



AXISYMMETRIC 3D EULER- α EQUATIONS WITHOUT SWIRL: EXISTENCE, UNIQUENESS, AND RADON MEASURE VALUED SOLUTIONS

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ABSTRACT. The global existence of weak solutions for the three-dimensional axisymmetric, without swirl, Euler- α (also known as Lagrangian-averaged Euler- α) equations of an ideal second-grade viscoelastic fluid model is established, whenever the initial unfiltered velocity v_0 satisfies $\frac{\nabla \times v_0}{r}$ is a finite Radon measure with compact support. Furthermore, the global existence and uniqueness, is also established in this case provided $\frac{\nabla \times v_0}{r} \in L^p_c(\mathbb{R}^3)$, with $p > \frac{3}{2}$. It is worth mention that no such results are known to be available, so far, for the three-dimensional Euler equations of ideal incompressible flows, in particular, in the Radon measure case, which covers the axisymmetric vortex sheet solutions.

1. INTRODUCTION

The Euler- α (also known as the Lagrangian-averaged Euler- α) equations, introduced by Holm, Marsden and Ratiu [23], [24] (see also [22]) are an inviscid regularization of the Euler equations that describe the motion of an ideal incompressible fluid. Moreover, these same equations also govern the motion of inviscid (non-viscous, ideal) three-dimensional second-grade incompressible non-Newtonian fluid (see, e.g., [18], [19], [45]). The equations are given by the system:

$$(1.1) \quad \begin{cases} \partial_t v + u \cdot \nabla v + \sum_j v_j \nabla u_j + \nabla p = 0, \\ v = (1 - \alpha^2 \Delta)u, \\ \operatorname{div} u = 0 \end{cases}$$

in \mathbb{R}^3 , where u, p represent the velocity vector field and the pressure of the fluid, respectively. The length scale $\alpha > 0$ is the regularization parameter, and one recovers, formally, the Euler equations when $\alpha = 0$.

The initial data for (1.1) are imposed as

$$(1.2) \quad u(t = 0, x) = u_0(x).$$

2010 *Mathematics Subject Classification.* 76B47, 35Q30.

Key words and phrases. Euler- α equations, ideal second-grade viscoelastic fluids, Lagrangian-averaged Euler- α equations, axisymmetric fluids, existence and uniqueness, axisymmetric vortex sheets.

Denoting by $q = \operatorname{curl}v$, the vorticity, it follows from (1.1) that q satisfies the system:

$$(1.3) \quad \begin{cases} \partial_t q + u \cdot \nabla q = q \cdot \nabla u, \\ u = K_\alpha * q, \\ q(t=0, x) = q_0(x) \end{cases}$$

in \mathbb{R}^3 , where K_α is the integral kernel of the inverse of the operator $(1 - \alpha^2 \Delta) \operatorname{curl}$, which is the curl of the Bessel potential. Equation (1.3) resembles the equation of motion of the vorticity in the three-dimensional Euler equations. However, the vorticity stretching term, which is the main obstacle for proving the global regularity for the three-dimensional Euler equations, is replaced above by the milder term $q \cdot \nabla u$. Despite this mollification of the vorticity stretching term the question of global regularity for the three-dimensional Euler- α is still, as in the case of the three-dimensional Euler equations, a challenging open problem.

In the following, $\operatorname{curl}u_0$ and $\operatorname{curl}v_0$ will denote the vorticity of u_0 and of $v_0 = (1 - \alpha^2 \Delta)u_0$ respectively. $\operatorname{curl}v_0$ is also called the initial unfiltered vorticity, while $\operatorname{curl}u_0$ is called the filtered initial vorticity.

There has been a lot of mathematical progress made on the Euler- α equations recently. For the two-dimensional case, Kouranbaeva and Oliver [32] obtained global existence and uniqueness of (1.1)-(1.2) for the initial unfiltered vorticity of class L^2 . The artificial viscosity method is applied in [32]. Furthermore, Oliver and Shkoller [46] proved the global existence and uniqueness of weak solutions for initial unfiltered vorticity $\operatorname{curl}v_0$ in $M(\mathbb{R}^2)$, which is the space of finite Radon measures. Ambrose, Lopes Filho and Nussenzweig Lopes [1] discussed with the weak solutions of the two-dimensional Euler- α equations in the full. The convergence results to Euler equations can be referred to [9, 39, 40] for different boundary conditions when α vanishes. For the three-dimensional axisymmetric case, Busuioc and Ratiu [10] proved the global existence and uniqueness of classical solution for the three-dimensional axisymmetric Euler- α equations without swirl. Specifically, in the case of the whole space \mathbb{R}^3 , the restrictions on the initial data in [10] are: $u_0 \in H^3(\mathbb{R}^3)$, $\operatorname{curl}v_0/r \in L^2(\mathbb{R}^3)$ and $\operatorname{curl}v_0 \in L^p(\mathbb{R}^3)$, for some $p \in [1, 2]$. For the general three-dimensional case, a blow-up criterion of the smooth solutions, in the spirit of the Beale-Kato-Majda [6], was presented in [26] (see also [47]). The local existence and a blow-up criterion in Besov spaces for the 3D Lagrangian averaged Euler equations was given in [36]. Zang [53] proved the local well-posedness of Euler- α equations with periodic boundary conditions and the convergence to the solutions of Euler equations in H^s sense, depending on the regularity of the initial data.

