Volume 4, Number 3, 2019, 559–572

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

SOME PROPERTIES OF A HILBERTIAN NORM FOR PERIMETER

FELIPE HERNÁNDEZ

Abstract. We investigate a relationship first described in [3] between the perimeter of a set and a related fractional Sobolev norm. In particular, we derive a new characterization of sets of finite perimeter, and demonstrate that the fractional Sobolev norm does not recover the BV norm but rather a certain quadratic integral.

In a recent paper of Jerison and Figalli [3], a relationship is developed between the perimeter of a set and a fractional Sobolev norm of its indicator function. More precisely, letting $\gamma(x)$ denote the standard Gaussian in $\boldsymbol{R}^n$, and defining the scaled Gaussian $\gamma_\varepsilon(x) = \varepsilon^{-n}\gamma(x/\varepsilon)$, Jerison and Figalli showed that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{|\log\varepsilon|}\|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} \simeq \liminf_{\varepsilon\to 0^+} \frac{1}{|\log\varepsilon|}\|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} \simeq P(E). \tag{0.1}$$

Here E is a set of finite perimeter and $\mathbf{1}_E$ is its indicator function. The definition of the $H^{1/2}$ norm that we shall use for the paper is given in Section 1. The formula is remarkable for its quadratic scaling and for its apparent Fourier-analytic nature. The motivation for writing down this expression for the perimeter actually came from a purely geometric question about characterizing convex sets in terms of there marginals.

In this paper we discuss extensions of Equation (0.1). In particular, first we discuss the validity of the result for sets E that do not necessarily have finite perimeter. Second we demonstrate a strengthening of (0.1) which in particular implies that the limit exists and that one can recover the perimeter exactly from the $H^{1/2}$ norm. This strengthening is a consequence of a much stronger result of Poliakovsky [8], and was pointed out to me by an anonymous reviewer.

A very similar quantity has appeared in the literature before, in the foundational work of Bourgain, Brezis, and Mironescu [2]. There the one-dimensional expression

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|} \int\int_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|^2}{|x-y|^2}\,dxdy$$

appears as a remark referencing an earlier unpublished work of Mironescu and Shafrir. The motivation for studying this functional came from studying Γ-limits

2010 *Mathematics Subject Classification.* 28A75,42B35.
Key words and phrases. Sets of finite perimeter, bounded variation, Fourier analysis.

of Ginzburg-Landau type functionals, as for example in [5, 6]. More recently Poliakovsky introduced a similar functional in connection to the Γ-limit of the Aviles-Gila problem [9]. Poliakovsky introduced the notion of BV^q spaces and showed that a certain nonlocal functional very similar to that appearing in Equation (0.1) captures the L^q norm of the jump set of a function.

We mention also several other works which relate perimeter and total variation to nonlocal functionals. A paper of Leoni and Spector [7] studies a fractional Sobolev expression that recovers the total variation of a function. The main difference is that the integrand in their expression scales linearly in the function u, whereas the formula (0.1) scales quadratically. Similarly, a recent paper of Ambrosio, Bourgain, Brezis, and Figalli [1] introduced a very interesting BMO-type norm which recovers the perimeter of a set. In [4] this norm was shown to also give the total variation of functions in SBV.

0.1. **Summary of Results.** The first question we investigate in this paper is what happens to sets that do not have finite perimeter. The answer comes in two parts. First, we show in Theorem 2.1 that there is a set $E \subset [0,1]^n$ for which the limit inferior in (0.1) vanishes. We do this by showing it is possible to construct a set of infinite perimeter such that, for an sequence $\varepsilon_k \to 0$ the functions $\gamma_{\varepsilon_k} * \mathbf{1}_E$ are much smoother than γ_{ε_k}. The construction is presented in Section 2.

The second part of the answer is that the limit superior does characterize sets of finite perimeter. This is proven in Section 3 using in a strong way the L^2 structure of the norm. The proof relies on the characterization of sets of finite perimeter provided in [1].

The second purpose of this paper is to bring some light to a generalization of (0.1) that was already available in the literature. This generalization is described in Section 4.

0.2. **A note on organization.** Preliminary results and basic definitions appear in Section 1. Each of the other sections can be read independently of each other, so the reader should feel free to skip to the result that is most interesting to them.

0.3. **Acknowledgements.** I am grateful to David Jerison, who suggested this problem to me and helped me to focus the investigation. Daniel Spector pointed out to me the very interesting paper [9], and a reviewer showed me the argument of Section 4 which was very helpful. The author is supported by the John and Fannie Hertz Foundation Fellowship.

