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THE ASYMPTOTICAL BEHAVIOUR OF EMBEDDED EIGENVALUES FOR PERTURBED PERIODIC OPERATORS

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Abstract. Let H_0 be a periodic operator on $\mathbb{R}^+$(or periodic Jacobi operator on $\mathbb{N}$). It is known that the absolutely continuous spectrum of H_0 is consisted of spectral bands $\cup[\alpha_l,\beta_l]$. Under the assumption that $\limsup_{x\to\infty} x|V(x)|<\infty$ ($\limsup_{n\to\infty} n|V(n)|<\infty$), the asymptotical behaviour of embedded eigenvalues approaching to the spectral boundary is established.

1. Introduction and main results

We consider the continuous Schrödinger equation on $\mathbb{R}$ ($\mathbb{R}^+$),

$$Hu = -u'' + (V(x) + V_0(x))u = Eu, \tag{1.1}$$

where $V_0(x)$ is 1-periodic and $V(x)$ is a decaying perturbation.

In this paper, we always assume V_0 is 1-periodic and in $L^1[0,1]$. Without loss of generality, we only consider the half-line equation.

When $V\equiv 0$, we have a 1-periodic Schrödinger equation,

$$H_0\varphi = -\varphi'' + V_0(x)\varphi = E\varphi. \tag{1.2}$$

It is known that the absolutely continuous spectrum (essential spectrum) of H_0 is consisted of a union of closed intervals (often referred to as spectral bands). We denote by

$$\sigma_{\rm ac}(H_0) = \sigma_{\rm ess}(H_0) = \bigcup_l [\alpha_l,\beta_l].$$

The bands may collapse at some boundaries. Putting the collapsed bands together, we denote by

$$\sigma_{\rm ac}(H_0) = \sigma_{\rm ess}(H_0) = \bigcup_l [\tilde\alpha_l,\tilde\beta_l].$$

The last band may have the form as $[c,\infty)$. In order to make the difference, we call $[\alpha_l,\beta_l]$ a *standard spectral band* and $[\tilde\alpha_l,\tilde\beta_l]$ a *non-standard spectral band*.

By Floquet theory, for any $E\in(\alpha_l,\beta_l)$, there exists φ of (1.2), which has the following form

$$\varphi(x,E) = p(x,E)e^{ik(E)x} \tag{1.3}$$

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where $k(E)$ is the quasimomentum, and $p(x,E)$ is 1-periodic. In each standard spectral band $[\alpha_l,\beta_l]$, $k(E)$ is monotonically increasing from 0 to π or monotonically decreasing from π to 0.

Similarly, we consider the periodic Jacobi equation,

$$(H_0u)(n) := a_{n+1}u(n+1) + a_nu(n-1) + b_{n+1}u(n) = Eu(n), n \geq 0, \tag{1.4}$$

where the $\{a_j, b_j\}$ are real sequences with q-period, and a_j is positive for all possible j.

Likewise, we consider a class of the perturbed periodic Jacobi equation,

$$(Hu)(n) = a_{n+1}u(n+1) + a_nu(n-1) + (b_{n+1} + V(n+1))u(n) = Eu(n), n \geq 0, \tag{1.5}$$

where $V(n)$ is an real sequences. Now we will extend the notations for the continuous case to the discrete one. For $E \in (\alpha_l, \beta_l)$, let φ be the Floquet solution of (1.4), namely,

$$\varphi(n,E) = p(n,E)e^{i\frac{k(E)}{q}n}, \tag{1.6}$$

where $p(n,E)$ is a q-periodic sequence and $k(E) \in (0,\pi)$ is the quasimomentum (q is the period for a_n and b_n). We also have the standard bands and non-standard bands notations,

$$\sigma_{\text{ac}}(H_0) = \sigma_{\text{ess}}(H_0) = \cup_l[\alpha_l, \beta_l] = \cup_l[\tilde{\alpha}_l, \tilde{\beta}_l].$$

It is a very interesting problem to study phenomenons of eigenvalues/singular continuous spectrum embedded into the absolutely continuous (or essential) spectrum based on the decaying perturbations, for example [4, 6–8, 10–17, 22–25, 27, 30, 31]. We care about the asymptotical behavior of eigenvalues approaching to the spectral boundaries. The asymptotical behavior of eigenvalues lying outside spectral bands has been well studied [1–3, 9, 29].

Our goal is to investigate the asymptotical behavior of eigenvalues approaching to the spectral boundaries from the inside part. In our previous note, we show that there is no embedded eigenvalues in the spectral band if $V(x) = \frac{o(1)}{1+x}$. If we allow $V(x) = \frac{h(x)}{1+x}$ for some positive function $h(x)$ going to ∞, $H_0 + V$ can have the desired countably embedded eigenvalues [21]. In those two cases, it is trivial to study the asymptotical behavior of embedded eigenvalues. The purpose of this paper is to study the remaining case, namely $V(x) = \frac{O(1)}{1+x}$. If $V_0 = 0$ (it is reduced to the free case), both the continuous and discrete cases have been studied in [11] and [19] respectively. Starting from the Floquet solutions for periodic operators, we obtained a universal and optimal result for the asymptotical behavior. It leads to various applications. See Remark 1.9 for the details. In another two papers, we will show that there is no singular continuous spectrum if $V(x) = \frac{O(1)}{1+x}$ [18, 20].

