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ONE-DIMENSIONAL VARIATIONAL OBSTACLE PROBLEMS

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Abstract. In this paper we prove partial regularity for solutions of one-dimensional variational obstacle problems. If the obstacles are sufficiently close then there exists a solution and it is regular.

1. Introduction

Consider a continuous Lagrangian $L = L(x, u, v)\colon \mathbb{R}^3 \to \mathbb{R}$ which is convex in v. We shall consider the following precise assumptions (H) on L: that for all compact sets $K \subseteq \mathbb{R}^3$,

(H1) there exist constants $c, \alpha > 0$ such that
$$|L(x_1, u_1, v_1) - L(x_2, u_2, v_2)| \le c\,(|x_1 - x_2| + |u_1 - u_2| + |v_1 - v_2|)^\alpha$$
for all $(x_1, u_1, v_1), (x_2, u_2, v_2) \in K$; and

(H2) there exist constants $\mu > 0$ and $p > 1$ such that for all $(x, u, v_1) \in K$, there exists $l \in \partial L_v(x, u, v_1)$ such that for all $(x, u, v_2) \in K$, we have
$$L(x, u, v_2) - L(x, u, v_1) - (l, v_2 - v_1) \ge \mu |v_2 - v_1|^p.$$

For subintervals $[a, b]$ of $\mathbb{R}$, and appropriate continuous functions $f, g\colon [a, b] \to \mathbb{R}$ satisfying $g < f$ on (a, b), we consider the following "obstacle problem":

$$\text{minimize} \quad J(u) := \int_a^b L(x, u(x), \dot{u}(x))\, dx \tag{1.1}$$

$$\text{over all } u \text{ such that } \begin{cases} u \in W^{1,1}(a, b); \\ g \le u \le f \quad \text{on } [a, b]; \text{ and} \\ u(a) = A, u(b) = B; \end{cases} \tag{1.2}$$

where $g(a) \le A \le f(a)$ and $g(b) \le B \le f(b)$ are fixed boundary conditions.

Definition 1.1. Consider a class of functions
$$\Xi = \left\{\xi\colon [a_\xi, b_\xi] \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}\right\},$$
where each $\xi\colon [a_\xi, b_\xi] \to \overline{\mathbb{R}}$ is continuous. We say the family Ξ is *conditionally equacontinuous* if for every $M > 0$ and $\epsilon > 0$, there exists a $\delta = \delta(M, \epsilon) > 0$ such that if $|\xi(x_0)| \le M$, then $|x - x_0| \le \delta$ implies that $|\xi(x) - \xi(x_0)| \le \epsilon$.

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Theorem 1.2. *Let $f, g \in C^{1,\sigma}[a,b]$ satisfy $g < f$ on (a,b), and let L satisfy condition (H).*

Let U be the set of solutions of the problem (1.1)–(1.2) for all $A \in [g(a), f(a)]$ and $B \in [g(b), f(b)]$. Then the set of derivatives of elements of U forms a conditionally equa-continuous family.

Remark 1.3. It follows from the proof of the theorem that the same conclusion holds considering a family of integrands L, provided the constants $c(K)$, $\alpha(K)$, $\mu(K)$, $p(K)$ may be chosen uniformly for all L, and obstacles f, g that are bounded in $C^{1,\sigma}$-norm.

The fact that the derivatives of solutions of obstacle problems are continuous with values in $\overline{\mathbb{R}}$ was established recently by Mandallena [3] provided L is elliptic in v ($L_{vv} \geq \mu > 0$).

Theorem 1.4. *Let L satisfy condition (H) and $\|f\|_{C^{1,\sigma}}, \|g\|_{C^{1,\sigma}} \leq const$. Then there exists $\eta > 0$ such that if additionally $\|f - g\|_C \leq \eta$ then solutions of admissible obstacle problems exist and are $C^{1,\gamma}$-regular functions.*

In the case in which $p = 2$, the existence of solutions in the class of Lipschitz functions was established by Sychëv [4].

2. Existence and regularity "in small"

We prove the following theorem, which is an adaptation to obstacle problems of a previously established result [2, theorem 1.1].

Theorem 2.1. *Let L satisfy (H), $[a_0, b_0]$ be fixed, and $f, g \in C^{1,\sigma}[a_0, b_0]$. Let $G \subseteq \{(x,u) : x \in [a_0,b_0], g(x) \leq u \leq f(x)\}$ be compact. Then for all $M \geq \max\{\|f\|_{C^1[a_0,b_0]}, \|g\|_{C^1[a_0,b_0]}\}$, there exist $\epsilon_0, \delta_0 > 0$ such that for every $\epsilon \leq \epsilon_0$ and $\delta \leq \delta_0$, for all $(a,A) \in G$, for any $(b,B) \in G$ satisfying $|B-A|/|b-a| \leq M$ and $|b-a| \leq \delta$, the obstacle problem (1.1)–(1.2) is solvable over all those u satisfying further the condition $|u(x) - A| \leq \epsilon$ for all $x \in [a,b]$. Moreover, the solutions are bounded in $C^{1,\gamma}[a,b]$, where $\gamma = \gamma(M)$ does not depend on δ, ϵ, or (a,A).*

