}^{p,l}\right)^n \\ &= \left(l_p^A\right)^n \otimes \left(\delta_{k,l} T_{a^n,k}^{p,k}\right) \\ &= \delta_{k,l}\left(\left(l_p^A\right)^n \otimes T_{a^n,k}^{p,k}\right), \end{aligned} \tag{7.32}$$

for all $n \in \mathbb{N}$, by (7.16) (Also, see (6.19)).

By (7.32), one can realize that, under our dynamical system, if $k \neq l$ in $\mathbb{Z}$, a generating operator $X_{a,k}^{p,l}$ satisfies

$$\left(X_{a,k}^{p,l}\right)^n = \begin{cases} X_{a,k}^{p,l} & \text{if } n = 1 \\ 0_{\mathfrak{L}\mathfrak{S}_p^A} & \text{otherwise,} \end{cases}$$

meanwhile, if $k = l$ in $\mathbb{Z}$, then

$$\left(X_{a,k}^{p,k}\right)^n = \left(l_p^A\right)^n \otimes T_{a^n,k}^{p,k},$$

for all $n \in \mathbb{N}$, where $0_{\mathfrak{L}\mathfrak{S}_p^A}$ is the zero element of $\mathfrak{L}\mathfrak{S}_p^A$.

Theorem 7.13. *Fix $p \in \mathcal{P}$, and $j \in \mathbb{Z}$, and let $\mathfrak{L}\mathfrak{S}_p^A(j)$ be the j-th filtered A-dynamical p-probability space. Let*

$$X_{a,j}^{p,j} = l_p^A \otimes T_{a,j}^{p,j} = l_p^A \otimes ((a \otimes P_{p,j}) \otimes P_{p,j})$$

be a generating operator of the A-dynamical j-th filtered p-probability space $\mathfrak{L}\mathfrak{S}_p^A(j)$ of (7.30) *for $a \in (A, \psi)$. Then*

$$\tau_{p,j}^A\left(\left(X_{a,j}^{p,j}\right)^n\right) = \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a^n)), \tag{7.33}$$

where

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

for all $n \in \mathbb{N}$, and c_m are the m-th Catalan numbers,

$$c_m = \frac{1}{m+1}\binom{2m}{m} = \frac{(2m)!}{m!(m+1)!}, \text{ for all } m \in \mathbb{N}_0.$$

Proof. If $X_{a,j}^{p,j}$ is a given generating operator in the sense of (7.31) in $\mathfrak{L}\mathfrak{S}_p^A(j)$, for $a \in (A, \psi)$, and $j \in \mathbb{Z}$, then

by (7.29)

$$\tau_{p,j}^A\left(\left(X_{a,j}^{p,j}\right)^n\right) = \left(\left(\frac{1}{\phi(p)}\psi_{p,j}\right) \circ E_{p,j}^A\right)\left(\left(X_{a,j}^{p,j}\right)^n\right)$$

by (7.32)

$$= \left(\frac{1}{\phi(p)}\psi_{p,j}\right)\left(E_{p,j}^A\left(\left(l_p^A\right)^n \otimes T_{a^n,j}^{p,j}\right)\right)$$

by (7.28)

$$= \left(\frac{1}{\phi(p)}\psi_{p,j}\right)\left(\frac{\left(p^{j+1}\right)^{n+1}}{[\frac{n}{2}]+1}\left(l_p^A\right)^n\left(T_{a^n,j}^{p,j}\right)\right)$$

$$= \left(\frac{\left(p^{j+1}\right)^{n+1}}{[\frac{n}{2}]+1}\right)\left(\left(\frac{1}{\phi(p)}\psi_{p,j}\right)\left(\left(l_p^A\right)^n\left(T_{a^n,j}^{p,j}\right)\right)\right)$$

$$= \left(\frac{\left(p^{j+1}\right)^{n+1}}{\left([\frac{n}{2}]+1\right)\phi(p)}\right)\psi_{p,j}\left(\sum_{k=0}^{n}\binom{n}{k}\left(c_p^A\right)^k\left(a_p^A\right)^{n-k}\left(T_{a,j}^{p,j}\right)\right), \tag{7.34}$$

by (7.25).

Observe now that, for any $n \in \mathbb{N}$,

$$\left(l_p^A\right)^{2n-1} = \sum_{k=0}^{2n-1}\binom{2n-1}{k}\left(c_p^A\right)^k\left(a_p^A\right)^{2n-k-1}, \tag{7.35}$$

by (7.25). Thus, $\left(l_p^A\right)^{2n-1}$ does not contain $1_{\mathfrak{S}_p^A}$-terms by (7.35).

Similarly, for any $n \in \mathbb{N}$,

$$\left(l_p^A\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} \left(c_p^A\right)^k \left(a_p^A\right)^{2n-k}$$
$$= \binom{2n}{n} \left(c_p^A\right)^n \left(a_p^A\right)^n + [RestTerms]$$
$$= \binom{2n}{n} \cdot 1_{\mathfrak{S}_p^A} + [RestTerms], \tag{7.36}$$

by (7.23) and (7.24). So, $\left(l_p^A\right)^{2n}$ contains $\binom{2n}{n} \cdot 1_{\mathfrak{S}_p^A}$ term by (7.36).