1. Problem setup

First we define more clearly the Sobolev norm that we shall use. Given a smooth function $u \in \mathcal{S}(\boldsymbol{R}^n)$ with rapid decay, we define

$$\|u\|_{H^{1/2}}^2 = \int |\xi| |\widehat{u}(\xi)|^2 \, d\xi.$$

We would like to study the expression

$$\frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * u\|_{H^{1/2}}^2$$

for functions $u \in BV \cap L^\infty(\boldsymbol{R}^n)$ by decomposing the contributions from different scales. To do this, we write

$$\|\gamma_\varepsilon * u\|^2_{H^{1/2}} = \sum_{k=0}^{|\log \varepsilon|} \varepsilon^{-1} 2^{-k} \|\varphi_{\varepsilon 2^k} * u\|^2_{L^2} + O(\|u\|_{L^2})$$

with the convolution kernel φ chosen such that $\widehat{\varphi}(\xi)$ is nonnegative and

$$|\widehat{\varphi}(\xi)|^2 = |\xi| \left(|\widehat{\gamma}(\xi)|^2 - |\widehat{\gamma}(2\xi)|^2 \right). \tag{1.1}$$

Observe that $|\xi|^{-3/2}\widehat{\varphi}(\xi)$ is continuous and bounded, and that $\widehat{\varphi}(\xi)$ is smooth outside the origin. From this we deduce that f has the decay

$$|\varphi(x)| \leq C(1 + |x|)^{-\frac{3}{2}-n}.$$

In particular, $|x|\varphi \in L^1(\boldsymbol{R}^n)$. In addition, φ is smooth and satisfies the cancellation condition $\int \varphi = 0$. These three conditions are sufficient for us to prove the following elementary estimates that we will use throughout the paper.

2. An example with infinite perimeter.

In this section we show that $\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}}$ does not characterize sets of finite perimeter. Indeed we show that one can find a set with infinite perimeter for which the lim inf is zero.

Theorem 2.1. *For any $n > 0$ there exists $E \subset \boldsymbol{R}^n$ with $P(E) = \infty$ and such that*

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} = 0.$$

This will follow from the result in one dimension.

Lemma 2.2. *There exists a set $E \subset [0, 1]$ with $0 < |E| < 1$ and*

$$\liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} = 0.$$

We show now that Theorem 2.1 follows from the one-dimensional case.

Proof of Theorem 2.1 using Lemma 2.2. Choose $E \subset [0, 1]$ according to Lemma 2.2. For $n > 1$, consider the Cartesian product $E^n \subset \boldsymbol{R}^n$. That is, the indicator function can be written

$$\mathbf{1}_{E^n}(x_1, \ldots, x_n) = \mathbf{1}_E(x_1)\mathbf{1}_E(x_2) \cdots \mathbf{1}_E(x_n).$$

We can use the fact that the Gaussian separates to estimate $\gamma_\varepsilon * \mathbf{1}_{E^n}$:

$$\begin{aligned}
\|\gamma_\varepsilon * \mathbf{1}_{E^n}\|_{H^{1/2}}^2 &= \int \cdots \int \left(\sum_j \xi_j^2\right)^{1/2} \prod_{i=1}^n \left|\widehat{\gamma_\varepsilon}(\xi_i)\widehat{\mathbf{1}_E}(\xi_i)\right|^2 d\xi_i. \\
&\leq \sum_{j=1}^n \int \cdots \int |\xi_j| \prod_{i=1}^n \left|\widehat{\gamma_\varepsilon}(\xi_i)\widehat{\mathbf{1}_E}(\xi_i)\right|^2 d\xi_i. \\
&= n\|\gamma_\varepsilon * \mathbf{1}_E\|_{H^{1/2}}^2 \|\gamma_\varepsilon * \mathbf{1}_E\|_{L^2}^{2(n-1)} \\
&\leq n\|\gamma_\varepsilon * \mathbf{1}_E\|_{H^{1/2}}^2.
\end{aligned}$$

In the last step we used the fact that $|E| < 1$. □

The rest of the section will be devoted to proving Lemma 2.2.

2.1. **Plan for the construction.** The idea of the construction is to design a sequence of smooth functions ϕ_k that act as the smoothed versions of the set E at varying scales. That is, we would like

$$\gamma_\delta * \mathbf{1}_E \approx \gamma_\delta * \phi_k$$

for any $\delta \geq \delta_k$, where δ_k is a sequence of scales converging to zero. If the scales δ_k are sufficiently small, then because ϕ_k are smooth we should have

$$\frac{1}{|\log \varepsilon|}\|\gamma_{\delta_k} * \mathbf{1}_E\|_{H^{1/2}}^2 \approx \frac{1}{|\log \varepsilon|}\|\gamma_{\delta_k} * \phi_k\|_{H^{1/2}}^2 \approx 0.$$

To ensure that the smooth functions ϕ_k converge to a measurable set, we enforce that $0 \leq \phi_k \leq 1$ and that the sets $\{\phi_k = 0\}$ and $\{\phi_k = 1\}$ are strictly increasing. Moreover the functions ϕ_k face a compatibility condition whereby local averages of ϕ_{k+1} must match local averages of ϕ_k. The compatibility condition is of the form

$$\gamma_{\delta_k}\phi_{k+1} \approx \gamma_{\delta_k}\phi_k.$$

To reconcile this with our need for the set $\{\phi_{k+1} \in \{0,1\}\}$ to increase, we construct ϕ_{k+1} to be highly oscillatory compared to the scale δ_k, so that the δ_k smoothing recovers only the smoother function ϕ_k. A cartoon of the first step of the construction is given in Figure 1.

The following definition quantifies the conditions outlined above.

