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A SURVEY ON THE REPLICATOR DYNAMICS FOR GAMES WITH STRATEGIES IN METRIC SPACES

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Abstract. This survey concerns the replicator dynamics for games in which the strategy sets are metric spaces. In this case the replicator dynamics evolves in a Banach space. We provide a review on the existence of solutions for the replicator dynamics, characterizations on the relation between a Nash equilibrium of a normal form game and the replicator dynamics, stability criteria for the replicator dynamics with respect to different topologies and metrics, and approximation theorems for numerical implementation. These approximations are studied in both the weak topology and the strong topology. Some numerical examples illustrate our results.

1. Introduction

Evolutionary games form a class of noncooperative population games that study the population decisions as an interaction of the strategies using evolutionary ideas from two different approaches, static and dynamic. The static approach captures evolutionary concepts by defining and studying equilibrium terms. The dynamic approach, on the other hand, studies the interaction of strategies as a dynamical system, which is determined by a system of differential equations. By a population decision, we mean the decision of a (very) large number of small competitors, where "small" means that each competitor has a very little influence on the behavior of the overall system.

This survey concerns the dynamic approach with a specific dynamical system known as the *replicator dynamics* for evolutionary games where the set of strategies are measurable spaces. In game theory it is important to consider models with strategies in measurable spaces because this allows us to present in a unified manner essentially all standard model, namely, games with finite strategy sets, compact or unbounded intervals, and so on. This is the case of some models in oligopoly theory, international trade theory, war of attrition, and public goods, among many others.

An evolutionary game is said to be symmetric if there are two players only and, furthermore, they have the same strategy sets and the same payoff functions. This type of games models interactions of the strategies of a single population. In contrast, an asymmetric evolutionary game, also known as a multipopulation game, is

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a game with a finite set of players (or populations) each of which has a different set of strategies and different payoff functions.

If for an evolutionary game the pure strategies set of each player (or population) is a metric space, then consequently the replicator dynamics lives in a Banach space (a space of finite signed measures). In particular, if we have n players each of which has m_i strategies, for $i = 1, ..., n$, then the replicator dynamics lives in R^m, where $m = m_1 + ... + m_n$.

The main goal of this survey is to provide a general framework to study the replicator dynamics for games in which the strategy set is a measurable space—more precisely, a separable metric space. In fact, we are particularly interested in reviewing topics such as the existence of solutions to the replicator dynamics, characterizations of the relation between a Nash equilibrium of a certain normal form game and the replicator dynamics, stability criteria for the replicator dynamics, and finite-dimensional approximations for numerical implementation.

For characterizations on the relation between a Nash equilibrium of a normal form game and the replicator dynamics, see, for instance, relations 5.5 and Propositions 4.4, 4.5, and 5.4. With respect to stability criteria for the replicator dynamics for asymmetric games we refer to Theorems 4.7 and 4.10. For symmetric games, we also present results about stability on different topologies and metrics, as in Theorems 6.1, 6.2, and 6.3. We also review conditions under which a finite-dimensional dynamical system approximates the replicator dynamics for games with strategies in metric spaces. In this manner, we can use numerical techniques for finite-dimensional differential equations to approximate a solution to the replicator dynamics, which typically lives in an infinite-dimensional Banach space. See Theorems 8.1 and 8.4, and Sections 9.2 and 9.3.

Conditions for the existence of solutions to the replicator dynamics in measure spaces for asymmetric games are given by Mendoza-Palacios and Hernández-Lerma [31]; for symmetric games conditions are given by several authors, including Bomze [5], Oechssler and Riedel [35], and more generally (including dynamics different from the replicator equation) by Cleveland and Ackleh [8].

Similarly, conditions for dynamic stability of asymmetric games are given by Mendoza-Palacios and Hernández-Lerma [31] and Narang and Shaiju [33]. For symmetric games, conditions for dynamic stability have been developed with respect to different topologies, as in e.g. Bomze [4], Oechssler and Riedel [35] and [36], Eshel and Sansone [13], Veelen and Spreij [45], Cressman and Hofbauer [11]. We standardized different stability criteria with respect to various metrics and topologies in the probability-measures space and present a brief review of results on the stability of the replicator dynamics.