Given a function $u \colon [a,b] \to \mathbb{R}$, and a nontrivial subinterval $[x_1, x_2] \subseteq [a,b]$, we define $l_{x_1,x_2} \colon [x_1,x_2] \to \mathbb{R}$ to be the affine function such that $l_{x_1,x_2}(x_1) = u(x_1)$ and $l_{x_1,x_2}(x_2) = u(x_2)$. Thus we have

$$\dot{l}_{x_1,x_2}(x) = \dot{l}_{x_1,x_2} = \frac{u(x_2) - u(x_1)}{x_2 - x_1}$$

for all $x \in (x_1, x_2)$. We shall also adopt the notation

$$J(u; [x_1,x_2]) \coloneqq \int_{x_1}^{x_2} L(x, u(x), \dot{u}(x))\, dx.$$

The following lemma is an appropriate generalization of previous similar results [4, lemma 3.1], [2, lemma 2.1].

Lemma 2.2. *Let $G \subseteq \mathbb{R}^2$ be compact, and let $M > 0$. Suppose $u \in W^{1,\infty}(a,b)$ has $\|u\|_{W^{1,\infty}(a,b)} \leq M$ and has graph contained in G. Suppose further that $x_1 \leq x_3 \leq$*

$x_4 \le x_2$ are points of $[a,b]$ such that $|x_2 - x_1| \le 1$ and $|x_2 - x_1| \le 2|x_4 - x_3|$, and that the graphs of the functions l_{x_1,x_2} and l_{x_3,x_4} are contained in G.

Let L satisfy condition (H), with, for the compact set $K := G \times [-M, M]$, constants $c_1, \alpha > 0$ in the Hölder condition (H1), and constants $\mu > 0$ and $p > 1$ in the singular ellipticity condition (H2).

Suppose further that there exist constants $c_2, \beta > 0$ such that

$$J(u;[x_1,x_2]) \le J(l_{x_1,x_2};[x_1,x_2]) + c_2|x_2 - x_1|^{1+\beta}; \tag{2.1}$$

$$J(u;[x_3,x_4]) \le J(l_{x_3,x_4};[x_3,x_4]) + c_2|x_4 - x_3|^{1+\beta}. \tag{2.2}$$

Then

$$\left|\frac{u(x_2)-u(x_1)}{x_2-x_1} - \frac{u(x_4)-u(x_3)}{x_4-x_3}\right| \le 4\left(\frac{c_2 + 2c_1(1+M)^\alpha}{\mu}\right)^{1/p} |x_2 - x_1|^{\gamma/p}, \tag{2.3}$$

where $\gamma := \min\{\alpha, \beta\}$.

Proof. We first obtain estimates of the excess of the derivative on $[x_1, x_2]$ and $[x_3, x_4]$, i.e. estimate the integrals

$$\int_{x_1}^{x_2} \left|\dot{u}(x) - \dot{l}_{x_1,x_2}\right|^p dx \text{ and } \int_{x_3}^{x_4} \left|\dot{u}(x) - \dot{l}_{x_3,x_4}\right|^p dx.$$

By the Hölder condition (H1) on L, we have that

$$|L(x_1, u(x_1), \dot{u}(x)) - L(x, u(x), \dot{u}(x))| \le c_1 \left(|x_1 - x| + M|x_1 - x|\right)^\alpha, \text{ and}$$

$$\left|L(x_1, u(x_1), \dot{l}_{x_1,x_2}(x)) - L(x, l_{x_1,x_2}(x), \dot{l}_{x_1,x_2}(x))\right| \le c_1 \left(|x_1 - x| + M|x_1 - x|\right)^\alpha.$$

Therefore defining $\tilde{L}(\cdot) = L(x_1, u(x_1), \cdot)$, we have, by (2.1), writing $\tilde{J}$ for the integral functional corresponding to the integrand $\tilde{L}$, that

$$\begin{aligned}\tilde{J}(u;[x_1,x_2]) &\le J(u;[x_1,x_2]) + c_1(1+M)^\alpha|x_2 - x_1|^{1+\alpha}\\ &\le J(l_{x_1,x_2};[x_1,x_2]) + c_2|x_2 - x_1|^{1+\beta} + c_1(1+M)^\alpha|x_2 - x_1|^{1+\alpha}\\ &\le \tilde{J}(l_{x_1,x_2};[x_1,x_2]) + c_2|x_2 - x_1|^{1+\beta} + 2c_1(1+M)^\alpha|x_2 - x_1|^{1+\alpha}.\end{aligned} \tag{2.4}$$

The singular ellipticity condition (H2) implies that there exists $l \in \partial\tilde{L}(\dot{l}_{x_1,x_2})$ such that

$$\begin{aligned}&\tilde{J}(u;[x_1,x_2]) - \tilde{J}(l_{x_1,x_2};[x_1,x_2])\\ &= \int_{x_1}^{x_2} \left(\tilde{L}(\dot{u}(x)) - \tilde{L}(\dot{l}_{x_1,x_2}(x)) - (l, \dot{u}(x) - \dot{l}_{x_1,x_2}(x))\right) dx\\ &\ge \mu \int_{x_1}^{x_2} \left|\dot{u}(x) - \dot{l}_{x_1,x_2}(x)\right|^p dx.\end{aligned}$$

Therefore (2.4) implies that

$$\int_{x_1}^{x_2} \left|\dot{u}(x) - \dot{l}_{x_1,x_2}\right|^p dx \le \frac{c_2 + 2c_1(1+M)^\alpha}{\mu}|x_2 - x_1|^{1+\gamma},$$

where $\gamma := \min\{\alpha, \beta\}$.