Definition 2.3. Let $\phi_k \in C_c^\infty((0,1))$ be a sequence of smooth functions and let $\varepsilon_k > 0$ be a decreasing sequence of scales with $\varepsilon_k \to 0$. We say that ϕ_k is a *compatible sequence* if the following properties hold:

- *Nontriviality:*

$$0 < |\{\phi_1 = 1\}|.$$

- *Convergence to a set:*

$$\begin{aligned}
|\{\phi_k \notin \{0,1\}\}| &< (0.99)^k, \\
\{\phi_k = 1\} &\subset \{\phi_{k+1} = 1\}, \\
\{\phi_k = 0\} &\subset \{\phi_{k+1} = 0\}, \qquad \text{and} \\
0 \leq &\phi_k \leq 1.
\end{aligned} \tag{2.1}$$

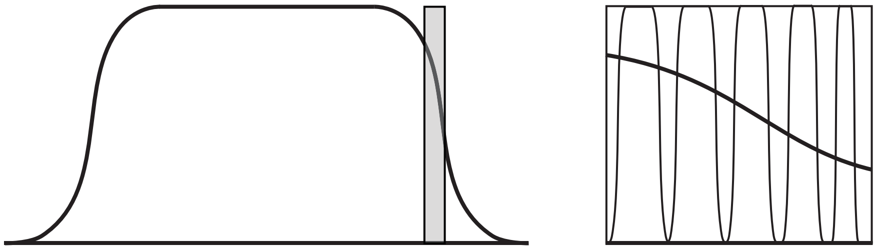

FIGURE 1. The beginning of the compatible sequence ϕ_k. On the left, a smooth function ϕ_1 is chosen which takes values in $[0,1]$. On the right, the function ϕ_2 is depicted on a magnified portion of the interval (colored in gray on the left). In this interval, ϕ_2 oscillates between 0 and 1 in such a way to preserve the local averages of ϕ_1.

- *Smoothness to scale ε_k:*

$$\frac{1}{|\log \varepsilon_k|}\|\gamma_{\varepsilon_k} * \phi_k\|^2_{H^{1/2}} < 2^{-k}. \tag{2.2}$$

- *Compatibility across scales:* One has

$$\|\gamma_r * (\phi_k - \phi_{k+1})\|_{L^2} < \varepsilon_k^3 2^{-k} \tag{2.3}$$

for all $r \geq \varepsilon_k$.

Lemma 2.4. *Let ϕ_k be a compatible sequence with scales ε_k. Then there exists a measurable set $E \subset [0,1]$ with $0 < |E| < 1$, $\phi_k \to E$ in L^p for all $p < \infty$, and*

$$\lim_{k\to\infty} \frac{1}{|\log \varepsilon_k|}\|\gamma_{\varepsilon_k} * \mathbf{1}_E\|^2_{H^{1/2}} = 0.$$

Proof. The existence of the limiting set E follows straightforwardly from the first two hypotheses on ϕ_k.

To show that $\|\gamma_{\varepsilon_k} * \mathbf{1}_E\|^2_{H^{1/2}}$ grow manageably, we write $\mathbf{1}_E$ as a telescoping sum and apply the triangle inequality:

$$\begin{aligned}\frac{1}{|\log \varepsilon_k|}\|\gamma_{\varepsilon_k} * \mathbf{1}_E\|^2_{H^{1/2}} &\leq \frac{2}{|\log \varepsilon_k|}\|\gamma_{\varepsilon_k} * \phi_k\|^2_{H^{1/2}} \\ &\quad + \frac{2}{|\log \varepsilon_k|}\left(\sum_{m=k}^{\infty}\|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|_{H^{1/2}}\right)^2\end{aligned}$$

The first term we bound using (2.2). For the terms in the sum we interpolate between L^2 and H^1,

$$\begin{aligned}\|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|_{H^{1/2}} &\leq \|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|^{1/2}_{L^2}\|\gamma_{\varepsilon_k} * (\phi_{m+1} - \phi_m)\|^{1/2}_{H^1} \\ &\leq (\varepsilon_m^3 2^{-m})^{1/2}(2\|\gamma_{\varepsilon_k}\|_{H^1})^{1/2} \\ &\leq C(\varepsilon_m^3 2^{-m})^{1/2}(\varepsilon_k)^{-3/2}\end{aligned}$$

The bound on the H^1 norm follows from the fact that $\|\phi_{m+1} - \phi_m\|_{L^1} < 2$. Since $\varepsilon_m < \varepsilon_k$ for $m > k$, this is bounded by $2^{-m/2}$, and is thus clearly summable over all $m \geq k$. □

Thus our task is reduced to showing the existence of a compatible set. This will be done inductively in the next subsection.

2.2. **Technical constructions.** In this section we prove three short facts that allow us to construct ϕ_{n+1} from ϕ_n. The first says that, in order to get the local approximation $\gamma_r * \phi_k \approx \gamma_r * \phi_{k+1}$, it suffices to demonstrate that the averages of ϕ_k and ϕ_{k+1} are equal on many short intervals. Then we show how to actually construct a function ϕ_{n+1} from ϕ_n such that the averages on short intervals are correct, with the constraint that ϕ_{n+1} takes values in $\{0,1\}$ more often. This is done with the help of a short proposition that takes care of the case of one single interval.