In the theory of evolutionary games there are several interesting dynamics, for instance, the imitation dynamics, the monotone-selection dynamics, the best-response dynamics, the Brown-von Neumann-Nash dynamics, and so forth (see, for instance, Hofbauer and Sigmund [21], [22], Sandholm [42]). Some of this evolutionary dynamics have been extended to games with strategies in a space of probability measures. For instance, Hofbauer, Oechssler and Riedel [20] extend the Brown-von Neumann-Nash dynamics; Lahkar and Riedel extend the logit dynamics [28]. These publications establish conditions for the existence of solutions and the stability of the corresponding dynamical systems. Cheung proposes a general theory for pairwise comparison dynamics [6] and for imitative dynamics [7]. M. Ruijgrok and T. Ruijgrok [41] extend the replicator dynamics with a mutation term.

We selected the replicator dynamics partly because it is the most studied dynamics for games with strategies in metric spaces, and partly because it has many interesting properties, as can be seen in Cressman [9], Hofbauer and Weibull [23], and many other references. In particular, with the replicator dynamics it is not difficult to construct a proof of the existence of Nash equilibria and, moreover, when the strategy set is finite, we can give a geometric characterization of the set of Nash equilibria; see Harsanyi [16], Hofbauer and Sigmund [21], Ritzberger [40].

The paper is organized as follows. Section 2 presents notation and technical requirements. Section 3 introduces a heuristic approach to the replicator dynamics and describes its relation with evolutionary games. Some important technical issues are also summarized. Section 4 establishes the relation between the replicator dynamics and a normal form game, using the concepts of *Nash equilibria* and other static equilibria such as a *strong uninvadible profile* and a *strong unbeatable profile.* Section 5 describes the replicator dynamics for symmetric games. Section 6 contains a review of results on the stability of the replicator dynamics. In this same section the different stability criteria are standardized with respect to various metrics and topologies in a space of probability measures.

Section 7 establishes a relationship between Nash equilibria and other stability concepts for the replicator dynamics. Section 8 proposes approximation theorems for the replicator dynamics on measure spaces, by means of dynamical systems on finite-dimensional spaces. Section 9 presents examples to illustrate our results. We conclude in section 10 with some general comments on possible extensions. An appendix contains results of some technical facts.

2. Technical preliminaries

2.1. Spaces of signed measures. Consider a separable metric space A and its Borel σ-algebra $\mathcal{B}(A)$. Let $\mathbb{M}(A)$ be the Banach space of finite signed measures μ on $\mathcal{B}(A)$ endowed with the total variation norm

$$\|\mu\| := \sup_{\|f\|\leq 1} \left| \int_A f(a)\mu(da) \right| = |\mu|(A), \tag{2.1}$$

where the supremum in (2.1) is taken over functions in the Banach space $\mathbb{B}(A)$ of real-valued bounded measurable functions on A, endowed with the supremum norm

$$\|f\| := \sup_{a\in A} |f(a)|. \tag{2.2}$$

Consider the subset $\mathbb{C}_B(A) \subset \mathbb{B}(A)$ of all real-valued continuous and bounded functions on A. Consider the dual pair $(\mathbb{C}_B(A), \mathbb{M}(A))$ given by the bilinear form $\langle \cdot, \cdot \rangle : \mathbb{C}_B(A) \times \mathbb{M}(A) \to \mathbb{R}$

$$\langle g, \mu \rangle = \int_A g(a)\mu(da). \tag{2.3}$$

We consider the *weak topology* on $\mathbb{M}(X)$ (induced by $\mathbb{C}_B(X)$),i.e., the topology under which all elements of $\mathbb{C}_B(X)$, when regarded as linear functionals $\langle g, \cdot\rangle$ on $\mathbb{M}(A)$ are continuous. In this topology a neighborhood of a point $\mu \in \mathbb{M}(A)$ is of the form

$$\mathcal{V}_\epsilon^{\mathcal{H}}(\mu) := \left\{\nu \in \mathbb{M}(A) : \ |\langle g, \nu - \mu\rangle| < \epsilon \ \forall g \in \mathcal{H}\right\} \tag{2.4}$$

for $\epsilon > 0$ and $\mathcal{H}$ a finite subset of $\mathbb{M}(A)$.