In order to state our main results, more notations are necessary. The Wronskian for the discrete case is given by

$$W(f,g)(n) = f(n)g(n+1) - f(n+1)g(n).$$

The Wronskian for the continuous case is given by

$$W(f,g)(x) = f(x)g'(x) - f'(x)g(x).$$

Since the Floquet solution $\varphi(x,E)$ ($\varphi(n,E)$) is a solution of $H_0u = Eu$, the Wronskian $W(\overline{\varphi},\varphi)$ is constant and there is an $\omega \in \mathbb{R}$ such that

$$W(\overline{\varphi},\varphi) = i\omega.$$

In the continuous case, define

$$\Gamma(E) = \int_0^1 \frac{4}{\omega^2}|\varphi(x,E)|^4 dx.$$

In the discrete case, define

$$\Gamma(E) = \frac{1}{q}\sum_{n=1}^{q} \frac{4}{\omega^2}|\varphi(n,E)|^4.$$

$\Gamma(E)$ is well defined for $E \in (\alpha_l,\beta_l)$ and blows up as $E \to \alpha_l$ or $E \to \beta_l$. In the following arguments, H_0 is always a continuous periodic operator or periodic Jacobi operator. $V(x)$ or $V(n)$ is the perturbed potential.

In the continuous case, define P as

$$P = \{E \in \mathbb{R} : -u'' + (V(x) + V_0(x))u = Eu \text{ has an } L^2(\mathbb{R}^+) \text{ solution }\}, \tag{1.7}$$

and in the discrete case, define P as

$$P = \{E \in \mathbb{R} : \text{ equation (1.5) has an } \ell^2(\mathbb{N}) \text{ solution }\}. \tag{1.8}$$

P is the collection of eigenvalues for operators H_0+V with all the possible boundary conditions at 0.

Theorem 1.1. *Suppose* $\limsup_{x\to\infty} x|V(x)| = A$ *(*$\limsup_{n\to\infty} n|V(n)| = A$*). For any standard spectral band* $[\alpha_l,\beta_l]$*, let* δ_l *be the unique point in* $[\alpha_l,\beta_l]$ *such that the quasimomentum* $k(\delta_l) = \frac{\pi}{2}$*. Then the set* $P\cap(\alpha_l,\beta_l)$ *is a countable set with two possible accumulation points* α_l *and* β_l*. Moreover,*

$$\sum_{E_i\in P\cap(\alpha_l,\delta_l)} \frac{1}{\Gamma(E_i)} \leq \frac{A^2}{2}, \tag{1.9}$$

and

$$\sum_{E_i\in P\cap(\delta_l,\beta_l)} \frac{1}{\Gamma(E_i)} \leq \frac{A^2}{2}.$$

For any $\lambda \in \cup_l\{\alpha_l,\beta_l\}$, we define $\kappa_\lambda = 1$ if λ is non-collapsed. Otherwise, $\kappa_\lambda = 2$. See the formal description of "non-collapsed" in Theorem 2.1 and the Remark after.

Based on Theorems 1.1, we can get a lot of Corollaries.

Corollary 1.2. *Suppose* $\limsup_{x\to\infty} x|V(x)| = A$ *(*$\limsup_{n\to\infty} n|V(n)| = A$*). Then for any standard spectral band* $[\alpha_l,\beta_l]$*,* $P\cap[\alpha_l,\beta_l]$ *is a countable set. Moreover, there is a constant* K *(depends on* k *uniformly in any bounded set) such that*

$$\sum_{E_i\in P\cap(\alpha_l,\beta_l)} \min\{|E_i-\alpha_l|^{\kappa_{\alpha_l}}, |E_i-\beta_l|^{\kappa_{\beta_l}}\} \leq KA^2.$$

In particular,

$$\sum_{E_i\in P\cap(\alpha_l,\beta_l)} \min\{|E_i-\alpha_l|^2, |E_i-\beta_l|^2\} \leq KA^2.$$

Corollary 1.3. *Suppose* $\limsup_{x\to\infty} x|V(x)| = A$ *(*$\limsup_{n\to\infty} n|V(n)| = A$*). Then for any non-standard spectral band* $[\tilde\alpha_l, \tilde\beta_l]$*, there exists some* ϵ *(only depends on* H_0 *not on* l*) and a constant* K *(depends on* l *uniformly in any bounded set) such that*

$$\sum_{E_i\in P\cap(\tilde\alpha_l,\tilde\alpha_l+\epsilon)} \min\{|E_i-\tilde\alpha_l|, |E_i-\tilde\beta_l|\} \le KA^2,$$

and

$$\sum_{E_i\in P\cap(\tilde\beta_l-\epsilon,\tilde\beta_l)} \min\{|E_i-\tilde\alpha_l|, |E_i-\tilde\beta_l|\} \le KA^2.$$

Remark 1.4. In the continuous case, $\beta_l - \alpha_l = 2\pi^2 l + O(1)$ as k goes to infinity [5]. Thus we can choose $\epsilon \asymp \pi^2 l$ for large l.