Proposition 2.5. *Let $\phi \in C_c^\infty((0,1))$ with $0 \leq \phi \leq 1$, $\delta > 0$ be a scale, $\varepsilon > 0$ be some tolerance, and $k > 0$ be an integer. Then for sufficiently large N we have the following: For every $\psi \in C_c^\infty((0,1))$ with $0 \leq \psi \leq 1$, if*

$$\phi(\frac{i}{N}) = N \cdot \int_{i/N}^{(i+1)/N} \psi(t)\, dt \tag{2.4}$$

for all but at most k values of $i \in [N]$, then

$$\|\gamma_r * (\phi - \psi)\|_{L^\infty} < \varepsilon.$$

for all $r > \delta$.

Proof. Let χ be the indicator function for the interval $[0,1]$, so that $(N/M)\chi_{M/N}$ is the function

$$(N/M)\chi_{M/N}(x) = \begin{cases} N/M, & 0 \leq x \leq M/N \\ 0, & else \end{cases}.$$

We will consider the function $g = (N/M)\chi_{M/N} * \psi$. We show that by choosing M large enough, and then N to be a sufficiently large multiple of M, we can enforce $\|g - \phi\|_\infty < \varepsilon/2$.

Indeed, since (2.4) holds on all but at most k intervals,

$$\left| g\left(\frac{i}{N}\right) - \frac{1}{M} \sum_{j=i}^{i+M-1} \phi\left(\frac{j}{N}\right) \right| \leq \frac{k}{M}.$$

Assuming we take M to be sufficiently large, and then N to be a sufficiently large multiple of M, we have $|g(i/N) - \phi(i/N)| < \varepsilon/10$. The definition of g also yields the Lipschitz bound $|g'(x)| \leq 2NM^{-1}$, so that we can conclude $\|g - \phi\|_\infty \leq \varepsilon/2$ as desired, provided again M is large enough.

Finally, we take N large enough that

$$\|\gamma_\delta - \chi_{M/N} * \gamma_\delta\|_{L^\infty} < \varepsilon/100$$

and

$$\|\phi - \chi_{M/N} * \phi\|_{L^\infty} < \varepsilon/100.$$

Then we conclude since

$$\begin{aligned}\|\gamma_\delta * (\phi - \psi)\|_{L^\infty} &\leq \|(\gamma_\delta - \gamma_\delta * \chi_{M/N}) * (\phi - \psi)\|_{L^\infty} \\ &\quad + \|\gamma_\delta * (\chi_{M/N} * (\phi - \psi))\|_{L^\infty} \\ &< \varepsilon.\end{aligned}$$

□

The next proposition lets us satisfy the local average condition on an interval with a function that looks more like an indicator function.

Proposition 2.6. *Let $a \in [0,1)$ be a target average. Then there exists a function $\psi \in C_c^\infty((0,1))$ satisfying $|\{\psi \in \{0,1\}| > 0.1$ and $\int_0^1 \psi = a$, and such that the sets $\{\psi = 0\}$ and $\{\psi = 1\}$ are unions of finitely many closed intervals.*

Proof. We will split into the cases $a > 1/2$ and $a < 1/2$. We begin with the case $a > 1/2$. Let σ be a smooth increasing function satisfying $\sigma(x) = 0$ for $x \leq 1/2$ and $\sigma(x) = 1$ for $x \geq 1$. k

Consider the following function $\psi_t \in C_c^\infty((0,1))$ defined for $0 < t \leq 1/2$:

$$\psi_t(x) = \begin{cases} \sigma(x/t), & x < t \\ 1, & x \in [t, 1-t] \\ \sigma((1-x)/t), & x > 1-t \end{cases}.$$

Each of ψ_t satisfy the condition $|\{\psi \in \{0,1\}\}| > 0.1$. Moreover $I(t) = \int_0^1 \psi_t$ is a continuous function in t with $I(1/2) < 1/2$ and $\lim_{t\to 0} I(t) = 1$. Thus by the intermediate value theorem we have that, for any $a \in (1/2, 1)$, there exists t such that $\int \psi_t = a$.

Now suppose $a \leq 1/2$. Observe that $I(0.1) > 1/2$, so the function $\psi = \frac{a}{I(0.1)}\psi_{1/10}$ satisfies the constraints. □

Finally we use Proposition 2.6 to construct a function satisfying the local average constraints of Proposition 2.5.

Proposition 2.7. *Let $\phi \in C_c^\infty((0,1))$ satisfy $0 \leq \phi \leq 1$, and let $N > 0$. Suppose that the sets $\{\phi = 1\}$ and $\{\phi = 0\}$ can be written as unions of at most k intervals. Then there exists $\psi \in C_c^\infty((0,1))$ satisfying the following constraints:*

- *$\{\phi = 1\} \subset \{\psi = 1\}$ and $\{\phi = 0\} \subset \{\psi = 0\}$.*
- *$|\{\psi \notin \{0,1\}\}| \leq 0.99 \cdot |\{\phi \notin \{0,1\}\}|$.*
- *The level sets $\{\psi = 0\}$ and $\{\psi = 1\}$ can each be written as a union of finitely many intervals.*
- *The function ψ satisfies the following local average constraints*

$$\phi\left(\frac{i}{N'}\right) = N' \int_{i/N'}^{(i+1)/N'} \psi(t)\,dt$$

for some $N' > N$, and on all but at most $2k$ intervals.