Formally, as we have already mentioned above, when $\alpha = 0$ in (1.1), the Euler- α equations become the classical Euler equations. A natural question is whether solutions of the Euler equations can be approximated properly by those of the corresponding Euler- α equations, especially for the vortex-sheets initial data. In particular, it has recently been shown in [35] that the solutions of three-dimensional Euler- α equations converge to the corresponding strong solutions of the three-dimensional Euler equations on the interval of existence of the latter. As a consequence, it is also shown that the life span of the solution of the three-dimensional

Euler- α is longer than that of one of the three-dimensional Euler equations corresponding to the same regular initial data. It is well-known, however, that when the initial data are a vortex-sheets data, *i.e.*, the initial vorticity is a finite Radon measure and the initial velocity is locally square-integrable, the two-dimensional time-dependent Euler equations have global (in time) weak solutions when the initial vorticity $\text{curl}u_0$ is of one-sign. This result was first proved by Delort [15] by regularizing the initial data to construct the approximate solutions (see also [38] for a slight generalization). Then the result of Delort was proved by different approaches (see, e.g., [21], [37], [41], [42], [48]). Specifically, the Navier-Stokes approximations were used in [41], [44], and the vortex-method approximations were applied in [37]. Notably, the Euler- α equations were proposed as an inviscid approximation to the Euler equations both in the two- and three-dimensional cases (see, e.g., the convergence results and error estimates in [35]). Specifically, it was shown in [2] and [3] the global regularity of the vortex sheet problem of the 2D Euler- α equations for a wider class of vortex-sheet and that their solutions exists globally. Moreover, it was shown that these solutions converge to one of the Delort solutions of the vortex sheet for the 2D Euler equations, as a subsequence of the regularization parameter $\alpha_j \rightarrow 0$. In addition, it was also shown in [2] and [3] that the solutions of the Birkhoff-Rott- α model for the vortex-sheet dynamics exists globally in time, and that it is equivalent to the unique weak solution of the two-dimensional Euler- α equations. In particular, it was also pointed out the role played by the regularizing parameter α in inhibiting the Kelvin-Helmholtz instability. Numerical implementation of the Birkhoff-Rott- α , and comparison to other numerical regularization of the Birkhoff-Rott (see, e.g., [33]), is reported in [25]. However, the vortex-sheet problem for the three-dimensional Euler equations remains unsolved even for the case of one-signed measure. In [5], it has been shown, by an example of 3D shear flow of Euler equations, the existence, and persistence for all time, of singular vortex sheet solutions. This is a fundamentally different behavior than in the 2D case (see, e.g., [11], [17], [33], [34], [52]). For some other works concerning the vortex sheet problem the reader is referred to [4], [7], [12], [13], [16], [28]-[30], [35].

The purpose of this paper is to prove the global existence and uniqueness of weak solutions for the three-dimensional axisymmetric Euler- α equations without swirl. Specifically, we will first obtain the global existence and uniqueness of the solutions when the initial unfiltered vorticity $\frac{\text{curl}v_0}{r}$ belongs to $L_c^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$, which is the usual L^p Lebesgue space with compact support. Then we will prove the existence of global weak solutions when the initial unfiltered vorticity $\frac{\text{curl}v_0}{r}$ belongs to $M_c(\mathbb{R}^3)$, which is the space of finite Radon measures with compact support; by this we improve the results reported in [10]. The uniqueness of the solutions is still not clear in the case of weak Radon measures valued solutions. In our analysis, the filtered velocity $u(t, x)$ will be recovered from the unfiltered vorticity $q = \text{curl } v$ by the expression $u = K_\alpha * q$, where K_α is the integral kernel of the inverse of $(1 - \alpha^2 \Delta)\text{curl}$ (see (3.3) for more details). As in [2], [3] and [46], properties of the kernel K_α near the origin and at the infinity would be essential to apply the singular integral estimates approach. More precisely, when $\frac{\text{curl}v_0}{r} \in L_c^p(\mathbb{R}^3)$ with $p > \frac{3}{2}$, we will be able to prove that the velocity $u(t, x)$ is Lipschitz continuous with respect to the space variables and uniformly continuous with respect to the time variable

so that we can prove the global existence and uniqueness of the solution. In the case where the initial unfiltered vorticity $\frac{\text{curl}v_0}{r}$ belongs to $M_c(\mathbb{R}^3)$ we obtain the global existence of the weak solutions. This is done by establishing appropriate bounds for the gradient and the Hessian of the approximate solutions and using standard compactness arguments. It should be noted that in this case the Lipschitz continuity of $u(t, x)$ with respect to the spatial variables and the uniform continuity with respect to the time variable are still open.

We will state our main results in Section 2 and the proof of our main results will be given in Section 3. Notably, an earlier version of this manuscript was posted on [27].

2. MAIN RESULTS

In the cylindrical coordinates r, θ, z , the velocity u is written as $u = u_r e_r + u_\theta e_\theta + u_z e_z$, where $e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)$, $e_\theta = (\frac{-x_2}{r}, \frac{x_1}{r}, 0)$, $e_z = (0, 0, 1)$, $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$. Axisymmetric flows without swirl are solutions for which the azimuthal (angular) component of the velocity field satisfies $u_\theta \equiv 0$, and u_r, u_z are independent of θ . In this case, we also have the unfiltered velocity $v = v_r e_r + v_z e_z$ in the cylindrical coordinate systems. It should be mentioned that the space of axisymmetric flows is invariant under the solution of three-dimensional Euler- α equations, as in the case of the Euler equations.

Let $q = \text{curl } v$ be the unfiltered vorticity. Then only the azimuthal (angular) component of q is non-zero in the cylindrical system, i.e.,

$$(2.1) \quad q = q^\theta(t, r, z)e_\theta,$$

where $q^\theta = \partial_z v_r - \partial_r v_z$. It follows that q solves equation (1.3) and direct calculations show that

$$(2.2) \quad \partial_t \left(\frac{q^\theta}{r} \right) + u \cdot \nabla \left(\frac{q^\theta}{r} \right) = 0.$$

That is, the scalar function q^θ/r satisfies the transport equation. This is a key and an important property of the invariant family of axisymmetric flows without swirl (see also [16], [28], [29], [20], [51]). As observed above, for this invariant family of flows the vorticity stretching term, i.e. the right-hand side of equation (1.3), $q \cdot \nabla u \equiv 0$. Consequently, it follows, formally, from the transport equation (2.2) that

$$(2.3) \quad \left\| \frac{q^\theta}{r} \right\|_{L^r} = \left\| \frac{q_0^\theta}{r} \right\|_{L^r},$$

for $r \in [1, \infty]$.