By (7.35) and (7.36), the formula (7.34) goes to

$$\tau_{p,j}^A \left(\left(X_{a,j}^{p,j}\right)^n\right) \left(\frac{\left(p^{j+1}\right)^{n+1}}{\left(\left[\frac{n}{2}\right]+1\right)\phi(p)}\right) \psi_{p,j} \left(\sum_{k=0}^{n} \binom{n}{k} \left(c_p^A\right)^k \left(a_p^A\right)^{n-k} \left(T_{a^n,j}^{p,j}\right)\right)$$
$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\left(\left[\frac{n}{2}\right]+1\right)\phi(p)}\right) \psi_{p,j} \left(\binom{n}{\frac{n}{2}} \left(T_{a^n,j}^{p,j}\right) + [\text{Rest Terms}] \left(T_{a^n,j}^{p,j}\right)\right) \tag{7.37}$$

by (7.29), where

$$l\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{7.38}$$

for all $n \in \mathbb{N}$, and hence, it goes to

$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\left(\frac{n}{2}+1\right)\phi(p)}\right) \psi_{p,j} \left(\binom{n}{\frac{n}{2}} \left(T_{a^n,j}^{p,j}\right)\right)$$
$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\left(\frac{n}{2}+1\right)\phi(p)}\right) \left(\frac{\frac{n}{2}+1}{\frac{n}{2}+1}\right) \binom{n}{\frac{n}{2}} \varphi_j^p \left(P_{p,j}\right) \psi\left(a^n\right)$$
$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)}\right) \left(c_{\frac{n}{2}}\right) \left(\frac{\phi(p)}{p^{j+1}}\right) \left(\psi(a^n)\right)$$

where $c_m = \frac{1}{m+1} \binom{2m}{m}$ are the m-th Catalan numbers for all $m \in \mathbb{N}_0$

$$= \left(\omega_n \left(p^{j+1}\right)^n c_{\frac{n}{2}}\right) \left(\psi(a^n)\right),$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (7.38). Therefore, the free-distributional data (7.33) holds. □

The above theorem provides a general tool to compute free distributions of arbitrary operators in the j-th filtered A-dynamical p-probability space $\mathfrak{L}\mathfrak{S}_p^A(j)$. By the definitions (7.28) and (7.29), one also obtains the following generalized result of (7.33).

Theorem 7.14. *Let* $X_{a,k}^{p,l}$ *be a generating operator* (7.31) *of* $\mathfrak{LS}_p^A(j)$, *for* $p \in \mathcal{P}$, *and* j, k, $l \in \mathbb{Z}$. *Then*

$$\tau_{p,j}^{A}\left(\left(X_{a,k}^{p,l}\right)^{n}\right) = \delta_{k,l}\delta_{k,j}\left(\omega_n\left(p^{2(j+1)}\right)^{\frac{n}{2}}c_{\frac{n}{2}}\right)(\psi(a^n)), \tag{7.39}$$

for all $n \in \mathbb{N}$.

Proof. Let $X_{a,k}^{p,l}$ be in the sense of (7.31) in $\mathfrak{LS}_p^A(j)$, for $a \in (A, \psi)$, and j, k, $l \in \mathbb{Z}$. If $j = k = l$ in $\mathbb{Z}$, then we obtain the formula (7.39) by (7.33). Meanwhile, if either $j \neq k$ or $k \neq l$ in $\mathbb{Z}$, then the free moments of $X_{a,k}^{p,l}$ vanish by (7.28) and (7.29). So, the formula (7.39) holds. □

From below, we focus on studying generating operators $X_{a,j}^{p,j}$ of j-th filtered A-dynamical p-probability spaces $\mathfrak{LS}_p^A(j)$ having non-zero free distributions, for all $j \in \mathbb{Z}$, for $p \in \mathcal{P}$.

8. Weighted-Semicircularity on $\mathfrak{LS}_p^A(j)$

In this section, we construct weighted-semicircular, and semicircular elements in the j-th filtered A-dynamical p-probability spaces $\mathfrak{LS}_p^A(j)$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$. As in the previous sections, let (A, ψ) be a fixed unital C^*-probability space, and $\mathfrak{LS}_p^A$, the A-dynamical p-radial-projection algebra.

8.1. **Semicircular and Weighted-Semicircular Elements.**

Let (B, φ_B) be an arbitrary topological $*$-probability space (C^*-probability space, or W^*-probability space, or Banach $*$-probability space, etc.) equipped with a topological $*$-algebra A (C^*-algebra, resp., W^*-algebra, resp., Banach $*$-algebra, etc.), and a (bounded, or unbounded) linear functional φ_B on B.

Definition 8.1. Let a be a self-adjoint free random variable in (B, φ_B). It is said to be even in (B, φ_B), if all odd free moments of a vanish, i.e.,

$$\varphi_B\left(a^{2n-1}\right) = 0, \text{ for all } n \in \mathbb{N}. \tag{8.1}$$

Let a be a "self-adjoint," and "even" free random variable of (B, φ_B) satisfying (8.1). Then a is said to be semicircular in (B, φ_B), if

$$\varphi_B(a^{2n}) = c_n, \text{for all } n \in \mathbb{N}, \tag{8.2}$$

where c_n are the n-th Catalan number,

$$c_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix} = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} = \frac{(2n)!}{n!(n+1)!},$$

for all $n \in \mathbb{N}_0$.

It is well-known that, if $k_n^B(...)$ is the *free cumulant on* B *in terms of a linear functional* φ_B (in the sense of [26]), then a self-adjoint free random variable a is *semicircular* in (B, φ_B), if and only if

$$k_n^B\left(\underbrace{a, a, \ ......, \ a}_{n\text{-times}}\right) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

for all $n \in \mathbb{N}$ (e.g., [26]). The above characterization (8.3) is obtained by the *Möbius inversion of* [26].