Proof. Choose $N' > N$ such that each interval $[i/N', (i+1)/N']$ contains at most point in $\partial\{\phi = 0\} \cup \partial\{\phi = 1\}$. Let I_i be the interval $[i/N', (i+1)/N']$. Let A be the set of indices such that their intervals contain such an endpoint, that is

$$A := \{i; I_i \cap (\partial\{\phi = 0\} \cup \partial\{\phi = 1\}) \neq \emptyset\}.$$

We will define functions $F_i \in C^\infty([0,1])$ for $0 \leq i \leq N$ and set

$$\psi(x) = F_{\lfloor N\cdot x \rfloor}(\mathrm{frac}(N \cdot x))$$

where $\mathrm{frac}(x)$ denotes the fractional part of x. We split the choice of F_i into three cases.

Case I: $i \notin A$ *and* $I_i \subset \{\phi = 1\}$. In this case we simply set $F_i = 1$.

Case II: $i \notin A$ *and* $\phi(i/N') < 1$. Simply use Proposition 2.6 to choose F_i such that $\int F_i = \phi(i/N')$.

Case III: $i \in A$. Choose any F_i subject to the constraints $0 \leq F_i \leq 1$, $\phi \in C_c^\infty$, $\{\phi = 1\} \subset \{\psi = 1\}$ and $\{\phi = 0\} \subset \{\psi = 0\}$.

Our choice of N' guarantees that the above three cases are exhaustive. The resulting function ψ satisfies all the conditions of the lemma. □

2.3. **The iterative algorithm.** In this section we combine the main lemmas above to inductively define a compatible sequence ϕ_n.

Proof of Lemma 2.2. Using Lemma 2.4, it suffices to construct a compatible sequence. We begin with any valid function $\phi_1 \in C_c^\infty((0,1))$ satisfying the nontriviality constraint $|\{\phi = 1\}| > 0$ and such that the sets $\{\phi_1 = 1\}$ and $\{\phi_1 = 0\}$ are finite unions of closed intervals. Since ϕ_1 is smooth, and thus in $H^{1/2}$ we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \phi_1\|_{H^{1/2}}^2 \to 0,$$

so we may choose ε_1 small enough to satisfy the smoothness constraint (2.2).

We now induct on k. Suppose that the sets $\{\phi_k = 1\}$ and $\{\phi_k = 0\}$ are unions of at most K intervals. Applying Proposition 2.5 with $\phi = \phi_k$, $\delta = \varepsilon_k$, $\varepsilon = \varepsilon_k^3 2^{-k}$, and $k = K$, we obtain a value N_k for which the interval average constraints (2.4) imply the compatibility bound (2.3). We can then use Proposition 2.7 with $\phi = \phi_k$ and N_k to construct ϕ_{k+1}. The function ϕ_{k+1} is smooth, so we can find ε_{k+1} to satisfy (2.2). Moreover, we have that the sets $\{\phi_{k+1} = 1\}$ and $\{\phi_{k+1} = 0\}$ are finite unions of closed intervals, so the induction is closed. □

3. Characterizing Sets of Finite Perimeter

3.1. **The lower bound.** In this section we prove the following characterization of sets of finite perimeter.

Theorem 3.1. *Let* $E \subset \boldsymbol{R}^n$ *be a set with* $P(E) = \infty$. *Then*

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \mathbf{1}_E\|_{H^{1/2}}^2 = \infty.$$

The proof of this theorem goes through an analysis of the smoothed functions $\gamma_\varepsilon * \mathbf{1}_E$. The difficulty is that these functions may be so smooth that $\|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}}$ could be very small. However, using a characterization of sets of finite perimeter due to Ambrosio, Bourgain, Brezis, and Figalli, we will be able to see that

$$\varepsilon^{-1}\|\mathbf{1}_E - \gamma_\varepsilon * \mathbf{1}_E\|^2_{L^2}$$

grows to be large if E is a set of infinite perimeter [1]. Decomposing the difference $\mathbf{1}_E - \gamma_\varepsilon * \mathbf{1}_E$ over many scales in the Fourier domain, we will see that there must be at least some wavelength $\varepsilon' < \varepsilon$ that contributes significantly to the difference. It is at this wavelength that $\|\gamma_{\varepsilon'} * \mathbf{1}_E\|^2_{H^{1/2}}$ is large.