Before stating our main results, we introduce the following definition of weak solutions. Let \mathcal{G} be the group of all homeomorphisms ϕ of \mathbb{R}^3 which preserve the Lebesgue measure.

Definition 2.1 (The case of $L^p(\mathbb{R}^3)$ ($p \geq 1$)). Let $T > 0$, the vector field $u \in C([0, T]; Lip(\mathbb{R}^3))$ and $q = \text{curl } v \in L^p(\mathbb{R}^3)$ with $p \geq 1$ are said to be a weak

solution of (1.3) if there exists a unique Lagrangian trajectory $y(t, x) \in C([0, T]; \mathcal{G})$ satisfying

$$(2.4) \quad y(t, x_0) = x_0 + \int_0^t u(s, y(s, x_0)) ds \quad x_0 \in \mathbb{R}^3,$$

$$(2.5) \quad u(t, x) = K_\alpha * q,$$

$$(2.6) \quad u(t = 0, x) = u_0(x),$$

and the first equation of (1.3) (i.e., equation (2.2)) is satisfied in the sense of distributions. In (2.5), K_α is, as before, the integral kernel representation of the inverse operator of $(1 - \alpha^2 \Delta)\text{curl}$ (see (3.3) for more details).

Remark 2.1 For the axisymmetric Euler- α equations without swirl, the first equation of (1.3) becomes (2.2) and in Definition 2.1 it should be satisfied in the following sense:

$$(2.7) \quad \int_{\mathbb{R} \times \mathbb{R}^3} (\partial_t \varphi + u \cdot \nabla \varphi) \frac{q^\theta}{r} dx dt = 0$$

for any $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^3)$.

Definition 2.2 (The case of $M(\mathbb{R}^3)$). Let $T > 0$, and let $\text{curl}v_0/r \in M_c(\mathbb{R}^3)$. The vector field $u \in L^\infty([0, T] \times \mathbb{R}^3)$ is said to be a weak solution of (1.1) if

- (1) $\nabla u \in L^2([0, T]; L^2_{loc}(\mathbb{R}^3)); D^2 u \in L^1([0, T]; L^1_{loc}(\mathbb{R}^3))$.
- (2) For every test function $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ with $\text{div} \varphi = 0$, equation (1.1) is satisfied in the following sense

$$(2.8) \quad \int_{[0, T] \times \mathbb{R}^3} [u(t, x) \cdot (1 - \alpha^2 \Delta) \partial_t \varphi(t, x) + (u \cdot \nabla) \varphi \cdot (1 - \alpha^2 \Delta) u] dx dt + \alpha^2 \int_{[0, T] \times \mathbb{R}^3} (\nabla \varphi : D^2) u \cdot u dx dt = - \int_{\mathbb{R}^3} u_0 \cdot (1 - \alpha^2 \Delta) \varphi(0, x) dx,$$

where $\nabla \varphi : D^2 = \sum_{i, k=1}^n \partial_i \varphi_k \partial_k \partial_i$.

Our main results are stated as:

Theorem 2.1. *Assume that the initial velocity is divergence free, axisymmetric without swirl and $\text{curl}v_0/r \in L^p_c$ with $p > \frac{3}{2}$. Then for any $T > 0$, there exists a unique solution of (1.3) in the sense of Definition 2.1, over the interval $[0, T]$.*

Theorem 2.2. *Assume that the initial velocity is divergence free, axisymmetric without swirl and $\text{curl}v_0/r \in M_c(\mathbb{R}^3)$. Then for any $T > 0$, there exists a global weak solution $u \in L^\infty([0, T] \times \mathbb{R}^3)$ of (1.1) in the sense of Definition 2.2. Moreover, we have that $\nabla u \in L^\infty((0, T); L^a + L^\infty)$ with $1 \leq a < 3$ and $D^2 u \in L^\infty((0, T); L^b + L^\infty)$ with $1 \leq b < \frac{3}{2}$.*

3. PROOF OF THE MAIN RESULTS

In this section, we will give the proof of Theorems 2.1 and 2.2.

For a smooth solution to (1.3), one can define a particle trajectory $y(t, x)$ by

$$\partial_t y(t, x_0) = u(t, y(t, x_0)),$$

$$(3.1) \quad y(0, x_0) = x_0,$$

where $x_0 \in \mathbb{R}^3$. It is noted that the transformation $y(t, x_0)$ on \mathbb{R}^3 preserves the measure due to the divergence free condition of u (see [43]). Moreover, one can recover the velocity $u(t, x)$ from the unfiltered vorticity $q = \text{curl } v$ through a precise expression of the integral kernel K_α in (3.3) as follows. Due to the divergence free condition of u , there exists a potential vector Ψ such that $\text{div} \Psi = 0$ and $u = \nabla \times \Psi$. Then by applying the curl operator to the second equation of (1.1) yields that

$$(3.2) \quad q = -(1 - \alpha^2 \Delta) \Delta \Psi.$$

Direct calculations show that the Green function associated with the operator $(1 - \alpha^2 \Delta) \Delta$ is (see, e.g., [31]) $G_\alpha(|x - y|) = \frac{1 - e^{-\frac{|x-y|}{\alpha}}}{4\pi|x-y|}$. Thus, we deduce that