Thus, the semicircular elements a of (B, φ_B) can be re-defined by the self-adjoint free random variables satisfying the free-cumulant characterization (8.3). We will use the free-moment definition (8.2), and the free-cumulant characterization (8.3) alternatively below.

Motivated by (8.3), one can define the *weighted-semicircularity.*

Definition 8.2. Let $a \in (B, \varphi_B)$ be a self-adjoint free random variable. It is said to be weighted-semicircular in (B, φ_B) with its weight t_0 (in short, t_0-semicircular), if there exists $t_0 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, such that

$$k_n^B \left(\underbrace{a, a, \ ...., \ a}_{n\text{-times}} \right) = \begin{cases} t_0 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{8.4}$$

for all $n \in \mathbb{N}$.

By the Möbius inversion of [26], one can get the following free-moment characterization (8.5) of the definition (8.4): A self-adjoint free random variable a of (B, φ_B) is t_0-*semicircular*, if and only if there exists $t_0 \in \mathbb{C}^\times$, such that

$$\varphi(a^n) = \omega_n t_0^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{8.5}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (7.38), and c_m are the m-th Catalan numbers for all $m \in \mathbb{N}_0$ (Also, see [8] and [12]).

Therefore, we will use the free-cumulant definition (8.4) and the free-moment characterization (8.5) alternatively as weighted-semicircularity below.

8.2. **Weighted-Semicircular Elements in $\mathfrak{LS}_p^A(j)$.** Fix $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, and let $\mathfrak{LS}_p^A(j)$ be the j-th filtered A-dynamical p-probability space, where (A, ψ) is a fixed unital C^*-probability space.

Theorem 8.3. *Let $X_{a,j}^{p,j} = l_p^A \otimes T_{a,j}^{p,j}$ be a generating operator* (7.31) *of $\mathfrak{LS}_p^A(j)$. Assume that a free random variable $a \in (A, \psi)$ satisfies that:*

(i) *a is self-adjoint in A,*
(ii) *$\psi(a) \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ in $\mathbb{C}^\times$,*
(iii) *$\psi(a^n) = \psi(a)^n$, for all $n \in \mathbb{N}$.*

Then the operator $X_{a,j}^{p,j}$ is $\left(p^{j+1}\psi(a)\right)^2$-semicircular in $\mathfrak{LS}_p^A(j)$.

Proof. Let $a \in (A, \psi)$ be a given self-adjoint free random variable satisfying the additional conditions (i), (ii) and (iii), and let $X_{a,j}^j$ be the corresponding generating operator,

$$X_{a,j}^{p,j} = l_p^A \otimes T_{a,j}^{p,j} = l_p^A \otimes ((a \otimes P_{p,j}) \otimes P_{p,j}) \in \mathfrak{LS}_p^A(j).$$

Then

$$\begin{aligned} \left(X_{a,j}^{p,j}\right)^* &= \left(l_p^A\right)^* \otimes \left(T_{a,j}^j\right)^* \\ &= l_p^A \otimes \left(T_{a^*,j}^{p,j}\right) = l_p^A \otimes T_{a,j}^{p,j} \\ &= X_{p,j}^{p,j}, \end{aligned} \tag{8.6}$$

by the θ^p-relation (6.13), and by the condition (i). Thus $X_{a,j}^{p,j}$ is self-adjoint in $\mathfrak{LS}_p^A(j)$ by (8.6).

Consider that, for any $n \in \mathbb{N}$,

$$\tau_{p,j}^A\left(\left(X_{a,j}^{p,j}\right)^n\right) = \left(\omega_n\left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a^n))$$

by (7.38) and (7.39)

$$= \left(\omega_n\left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)(\psi(a)^n)$$

by (iii)

$$= \left(\omega_n\left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)\left(\psi(a)^2\right)^{\frac{n}{2}}$$

by (ii)

$$= \omega_n\left(p^{2(j+1)}\psi(a)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}}$$

$$= \omega_n\left(\left(p^{j+1}\psi(a)\right)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}}. \tag{8.7}$$

Therefore, by the self-adjointness (8.6), and by (8.7), (8.5), (8.4), this operator $X_{a,j}^{p,j}$ is $\left(p^{j+1}\psi(a)\right)^2$-semicircular in $\mathfrak{LS}_p^A(j)$ under the conditions (i), (ii) and (iii). □

By the weighted-semicircularity (8.7) of the above theorem, one obtains the following corollary.

Corollary 8.4. *Let $X_{1_A,j}^{p,j}$ be in the sense of* (7.31) *in the j-th filtered A-dynamical p-probability space $\mathfrak{LS}_p^A(j)$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, where 1_A is the unit of (A, ψ). Then $X_{1_A,j}^{p,j}$ is $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_p^A(j)$.*

Proof. Let $X_{1_A,j}^{p,j}$ be given as above in $\mathfrak{LS}_p^A(j)$. Then the unit 1_A of A satisfies that:

(i)′ 1_A is self-adjoint in A,

(ii)′ $\psi(1_A) = 1 \in \mathbb{R}^\times$ in $\mathbb{C}$, and

(iii)′ $\psi\left(1_A^n\right) = 1 = \psi(1_A)^n$, for all $n \in \mathbb{N}$.