Analogously

$$\int_{x_3}^{x_4} \left|\dot{u}(x) - \dot{l}_{x_3,x_4}\right|^p \, dx \le \frac{c_2 + 2c_1(1+M)^\alpha}{\mu} |x_4 - x_3|^{1+\gamma}.$$

Then by Hölder's inequality we have that

$$\begin{aligned}
&\int_{x_3}^{x_4} \left|\dot{l}_{x_1,x_2} - \dot{l}_{x_3,x_4}\right| \, dx \\
&\le \int_{x_3}^{x_4} \left|\dot{l}_{x_3,x_4} - \dot{u}(x)\right| \, dx + \int_{x_1}^{x_2} \left|\dot{l}_{x_1,x_2} - \dot{u}(x)\right| \, dx \\
&\le \left(\int_{x_3}^{x_4} \left|\dot{l}_{x_3,x_4} - \dot{u}(x)\right|^p \, dx\right)^{1/p} |x_4 - x_3|^{(p-1)/p} \\
&\quad + \left(\int_{x_1}^{x_2} \left|\dot{l}_{x_1,x_2} - \dot{u}(x)\right|^p \, dx\right)^{1/p} |x_2 - x_1|^{(p-1)/p} \\
&\le \left(\frac{c_2 + 2c_1(1+M)^\alpha}{\mu}\right)^{1/p} \left(|x_4 - x_3|^{1+\gamma/p} + |x_2 - x_1|^{1+\gamma/p}\right) \\
&\le \left(\frac{c_2 + 2c_1(1+M)^\alpha}{\mu}\right)^{1/p} 2|x_2 - x_1|^{1+\gamma/p},
\end{aligned}$$

and therefore,

$$\left|\dot{l}_{x_1,x_2} - \dot{l}_{x_3,x_4}\right| \le \left(\frac{c_2 + 2c_1(1+M)^\alpha}{\mu}\right)^{1/p} 4|x_2 - x_1|^{\gamma/p},$$

so (2.3) is established. □

We state for reference the following lemma [2, proposition 2.2].

Lemma 2.3. *Let $G \subseteq \mathbb{R}^2$ be compact, and suppose L satisfies (H) and the inequality*

$$L(x,u,v) \ge \alpha_0|v| + \beta_0,$$

for some constants $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$, for all $(x,u,v) \in G \times \mathbb{R}$. Let $r_0 > 0$.

Then there exist real numbers $r_1, \delta, \alpha_1, \alpha_2$, and γ satisfying $r_1 > r_0$, $\alpha_2 > \alpha_1 > 0$, and $\delta > 0$, such that for each integer $k > r_1$, there exists $H_k\colon G \times \mathbb{R} \to \mathbb{R}$ such that the following conditions hold:

(2.3.1) H_k satisfies condition (H);
(2.3.2) $H_k(x,u,v) = L(x,u,v)$ whenever $|v| \le r_0$;
(2.3.3) $H_k(x,u,v) \le L(x,u,v) - \delta$ whenever $r_1 \le |v| \le k$;
(2.3.4) $H_k(x,u,v) \le \alpha_2|v| + \gamma$;
(2.3.5) $H_k(x,u,v) \ge \alpha_1|v| + \beta_0$; and
(2.3.6) $H_k(x,u,v) \le L(x,u,v) + k^{-1}$ whenever $r_0 \le |v| \le r_1$.

We finally need a lemma which allows us to approximate Sobolev functions admissible in our obstacle problem with admissible Lipschitz functions. The result for no obstacles was given by Clarke and Vinter [1, lemma 7.1].

Lemma 2.4. *Let $f, g \in W^{1,\infty}(a,b)$ satisfy $g < f$ on (a,b), and suppose $u \in W^{1,1}(a,b)$ satisfies $g \le u \le f$ on $[a,b]$. Let $\epsilon > 0$.*

Then there exists $v \in W^{1,\infty}(a,b)$ *such that* $v(a) = u(a)$, $v(b) = u(b)$, $g \le v \le f$ *on* $[a,b]$, *and*

$$\int_a^b |\dot v(x) - \dot u(x)|\, dx < \epsilon.$$

Proof. By passing to the connected components of $\{g < u < f\}$, we may assume that $g < u < f$ on (a,b).

Find $\delta \in (0, (b-a)/2)$ satisfying

$$\delta < \frac{\epsilon}{16\left(\|\dot f\|_{L^\infty(a,b)} + \|\dot g\|_{L^\infty(a,b)} + 1\right)}$$

and such that

$$\int_E |\dot u(x)|\, dx < \frac{\epsilon}{16}$$

whenever $E \subseteq [a,b]$ has $|E| \le \delta$.