$$(3.3) \quad \begin{aligned} u(t, x) &= \nabla \times \int_{\mathbb{R}^3} G_\alpha(|x - y|) q(t, y) dy \\ &= \int_{\mathbb{R}^3} f_\alpha(|x - y|) \frac{x - y}{|x - y|} \times q(t, y) dy, \end{aligned}$$

where $f_\alpha(|x - y|) := \frac{1}{\alpha^2} f(\frac{|x-y|}{\alpha})$, and $f(z) := \frac{(1+z)e^{-z} - 1}{4\pi z^2}$. Obviously, $f(z)$, $\nabla f(z)$, and $zf(z)$ are continuous and bounded functions for $z \in (0, +\infty)$. In addition, the kernel K_α in the second equation of (1.3) can be represented as $K_\alpha(x, y) = \nabla \times G_\alpha(|x - y|)$.

In view of (2.1), we obtain that for the axisymmetric Euler- α equations without swirl the velocity is recovered by

$$(3.4) \quad u(t, x) = \int_{\mathbb{R}^3} f_\alpha(|x - y|) \frac{x - y}{|x - y|} \times \frac{q^\theta(t, y)}{r} (y_2, -y_1, 0) dy.$$

Now we are ready to prove Theorem 2.1. Motivated by [43] and [46], we will utilize the particle trajectory (3.1) to construct the approximate solutions and prove the existence and uniqueness of solutions of the axisymmetric Euler- α equations without swirl. The key ingredient is to prove the Lipschitz continuity of the vector fields u with respect to the spatial variables.

Proof of Theorem 2.1. The sequences of the approximate solutions of (2.2) can be constructed as follows

$$(3.5) \quad \partial_t y^n(t, x_0) = u^n(t, y^n(t, x_0)),$$

$$(3.6) \quad y^n(0, x_0) = x_0,$$

$$(3.7) \quad y^0(t, x_0) = x_0,$$

$$(3.8) \quad u^n(t, x) = \int_{\mathbb{R}^3} K_\alpha(x, y) q^{n-1}(t, y) dy,$$

$$(3.9) \quad \frac{(q^n)^\theta}{r} (y^n(t, x_0), t) = \frac{q_0^\theta}{r} (x_0),$$

for $n \in \mathbb{N}$.

Thanks to (3.4), we have

$$\begin{aligned}
 |u^n(t, x)| &= \left| \int_{\mathbb{R}^3} f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right. \\
 &\quad \times (-x_2 + y_2^{n-1}(t, z), x_1 - y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \\
 &\quad \left. + \int_{\mathbb{R}^3} f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \times (x_2, -x_1, 0) \frac{q_0^\theta}{r}(z) dz \right| \\
 &\leq \int_{\mathbb{R}^3} |x - y^{n-1}(t, z)| |f_\alpha(|x - y^{n-1}(t, z)|)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 &\quad + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t, z)|)| \left| \frac{q_0^\theta}{r}(z) \right| dz |x| \\
 (3.10) \quad &\leq \frac{C}{\alpha^2} (1 + |x|) \left\| \frac{q_0^\theta}{r} \right\|_{L^1}.
 \end{aligned}$$

In the last inequality above, we have used the facts that $|x|f_\alpha(|x|)$ and $f_\alpha(|x|)$ in (3.3) are continuous and bounded functions in \mathbb{R}^3 . By the assumption that $\text{curl}v_0/r$ is compactly supported and $\text{curl}v_0/r \in L^p$, for $p > 3/2$, then $\text{curl}v_0/r \in L^1$. Thus, from (3.10) one has

$$(3.11) \quad |u^n(t, x)| \leq \frac{C}{\alpha^2} (1 + |x|),$$

where C is a constant depending on the L^1 norm of $\frac{q_0^\theta}{r}$ and the bounds of $f_\alpha(x)$ and $xf_\alpha(x)$ which are introduced in (3.3), for $x \in (0, +\infty)$.

Integrating (3.5) from 0 to t gives

$$(3.12) \quad y^n(t, x_0) = x_0 + \int_0^t u^n(s, y^n(s, x_0)) ds.$$

(3.11) and the Gronwall inequality imply that

$$(3.13) \quad |y^n(t, x_0)| \leq |x_0| + (1 + |x_0|)(e^{\frac{C}{\alpha^2}t} - 1) := L(t, |x_0|; \alpha),$$

for any $t \in [0, T]$. Using (3.4) again yields

$$\begin{aligned}
 |u^n(t, x)| &= \left| \int_{\mathbb{R}^3} f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right. \\
 &\quad \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \left. \right| \\
 &\leq \frac{C}{\alpha^2} \int_{\mathbb{R}^3} L(t, |z|; \alpha) \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 (3.14) \quad &\leq \frac{C_\alpha}{\alpha^2} \left\| \frac{q_0^\theta}{r} \right\|_{L^1},
 \end{aligned}$$

where C_α is a constant depending on the maximum of $L(t, |z|; \alpha)$ in the $\text{supp}\{|z| : z \in \text{supp} \frac{q_0^\theta}{r}\}$.