Thus, by (i)′, (ii)′, and (iii)′, one realizes that 1_A automatically satisfy the additional conditions (i), (ii) and (iii) of the above theorem. So, the operator $X_{1_A,j}^{j}$ is

$$\left[\left(p^{j+1}\psi(1_A)\right)^2 = p^{2(j+1)}\right]\text{-semicircular}$$

in $\mathfrak{LS}_p^A(j)$, by (7.9).

Therefore, the statement (7.10) holds true. □

The above corollary generalizes the weighted-semicircularity of [8] and [12] up to dynamical systems

8.3. **Semicircular Elements in $\mathfrak{LS}_p^A(j)$.** Let's fix $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, and let $\mathfrak{LS}_p^A(j)$ be the j-th filtered A-dynamical p-probability space.

Theorem 8.5. *Let $X_{a,j}^{p,j} = l_p^A \otimes T_{a,j}^{p,j} = l_p^A \otimes ((a \otimes P_{p,j}) \otimes P_{p,j})$ be a generating operator of $\mathfrak{LS}_p^A(j)$. Assume that a free random variable $a \in (A, \psi)$ satisfies that:*

(i) *a is self-adjoint in A,*

(ii) *$\psi(a) \in \mathbb{R}^\times$ in $\mathbb{C}$, and*

(iii) $\psi(a^n) = \psi(a)^n$, *for all* $n \in \mathbb{N}$.

Then the operator

$$Y_{a,j}^{p,j} = \frac{1}{p^{j+1}\psi(a)} X_{a,j}^{p,j} \in \mathfrak{LS}_p^A \tag{8.8}$$

is semicircular in $\mathfrak{LS}_p^A(j)$.

Proof. First of all, under the conditions (i), (ii) and (iii), the operator $Y_{a,j}^{p,j}$ of (8.8) is very well-defined in $\mathfrak{LS}_p^A$.

Let $X_{a,j}^{p,j}$ and $Y_{a,j}^{p,j}$ be given as above in $\mathfrak{LS}_p^A(j)$, where a free random variable $a \in (A,\ \psi)$ satisfies the additional conditions (i), (ii) and (iii). Now, let $k_n^{A,p,j}(...)$ be the free cumulant (of [26]) on $\mathfrak{LS}_p^A$ in terms of the linear functional $\tau_{p,j}^A$.

Note and recall that, under the conditions (i), (ii) and (iii), the generating operator $X_{a,j}^{p,j}$ is $\left(p^{j+1}\psi(a)\right)^2$-semicircular in $\mathfrak{LS}_p^A(j)$ by (8.7), (8.4) and (8.5).

Observe that

$$k_n^{A,p,j}\left(\underbrace{Y_{a,j}^{p,j}, Y_{a,j}^{p,j},\ ...,\ Y_{a,j}^{p,j}}_{n\text{-times}}\right) = \left(\frac{1}{p^{j+1}\psi(a)}\right)^n k_n^{A,p,j}\left(X_{a,j}^{p,j},\ ...,\ X_{a,j}^{p,j}\right)$$

by the bi-module-map property of free cumulants (e.g., [26])

$$\begin{aligned} &= \begin{cases} \left(\frac{1}{p^{j+1}\psi(a)}\right)^2 k_2^{A,p,j}\left(X_{a,j}^{p,j}, X_{a,j}^{p,j}\right) & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \left(\frac{1}{p^{j+1}\psi(a)}\right)^2 \left(p^{j+1}\psi(a)\right)^2 = 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{8.9}$$

for all $n \in \mathbb{N}$, by the $\left(p^{j+1}\psi(a)\right)^2$-semicircularity (7.9) of $X_{a,j}^{p,j}$ in $\mathfrak{LS}_p^A(j)$.

Therefore, by (8.9) and (8.3), this operator $Y_{a,j}^{p,j}$ is semicircular in $\mathfrak{LS}_p^A(j)$. □

The following result is a direct consequence of the above theorem.

Corollary 8.6. *Let* $Y_{1_A,j}^{p,j} = \frac{1}{p^{j+1}} X_{1_A,j}^{p,j} \in \mathfrak{LS}_p^A(j)$. *Then* $Y_{1_A,j}^{p,j}$ *is semicircular in* $\mathfrak{LS}_p^A(j)$.

Proof. Note and recall that $X_{1_A,j}^{p,j}$ is $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_p^A(j)$ by (8.7). So, with similar arguments, the semicircularity of $Y_{1_A,j}^{p,j}$ in $\mathfrak{LS}_p^A(j)$ is guaranteed. □

The above corollary generalizes the semicircularity of [8, 11] and [12].

8.4. **Summary.** In Sections 9.2 and 9.3, we showed that, for any fixed $p \in \mathcal{P}$, $j \in \mathbb{Z}$, a generating operator

$$X_{a,j}^{p,j} = a \otimes T_{a,j}^{p,j} = l_p^A \otimes ((a \otimes P_{p,j}) \otimes P_{p,j})$$

can be weighted-semicircular in the j-th filtered A-dynamical p-probability space $\mathfrak{LS}_p^A(j)$. i.e., starting from a semigroup $\sigma(p)$, one can construct weighted-semicircular elements.

More precisely, if a "self-adjoint" free random variable a of a fixed unital C^*-probability space $(A,\ \psi)$ satisfies

$$\psi(a) \in \mathbb{R}^{\times} \text{ in } \mathbb{C}, \text{ and } \psi(a^n) = \psi(a)^n, \text{ for all } n \in \mathbb{N}, \tag{8.10}$$

then $X_{a,j}^{p,j}$ is $\left(p^{j+1}\psi(a)\right)^2$-semicircular in $\mathfrak{LS}_p^A(j)$.