Corollary 1.5. *Suppose* $V_0(x) = 0$ *and* $\limsup_{x\to\infty} x|V(x)| = A$*. Then* $P\cap(0,\infty)$ *is a countable set. Moreover,*

$$\sum_{E_i\in P\cap(0,\infty)} E_i \le \frac{A^2}{2}.$$

We remark that Corollary 1.5 with exactly the same bound has been proved in [11]. It means our bounds in Theorem 1.1 are effective. In the discrete setting, Theorem 1.1 implies the following known result in [19].

Corollary 1.6. *Suppose* $V_0(n) = 0$ *and* $\limsup_{n\to\infty} n|V(n)| = A$*. Then* $P\cap(-2,2)$ *is a countable set. Moreover,*

$$\sum_{E_i\in P\cap(-2,2)} (4-E_i^2) \le 4A^2 + 4\min\{1, A\}.$$

For the discrete case, H_0 has q standard spectral bands and the set of spectrum is bounded. Then the constant K (ϵ) only depends on H_0.

Corollary 1.7. *Suppose* $\limsup_{n\to\infty} n|V(n)| = A$*. Let* $d(E) = \min_l \mathit{dist}(E, \{\alpha_l, \beta_l\})$*. Then there is a constant* K *such that*

$$\sum_{E_i\in P\cap\sigma_{\rm ess}(H_0)} d^2(E_i) \le KA^2.$$

Corollary 1.8. *Suppose* $\limsup_{n\to\infty} n|V(n)| = A$*. Then there exist constants* $\epsilon > 0$ *and* $K > 0$ *such that*

$$\sum_{E_i} \min\{|E_i-\tilde\alpha_l|, |E_i-\tilde\beta_l|\} \le KA^2,$$

where $\{E_i\}$ *goes over all the values in set* $P\cap\Big(\cup_l(\tilde\alpha_l, \tilde\alpha_l+\epsilon)\cup(\tilde\beta_l-\epsilon, \tilde\beta_l)\Big)$.

Remark 1.9.
- All the bounds go to 0 as A goes to zero. It implies that there is no $L^2(\mathbb{R}^+)$ (or $\ell^2(\mathbb{N})$) solution of $(H_0+V)u = Eu$ for $E\in\cup_l(\alpha_l,\beta_l)$ if $V(x) = \frac{o(1)}{1+x}$ ($V(n) = \frac{o(1)}{1+n}$). Thus under any fixed boundary condition at 0, there is no eigenvalues of the operator H_0+V embedded into its bands if $V(x) = \frac{o(1)}{1+x}$ ($V(n) = \frac{o(1)}{1+n}$).

- Fix l. Corollary 1.2 implies the speed of $E_i \in P$ going to the collapsed boundaries behaves like $|E_i - \alpha_l| \asymp \frac{1}{1+\sqrt{i}}$ ($|E_i - \beta_l| \asymp \frac{1}{1+\sqrt{i}}$). The speed going to non-collapsed boundaries behaves like $|E_i - \alpha_l| \asymp \frac{1}{1+i}$ ($|E_i - \beta_l| \asymp \frac{1}{1+i}$). Remling [28] shows that we can not improve it to $|E_i - \alpha_l| \asymp \frac{1}{1+i^{1+\epsilon}}$ in the continuous case. This means that our main results are optimal in some sense.

Our paper is organized in the following way. In Section 2, we will introduce the Floquet solution and generalized Prüfer transformation. In Section 3, we will give the proof of all the results in the continuous setting. In Section 4, we will give the proof of all the results in the discrete setting.

In the following, E is always in some spectral band (α_l, β_l). The constant C depends on E, and so does $O(1)$. K depends on E (or say l) uniformly in any compact set.

2. Floquet solution and generalized Prüfer transformation for the continuous Schrödinger operator

Let $T_0(E)$ be the transfer matrix of H_0 from 0 to 1, that is

$$T_0(E) \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \begin{pmatrix} u(1) \\ u'(1) \end{pmatrix}$$

for any solution u of $H_0 u = Eu$.

Let

$$T_0(E) = \begin{pmatrix} a(E) & b(E) \\ c(E) & d(E) \end{pmatrix}$$

and $D(E) = a(E) + d(E)$

Theorem 2.1 ([5, Theorem 2.3.3]). *Let $\Lambda = \cup_l \{\alpha_l, \beta_l\}$. Then for any $\lambda \in \Lambda$, $D(\lambda)$ is either 2 or -2. Furthermore, one of $D'(\lambda)$ and $D''(\lambda)$ must be non-zero.*

Remark 2.2. $D'(\lambda) \neq 0$ if and only if λ is non-collapsed.