Next we prove that u^n is Lipschitz continuous with respect to the spatial variables and is uniformly continuous in time. For any $x, x' \in \mathbb{R}^3$, it follows from (3.4)-(3.9) that

$$\begin{aligned}
& |u^n(t, x) - u^n(t, x')| \\
&= \left| \int_{\mathbb{R}^3} [f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} - f_\alpha(|x' - y^{n-1}(t, z)|) \right. \\
&\quad \left. \cdot \frac{x' - y^{n-1}(t, z)}{|x' - y^{n-1}(t, z)|}] \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \right| \\
&= \left| \int_{\mathbb{R}^3} [(f_\alpha(|x - y^{n-1}(t, z)|) - f_\alpha(|x' - y^{n-1}(t, z)|)) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right. \\
&\quad \left. + f_\alpha(|x' - y^{n-1}(t, z)|) \left(\frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} - \frac{x' - y^{n-1}(t, z)}{|x' - y^{n-1}(t, z)|} \right)] \right. \\
(3.15) \quad &\quad \left. \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \frac{q_0^\theta}{r}(z) dz \right|.
\end{aligned}$$

By the mean value theorem, there exists a point $x'' \in \mathbb{R}^3$ such that

$$\begin{aligned}
& |u^n(t, x) - u^n(t, x')| \\
&\leq |x - x'| \int_{\mathbb{R}^3} [|\nabla f_\alpha(|x'' - y^{n-1}(t, z)|)| + \frac{|f_\alpha(|x' - y^{n-1}(t, z)|)|}{|x' - y^{n-1}(t, z)|}] \\
&\quad \cdot |y^{n-1}(t, z)| \frac{q_0^\theta}{r}(z) dz \\
&\leq \frac{c}{\alpha^3} |x - x'| \int_{\{z | z \in \text{supp}\{\frac{q_0^\theta}{r}\}\}} \left(1 + \frac{1}{|x' - y^{n-1}(t, z)|}\right) L(t, |z|; \alpha) \frac{q_0^\theta}{r}(z) dz \\
&\leq \frac{c}{\alpha^3} |x - x'| \left(\left\| \frac{q_0^\theta}{r} \right\|_{L^1} + \left\| \frac{q_0^\theta}{r} \right\|_{L^p} \left\| \frac{1}{|x' - y^{n-1}(t, z)|} \right\|_{L^{\frac{p}{p-1}}} \right) \\
(3.16) \quad &\leq \frac{c_\alpha}{\alpha^3} |x - x'| \left(\left\| \frac{q_0^\theta}{r} \right\|_{L^1} + \left\| \frac{q_0^\theta}{r} \right\|_{L^p} \right),
\end{aligned}$$

where $p > \frac{3}{2}$, and c_α depends on the bounds of $\nabla f(|x|)$ and $L(T, |z|; \alpha)$ (see (3.13)) for $z \in \text{supp}\{\frac{q_0^\theta}{r}\}$.

To show the uniform continuity with respect to the time variable, we have, for any $t, t' \in [0, T]$,

$$\begin{aligned}
& |u^n(t, x) - u^n(t', x)| \\
&= \left| \int_{\mathbb{R}^3} [f_\alpha(|x - y^{n-1}(t, z)|) \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \times (y_2^{n-1}(t, z), -y_1^{n-1}(t, z), 0) \right. \\
&\quad \left. - f_\alpha(|x - y^{n-1}(t', z)|) \frac{x - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} \times (y_2^{n-1}(t', z), -y_1^{n-1}(t', z), 0)] \frac{q_0^\theta}{r}(z) dz \right| \\
&\leq \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t, z)|)|
\end{aligned}$$

$$\begin{aligned}
 & - f_\alpha(|x - y^{n-1}(t', z)|) \left| \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right| |y^{n-1}(t, z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t', z)|)| \left| \frac{x - y^{n-1}(t, z)}{|x - y^{n-1}(t, z)|} \right| \\
 (3.17) \quad & - \frac{x - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} |y^{n-1}(t, z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & + |f_\alpha(|x - y^{n-1}(t', z)|)| \left| \frac{x - y^{n-1}(t', z)}{|x - y^{n-1}(t', z)|} \right| |y^{n-1}(t, z) \\
 & - y^{n-1}(t', z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & \leq \|\nabla f_\alpha\|_{L^\infty} \int_{\mathbb{R}^3} |y^{n-1}(t, z) - y^{n-1}(t', z)| |y^{n-1}(t, z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t', z)|)| \left| \frac{|y^{n-1}(t, z) - y^{n-1}(t', z)|}{|x - y^{n-1}(t', z)|} \right| |y^{n-1}(t', z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & + \int_{\mathbb{R}^3} |f_\alpha(|x - y^{n-1}(t', z)|)| |y^{n-1}(t, z) - y^{n-1}(t', z)| \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & \leq \frac{C}{\alpha^3} \int_{\mathbb{R}^3} |y^{n-1}(t, z) - y^{n-1}(t', z)| \left[\frac{1 + |x - y^{n-1}(t', z)|}{|x - y^{n-1}(t', z)|} L(T, z; \alpha) + 1 \right] \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & \leq \frac{C_\alpha}{\alpha^3} |t - t'| \int_{\mathbb{R}^3} \sup_{s \in [0, T]} |u^{n-1}(s, z)| \left[\frac{1 + |x - y^{n-1}(t', z)|}{|x - y^{n-1}(t', z)|} + 1 \right] \left| \frac{q_0^\theta}{r}(z) \right| dz \\
 & \leq \frac{C_\alpha}{\alpha^3} |t - t'| (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p})
 \end{aligned}$$

for any $p > \frac{3}{2}$, where constant C_α depends on the bounds of $u^{n-1}(t, z)$ and $y^{n-1}(t, z)$ for z lies in the support of $\frac{q_0^\theta}{r}$ by virtue of (3.13) and (3.14), respectively. Since u^n is Lipschitz continuous with respect to the spatial variables and is uniformly continuous with respect to time, and u^n is uniformly bounded by (3.14), then the map $y^n \in C^1([0, T]; C(\mathbb{R}^3))$, for every $T > 0$. Moreover, we have that $y^n(t, x) - x \in C^1([0, T]; C_B(\mathbb{R}^3))$ in which $C_B(\mathbb{R}^3)$ means the space of bounded and continuous functions.