2.2. **Metrics on $\mathbb{P}(A)$.** There are many metrics that metrize the weak topology on the set of probability measures. Here we use the Kantorovich-Rubinstein metric and the Wasserstein distance. Let (A, ϑ) be a separable metric space, and $\mathbb{P}(A)$ the set of probability measures on A. For any $\mu, \nu \in \mathbb{P}(A)$ we define the **Kantorovich-Rubinstein metric** r_{kr}, as

$$r_{kr}(\mu,\nu) := \sup_{f\in\mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) - \int_A f(a)\nu(da) \ : \ \|f\|_L \le 1 \right\}, \tag{2.5}$$

where $(\mathbb{L}(A), \|\cdot\|_L)$ is the space of continuous real-valued functions on A that satisfy the Lipschitz condition

$$\|f\|_L := \sup |f(a) - f(b)|/\vartheta(a, b) < \infty \ \ \forall\, a, b \in A, \ a \ne b. \tag{2.6}$$

Let a_0 be a fixed point in A and

$$\mathbb{M}_K(A) := \left\{ \mu \in \mathbb{M}(A) : \sup_{f\in\mathbb{L}(A)} \int_A |f(a)|\mu(da) < \infty \right\}. \tag{2.7}$$

The Kantorovich-Rubinstein metric r_{kr} can be extended as a norm on $\mathbb{M}_K(A)$ defined as

$$\|\mu\|_{kr} := |\mu(A)| + \sup_{f\in\mathbb{L}(A)} \left\{ \int_A f(a)\mu(da) \ : \ \|f\|_L \le 1, \ f(a_0) = 0 \right\} \tag{2.8}$$

for any μ in $\mathbb{M}(A)$ (see Bogachev [3], chapter 8). Note that for any $\mu, \nu \in \mathbb{P}(A)$, $r_{kr}(\mu, \nu) = \|\mu - \nu\|_{kr}$.

Let us suppose that the separable metric space A is also complete (that is, A is a so-called Polish space), and let a_0 be a fixed point in A. For each p with $1 \le p < \infty$, we define the space $\mathbb{P}_p(A)$ as

$$\mathbb{P}_p(A) := \left\{ \mu \in \mathbb{P}(A) : \ \int_A [\vartheta(a, a_0)]^p \mu(da) < \infty \right\}.$$

The **L^p-Wasserstein distance** r_{w_p} between μ and ν in $\mathbb{P}_p(A)$ is defined by

$$r_{w_p}(\mu,\nu) := \left[\inf_{\pi\in\Pi} \int_A \int_A \vartheta(a, b)\pi(da, db) \right]^{\frac{1}{p}}, \tag{2.9}$$

where Π is the set of probability measures on $A \times A$ with marginals μ and ν. In particular, when $p = 1$ we write the **L^1-Wasserstein distance** r_{w_1} as r_w and in addition we have that $r_w = r_{kr}$ on $\mathbb{P}(A)$.

Remark 2.1. There are some metrics that metrize the weak topology on $\mathbb{P}(A)$ that are particularly useful, for instance, the **Prokhorov metric** r_p, the **bounded Lipschitz metric** r_{bl}, the **Kantorovich-Rubinstein metric** r_{kr}, and the L^p**-Wasserstein distance** r_{w_p}. (For details see, for instance, Shiryaev [43], Billingsley [1] or Villani [46].) In the rest of this paper we will denote by r_{w^*} **any metric that metrizes the weak topology** on $\mathbb{P}(A)$ (not to be confused with the notation r_w of the L^1-Wasserstein distance). Moreover, we denote by r **any metric on** $\mathbb{P}(A)$ that is either the total variation norm (2.1) or any distance that metrizes the weak topology. An *open ball* in the metric space $(\mathbb{P}(A), r)$ is defined in the classical form

$$\mathcal{V}^r_\alpha(\mu) := \Big\{\nu \in \mathbb{P}(A) : \; r(\nu,\mu) < \alpha\Big\} \tag{2.10}$$

where $\alpha > 0$.

Remark 2.2. Let A be a separable metric space, and r_{w^*} any distance that metrizes the weak topology τ_{w^*} in $\mathbb{P}(A)$. Let μ be any measure in $\mathbb{P}(A)$, and consider the family $\mathcal{V}^H(\mu)$ of neighborhoods $\mathcal{V}^{\mathcal{H}}_\epsilon(\mu)$ of the form (2.4). In addition, consider the family $\mathcal{V}^{r_{w^*}}(\mu)$ of the open balls $\mathcal{V}^{r_{w^*}}_\alpha(\mu)$ of the form (2.10). Both families $\mathcal{V}^{\mathcal{H}}(\mu)$ and $\mathcal{V}^{r_{w^*}}(\mu)$ are neighborhood basis for μ in the space $(\mathbb{P}(A), \tau_{w^*})$. For details see Pedersen [38], chapters I-II.