For $E \in (\alpha_l, \beta_l)$, let $k(E) \in (0, \pi)$ be the quasimomentum. Let $\varphi(x, E)$ be a solution of $H_0 u = Eu$ with $E \in (\alpha_l, \beta_l)$ and satisfies boundary condition $\varphi(0) = -b(E)$ and $\varphi'(0) = a(E) - e^{ik(E)}$. It is easy to check that

$$T_0(E) \begin{pmatrix} \varphi(0) \\ \varphi'(0) \end{pmatrix} = e^{ik(E)} \begin{pmatrix} \varphi(0) \\ \varphi'(0) \end{pmatrix}.$$

We get a Floquet solution $\varphi(x, E)$.

Since $\varphi(x, E)$ is a solution of $H_0 u = Eu$, the Wronskian $W(\overline{\varphi}, \varphi)$ is constant and

$$W(\overline{\varphi}, \varphi) = 2i \text{ Im } (\overline{\varphi} \varphi')$$

Let $\omega(E) \in \mathbb{R}$ be such that

$$2i \text{ Im } (\overline{\varphi(x, E)} \varphi'(x, E)) = i\omega(E).$$

Thus

$$\omega(E) = 2b(E) \sin k(E).$$

Without loss of generality, assume $\omega(E) > 0$ (i.e., $b(E) > 0$).

Let $\gamma(x,E)$ be such that

$$\varphi(x,E) = |\varphi(x,E)|e^{i\gamma(x,E)}.$$

Then (See [12, Prop.2.1])

$$\gamma'(x,E) = \frac{\omega(E)}{2|\varphi(x,E)|^2}. \tag{2.1}$$

Proposition 2.3 (Proposition 2.2 and Theorem 2.3bc of [12])**.** *Suppose u is a real solution of* (1.1). *Then there exist real functions $R(x,E) > 0$ and $\theta(x,E)$ such that*

$$[\ln R(x,E)]' = \frac{V(x)}{2\gamma'(x,E)} \sin 2\theta(x,E) \tag{2.2}$$

and

$$\theta(x,E)' = \gamma'(x,E) - \frac{V(x)}{2\gamma'(x,E)} \sin^2 \theta(x,E). \tag{2.3}$$

Moreover, there exists a constant C (depending on E) such that

$$\frac{|u(x,E)|^2 + |u'(x,E)|^2}{C} \leq R(x,E)^2 \leq C(|u(x,E)|^2 + |u'(x,E)|^2). \tag{2.4}$$

$R(x,E) > 0$ and $\theta(x,E)$ are usually referred as generalized Prüfer variables.

Lemma 2.4. *The following estimate holds,*

$$\frac{1}{\gamma'(x,E)} \leq \frac{K(E)}{\sin k(E)}, \tag{2.5}$$

and then (in the continuous case)

$$\Gamma(E) \leq \frac{K(E)}{|\sin k(E)|^2},$$

where $K(E)$ is uniformly bounded on any compact set of E.

Proof. By (2.1), it suffices to show that

$$\frac{|\varphi(x,E)|^2}{\omega} \leq \frac{K(E)}{\sin k(E)}. \tag{2.6}$$

Since $|\varphi(x,E)|$ is 1-periodic, it suffices to prove (2.6) for $0 \leq x \leq 1$. By the fact that $V_0 \in L^1[0,1]$ and Grönwall's inequality, one has

$$|\varphi(x,E)|^2 \leq K(E)(|\varphi(0,E)|^2 + |\varphi'(0,E)|^2).$$

Thus it suffices to show

$$\frac{|\varphi(0,E)|^2 + |\varphi'(0,E)|^2}{\omega} \leq \frac{K(E)}{\sin k(E)}.$$

By the definition,

$$\frac{|\varphi(0,E)|^2 + |\varphi'(0,E)|^2}{\omega} = \frac{b^2(E) + (a(E) - \cos k(E))^2 + \sin^2 k(E)}{2b \sin k(E)}.$$

Using $a(E)+d(E)=2\cos k(E)$ and $a(E)d(E)-b(E)c(E)=1$, one has

$$\begin{aligned} b^2(E)+(a(E)-\cos k(E))^2+\sin^2 k(E) &= b^2(E)+a^2(E)-2a(E)\cos k(E)+1 \\ &= b^2(E)-a(E)d(E)+1 \\ &= b^2(E)-b(E)c(E). \end{aligned}$$

Since $||T_0|| \leq K(E)$, one has $|b(E)| \leq K, |c(E)| \leq K$. It implies

$$\frac{|\varphi(0,E)|^2+|\varphi'(0,E)|^2}{\omega} = \frac{b(E)-c(E)}{2\sin k(E)} \leq \frac{K(E)}{\sin k(E)}.$$

We finish the proof.