There exists a constant $\eta \in (0,\epsilon)$ such that $u - g \ge \eta$ and $f - u \ge \eta$ on $[a+\delta, b-\delta]$. So any function v satisfying $|v - u| \le \eta$ on $[a+\delta, b-\delta]$ fits the obstacles on this interval.

Choose $\tilde w \in L^\infty(a+\delta, b-\delta)$ such that

$$\int_{a+\delta}^{b-\delta} |\tilde w(x) - \dot u(x)|\, dx < \frac{\eta}{4},$$

and define

$$w(x) := \tilde w(x) + \frac{1}{b-a-2\delta}\int_{a+\delta}^{b-\delta} \dot u(t) - \tilde w(t)\, dt.$$

Then

$$\int_{a+\delta}^{b-\delta} |w(x) - \dot u(x)|\, dx \le 2\int_{a+\delta}^{b-\delta} |\tilde w(x) - \dot u(x)|\, dx < \frac{\eta}{2},$$

and

$$\int_{a+\delta}^{b-\delta} w(x)\, dx = \int_{a+\delta}^{b-\delta} \dot u(x)\, dx.$$

Defining

$$v(x) := u(a+\delta) + \int_{a+\delta}^{x} w(t)\, dt$$

for $x \in [a+\delta, b-\delta]$ then defines $v \in W^{1,\infty}(a+\delta, b-\delta)$ such that

$$\int_{a+\delta}^{b-\delta} |\dot v(x) - \dot u(x)|\, dx < \frac{\eta}{2} < \frac{\epsilon}{2}, \text{ and}$$
$$v(a+\delta) = u(a+\delta),\ v(b-\delta) = u(b-\delta).$$

Furthermore, on $[a+\delta, b-\delta]$ we have

$$|v(x) - u(x)| \le \int_{a+\delta}^{x} |\dot v(t) - \dot u(t)|\, dt = \int_{a+\delta}^{x} |w(t) - \dot u(t)|\, dt < \frac{\eta}{2} < \eta,$$

so $g \le v \le f$ on $[a+\delta, b-\delta]$.

It remains to deal with the intervals $[a, a+\delta]$ and $[b-\delta, b]$. We shall consider only $[a, a+\delta]$; the other may be treated analogously.

For $x \in [a, a+\delta]$, define

$$v(x) := \begin{cases} l_{a,a+\delta}(x) & g \le l_{a,a+\delta} \le f, \\ g(x) & l_{a,a+\delta} < g, \\ f(x) & f < l_{a,a+\delta}. \end{cases}$$

Then v satisfies $g \le v \le f$ on $[a, a+\delta]$, $v(a) = u(a)$, and $v(a+\delta) = u(a+\delta)$. Furthermore, $\dot{v} = \dot{l}_{a,a+\delta}$, $\dot{v} = \dot{f}$, or $\dot{v} = \dot{g}$ almost everywhere. Since

$$\left|\dot{l}_{a,a+\delta}\right| = \left|\frac{u(a+\delta) - u(a)}{\delta}\right| \le \delta^{-1} \int_a^{a+\delta} |\dot{u}(x)|\, dx,$$

we have that

$$\begin{aligned} \int_a^{a+\delta} |\dot{v}(x)|\, dx &\le \int_a^{a+\delta} \left(\left|\dot{l}_{a,a+\delta}\right| + |\dot{f}(x)| + |\dot{g}(x)|\right)\, dx \\ &\le \int_a^{a+\delta} |\dot{u}(x)|\, dx + \delta\left(\|\dot{f}\|_{L^\infty(a,b)} + \|\dot{g}\|_{L^\infty(a,b)}\right) \\ &\le \frac{\epsilon}{16} + \frac{\epsilon}{16} \\ &= \frac{\epsilon}{8}. \end{aligned}$$

Thus

$$\int_a^{a+\delta} |\dot{v}(x) - \dot{u}(x)|\, dx \le \int_a^{a+\delta} |\dot{v}(x)|\, dx + \int_a^{a+\delta} |\dot{u}(x)|\, dx \le \frac{\epsilon}{8} + \frac{\epsilon}{16} \le \frac{\epsilon}{4}.$$

Analogously v may be defined on $[b-\delta, b]$ so that v is Lipschitz, $v(b-\delta) = u(b-\delta)$, $v(b) = u(b)$, and v fits the obstacles on this interval, and

$$\int_{b-\delta}^{b} |\dot{v}(x) - \dot{u}(x)|\, dx \le \frac{\epsilon}{4}.$$

Having been defined now on the whole interval $[a, b]$, we have that $v \in W^{1,\infty}(a, b)$, $g \le v \le f$ on $[a, b]$, $v(a) = u(a)$, $v(b) = u(b)$, and

$$\begin{aligned} \int_a^b |\dot{v}(x) - \dot{u}(x)|\, dx &= \int_a^{a+\delta} |\dot{v}(x) - \dot{u}(x)|\, dx + \int_{a+\delta}^{b-\delta} |\dot{v}(x) - \dot{u}(x)|\, dx \\ &\quad + \int_{b-\delta}^{b} |\dot{v}(x) - \dot{u}(x)|\, dx \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} \\ &= \epsilon, \end{aligned}$$