Similar to the estimates in (3.16) and (3.17), one will prove that $\{y^n(t, x) - x\}$ is a Cauchy sequence in $C([0, T]; C_B(\mathbb{R}^3))$ and furthermore $\{y^n\}$ is a Cauchy sequence in $C([0, T]; \mathcal{G})$. For simplicity, the time dependence of sequences $\{u^n\}$ and $\{y^n\}$, $n \in \mathbb{N}$, is dropped in the following estimates. Note that

$$\begin{aligned}
 & |y^n(t, x) - y^{n-1}(t, x)| \\
 & \leq \int_0^t |u^n(s, y^n(s, x)) - u^{n-1}(s, y^{n-1}(s, x))| ds \\
 & \leq \int_0^t \left| \int_{\mathbb{R}^3} f_\alpha(|y^n(x) - y^{n-1}(z)|) \frac{y^n(x) - y^{n-1}(z)}{|y^n(x) - y^{n-1}(z)|} \times (y_2^{n-1}, -y_1^{n-1}, 0) \frac{q_0^\theta}{r} \right.
 \end{aligned}$$

$$\begin{aligned}
 & - f_\alpha(|y^{n-1}(x) - y^{n-2}(z)|) \frac{y^{n-1}(x) - y^{n-2}(z)}{|y^{n-1}(x) - y^{n-2}(z)|} \times (y_2^{n-2}, -y_1^{n-2}, 0) \frac{q_0^\theta}{r} dz ds \\
 & \leq \frac{C}{\alpha^3} \int_0^t \int_{\mathbb{R}^3} |y^n(x) - y^{n-1}(x)| |y^{n-1}(z)| \frac{q_0^\theta}{r} |dz ds \\
 & \quad + \frac{C}{\alpha^2} \int_0^t \int_{\mathbb{R}^3} \frac{|y^n(x) - y^{n-1}(x)|}{|y^{n-1}(x) - y^{n-1}(z)|} |y^{n-1}| \frac{q_0^\theta}{r} |dz ds \\
 & \quad + \frac{C}{\alpha^2} \int_0^t \int_{\mathbb{R}^3} |y^{n-1} - y^{n-2}| \frac{q_0^\theta}{r} |dz ds \\
 (3.18) \quad & \leq \frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) \int_0^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds
 \end{aligned}$$

for $p > \frac{3}{2}$, where $C_\alpha > 0$ is a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. Define $g^N(t) = \sup_{n \geq N} \sup_{x \in \mathbb{R}^3} |y^n(x, t) - y^{n-1}(x, t)|$. Then (3.14) implies that $g^N(t) \leq \frac{C_\alpha}{\alpha^3} \| \frac{q_0^\theta}{r} \|_{L^1} t < \infty$ for $t \in [0, T]$, where $C_\alpha > 0$ is a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. Choose $T_1 > 0$ small enough such that $\frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) T_1 \leq 1/2$. Then it follows from (3.18) that

$$(3.19) \quad g^N(t) \leq k \int_0^t g^{N-1}(s) ds, \quad t \in [0, T_1],$$

where $k := \frac{2C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p})$, with $C_\alpha > 0$ a constant depending on $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. Similar as Lemma 3.2 of Chapter 2 in [43], we obtain that

$$(3.20) \quad \lim_{N \rightarrow \infty} g^N(t) \rightarrow 0$$

uniformly on $[0, T_2]$, for some $T_2 \in (0, T_1]$ sufficient small. Actually, it can be proved that $g^N(t)$ can be bounded by the terms of a convergent geometrical series for $t \in [0, T_2]$. This implies that $\{y^n - x\}$ is a Cauchy sequence in $C([0, T_0]; C_B(\mathbb{R}^3))$, where $T_0 = \min\{T_1, T_2\}$ depends on $\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}$ and $\sup\{|z| : z \in \text{supp } \frac{q_0^\theta}{r}\}$. To continue the procedure above to the interval $[T_0, 2T_0]$, we have

$$\begin{aligned}
 & |y^n(t, x) - y^{n-1}(t, x)| \\
 (3.21) \quad & \leq \int_{T_0}^t |u^n(y^n) - u^{n-1}(y^{n-1})| ds + |y^n(T_0, x) - y^{n-1}(T_0, x)|
 \end{aligned}$$

for $t \geq T_0$. Due to (3.20), for any $\epsilon > 0$, there exists a $N_1 > 0$ such that $\sup_{x \in \mathbb{R}^3} |y^n(T_0, x) - y^{n-1}(T_0, x)| < \epsilon$, for $n \geq N_1$. Therefore, similar to (3.18), for all $n \geq N_1$, one has

$$\begin{aligned}
 & |y^n(t, x) - y^{n-1}(t, x)| \\
 & \leq \int_{T_0}^t |u^n(y^n) - u^{n-1}(y^{n-1})| ds + \epsilon
 \end{aligned}$$

$$(3.22) \leq \frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) \int_{T_0}^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds + \epsilon,$$

for $t \in [T_0, 2T_0]$. It follows that

$$\begin{aligned} & \int_{T_0}^t \sup_{x \in \mathbb{R}^3} |y^n(s, x) - y^{n-1}(s, x)| ds \\ & \leq \frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) (t - T_0) \int_{T_0}^t (\sup_{x \in \mathbb{R}^3} |y^n - y^{n-1}| + \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds \\ & \quad + \epsilon(t - T_0), \end{aligned}$$

for $t \in [T_0, 2T_0]$. This implies that

$$(3.23) \quad \begin{aligned} & \int_{T_0}^t \sup_{x \in \mathbb{R}^3} |y^n(s, x) - y^{n-1}(s, x)| ds \\ & \leq \int_{T_0}^t \sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}| ds + 2\epsilon(t - T_0), \end{aligned}$$