□

Lemma 2.5 ([11, Lemma 4.4]). *Let $\{e_i\}_{i=1}^N$ be a set of unit vectors in a Hilbert space $\mathcal{H}$ so that*

$$\alpha = N \sup_{i\neq j} |\langle e_i, e_j\rangle| < 1.$$

Then

$$\sum_{i=1}^N |\langle g, e_i\rangle|^2 \leq (1+\alpha)||g||^2. \tag{2.7}$$

3. Proof of the results in continuous settings

Although the following Lemma is proved for a special class of functions in [21], it works for all $L^2[0,1]$ functions.

Lemma 3.1. *Suppose $V(x) = \frac{O(1)}{1+x}$ and $f \in L^2([0,1])$ is a 1-periodic function. Then*

$$\left| \int_0^x f(t)\frac{\cos 4\theta(t,E)}{1+t}dt \right| = O(1), \tag{3.1}$$

for $E \in \cup_l(\alpha_l,\beta_l)$ with $k(E) \neq \frac{\pi}{2}$. Suppose $E_1, E_2 \in \cup_l(\alpha_l,\beta_l)$ satisfy $k(E_1) \neq k(E_2)$ and $k(E_1)+k(E_2) \neq \pi$. Then we have

$$\left| \int_0^x f(t)\frac{\sin 2\theta(t,E_1)\sin 2\theta(t,E_2)}{1+t}dt \right| = O(1). \tag{3.2}$$

Proof of Theorem 1.1 in the continuous setting. By the assumption of Theorem 1.1, for any $M > A$, we have

$$|V(x)| \leq \frac{M}{1+x}$$

for large x. By shifting the equation, we can assume

$$|V(x)| \leq \frac{M}{1+x} \tag{3.3}$$

for all $x > 0$.

Let us take any standard spectral band $(\alpha,\beta) = (\alpha_l,\beta_l)$ into consideration. Let $\delta \in (\alpha,\beta)$ be such that the quasimomentum $k(\delta) = \frac{\pi}{2}$.

Suppose $E_1, E_2, \ldots E_N \in (\alpha,\delta)\cap P$. By (2.4) (also use $u\in L^2$ implies $u'\in L^2$), we have

$$\sum_{i=1}^{N} R(x,E_i)\in L^2(\mathbb{R}^+),$$

and then there exists $B_j\to\infty$ such that

$$R(B_j,E_i)\leq B_j^{-\frac{1}{2}}, \tag{3.4}$$

for all $i=1,2,\ldots,N$.

By (2.2), one has

$$\ln R(x,E)-\ln R(0,E)=\int_0^x \frac{V(t)}{2\gamma'(t,E)}\sin 2\theta(t,E)dt. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\int_0^{B_j}\frac{V(t)}{\gamma'(t,E_i)}\sin 2\theta(t,E_i)dt\leq -B_j+O(1), \tag{3.6}$$

for all $i=1,2,\ldots,N$.

Now let us consider the Hilbert spaces

$$\mathcal{H}_j=L^2((0,B_j),(1+x)dx).$$

In $\mathcal{H}_j$, by (3.3) we have

$$||V||^2_{\mathcal{H}_j}\leq M^2\log(1+B_j). \tag{3.7}$$

Let

$$e_i^j(x)=\frac{1}{\sqrt{A_i^j}}\frac{\sin 2\theta(x,E_i)}{\gamma'(x,E_i)(1+x)}\chi_{[0,B_j]}(x),$$

where A_i^j is chosen so that e_i^j is a unit vector in $\mathcal{H}_j$.

Obviously,

$$\begin{aligned} A_i^j &= \int_0^{B_j}\frac{\sin^2 2\theta(x,k_i)}{|\gamma'(x,E_i)|^2(1+x)}dx\\ &= \int_0^{B_j}\frac{1}{2|\gamma'(x,E_i)|^2(1+x)}dx-\int_0^{B_j}\frac{\cos 4\theta(x,k_i)}{|\gamma'(x,E_i)|^2(1+x)}dx. \end{aligned} \tag{3.8}$$

By (3.1), one has

$$\left|\int_0^{B_j}\frac{\cos 4\theta(x,k_i)}{|\gamma'(x,E_i)|^2(1+x)}dx\right|=O(1), \tag{3.9}$$

for all $i=1,2,\ldots,N$.

Direct computation shows that

$$\begin{aligned} \int_0^{B_j}\frac{1}{|\gamma'(x,E_i)|^2(1+x)}dx &= O(1)+\sum_{n=0}^{B_j-1}\int_n^{n+1}\frac{1}{|\gamma'(x,E_i)|^2(1+n)}dx\\ &= O(1)+\Gamma(E_i)\log B_j. \end{aligned} \tag{3.10}$$

By (3.8), (3.9) and (3.10), we have

$$A_i^j = \frac{1}{2}\Gamma(E_i)\log B_j + O(1). \tag{3.11}$$

Since $E_i \in (\alpha, \delta)$ $(k(E_i) \in (0, \frac{1}{2}))$ for all $i = 1, 2, \ldots, N$, one has

$$k_i + k_g \neq \pi,$$

for all $1 \leq i, g \leq N$.