To make this analysis convenient, we will make our smoothing kernels compactly supported in Fourier space. That is, let $\psi \in C_c^\infty(\boldsymbol{R})$ have support in $[-1,1]$ with $\psi(\xi) = 1$ for $|\xi| < 1/2$. Then by construction, the differences $(\psi_r - \psi_{r/2}) * \mathbf{1}_E$ and $(\psi_h - \psi_{h/2}) * \mathbf{1}_E$ are orthogonal so long as $r \notin (h/4, 4f)$. From this we deduce the following approximate orthogonality property:

$$\|\psi_r * \mathbf{1}_E - \mathbf{1}_E\|^2_{L^2} \le C \sum_{k=0}^{\infty} \|\psi_{r/2^k} * \mathbf{1}_E - \psi_{r/2^{k+1}} * \mathbf{1}_E\|^2_{L^2}. \tag{3.1}$$

Next we demonstrate the connection between these differences and the quantity $\|\gamma_\varepsilon * \mathbf{1}_E\|_{H^{1/2}}$ via the kernel described in (1.1).

Proposition 3.2. *With φ as defined by Equation* (1.1)*, we have*

$$\|(\psi_r - \psi_{r/2}) * \mathbf{1}_E\|^2_{L^2} \le C\|\varphi_r * \mathbf{1}_E\|^2_{L^2}$$

for any measurable set $E \subset \boldsymbol{R}^n$.

Proof. By Plancherel's theorem and homogeneity it suffices to check that there exists some constant C such that

$$\left|\widehat{\psi}(\xi) - \widehat{\psi}(\xi)\right|^2 \le C\,|\widehat{\varphi}(\xi)|^2$$

for all $\xi \in \boldsymbol{R}^n$. By construction of φ, the left hand side has support in the annulus $\frac{1}{4} \le r|\xi| \le 1$. The result follows from the compactness of the annulus and the positivity of $\widehat{\varphi}$ on $\boldsymbol{R}^n \setminus \{0\}$. □

Lemma 3.3. *Suppose that $E \subset \boldsymbol{R}^n$ satisfies*

$$\limsup_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|}\|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} < \infty.$$

Then

$$\liminf_{n\to\infty} 2^n \sum_{k=n}^{\infty} \|\varphi_{2^{-k}} * \mathbf{1}_E\|^2_{L^2} < \infty. \tag{3.2}$$

Proof. Using the definition of φ, the condition on E implies that there exists C such that for every integer $n > 0$,

$$\sum_{k=1}^{n} 2^k \|\varphi_{2^{-k}} * \mathbf{1}_E\|^2_{L^2} < Cn. \tag{3.3}$$

We first use this inequality to bound the infinite sum in (3.2) in terms of a finite one. Indeed, by grouping the infinite sum into dyadic pieces and applying the bound above in each piece, we have

$$\begin{aligned}2^n \sum_{k=2n}^{\infty} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 &\le 2^n \sum_{N=1}^{\infty} \sum_{k=n2^N}^{n2^{N+1}} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 \\ &\le 2^n \sum_{N=1}^{\infty} 2^{-n2^N} \sum_{k=n2^N}^{n2^{N+1}} 2^k \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 \\ &\le C2^n \sum_{N=1}^{\infty} 2^{-n2^N} n2^{N+1}\end{aligned}$$

which is clearly bounded independently of n. Thus, it suffices to show that

$$\liminf_{n\to\infty} 2^n \sum_{k=n}^{2n} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 < \infty.$$

We do this by finding, for each $n > 0$, a suitable scale $n \le m \le 2n$. To see that at least one m suffices we average over all such scales:

$$\begin{aligned}\frac{1}{n} \sum_{m=n}^{2n} 2^m \sum_{k=m}^{2m} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 &\le \frac{1}{n} \sum_{k=n}^{4n} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 \sum_{m=0}^{k} 2^m \\ &\le \frac{2}{n} \sum_{k=n}^{4n} 2^k \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2.\end{aligned}$$

The last sum is bounded by using again (3.3). Thus it is possible to find $m > n$ such that

$$2^m \sum_{k=m}^{2m} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2$$

is bounded independent of n. □

The following lemma is a characterization of sets of finite perimeter that appears in [1] that we will rely on. A δ-cube is any cube in $\boldsymbol{R}^n$ with side length δ.

Lemma 3.4 ([1, Lemma 3.2])**.** *Let $K > 0$ and $E \subset \boldsymbol{R}^n$ be a measurable set with $P(E) = \infty$. Then there exists $\delta_0 = \delta_0(K, A)$ such that for every $\delta < \delta_0$ it is possible to find a disjoint collection $\mathcal{U}_\delta$ of δ-cubes Q' with $\#\mathcal{U}_\delta > K\delta^{-n+1}$ and*

$$2^{-n-1} \le \frac{|Q' \cap E|}{|E|} \le 1 - 2^{-n-1}$$

for every $Q' \in \mathcal{U}_\delta$.