So, under the condition (8.10), the operator

$$Y_{a,j}^{p,j} = \frac{1}{p^{j+1}\psi(a)} X_{a,j}^{p,j}$$

is semicircular in $\mathfrak{LS}_p^A(j)$.

9. The A-Dynamical Free Filterization $\mathfrak{LS}_A$

In this section, we use free product to construct the universalized, or globalized free-probabilistic structures from our j-th filtered A-dynamical p-probability spaces $\mathfrak{LS}_p^A(j)$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$. As before, let $(A,\ \psi)$ be a fixed unital C^*-probability space, and let $\mathfrak{LS}_p^A$ be the A-dynamical p-radial-projection algebras for all $p \in \mathcal{P}$.

Define the *free product Banach $*$-probability space*,

$$\begin{aligned} \mathfrak{LS}_A = (\mathfrak{LS}_A, \tau_A) &\overset{def}{=} \underset{p\in\mathcal{P}, j\in\mathbb{Z}}{\star} \mathfrak{LS}_p^A(j) \\ &= \left(\underset{p\in\mathcal{P}, j\in\mathbb{Z}}{\star} \mathfrak{LS}_p^A,\ \underset{p\in\mathcal{P}, j\in\mathbb{Z}}{\star} \tau_{p,j}^A \right), \end{aligned} \tag{9.1}$$

by the free product $*$-probability space (e.g., [26] and [35]).

Definition 9.1. We call this free product Banach $*$-probability space $\mathfrak{LS}_A = (\mathfrak{LS}, \tau_A)$ of (9.1), the *A-dynamical free filterization*.

Let $(B,\ \varphi_B)$ be an arbitrary topological $*$-probability space, and let $\mathcal{F} = \{b_k\}_{k\in\Lambda}$ be a subset of B, where Λ is a countable (finite, or infinite) index set. Such a family $\mathcal{F}$ is said to be a *free family*, if all elements b_k's of $\mathcal{F}$ are mutually free from each other in $(B,\ \varphi_B)$. i.e., whenever $k_1 \neq k_2$ in Λ, the elements b_{k_1} and b_{k_2} of $\mathcal{F}$ are free in $(B,\ \varphi_B)$, if and only if all "mixed" free cumulants of

$$\{b_{k_1}, b_{k_1}^*\} \cup \{b_{k_2}, b_{k_2}^*\}$$

vanish (e.g., [26]), if and only if the minimal free summands $B[b_{k_1}]$ and $B[b_{k_2}]$ of B containing b_{k_1}, respectively, b_{k_2} (in the sense of Section 6) are distinct in B (e.g., [8]).

Suppose $\mathcal{F}$ is a free family of $(B,\ \varphi_B)$, and assume that all elements of $\mathcal{F}$ are (weighted-)semicircular in $(B,\ \varphi_B)$. Then this free family $\mathcal{F}$ is said to be a *free (weighted-)semicircular family.*

Let $a \in (A,\ \psi)$ be a self-adjoint free random variable satisfying the additional condition (8.10). Construct subsets,

$$\begin{aligned} \mathfrak{X}_a &= \left\{ X_{a,j}^{p,j} \in \mathfrak{LS}_p^A(j) \,|p \in \mathcal{P}, j \in \mathbb{Z} \right\}, \text{ and} \\ \mathfrak{Y}_a &= \left\{ Y_{a,j}^{p,j} = \frac{1}{p^{j+1}\psi(a)} X_{a,j}^{j} \in \mathfrak{LS}_p^A(j) \,|p \in \mathcal{P}, j \in \mathbb{Z} \right\} \end{aligned} \tag{9.2}$$

in the A-dynamical free filterization $\mathfrak{LS}_A$, where

$$X_{a,j}^{p,j} = l_p^A \otimes T_{a,j}^{p,j} = l_p^A \otimes ((a \otimes P_{p,j}) \otimes P_{p,j})$$

are the generating operators of $\mathfrak{LS}_p^A$, in the sense of Section 8, for all $p \in \mathcal{P}$.

Theorem 9.2. *Let $\mathfrak{LS}_A$ be the A-dynamical free filterization (9.1), and let $\mathfrak{X}_a$ and $\mathfrak{Y}_a$ be the subsets (9.2), where a self-adjoint element $a \in (A, \psi)$ satisfies* (8.10).

(9.3) The family $\mathfrak{X}_a$ is a free weighted-semicircular family in $\mathfrak{LS}_A$

(9.4) The family $\mathfrak{Y}_a$ is a free semicircular family in $\mathfrak{LS}_A$.

Proof. Let $\mathfrak{X}_a$ be in the sense of (9.2) in the A-dynamical free filterization $\mathfrak{LS}_A$, where a self-adjoint free random variable $a \in (A, \psi)$ satisfies the condition (8.10). By (9.1) and (9.2), all elements $X_{a,j}^{p,j}$'s of $\mathfrak{X}_a$ are mutually free from each other in $\mathfrak{LS}_A$, because all elements $X_{a,j}^{p,j}$ of $\mathfrak{X}_a$ are from mutually distinct free blocks $\mathfrak{LS}_p^A(j)$ of $\mathfrak{LS}_A$. So, this family $\mathfrak{X}_a$ forms a free family in $\mathfrak{LS}_A$.