By (3.2), we have

$$\frac{O(1)}{\log B_j} \leq \langle e_i^j, e_g^j \rangle \leq \frac{O(1)}{\log B_j}, \tag{3.12}$$

for all $1 \leq i, g \leq N$ and $i \neq g$.

By (3.11) and (3.6)

$$\langle V, e_i^j \rangle_{\mathcal{H}_j} \leq -\frac{\sqrt{2}}{\sqrt{\Gamma(E_i)}}\sqrt{\log B_j} + O(1), \tag{3.13}$$

for large j. By (2.7) and (3.12), one has

$$\sum_{i=1}^{N} |\langle V, e_i^j \rangle_{\mathcal{H}_j}|^2 \leq (1 + \frac{O(1)}{\log B_j}||V||_{\mathcal{H}_j}. \tag{3.14}$$

By (3.13), (3.14) and (3.7), we have

$$\sum_{i=1}^{N} \frac{2}{\Gamma(E_i)} \log B_j \leq M^2 \log B_j + O(1).$$

Let $j \to \infty$, we get

$$\sum_{i=1}^{N} \frac{1}{\Gamma(E_i)} \leq \frac{M^2}{2},$$

for any $M > A$. This implies

$$\sum_{i=1}^{N} \frac{1}{\Gamma(E_i)} \leq \frac{A^2}{2}. \tag{3.15}$$

By (2.5), one has

$$\Gamma(E) \leq \frac{K(E)}{|\sin k(E)|^2}.$$

It implies

$$\sum_{i=1}^{N} |\sin k(E_i)|^2 \leq KA^2. \tag{3.16}$$

$(\alpha, \delta) \cap P$ is a countable set with possible accumulation point α. Similarly, $(\delta, \beta) \cap P$ is a countable set with possible accumulation point β, and the bounds as in (3.15) and (3.16) hold. We finish the proof. □

We need another lemma to treat the special situation $E = \delta$ separately.

Lemma 3.2 ([21])**.** *Fix any spectral band $[\alpha_l, \beta_l]$. Let $\delta_l \in [\alpha_l, \beta_l]$ be such that $k(\delta_l) = \frac{\pi}{2}$. Then there exists some $K > 0$ such that if* $\limsup_{x\to\infty} x|V(x)| \leq \frac{1}{K}$, *then $\delta_l \notin P$.*

Proof of Corollary 1.2 in the continuous setting. By (3.16) and also taking $(\delta, \beta) \cap P$ into consideration, we have

$$\sum_{E_i \in P\cap(\alpha,\beta), E_i\neq\delta} |\sin k(E_i)|^2 \leq KA^2. \tag{3.17}$$

Combining with Lemma 3.2, we have,

$$\sum_{E_i \in P\cap(\alpha,\beta)} |\sin k(E_i)|^2 \leq KA^2. \tag{3.18}$$

By the fact that $2\cos \pi k(E_i) = D(E_i)$, one has

$$\sum_{E_i \in P\cap(\alpha,\beta)} (4 - D(E_i)^2) \leq KA^2. \tag{3.19}$$

Without loss of generality, assume $D(\alpha) = 2$ and $D(\beta) = -2$. If α is non-collapsed,

$$2 - D(E_i) \geq \frac{1}{K}|E_i - \alpha|. \tag{3.20}$$

If α is collapsed, by Theorem 2.1, one has

$$2 - D(E_i) \geq \frac{1}{K}|E_i - \alpha|^2. \tag{3.21}$$

Similarly, if β is non-collapsed,

$$2 + D(E_i) \geq \frac{1}{K}|E_i - \beta|. \tag{3.22}$$

If β is collapsed, by Theorem 2.1, one has

$$2 + D(E_i) \geq \frac{1}{K}|E_i - \beta|^2. \tag{3.23}$$

By (3.18), (3.20), (3.21), (3.22) and (3.23), we have

$$\sum_{E_i \in P\cap(\alpha,\beta)} \min\{|E_i - \alpha|^{\kappa_\alpha}, |E_i - \beta|^{\kappa_\beta}\} \leq KA^2. \tag{3.24}$$

□

Proof of Corollary 1.3 in the continuous setting. Since $\tilde{\alpha}_l$ is non-collapsed, there exists some standard spectral band $[\alpha, \beta] \subset [\tilde{\alpha}_l, \beta_l]$ such that $\alpha = \tilde{\alpha}$ and $D'(\alpha) \neq 0$. Let $\epsilon = \frac{\beta-\alpha}{2}$. Obviously, for any $E \in (\alpha, \alpha + \epsilon) \cap P$,

$$|E - \beta| \geq \frac{1}{K}.$$

Then

$$\min\{|E - \beta|, |E - \alpha|\} \geq \frac{|E - \alpha|}{K}. \tag{3.25}$$

Notice that $\kappa_\alpha = 1$. By (3.24) and (3.25), we have

$$\sum_{E_i \in P \cap (\tilde{\alpha}_l, \tilde{\alpha}_l + \epsilon]} |E_i - \tilde{\alpha}_l| \leq KA^2.$$