Proposition 3.5. *Let $Q = (-\frac{1}{2}, \frac{1}{2})^n \subset \boldsymbol{R}^n$ be the unit cube. Suppose that $E \subset \boldsymbol{R}^n$ is a measurable set with $2^{-n-1} \le |E \cap Q| \le 1 - 2^{-n-1}$. Then there exists constants $c_n, r_n > 0$ such that*

$$\|\psi_{r_n} * \mathbf{1}_E - \mathbf{1}_E\|_{L^2(Q)}^2 > c_n.$$

Proof. We choose r_n so small that

$$\|\psi_{r_n} * \mathbf{1}_Q - \mathbf{1}_Q\|_{L^1} < 2^{-n-2}.$$

With this choice for r_n,

$$\begin{aligned}\left|\int_Q \psi_{r_n} * \mathbf{1}_E(x)\,dx - |E \cap Q|\right| &= \left|\int \mathbf{1}_E(x)\left(\psi_{r_n} * \mathbf{1}_Q(x) - \mathbf{1}_Q(x)\right)\,dx\right| \\ &< 2^{-n-2}.\end{aligned}$$

It follows from the continuity of $\psi_{r_n} * \mathbf{1}_E$ that for some point $x_0 \in Q$, $\psi_{r_n} * \mathbf{1}_E(x_0) \in (2^{-n-2}, 1 - 2^{-n-2})$. Since $\|\nabla \psi_{r_n}\|_{L^\infty} < r_n^{-1}$, one has

$$2^{-n-2} \le \psi_{r_n} * \mathbf{1}_E(y) < 1 - 2^{-n-2}$$

for any $|y - x_0| < 2^{-n-2} r_n$. In particular,

$$|\psi_{r_n} * \mathbf{1}_E(y) - \mathbf{1}_E(y)| > 2^{-n-2}$$

for $y \in B_{2^{-n-2}r_n}$. The claim follows upon integrating the above bound over $B_{2^{-n-2}r_n} \cap Q$. □

The above lemmas combine in a straightforward manner to prove our main result for this section.

Proof of Theorem 3.1. Suppose that $E \subset \boldsymbol{R}^n$ is a set with

$$\limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} < \infty.$$

We will then show that

$$\liminf_{\delta > 0} \delta^{-1} \|\psi_\delta * \mathbf{1}_E - \mathbf{1}_E\|^2_{L^2} < \infty. \tag{3.4}$$

To show that this implies that E is a set of finite perimeter, let $\delta > 0$ be very small and such that

$$\delta^{-1} \|\psi_\delta * \mathbf{1}_E - \mathbf{1}_E\|^2_{L^2} < C.$$

Consider a collection $\mathcal{U}_{\delta/r_n}$ of δ/r_n-cubes such that $|Q' \cap E|/|Q'| \in (2^{-n-1}, 1-2^{-n-1})$ for all $Q' \in \mathcal{U}_\delta$. Appropriately scaling the conclusion of Proposition 3.5,

$$\|\psi_\delta * \mathbf{1}_E - \mathbf{1}_E\|^2_{L^2(Q')} > \delta^n c_n$$

for all $Q' \in \mathcal{U}_\delta$. In particular,

$$\delta^n c_n \# \mathcal{U}_\delta < \|\psi_\delta * \mathbf{1}_E - \mathbf{1}_E\|^2_{L^2} < \delta C.$$

Thus $\#\mathcal{U}_\delta < K\delta^{1-n}$ for some K. Since this holds for arbitrarily small δ, it follows from Lemma 3.4 that $P(E) < \infty$.

Now we prove (3.4). Indeed, according to Lemma 3.3, we may find $C > 0$ and arbitrarily large n such that

$$2^n \sum_{k=n}^{\infty} \|\varphi_{2^{-k}} * \mathbf{1}_E\|^2_{L^2} < C.$$

Setting $\delta = 2^{-n}$ and applying Proposition 3.2 and the orthogonality property (3.1) we obtain

$$\begin{aligned}\|\psi_{2^{-n}} * \mathbf{1}_E - \mathbf{1}_E\|_{L^2}^2 &\le C \sum_{k=n}^{\infty} \|\psi_{2^{-k}} * \mathbf{1}_E - \psi_{2^{-k-1}} * \mathbf{1}_E\|_{L^2}^2 \\ &\le C \sum_{k=n}^{\infty} \|\varphi_{2^{-k}} * \mathbf{1}_E\|_{L^2}^2 \\ &\le 2^{-n} C\end{aligned}$$

as desired. □

4. Generalization to bounded functions of bounded variation

In this section we derive a generalization of (0.1) as a consequence of the work of Poliakovsky [8]. The argument provided here was suggested by an anonymous referee. Applying [8, Proposition 3.2] with $W \in C^1(\boldsymbol{R}, \boldsymbol{R})$ defined by $W(x) = |x|^2$, we obtain the following.

Proposition 4.1. *Let $\eta \in C_c^2(\boldsymbol{R}^n)$ satisfy $\int \eta = 0$, and let $u \in BV \cap L^\infty(\boldsymbol{R}^n)$. With u_ε defined as*

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{\boldsymbol{R}^n} \eta(\frac{y-x}{\varepsilon}) u(y)\, dy,$$

we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int |u_\varepsilon(x)|^2\, dx = \int_{J_u} \left\{ \int_{-\infty}^{+\infty} |\Gamma(t,x)|^2\, dt \right\} d\mathcal{H}^{n-1}(x) \tag{4.1}$$

where

$$\Gamma(t,x) = \left(\int_{-\infty}^{t} P(s,x)\, ds \right) u^-(x) + \left(\int_{t}^{+\infty} P(s,x)\, ds \right) u^+(x)$$

with

$$P(t,x) = \int_{H^0_{\boldsymbol{\nu}(x)}} \eta(t\boldsymbol{\nu}(x) + y)\, d\mathcal{H}^{n-1}(y)$$

where

$$H^0_{\boldsymbol{\nu}} = \{ y \in \boldsymbol{R}^n \mid y \cdot \boldsymbol{\nu} = 0 \},$$

$\boldsymbol{\nu}(x)$ is the jump vector of u at x, and J_u is the jump set of u.