as required. □

Proof of theorem 2.1. This is an adaptation to the case of obstacle problems of the proof of the analogous result for usual minimization problems [2]. Select $\epsilon_0 > 0$ in such a way that if $(a, A) \in G$, there exists an $l \in \partial_v L(a, A, 0)$ such that the integrand $\tilde{L}(x, u, v) := L(x, u, v) - (l, v)$ achieves its minimum in v for each $(x, u) \in [a-\epsilon_0, a+\epsilon_0] \times [A-\epsilon_0, A+\epsilon_0]$. In this case we then have that $\tilde{L}(x, u, v) \ge \alpha_0|v| + \beta_0$ for some $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$, for all such (x, u). Note that minimizers of the problem (1.1)–(1.2) with the integrands L and $\tilde{L}$ are the same, since the integral of the $(l, \dot{u})$ term is constant on all functions u with the same boundary conditions. We will choose $\delta_0 \leq \epsilon_0$ later.

Let $N \geq M$ and consider the obstacle problem (1.1)–(1.2) for $(a, A) \in G$, $(b, B) \in G$, $|a-b| \leq \delta \leq \delta_0$, $|B-A|/|b-a| \leq M$, over the class of Lipschitz functions u with $\|u\|_{W^{1,\infty}(a,b)} \leq N$, satisfying also $|u(\cdot) - A| \leq \epsilon \leq \epsilon_0$ on $[a, b]$. Solutions in such a class certainly exist; let u be one such solution. We study the regularity of u.

Consider $[x_1, x_2] \subseteq [a, b]$. We want to verify inequality (2.1), so that we may consider using lemma 2.2. The affine function l_{x_1,x_2} will not necessarily fit the obstacles, so we cannot make an immediate comparison, but the following argument due to Sychëv [4] shows us how to make the required modifications to obtain the inequalities we need. Define $w \colon [x_1, x_2] \to \mathbb{R}$ by

$$w(x) := \begin{cases} l_{x_1,x_2}(x) & g \leq l_{x_1,x_2} \leq f, \\ g(x) & l_{x_1,x_2} < g, \\ f(x) & f < l_{x_1,x_2}. \end{cases}$$

Then w fits the obstacles on $[x_1, x_2]$, $\|w\|_{W^{1,\infty}(x_1,x_2)} \leq N$, and $|w(x) - A| \leq \epsilon$ on $[x_1, x_2]$, so therefore $J(u; [x_1, x_2]) \leq J(w; [x_1, x_2])$. We estimate the difference $|J(w; [x_1, x_2]) - J(l_{x_1,x_2}; [x_1, x_2])|$ by estimating the pointwise difference

$$\left| L(x, w(x), \dot{w}(x)) - L(x, l_{x_1,x_2}(x), \dot{l}_{x_1,x_2}(x)) \right|$$

on each connected component of the set $\{x \in [x_1, x_2] : w(x) \neq l_{x_1,x_2}(x)\}$. Let (y_1, y_2) be such a component. Then $w(y_1) = l_{x_1,x_2}(y_1)$ and $w(y_2) = l_{x_1,x_2}(y_2)$, and so since w is defined on $[y_1, y_2]$ to be either identically equal to f or to g, both of which are $C^{1,\sigma}$, the mean value theorem implies that $\dot{w} = \dot{l}_{x_1,x_2}$ at some interior point of $[y_1, y_2]$. Thus by the Hölder condition on the derivatives of f and g, there exists $\tilde{c} > 0$ such that

$$|\dot{w}(x) - \dot{l}_{x_1,x_2}| \leq \tilde{c}|y_2 - y_1|^{\sigma}$$

for all $x \in (y_1, y_2)$. Therefore we have for such x that

$$|w(x) - l_{x_1,x_2}(x)| \leq \tilde{c}|y_2 - y_1|^{1+\sigma}.$$

We consider condition (H) on L applied to the compact set $\tilde{K} := \tilde{G} \times [-N, N]$, where $\tilde{G} := \{(x, u) : |x-a| \leq \epsilon_0,\ |u - A| \leq \epsilon_0,\ (a, A) \in G\}$. Then there exist $c > 0$ and $\alpha > 0$ such that

$$\begin{aligned} &\left| L(x, w(x), \dot{w}(x)) - L(x, l_{x_1,x_2}(x), \dot{l}_{x_1,x_2}(x)) \right| \\ &\leq c \Big(|w(x) - l_{x_1,x_2}(x)| + |\dot{w}(x) - \dot{l}_{x_1,x_2}(x)| \Big)^{\alpha}. \end{aligned}$$

Then

$$\begin{aligned} \left| L(x, w(x), \dot{w}(x)) - L(x, l_{x_1,x_2}(x), \dot{l}_{x_1,x_2}(x)) \right| &\leq c|x_2 - x_1|^{\alpha\sigma} \tilde{c}^{\alpha} \left(1 + |x_2 - x_1|\right)^{\alpha} \\ &\leq c\tilde{c}^{\alpha} \left(1 + |b-a|\right)^{\alpha} |x_2 - x_1|^{\alpha\sigma}. \end{aligned}$$

Therefore (2.1) holds with $c_2 = c_2(N) := c\tilde{c}^{\alpha} \left(1 + |b-a|\right)^{\alpha}$ and $\beta = \beta(N) := \alpha\sigma$, for any subinterval $[x_1, x_2] \subseteq [a, b]$.