Similarly,

$$\sum_{E_i \in P \cap (\tilde{\beta}_l - \epsilon, \tilde{\beta}_l)} \min\{|E_i - \tilde{\beta}_l|\} \leq KA^2.$$

□

Proof of Corollary 1.5. For the free Schrödinger equation, we can choose Floquet solution $\varphi(x, E) = e^{i\sqrt{E}x}$ for $E > 0$. Direct computation and by (2.1), one has

$$\gamma'(x, E) = \sqrt{E}. \tag{3.26}$$

and then

$$\Gamma(E) = \frac{1}{E}. \tag{3.27}$$

In this case, let $\alpha = 0$, $\beta = \infty$ and $\delta = \infty$. Following the proof of (3.15), we have

$$\sum_{E_i \in P \cap (0, \infty)} E_i \leq \frac{A^2}{2}.$$

□

4. Proof of the results in discrete settings

In this section, all the equations are discrete. Similarly, we can also introduce the generalized Prüfer variables $R(n, E)$ and $\theta(n, E)$ for Jacobi matrices. See [26] (also [21]) for details.

$$\frac{R(n+1, E)^2}{R(n, E)^2} = 1 - V(n+1)\frac{2}{\omega} \sin 2\theta(n, E)|\varphi(n, E)|^2 + \frac{4V(n+1)^2|\varphi(n, E)|^4}{\omega^2} \sin^2 \theta(n, E).$$

Let us take one standard spectral band $[\alpha, \beta] = [\alpha_l, \beta_l]$ into consideration. Let $\delta \in (\alpha, \beta)$ be such that the quasimomentum $k(\delta) = \frac{\pi}{2}$.

Lemma 4.1 ([21]). *Suppose $V(n) = \frac{O(1)}{1+n}$ and $f(n)$ is q periodic. Let $E_1, E_2 \in (\alpha, \beta)$ be such that $k(E_1) \neq k(E_2)$ and $k(E_1) + k(E_2) \neq \pi$. Assume $k(E_1) \neq \frac{\pi}{2}$. Then for any $\varepsilon > 0$, there exist $D(E_1, E_2, \varepsilon)$ and $D(E_1, \varepsilon)$ such that*

$$\left| \sum_{t=1}^{n} f(t) \frac{\cos 4\theta(t, E_1)}{1+t} \right| \leq D(E_1, \varepsilon) + \varepsilon \ln n, \tag{4.1}$$

and

$$\left| \sum_{t=1}^{n} f(t) \frac{\sin 2\theta(t, E_1) \sin 2\theta(t, E_2)}{1+t} \right| \leq D(E_1, E_2, \varepsilon) + \varepsilon \ln n. \tag{4.2}$$

Proof of Theorem 1.1 in the discrete setting. Replacing Lemma 3.1 by Lemma 4.1 and using generalized Prüfer variables for Jacobi matrices in [26], the proof of discrete case can be proceeded in a similar way as that of continuous case. We omit the details. □

Lemma 4.2 ([21])**.** *There exists some $K > 0$ such that if $\limsup_{n\to\infty} n|V(n)| \leq \frac{1}{K}$, then $\delta \notin P$.*

Proof of Corollaries 1.2 and 1.3 in the discrete setting. The proof follows from the continuous case. We omit the details here. □

In order to prove Corollary 1.6, we also need one Lemma.

Lemma 4.3 ([19, Theorem 2.1])**.** *If $V_0 = 0$ and $\limsup_{n\to\infty} n|V(n)| < 1$, then 0 can not be in P.*

Proof of Corollary 1.6. In this case, we only have one spectral band $[-2, 2]$. Moreover, $\varphi(n, E) = e^{ik(E)n}$, $2\cos k(E) = E$ for $E \in (-2, 2)$ and $\Gamma(E) = \frac{1}{\sin^2 k(E)}$. By Theorem 1.1($\delta = 0$),

$$\sum_{E_i\in P\cap(-2,2);E_i\neq 0} \frac{1}{\Gamma(E_i)} \leq A^2.$$

Thus

$$\sum_{E_i\in P\cap(-2,2);E_i\neq 0} (4 - E_i^2) \leq 4A^2. \tag{4.3}$$

We may add 4 in the bound of (4.3) if 0 is in P. However, if $A < 1$, by Lemma 4.2, 0 can not be in P. We finish the proof. □

Proof of Corollaries 1.7 and 1.8. It follows from Corollaries 1.2 and 1.3 immediately since the discrete operator has finitely many spectral bands. □

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References

[1] D. Damanik, D. Hundertmark, R. Killip, and B. Simon, *Variational estimates for discrete Schrödinger operators with potentials of indefinite sign*, Comm. Math. Phys. **238** (2003), 545–562.

[2] D. Damanik and R. Killip, *Half-line Schrödinger operators with no bound states*, Acta Math. **193** (2004), 31–72.