Furthermore, by the condition (8.10) for $a \in (A, \psi)$, one has

$$\begin{aligned}
\tau_A\left(\left(X_{a,j}^{p,j}\right)^n\right) &= \tau_A\left(\left(X_{a,j}^{p,j}\right)^n\right) \\
&= \tau_{p,j}^A\left(\left(X_{a,j}^{p,j}\right)^n\right) \\
&= \omega_n\left(\left(p^{j+1}\psi(a)\right)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},
\end{aligned}$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, by (8.7).

So, all elements of $\mathfrak{X}_a$ are weighted-semicircular in $\mathfrak{LS}_A$. Therefore, $\mathfrak{X}_a$ is a free weighted-semicircular family in $\mathfrak{LS}_A$, equivalently, the statement (9.3) holds true.

Similarly, one can conclude that the family $\mathfrak{Y}_a$ is a free semicircular family in $\mathfrak{LS}_A$, i.e., the statement (9.4) holds. $\square$

By (9.3) and (9.4), one can get the following result.

Corollary 9.3. *Let $\mathfrak{LS}_A$ be the A-dynamical free filterization, and let $\mathfrak{X}_{1_A}$ and $\mathfrak{Y}_{1_A}$ be in the sense of (9.2) in $\mathfrak{LS}_A$, where 1_A is the unit of (A, ψ). Then*

(9.5) The family $\mathfrak{X}_{1_A}$ is a free weighted-semicircular family in $\mathfrak{LS}_A$.

(9.6) The family $\mathfrak{Y}_{1_A}$ is a free semicircular family in $\mathfrak{LS}_A$.

Proof. The proofs of the statements (9.5) and (9.6) are done by (9.3), respectively, (9.4), with help of the weighted-semicircularity (8.7) and the semicircularity (8.9), because the unit 1_A satisfies the condition (8.10) in (A, ψ). $\square$

The above corollary generalizes the main results of [8] up to dynamical systems.

10. Free Distributions of Free Reduced Words in $\mathfrak{LS}_A$

Let (A, ψ) be a fixed unital C^*-probability space, and let

$$\mathfrak{LS}_A = (\mathfrak{LS}_A, \tau_A)$$

be the A-dynamical free filterization (9.1).

Notation and Assumption. In this section, we automatically assume $a \in (A, \psi)$ is a self-adjoint free random variable satisfying the additional conditions (8.10) for convenience. i.e.,

$$\psi(a) \in \mathbb{R}^{\times} \text{ and } \psi(a^n) = \psi(a)^n,$$

for all $n \in \mathbb{N}$.

Let $\mathfrak{X}_a$ and $\mathfrak{Y}_a$ be the corresponding free weighted-semicircular family (9.3), respectively, free semicircular family (9.4) in $\mathfrak{LS}_A$ under the condition (8.10). In this section, we consider free reduced words W of $\mathfrak{LS}_A$ generated by $\mathfrak{X}_a$ or $\mathfrak{Y}_a$. In particular, we are interested in the free-distributional data of such elements W in $\mathfrak{LS}_A$. It shows how our (weighted-)semicircularity affects the free distributions of W with respect to those on (A, ψ), and p-adic analytic data over primes p.

10.1. **Free Reduced Words of $\mathfrak{LS}_A$ in $\mathfrak{X}_a$.** Let $\mathfrak{LS}_A$ be the A-dynamical free filterization (9.1), and let $a \in (A, \psi)$ be a self-adjoint free random variable (under (8.10)). And let $\mathfrak{X}_a$ be the free weighted-semicircular family (9.3) in $\mathfrak{LS}_A$, consisting of $\left(p^{j+1}\psi(a)\right)^2$-semicircular elements

$$X_{a,j}^{p,j} = l_p^A \otimes T_{a,j}^{p,j} = l_p^A \otimes \left((a \otimes P_{p,j}) \otimes P_{p,j} \right),$$

contained in the free blocks $\mathfrak{LS}_p^A(j)$ of $\mathfrak{LS}_A$, for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$.

Since we already characterize the free distributions of all free reduced words $X_{a,j}^{p,j}$ with their "length-1" in $\mathfrak{X}_a$ under the weighted-semicircularity (9.3), we focus on here free reduced words with their lengths > 1 in the A-dynamical free filterization $\mathfrak{LS}_A$.

By the freeness on the family $\mathfrak{X}_a$ in $\mathfrak{LS}_A$, free reduced words W (with their lengths > 1) in $\mathfrak{X}_a$ have their forms,

$$W = \prod_{l=1}^{N} \left(X_{a,j_{k_l}}^{p_{k_l}, j_{k_l}} \right)^{n_l}, \text{ with } X_{a,j_{k_l}}^{p,j_{k_l}} \in \mathfrak{X}_a, \text{ where} \tag{10.1}$$
$$p_{k_l} \in \mathcal{P}, j_{k_l} \in \mathbb{Z}, n_l \in \mathbb{N}, \text{ for } l = 1, ..., N,$$

for $N \in \mathbb{N} \setminus \{1\}$, such that either

$$(p_{k_1}, \ ..., \ p_{k_N}) \in alt\left(\mathcal{P}^N\right), \text{ or } (j_{k_1}, \ ..., \ j_{k_N}) \in alt\left(\mathbb{Z}^N\right), \tag{10.2}$$

where

$$(n_1, \ ..., \ n_N) \in \mathbb{N}^N,$$

which is not necessarily alternating in $\mathbb{N}$, where

$$alt\left(\mathcal{P}^N\right) = \left\{ (p_1, ..., p_N) \middle| \begin{array}{c} (p_1, ..., p_N) \in \mathcal{P}^N, \text{ satisfying} \\ p_1 \neq p_2, p_2 \neq p_3, \\ ..., p_{N-1} \neq p_N, \text{ in } \mathcal{P} \end{array} \right\},$$

and

$$alt\left(\mathbb{Z}^N\right) = \left\{ (j_1, ..., j_N) \middle| \begin{array}{c} (j_1, ..., j_N) \in \mathbb{Z}^N, \text{ satisfying} \\ j_1 \neq j_2, j_2 \neq j_3, \\ ..., j_{N-1} \neq j_N, \text{ in } \mathbb{Z} \end{array} \right\}.$$