So now consider $[x_3, x_4] \subseteq [x_1, x_2] \subseteq [a, b]$. Then there exists $k \in \mathbb{N}$ such that

$$\frac{|x_2 - x_1|}{2^k} \leq |x_4 - x_3| \leq \frac{|x_2 - x_1|}{2^{k-1}}.$$

We may then select a collection of intervals $[x_1^i, x_2^i]$ for $i = 0, \ldots, k-1$ such that $x_1^0 = x_1$, $x_2^0 = x_2$, $|x_2^i - x_1^i| = |x_2^{i-1} - x_1^{i-1}|/2$ for all i, and $[x_3, x_4] \subseteq [x_1^{k-1}, x_2^{k-1}]$, where $|x_4 - x_3| \geq |x_1^{k-1} - x_2^{k-1}|/2$. Since the assumptions (2.1), (2.2) of lemma 2.2 hold for any subintervals of $[a, b]$, as discussed above, the conclusion (2.3) applies, and we infer that

$$\begin{aligned}
&\left|\frac{u(x_2) - u(x_1)}{x_2 - x_1} - \frac{u(x_4) - u(x_3)}{x_4 - x_3}\right| \\
&\leq \left(\sum_{i=1}^{k-1}\left|\frac{u(x_2^i) - u(x_1^i)}{x_2^i - x_1^i} - \frac{u(x_2^{i-1}) - u(x_1^{i-1})}{x_2^{i-1} - x_1^{i-1}}\right|\right) \\
&\quad + \left|\frac{u(x_2^{k-1}) - u(x_1^{k-1})}{x_2^{k-1} - x_1^{k-1}} - \frac{u(x_4) - u(x_3)}{x_4 - x_3}\right| \\
&\leq \sum_{i=1}^{k} 4\left(\frac{c_2(N) + 2c_1(N)(1+N)^{\alpha(N)}}{\mu(N)}\right)^{1/p(N)} \left|\frac{x_2 - x_1}{2^{i-1}}\right|^{\gamma(N)/p(N)} \\
&\leq \sum_{i=1}^{k} 4\left(\frac{c_2(N) + 2c_1(N)(1+N)^{\alpha(N)}}{\mu(N)}\right)^{1/p(N)} \frac{1}{2^{(i-1)\gamma(N)/p(N)}}|x_2 - x_1|^{\gamma(N)/p(N)},
\end{aligned}$$

where the constants $c_1(N)$, $\gamma(N)$, $p(N)$, $\mu(N)$ correspond to applying condition (H) on $\tilde{K}$.

Finally then we obtain that for $[x_3, x_4] \subseteq [x_1, x_2]$, we have that

$$\left|\frac{u(x_2) - u(x_1)}{x_2 - x_1} - \frac{u(x_4) - u(x_3)}{x_4 - x_3}\right| \leq c(N)|x_2 - x_1|^{\gamma(N)/p(N)}, \tag{2.5}$$

where

$$c(N) := \sum_{i=1}^{\infty}\left(\frac{c_2(N) + 2c_1(N)(1+N)^{\alpha(N)}}{\mu(N)}\right)^{1/p(N)} \frac{1}{2^{(i-1)\gamma(N)/p(N)}}.$$

We therefore see that u is differentiable everywhere, and in fact from (2.5) we see that

$$|\dot{u}(y) - \dot{u}(x)| \leq 2c(N)|y - x|^{\gamma(N)/p(N)}, \tag{2.6}$$

in particular that $u \in C^{1,\gamma(N)/p(N)}$.

We now claim that for $\delta_0 > 0$ sufficiently small, we have $\|\dot{u}\|_\infty \leq M + 1$. Since $|B - A|/|b - a| \leq M$, there exists $x_0 \in [a, b]$ such that $|\dot{u}(x_0)| \leq M$. We consider the largest interval $[x_1, x_2] \subseteq [a, b]$ on which $|\dot{u}(x)| \leq M + 1$. For points x in this interval, we have, by (2.6), that

$$|\dot{u}(x) - \dot{u}(x_0)| \leq 2c(M+1)|x_2 - x_1|^{\gamma(M+1)/p(M+1)},$$

thus

$$|\dot{u}(x)| \leq M + 2c(M+1)|x_2 - x_1|^{\gamma(M+1)/p(M+1)}.$$

Therefore choosing

$$\delta_0 \leq \left(\frac{1}{2c(M+1)}\right)^{p(M+1)/\gamma(M+1)} \tag{2.7}$$

gives us δ_0 as claimed.

Gathering our information, we have proved that for $\delta \leq \delta_0$, where $\delta_0 \leq \epsilon_0$ satisfies (2.7), we have that all solutions of the obstacle problem under consideration have derivatives which are bounded in C-norm by $M+1$, and are themselves bounded in $C^{1,\gamma(M+1)/p(M+1)}$-norm, independent of $N \geq M$. Therefore solutions of the obstacle problem over the class of Lipschitz functions exist, and for each such solution u, we have that $\|\dot{u}\|_\infty \leq M+1$. It just remains to prove that solutions over the class of Lipschitz functions are the only solutions over the wider class of $W^{1,1}$ functions, the strategy of which has been previously established [1, 2].