[3] D. Damanik and C. Remling, *Schrödinger operators with many bound states*, Duke Math. J. **136** (2007), 51–80.

[4] S. A. Denisov and A. Kiselev, *Spectral properties of Schrödinger operators with decaying potentials*, In *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, volume 76 of *Proc. Sympos. Pure Math.*, pages 565–589, Amer. Math. Soc., Providence, RI, 2007.

[5] M. S. P. Eastham, *The spectral theory of periodic differential equations*, Texts in Mathematics (Edinburgh), Scottish Academic Press, Edinburgh; Hafner Press, New York, 1973.

[6] E. Judge, S. Naboko, and I. Wood, *Spectral results for perturbed periodic Jacobi matrices using the discrete Levinson technique*, Studia Math. **242** (2018), 179–215.

[7] E. Judge, S. Naboko, and I. Wood, *Embedded eigenvalues for perturbed periodic Jacobi operators using a geometric approach*, J. Difference Equ. Appl. **24** (2018), 1247–1272.

[8] R. Killip, *Perturbations of one-dimensional Schrödinger operators preserving the absolutely continuous spectrum*, Int. Math. Res. Not. **38** (2002), 2029–2061.

[9] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. (2) **158** (2003), 253–321.

[10] A. Kiselev, *Imbedded singular continuous spectrum for Schrödinger operators*, J. Amer. Math. Soc. **18** (2005), 571–603.

[11] A. Kiselev, Y. Last, and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Comm. Math. Phys. **194** (1998), 1–45.

[12] A. Kiselev, C. Remling, and B. Simon, *Effective perturbation methods for one-dimensional Schrödinger operators* J. Differential Equations **151** (1999), 290–312.

[13] H. Krüger, *On the existence of embedded eigenvalues*, J. Math. Anal. Appl. **395** (2012), 776–787.

[14] P. Kuchment, *An overview of periodic elliptic operators*, Bull. Amer. Math. Soc. (N.S.) **53** (2016), 343–414.

[15] P. Kuchment and B. Vainberg, *On embedded eigenvalues of perturbed periodic Schrödinger operators*, In *Spectral and scattering theory (Newark, DE, 1997)*, pages 67–75. Plenum, New York, 1998.

[16] P. Kuchment and B. Vainberg, *On absence of embedded eigenvalues for Schrödinger operators with perturbed periodic potentials*, Comm. Partial Differential Equations **25** (2000), 1809–1826.

[17] Y. Last and M. Lukic, *ℓ^2 bounded variation and absolutely continuous spectrum of Jacobi matrices*, Comm. Math. Phys. **359** (2018), 101–119.

[18] W. Liu, *Absence of the singular continuous spectrum for perturbed discrete Schrödinger operators*, J. Math. Anal. Appl. **472** (2019), 1420–1429.

[19] W. Liu, *Criteria for embedded eigenvalues for discrete Schrödinger operators*, arXiv preprint arXiv:1805.02817, 2018.

[20] W. Liu, *WKB and absence of the singular continuous spectrum for perturbed periodic Schrödinger operators*, Preprint, 2018.

[21] W. Liu and D. C. Ong, *Sharp spectral transition for eigenvalues embedded into the spectral bands of perturbed periodic operators*, J. Anal. Math. to appear.

[22] V. Lotoreichik and S. Simonov, *Spectral analysis of the half-line Kronig-Penney model with Wigner–Von Neumann perturbations*, Rep. Math. Phys. **74** (2014), 45–72.

[23] M. Lukic, *Schrödinger operators with slowly decaying Wigner-von Neumann type potentials*, J. Spectr. Theory **3** (2013), 147–169.

[24] M. Lukic, *A class of Schrödinger operators with decaying oscillatory potentials*, Comm. Math. Phys. **326** (2014), 441–458.

[25] M. Lukic and D. C. Ong, *Wigner-von Neumann type perturbations of periodic Schrödinger operators*, Trans. Amer. Math. Soc. **367** (2015), 707–724.

[26] M. Lukic and D. C. Ong, *Generalized Prüfer variables for perturbations of Jacobi and CMV matrices*, J. Math. Anal. Appl. **444** (2016), 1490–1514.

[27] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials*, Comm. Math. Phys. **193** (1998), 151–170.

[28] C. Remling, *Schrödinger operators with decaying potentials: some counterexamples*, Duke Math. J. **105** (2000), 463–496.

[29] B. Simon and A. Zlatoš, *Sum rules and the Szegő condition for orthogonal polynomials on the real line*, Comm. Math. Phys. **242** (2003), 393–423.

[30] S. Simonov, *Zeroes of the spectral density of discrete Schrödinger operator with Wigner-von Neumann potential*, Integral Equations Operator Theory **73** (2012), 351–364.

[31] S. Simonov, *Zeroes of the spectral density of the Schrödinger operator with the slowly decaying Wigner–von Neumann potential* Math. Z. **284** (2016), 335–411.

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