For example,

$$(2, 3, 2, 5, 2, 5, 7) \in alt\left(\mathcal{P}^7\right),$$

but

$$(2, 3, 3, 5, 2, 7, 2) \notin alt\left(\mathcal{P}^7\right).$$

The finite sequences, or the N-tuples in $alt\left(\mathcal{P}^N\right)$ (or, in $alt\left(\mathbb{Z}^N\right)$) are called *alternating N-tuples* in $\mathcal{P}$ (resp., in $\mathbb{Z}$) (e.g., see [26] and [35]).

Define the subsets $alt_{MD}\left(\mathcal{P}^N\right)$ and $alt_{MD}\left(\mathbb{Z}^N\right)$ by

$$alt_{MD}\left(\mathcal{P}^N\right) = \left\{ (p_1, ..., p_N) \left| \begin{array}{c} (p_1, ..., p_N) \in alt\left(\mathcal{P}^N\right), \\ \text{and the entries } p_1, ..., p_N \\ \text{are mutually distinct in } \mathcal{P}. \end{array} \right. \right\},$$

respectively

$$alt_{MD}\left(\mathbb{Z}^N\right) = \left\{ (j_1, ..., j_N) \left| \begin{array}{c} (j_1, ..., j_N) \in alt\left(\mathbb{Z}^N\right), \\ \text{and the entries } j_1, ..., j_N \\ \text{are mutually distinct in } \mathbb{Z}. \end{array} \right. \right\},$$

for all $N \in \mathbb{N}$.

For example,

$$(2,3,2,5,2,5,7) \in alt\left(\mathcal{P}^7\right),$$

but

$$(2,3,2,5,2,5,7) \notin alt_{MD}\left(\mathcal{P}^7\right).$$

The finite sequences in $alt_{MD}\left(\mathcal{P}^N\right)$ (or, in $alt_{MD}\left(\mathbb{Z}^N\right)$) are alternating N-tuples in $\mathcal{P}$ (resp., in $\mathbb{Z}$) consisting of mutually distinct entries.

Notation and Assumption 10.1 (in short, **NA 10.1** below)**.** Let W be a free reduced word (10.1) with its length-$N > 1$, satisfying (10.2) in the A-dynamical free filterization $\mathfrak{LS}_A$. Assume further that

$$P \overset{denote}{=} (p_{k_1},\ ...,\ p_{k_N}) \in alt_{MD}\left(\mathcal{P}^N\right),$$

$$J \overset{denote}{=} (j_{k_1},\ ...,\ j_{k_N}) \in alt_{MD}\left(\mathbb{Z}^N\right),$$

and

$$\eta \overset{denote}{=} (n_1,\ ...,\ n_N) \in \mathbb{N}^N.$$

If the finite sequences P, J and η are chosen as above, then we denote W by $W_{P,J}^{\eta}$, to emphasize the choices of such P, J and η. $\square$

Theorem 10.1. *Let $W = W_{P,J}^{\eta}$ be a free reduced word of $\mathfrak{LS}_A$ in the sense of **NA 10.1**. Then*

$$\tau_A(W) = \tau_A\left(W^*\right) = \prod_{l=1}^{N} \left(\omega_{n_l} \left(p_{k_l}^{2(j_{k_l}+1)} \psi(a)^2 \right)^{\frac{n_l}{2}} c_{\frac{n_l}{2}} \right). \tag{10.3}$$

Proof. Let $W = W_{P,J}^{\eta}$ be a free reduced word in $\mathfrak{LS}_A$ in the sense of **NA 10.1**. Then one obtains

$$W^* = \prod_{l=1}^{N} X_{a,j_{N-k_l+1}}^{p_{N-k_l+1},j_{N-k_l+1}} \in \mathfrak{LS}_A, \tag{10.4}$$

by the self-adjointness of $X_{a,j}^{p,j} \in \mathfrak{X}_a$ in $\mathfrak{LS}_A$.

Observe that

$$\tau_A(W) = \prod_{l=1}^{N} \tau_{p_{k_l},j_{k_l}}^{A} \left(X_{a,j_{k_l}}^{p_{k_l},j_{k_l}} \right)$$

since W is a free reduced word of $\mathfrak{LS}_A$ in $\mathfrak{X}_a$, by (9.1), and the free generators of W are mutually free from each other

$$= \prod_{l=1}^{N} \left(\omega_{n_l} \left(p_{k_l}^{2(j_{k_l}+1)} \psi(a)^2 \right)^{\frac{n_l}{2}} c_{\frac{n_l}{2}} \right)$$

by the $\left(p_{k_l}^{j_{k_l}+1} \psi(a)\right)^2$-semicircularity (9.3) on $\mathfrak{X}_a$ in $\mathfrak{LS}_A$, for all $l = 1, ..., N$

$$= \prod_{l=1}^{N} \left(\omega_{n_l} \left(p_{N-k_l+1}^{2(j_{N-k_l+1}+1)} \psi(a)^2 \right)^{\frac{n_l}{2}} c_{\frac{n_l}{2}} \right)$$

$$= \prod_{l=1}^{N} \tau_{p_{N-k_l+1}, N-j_{k_l+1}}^{A} \left(X_{a, j_{N-k_l+1}}^{p_{N0k_l+1}, j_{N-k_l+1}} \right)$$

$$= \tau_A(W^*),$$

by (10.4). Therefore, the formula (10.3) holds true. □

The following corollary is a direct consequence of the above theorem.