The formula 4.1 can be specialized further to the case that the mollifier η is radial, in which case we obtain

Corollary 4.2. *Let $\eta \in C_c^2(\boldsymbol{R}^n)$ be a* radial *mollifier $\eta(z) := \eta_0(|z|)$ satisfying $\int \eta = 0$, and let $u \in BV \cap L^\infty(\boldsymbol{R}^n)$. With u_ε defined as*

$$u_\varepsilon(x) = \varepsilon^{-n} \int_{\boldsymbol{R}^n} \eta(\frac{y-x}{\varepsilon}) u(y)\, dy,$$

we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int |u_\varepsilon(x)|^2\, dx = C_\eta \int_{J_u} |u^+(x) - u^-(x)|^2\, d\mathcal{H}^{n-1}(x) \tag{4.2}$$

where

$$C_\eta := \int_{-\infty}^{\infty} \left(\int_t^\infty P_0(s)\,ds \right)^2 dt = -\int_{-\infty}^{\infty} 2 \left(\int_t^\infty P_0(s)\,ds \right) P_0(t) t\,dt$$

with

$$P_0(t) = \int_{\boldsymbol{R}^{n-1}} \eta_0(\sqrt{t^2+w^2})\,dw.$$

Moreover, one can remove the constraint in Corollary (4.2) that η is compactly supported so long as $\eta(z)(|z|+1) \in L^1$. To see this, for each R let $\eta_R(z)$ satisfy $\eta_R(z) = \eta(z)$ for $|z| < R$, $\operatorname{supp} \eta_R(z) \subset B_{2R}$, and $\int \eta_R = 0$. Define now

$$u_\varepsilon(x) = \int_{\boldsymbol{R}^n} \eta(z) u(x+\varepsilon z)\,dz$$
$$u_{\varepsilon,R}(x) = \int_{\boldsymbol{R}^n} \eta_R(z) u(x+\varepsilon z)\,dz.$$

Then

$$\begin{aligned}
\varepsilon^{-1} \int |u_\varepsilon(x) - u_{\varepsilon,R}(x)|\,dx &= \int \varepsilon^{-1} \left| \int (\eta(z) - \eta_R(z)) u(x+\varepsilon z)\,dz \right| dx \\
&\le \varepsilon^{-1} \int \int |\eta(z) - \eta_R(z)||u(x) - u(x+\varepsilon z)|\,dz\,dx \\
&\le \varepsilon^{-1} \int \int |\eta(z) - \eta_R(z)||u(x) - u(x+\varepsilon z)|\,dz\,dx \\
&\le C\|u\|_{BV} \int |z||\eta(z) - \eta_R(z)|\,dz.
\end{aligned}$$

The right hand side goes to zero because $|z|\eta(z) \in L^1$, and we have used the inequality

$$\int_{\boldsymbol{R}^n} \frac{1}{\varepsilon|z|} |u(x+\varepsilon z) - u(x)|\,dz \le C\|u\|_{BV}.$$

As an application, choosing for η the kernel (1.1), we may derive that

$$\lim_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} \|\gamma_\varepsilon * \mathbf{1}_E\|^2_{H^{1/2}} = C(n)P(E),$$

so in particular the limit exists.

References

[1] L. Ambrosio, J. Bourgain, H. Brezis, and A. Figalli. *BMO-type norms related to the perimeter of sets*, Communications on Pure and Applied Mathematics **69** (2016), 1062–1086.

[2] J. Bourgain, H. Brezis, and P. Mironescu. *Another look at Sobolev spaces*, 2001.

[3] A. Figalli and D. Jerison, *How to recognize the convexity of a set from its marginals* Journal of Functional Analysis **266** (2014), 1685–1701.

[4] N. Fusco, G. Moscariello, and C. Sbordone. *A formula for the total variation of SBV functions* Journal of Functional Analysis **270** (2016) 419–446.

[5] M. Kurzke *Boundary vortices in thin magnetic films*, Calculus of Variations and Partial Differential Equations **26** (2006) 1–28.

[6] M. Kurzke *A nonlocal singular perturbation problem with periodic well potential* ESAIM: Control, Optimization, and Calculus of Variations **12** (2006) 52–63.

[7] G. Leoni and D. Spector, *Characterization of Sobolev and BV spaces* Journal of Functional Analysis **261** (2011) 2926–2958.
[8] A. Poliakovsky *A general technique to prove upper bounds for singular perturbation problems* Journal d'Analyse Mathématique **104** (2008) 247–290.
[9] A. Poliakovsky *Jumps detection in Besov spaces via a new BBM formula. Applications to Aviles-Gila type functionals*, 2017.

Manuscript received October 23 2017
revised June 27 2018

F. Hernández
450 Serra Mall, Stanford, CA
E-mail address: `felipeh@alum.mit.edu`