Putting (3.23) into (3.22), we get

$$(3.24) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^3} |y^n(t, x) - y^{n-1}(t, x)| \\ & \leq 2 \frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) \int_{T_0}^t (\sup_{x \in \mathbb{R}^3} |y^{n-1} - y^{n-2}|) ds \\ & \quad + 2\epsilon(t - T_0) \frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) + \epsilon. \end{aligned}$$

Since $\frac{C_\alpha}{\alpha^3} (\| \frac{q_0^\theta}{r} \|_{L^1} + \| \frac{q_0^\theta}{r} \|_{L^p}) (t - T_0) \leq 1/2$, for all $t \in [T_0, 2T_0]$, it follows that

$$(3.25) \quad g^N(t) \leq k \int_{T_0}^t g^{N-1}(s) ds + 2\epsilon, \quad t \in [T_0, 2T_0],$$

where k is same as in (3.19). By the arbitrariness of ϵ , similar to the previous procedure (see also Lemma 3.2 of Chapter 2 in [43]), we can prove that $\{y^n - x\}$ is a Cauchy sequence in $C([T_0, 2T_0]; C_B(\mathbb{R}^3))$, and furthermore in $C([0, T]; C_B(\mathbb{R}^3))$; and $\{y^n\}$ is a Cauchy sequence in $C([0, T]; \mathcal{G})$, for every $T > 0$. The limit of $\{y^n(t, x)\}$ in $C([0, T]; \mathcal{G})$ is denoted by $y(t, x)$.

Define $q(t, x) = \frac{q^\theta}{r}(x, t)(x_2, -x_1, 0)$ satisfying $\frac{q^\theta}{r}(y(t, x), t) = \frac{q^\theta}{r}(x, t)$. Then it is straightforward to prove that

$$q^n(t, x) = \frac{(q^n)^\theta}{r}(x, t)(x_2, -x_1, 0) \rightharpoonup q(t, x)$$

in the sense of weakly- $*$ convergence in $L^\infty([0, T]; L^p(\mathbb{R}^3))$, with $p > 3/2$, and

$$u^n(t, x) \rightarrow u(t, x)$$

in $C_B([0, T] \times \mathbb{R}^3)$. Moreover, y, u, q solve (3.22), (2.5) and satisfy

$$(3.26) \quad \int_{\mathbb{R} \times \mathbb{R}^3} (\partial_t \varphi + u \cdot \nabla \varphi) \frac{q^\theta}{r} dx dt = 0$$

for any $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^3)$.

The uniqueness can be shown by the direct estimate on the difference of two flow maps. The estimates are similar to the previous ones and we omit the details here. The proof of the theorem is finished.

Proof of Theorem 2.2. Mollifying the initial potential vorticity, we can construct the approximate solutions of (1.3) by solving the following problem:

$$(3.27) \quad \begin{cases} \partial_t q^\epsilon + u^\epsilon \cdot \nabla q^\epsilon = q^\epsilon \cdot \nabla u^\epsilon, \\ u^\epsilon = K_\alpha * q^\epsilon, \\ q^\epsilon(t = 0, x) = q_0^\epsilon, \end{cases}$$

where q_0^ϵ is a smooth vector with compact support which converges to q_0 in $M(\mathbb{R}^3)$ and $\|q_0^\epsilon\|_{L^1} \leq \|q_0\|_M$. Then there exists a unique smooth solution (u^ϵ, q^ϵ) to (3.27) (see [10] and references therein). Moreover, there is a smooth pressure p^ϵ such that u^ϵ satisfies

$$(3.28) \quad \begin{cases} \partial_t v^\epsilon + u^\epsilon \cdot \nabla v^\epsilon + \sum_j v_j^\epsilon \nabla u_j^\epsilon + \nabla p^\epsilon = 0, \\ v^\epsilon = (1 - \alpha^2 \Delta) u^\epsilon, \\ \operatorname{div} u^\epsilon = 0. \end{cases}$$

Therefore, for any test function $\varphi \in C_0^\infty([0, T], \mathbb{R}^3)$, satisfying $\operatorname{div} \varphi = 0$, integration by parts yields

$$(3.29) \quad \begin{aligned} & \int_{[0, T] \times \mathbb{R}^3} [u^\epsilon(t, x)(1 - \alpha^2 \Delta) \partial_t \varphi(t, x) + (u^\epsilon \cdot \nabla) \varphi \cdot (1 - \alpha^2 \Delta) u^\epsilon] dx dt \\ & + \alpha^2 \int_{[0, T] \times \mathbb{R}^3} (\nabla \varphi : D^2) u^\epsilon \cdot u^\epsilon dx dt = - \int_{\mathbb{R}^3} u_0^\epsilon (1 - \alpha^2 \Delta) \varphi(0, x) dx. \end{aligned}$$

Similar analysis as for (3.14) shows that

$$(3.30) \quad \|u^\epsilon\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq \frac{C_\alpha}{\alpha^2} \left\| \frac{q_0^\theta}{r} \right\|_M,$$

where $C_\alpha > 0$ is a constant depending on T and the Lebesgue measure of the support of $\frac{q_0^\theta}{r}$.