Corollary 10.2. *Let $\mathfrak{X}_{1_A}$ be the free weighted-semicircular family (9.5) in the A-dynamical free filterization $\mathfrak{LS}_A$, where 1_A is the unit of (A, ψ). Let*

$$W = W_{P,J}^{\eta} = \prod_{l=1}^{N} \left(X_{1_A, j_{k_l}}^{p_{k_l}, j_{k_l}} \right)^{n_l} \tag{10.5}$$

*be a free reduced word of $\mathfrak{LS}_A$ in $\mathfrak{X}_{1_A}$ under **NA 10.1**. Then*

$$\tau_A(W) = \tau_A(W^*) = \prod_{l=1}^{N} \left(\omega_{n_l} \left(p_{k_l}^{2(j_l+1)} \right)^{\frac{n_l}{2}} c_{\frac{n_l}{2}} \right). \tag{10.6}$$

Proof. Let W be a free reduced word (10.5) of $\mathfrak{LS}_A$ in the free weighted-semicircular family $\mathfrak{X}_{1_A}$ of (9.5) in $\mathfrak{LS}_A$. Since 1_A automatically satisfies the condition (8.10), we obtain the free-distributional data (10.6) of W by (10.3). □

The above free-distributional data (10.5) generalizes the main results of [8] up to dynamical systems.

10.2. **Free Reduced Words of $\mathfrak{LS}_A$ in $\mathfrak{Y}_a$.** Let (A, ψ) be a fixed unital C^*-probability space, and $a \in (A, \psi)$, a self-adjoint free random variable satisfying the condition (8.10), and let $\mathfrak{LS}_A$ be the A-dynamical free filterization (9.1). Also, let $\mathfrak{Y}_a$ be the free semicircular family (9.4) in $\mathfrak{LS}_A$. Define an operator U of $\mathfrak{LS}_A$ by

$$U = \prod_{l=1}^{N} \left(Y_{a, j_{k_l}}^{p_{k_l}, j_{k_l}} \right)^{n_l}, \tag{10.7}$$

for some $N \in \mathbb{N} \setminus \{1\}$, where

$$Y_{a, j_{k_l}}^{p_{k_l}, j_{k_l}} = \left(\frac{1}{p_{k_l}^{j_{k_l}+1} \psi(a)} \right) X_{p_{k_l}, j_{k_l}}^{a} \in \mathfrak{Y}_a, \tag{10.8}$$

for all $l = 1, ..., N$.

As in **NA 10.1**, we denote the operator U of (10.7), by

$$U = U_{P,J}^{\eta} \tag{10.9}$$

if

$$P = (p_{k_1},\ ...,\ p_{k_N}) \in alt_{MD}\left(\mathcal{P}^N\right),$$

or

$$J = (j_{k_1},\ ...,\ j_{k_N}) \in alt_{MD}\left(\mathbb{Z}^N\right),$$

and

$$\eta = (n_1,\ ...,\ n_N) \in \mathbb{N}^N.$$

Theorem 10.3. *Let $U = U_{P,J}^{\eta}$ be an operator* (10.9) *in $\mathfrak{LS}_A$. Then*

$$\tau_A(U) = \tau_A(U^*) = \prod_{l=1}^{N}\left(\omega_{n_l} c_{\frac{n_l}{2}}\right), \tag{10.10}$$

Proof. Let $U = U_{P,J}^{\eta}$ be in the sense of (10.9) in $\mathfrak{LS}_A$ generated by the free semicircular family $\mathfrak{Y}_a$ of (9.4).

$$\begin{aligned}\tau_A(U) &= \tau_A\left(\prod_{l=1}^{N} Y_{a,j_{k_l}}^{p_{k_l},j_{k_l}}\right) = \prod_{l=1}^{N}\tau_{p_{k_l},j_{k_l}}^{A}\left(Y_{a,j_{k_l}}^{p_{k_l},j_{k_l}}\right) \\ &= \prod_{l=1}^{N}\left(\omega_{nn_l} c_{\frac{nn_l}{2}}\right) = \prod_{l=1}^{N}\tau_{p_{k_l},j_{k_l}}^{A}\left(Y_{a,j_{N-k_l+1}}^{p_{N-k_l+1},j_{N-k_l+1}}\right) \\ &= \tau_A\left(\prod_{l=1}^{N} Y_{a,j_{N-k_l+1}}^{p_{N-k_l+1},j_{N-k_l+1}}\right) = \tau_A(U^*),\end{aligned}$$

by the semicircularity and the freeness on $\mathfrak{Y}_a$ in $\mathfrak{LS}_A$. Therefore, the free-distributional data (10.10) holds true. □

The above corollary generalize the main results of [8] up to dynamical systems.

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