Suppose for a contradiction that there exists an admissible $\tilde{u} \in W^{1,1}(a,b)$ such that $J(\tilde{u}) \leq J(u)$ and $\|\dot{\tilde{u}}\|_{L^\infty(a,b)} = \infty$. We recall that we may assume that there exist $\alpha_0 > 0$ and $\beta_0 \in \mathbb{R}$ such that

$$L(x,u,v) \geq \alpha_0|v| + \beta_0$$

for all $(x,u) \in [a-\delta_0, a+\delta_0] \times [A-\epsilon_0, A+\epsilon_0]$. We apply lemma 2.3 with these values of α_0, β_0, the compact set $[a-\delta_0, a+\delta_0] \times [A-\epsilon_0, A+\epsilon_0]$, and $r_0 \geq M+1$, to get functions $H_k\colon [a-\delta_0, a+\delta_0] \times [A-\epsilon_0, A+\epsilon_0] \times \mathbb{R} \to \mathbb{R}$ as in the lemma, for $k > r_1 > r_0$.

Mimicking the proof for L with the integrand H_k, we see that a solution u_k to the obstacle problem exists over the class of Lipschitz functions $u\colon [a,b] \to \mathbb{R}$ satisfying $u(a) = A$, $u(b) = B$, and $|u(x) - A| \leq \epsilon$ of the functional $\int_a^b H_k(x,u,\dot{u})$, and satisfies $\|\dot{u}_k\|_\infty \leq M+1$. Using condition (2.3.2) we see that

$$\begin{aligned}\int_a^b H_k(x,u_k,\dot{u}_k) \leq \int_a^b H_k(x,u,\dot{u}) = \int_a^b L(x,u,\dot{u}) \leq \int_a^b L(x,u_k,\dot{u}_k)\\ = \int_a^b H_k(x,u_k,\dot{u}_k),\end{aligned}$$

so $\int_a^b H_k(x,u_k,\dot{u}_k) = \int_a^b H_k(x,u,\dot{u}) = \int_a^b L(x,u,\dot{u})$, and u is a minimizer for the integrand H_k over admissible Lipschitz functions. By lemma 2.4 there exist admissible Lipschitz functions $v_i \in W^{1,\infty}(a,b)$ such that $v_i \to \tilde{u}$ in $W^{1,1}(a,b)$ and, owing to the linear growth conditions

$$\alpha_1|v| + \beta_0 \leq H_k(x,u,v) \leq \alpha_2|v| + \gamma,$$

we have as $i \to \infty$ that

$$\int_a^b H_k(x,v_i,\dot{v}_i) \to \int_a^b H_k(x,\tilde{u},\dot{\tilde{u}}).$$

Now, by our assumption on the derivative of $\tilde{u}$, we know that the set $\{|\dot{\tilde{u}}| > r_1\}$ has positive measure. Then by conditions (2.3.6), (2.3.2), (2.3.3), and (2.3.4), we see that

$$\begin{aligned}\int_a^b H_k(x,\tilde{u},\dot{\tilde{u}}) &\le \int_{\{|\dot{\tilde{u}}|\le r_1\}} \big(L(x,\tilde{u},\dot{\tilde{u}})+k^{-1}\big) + \int_{\{r_1<|\dot{\tilde{u}}|\le k\}} \big(L(x,\tilde{u},\dot{\tilde{u}})-\delta\big)\\ &\quad + \int_{\{|\dot{\tilde{u}}|>k\}} \big(\alpha_2|\dot{\tilde{u}}|+\gamma\big)\\ &\to \int_{\{|\dot{\tilde{u}}|\le r_1\}} L(x,\tilde{u},\dot{\tilde{u}}) + \int_{\{|\dot{\tilde{u}}|>r_1\}} \big(L(x,\tilde{u},\dot{\tilde{u}})-\delta\big) \quad \text{as } k\to\infty\\ &= \int_a^b L(x,\tilde{u},\dot{\tilde{u}}) - \delta\left|\{|\dot{\tilde{u}}|>r_1\}\right|.\end{aligned}$$

But on the other hand,

$$\int_a^b L(x,u,\dot{u}) = \int_a^b H_k(x,u_k,\dot{u}_k) \le \int_a^b H_k(x,v_i,\dot{v}_i)$$

for each i and each k. So

$$\begin{aligned}\int_a^b L(x,u,\dot{u}) \le \lim_{i\to\infty}\int_a^b H_k(x,v_i,\dot{v}_i) &= \int_a^b H_k(x,\tilde{u},\dot{\tilde{u}})\\ &\le \int_a^b L(x,\tilde{u},\dot{\tilde{u}}) - \delta\left|\{|\dot{\tilde{u}}|>r_1\}\right|,\end{aligned}$$

which contradicts the choice of $\tilde{u}$ as a solution over $W^{1,1}(a,b)$. This completes the proof of the theorem. □