We now estimate ∇u^ϵ and $D^2 u^\epsilon$. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function satisfying

$$\chi(s) = \begin{cases} 1, & |s| < 1, \\ 0, & |s| > 2. \end{cases}$$

Then it follows from (3.4) that

$$(3.31) \quad \begin{aligned} & \partial_{x_i} u^\epsilon(t, x) \\ & = \int_{\mathbb{R}^3} \partial_{x_i} [f_\alpha(|x - y^\epsilon|) \frac{x - y^\epsilon}{|x - y^\epsilon|}] (1 - \chi(|x - y^\epsilon|)) \times \frac{q^\theta(t, y^\epsilon)}{r}(y_2^\epsilon, -y_1^\epsilon, 0) dy \\ & \quad + \int_{\mathbb{R}^3} \partial_{x_i} [f_\alpha(|x - y^\epsilon|) \frac{x - y^\epsilon}{|x - y^\epsilon|}] \chi(|x - y^\epsilon|) \times \frac{q^\theta(t, y^\epsilon)}{r}(y_2^\epsilon, -y_1^\epsilon, 0) dy \\ & \equiv G_1(t, x) + G_2(t, x), \end{aligned}$$

for $i = 1, 2, 3$. It is clear that

$$\begin{aligned} |G_1(t, x)| &\leq \frac{C}{\alpha^3} \int_{\mathbb{R}^3} \frac{1}{|x - y^\epsilon(t, z)|} (1 - \chi(|x - y^\epsilon(t, z)|)) \left| \frac{(q_0^\epsilon)^\theta}{r}(z) \right| |y^\epsilon(t, z)| dz \\ &\leq \frac{C_\alpha}{\alpha^3} \left\| \frac{q_0^\theta}{r} \right\|_M, \end{aligned}$$

for $(t, x) \in [0, T] \times \mathbb{R}^3$. One can use Young's inequality for convolutions to obtain

$$\begin{aligned} &\|G_2(t, x)(t, x)\|_{L^\infty((0, T); L^a)} \\ &\leq \sup_{t \in [0, T]} \left\| \int_{\mathbb{R}^3} \frac{1}{|x - y^\epsilon(t, z)|} \chi(|x - y^\epsilon(t, z)|) \left| \frac{(q_0^\epsilon)^\theta}{r}(z) \right| |y^\epsilon(t, z)| dz \right\|_{L^a} \\ (3.32) \quad &\leq \frac{C_\alpha}{\alpha^3} \left\| \frac{(q_0^\epsilon)^\theta}{r} \right\|_{L^1} \leq \frac{C_\alpha}{\alpha^3} \left\| \frac{q_0^\theta}{r} \right\|_M \end{aligned}$$

for $1 \leq a < 3$. Thus $\nabla u^\epsilon(t, x)$ is bounded in $L^\infty((0, T); L^a + L^\infty)$ and

$$(3.33) \quad \|\nabla u^\epsilon(t, x)\|_{L^\infty((0, T); L^a + L^\infty)} \leq \frac{C_\alpha}{\alpha^3} \left\| \frac{q_0^\theta}{r} \right\|_M$$

for $1 \leq a < 3$. Similarly, we can prove that $D^2 u^\epsilon(t, x)$ is bounded in $L^\infty((0, T); L^b + L^\infty)$ and that

$$(3.34) \quad \|D^2 u^\epsilon(t, x)\|_{L^\infty((0, T); L^b + L^\infty)} \leq \frac{C_\alpha}{\alpha^4} \left\| \frac{q_0^\theta}{r} \right\|_M$$

for $1 \leq b < \frac{3}{2}$. In (3.33) and (3.34), the constant C_α is a positive constant depending on $\left\| \frac{q_0^\theta}{r} \right\|_M$ and the support of $\frac{q_0^\theta}{r}$. In view of (3.30), (3.33) and (3.34), it is easy to obtain that the terms $u^\epsilon \cdot \nabla v^\epsilon$, $\sum_j v_j^\epsilon \nabla u_j^\epsilon$ and ∇p^ϵ are bounded in

$L^\infty((0, T); W_{loc}^{-2,2}(\mathbb{R}^3))$ and hence it follows from (3.28) that $\partial_t u^\epsilon$ is bounded in $L^\infty((0, T); L_{loc}^2(\mathbb{R}^3))$. Note that u^ϵ is bounded in $L^\infty((0, T); W_{loc}^{1,a}(\mathbb{R}^3))$ for any $1 \leq a < 3$. By the Aubin-Lions Lemma (see, e.g., [14], [49]), we obtain that (up to a subsequence) that $u^\epsilon \rightarrow u$ strongly in $C([0, T]; L_{loc}^c(\mathbb{R}^3))$, with $1 \leq c < \frac{3a}{3-a}$, where $u \in L^\infty([0, T] \times \mathbb{R}^3)$ is a function satisfying $\nabla u \in L^\infty((0, T); L^a + L^\infty)$, with $1 \leq a < 3$, and $D^2 u \in L^\infty((0, T); L^b + L^\infty)$, with $1 \leq b < \frac{3}{2}$. Thanks to (3.30), we have that $u^\epsilon \rightarrow u$ strongly in $L^\sigma([0, T]; L_{loc}^\sigma(\mathbb{R}^3))$, for any $\sigma \in (1, \infty)$. Moreover, it is clear that $D^2 u^\epsilon \rightharpoonup D^2 u$ weakly-* convergence in $L^\infty((0, T); L_{loc}^b)$, for any $1 \leq b < 3/2$. Thus, it follows from (3.29) that

$$\begin{aligned} &\int_{[0, T] \times \mathbb{R}^3} [u(t, x)(1 - \alpha^2 \Delta) \partial_t \varphi(t, x) + (u \cdot \nabla) \varphi \cdot (1 - \alpha^2 \Delta) u] dx dt \\ &+ \alpha^2 \int_{[0, T] \times \mathbb{R}^3} (\nabla \varphi : D^2) u \cdot u dx dt = - \int_{\mathbb{R}^3} u_0 (1 - \alpha^2 \Delta) \varphi(0, x) dx \end{aligned}$$

for any $\varphi \in C_0^\infty([0, T], \mathbb{R}^3)$ satisfying $\operatorname{div} \varphi = 0$. This completes the proof of Theorem 2.2.

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Manuscript received March 18 2019

revised April 4 2019

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