3. Proofs of theorems 1.2 and 1.4

Proof of theorem 1.2. Let u be a solution of such an obstacle problem, and $x_0 \in [a,b]$. Suppose $x_1^n \to x_0$ and $x_2^n \to x_0$ such that $x_0 \in [x_1^n, x_2^n]$ for all $n \in \mathbb{N}$, and

$$\liminf_{n\to\infty}\left|\frac{u(x_2^n)-u(x_1^n)}{x_2^n-x_1^n}\right| \le M < \infty. \tag{3.1}$$

Then by theorem 2.1, u is C^1-regular in a neighbourhood of x_0. Hence u is C^1-regular on an open set of full measure. At other points x_0, where (3.1) does not hold, we obviously have either that $\dot{u}(x_0)=\infty$ or $\dot{u}(x_0)=-\infty$. To prove that $\dot{u}\colon [a,b]\to \bar{\mathbb{R}}$ is continuous, it suffices to show that if $\dot{u}(x_0)=\infty$, then $\dot{u}(x)\to\infty$ as $x\to x_0$, since the case of $-\infty$ follows analogously. Suppose for a contradiction that $x_n \to x_0$ and $\dot{u}(x_n) \le M < \infty$ for all $n\in\mathbb{N}$; for notational purposes we assume without loss of generality that $x_n < x_0$ for all $n\in\mathbb{N}$. Since $((u(x_0)-u(x_n))/(x_0-x_n)\to\infty$, we can infer the existence of $y_n \in [x_n, x_0]$ such that $((u(y_n)-u(x_n))/(y_n-x_n) = M+2$ for all sufficiently large $n\in\mathbb{N}$. Since $|y_n - x_n| \to 0$, theorem 2.1 implies that for sufficiently large n, the oscillations of the derivative $\dot{u}$ do not exceed 1 on the interval $[x_n, y_n]$. So $\dot{u}(y) \ge M+1$ for $y\in[x_n,y_n]$ for such n. But $\dot{u}(x_n)\le M$, which is a contradiction. So in fact we must have that $\dot{u}(x_n)\to\infty$ as $x_n\to x_0$, as required.

We are finally required to prove that the derivatives of our family of solutions form an equa-continuous family. So let $M>0$ and $\epsilon>0$ be given, and suppose $|\dot{u}(x_0)|\le M$. Consider $|y-x_0|\le\delta$, where $\delta>0$ is to be chosen below, and consider the largest interval I containing x_0 on which $|\dot{u}| \leq M+1$. By theorem 2.1, we have that

$$|\dot{u}(y) - \dot{u}(x_0)| \leq c|y - x_0|^\gamma$$

for small $|y - x_0|$. In particular if $|I| \leq (1/c)^{1/\gamma}$, then $|\dot{u}(y) - \dot{u}(x_0)| \leq 1$. If we choose

$$\delta = \delta(M, \epsilon) \coloneqq \min\left\{(1/c)^{1/\gamma}, (\epsilon/c)^{1/\gamma}\right\},$$

then we see in fact that $|y - x_0| \leq \delta$ implies that $|\dot{u}(y) - \dot{u}(x_0)| \leq \epsilon$, as required. □

Proof of theorem 1.4. The fact that an obstacle problem for sufficiently small $\eta > 0$ admits a solution u_0 in the class of Lipschitz functions can be proved as by Sychëv [4]. Moreover these solutions have derivatives bounded in modulus by $M+1$, where $M = \max\{\|\dot{f}\|_C, \|\dot{g}\|_C\}$. Therefore to prove the theorem we need to show that for an admissible non-Lipschitz function w we have

$$J(w) > J(u_0).$$

We proceed by contradiction. Assume first that the Lavrentiev phenomenon occurs (that is, for some Sobolev function w we have $J(w) < J(u_0)$). Consider a non-negative integrand with superlinear growth $\theta(v) \in C^\infty$ such that $\theta(v) = 0$ on $[-M-4, M+4]$ and $\theta_{vv} \geq 0$, where $\int_a^b \theta(\dot{w}(x))dx < \infty$. Then for sufficiently small $\mu_1 > 0$ we have $J_{\mu_1}(w) < J_{\mu_1}(u_0)$, where

$$J_{\mu_1}(u) = \int_a^b L(x, u, \dot{u}) + \mu_1 \theta(\dot{u})dx.$$

Then a solution u_{μ_1} for the obstacle problem for functional J_{μ_1} exists and is a singular (non-Lipschitz) function. By Lagrange's theorem, there exists $x_0 \in [a, b]$ such that $|\dot{u}_{\mu_1}(x_0)| \leq M+1$. Then by continuity there exists $y \in [a, b]$ such that $|\dot{u}_{\mu_1}(y)| = M+2$. If $\delta(M, \epsilon)$ is the modulus of conditional equa-continuity given by theorem 1.2 then $|\dot{u}_{\mu_1}(x)| \geq M+1$ for $|x - y| \leq \delta(M+2, 1)$. For sufficiently small η then the function u_{μ_1} does not fit the obstacles. This contradiction proves that there is no Lavrentiev phenomenon in the obstacle problem.

If there is no Lavrentiev phenomenon in the obstacle problem, but there is a singular solution in the problem, by the same arguments we can show that the singular solution does not fit the obstacles for sufficiently small η. This proves the theorem. □

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