Then, a neighborhood $\mathcal{V}^{\mathcal{H}}_\epsilon(\mu)$ for μ is contained in some open ball $\mathcal{V}^{r_{w^*}}_\alpha(\mu)$ with center μ. The inverse is also true, i.e., any open ball $\mathcal{V}^{r_{w^*}}_\alpha(\mu)$ is contained in some neighborhood $\mathcal{V}^{\mathcal{H}}_\epsilon(\mu)$.

2.3. **Differentiability.**

Definition 2.3. Let A be a separable metric space. We say that a mapping $\mu : [0,\infty) \to \mathbb{M}(A)$ is strongly differentiable if there exists $\mu'(t) \in \mathbb{M}(A)$ such that, for every $t > 0$,

$$\lim_{\epsilon\to 0}\left\|\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon} - \mu'(t)\right\| = 0. \tag{2.11}$$

Note that, by (2.1), the left-hand side in (2.11) can be expressed as

$$\lim_{\epsilon\to 0}\sup_{\|g\|\le 1}\left|\frac{1}{\epsilon}\left[\int_A g(a)\mu(t+\epsilon,da) - \int_A g(a)\mu(t,da)\right] - \int_A g(a)\mu'(t,da)\right|.$$

For weak differentiability, see Remark 8.3, below.

2.4. **Product Spaces.** Consider a finite family of metric spaces $\{X_i\}_{i=1}^n$ and their σ-algebras $\mathcal{B}(X_i)$, as well as the Banach spaces $(\mathbb{M}(X_i), \|\cdot\|)$ and $(\mathbb{M}_K(X_i), \|\cdot\|_{kr})$. For $i = 1,...,n$, let $\mu_i \in \mathbb{M}(X_i)$ and consider the elements $\mu = (\mu_1,...,\mu_n)$ in the product space $\mathbb{M}(X_1)\times ... \times \mathbb{M}(X_n)$ with the norm

$$\|\mu\|_\infty := \max_{1\le i\le n}\|\mu_i\| < \infty. \tag{2.12}$$

These elements form the Banach space $(\mathbb{M}(X_1)\times ... \times \mathbb{M}(X_n), \|\cdot\|_\infty)$. We can similarly define the Banach space $(\mathbb{M}_K(X_1)\times ... \times \mathbb{M}_K(X_n), \|\cdot\|^{kr}_\infty)$, where

$$\|\mu\|^{kr}_\infty := \max_{1\le i\le n}\|\mu_i\|_{kr} < \infty. \tag{2.13}$$

3. Replicator dynamics for asymmetric games

3.1. **A heuristic approach to the replicator dynamics.** Let $I =: \{1, 2, ..., n\}$ be a set of different species (or players). Each individual of the species $i \in I$ can choose a single element a_i in a set of characteristics (the set of pure strategies or pure actions) A_i, which is a separable metric space. For every $i \in I$ and every vector $a := (a_1, ..., a_n)$ in the Cartesian product $A_1 \times \cdots \times A_n$ we write a as (a_i, a_{-i}) where $a_{-i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$ is in

$$A_{-i} := A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n.$$

For each $i \in I$, let $\mathcal{B}(A_i)$ be the Borel σ-algebra of A_i, and $N_i \in \mathbb{M}(A_i)$ a positive measure such that for each E_i in $\mathcal{B}(A_i)$, $N_i(E_i)$ assigns the *number* (or *mass*) of individuals using pure strategies a_i in E_i. Then the total population of the species i is $N_i(A_i)$ and the proportion of individuals using strategies in E_i is

$$\mu_i(E_i) := \frac{N_i(E_i)}{N_i(A_i)}, \tag{3.1}$$

so μ_i is a population distribution over the set of actions A_i and it is an element of $\mathbb{P}(A_i)$, the set of probability measures on A_i, also known as the set of *mixed strategies*. For every $i \in I$ and vector $\mu := (\mu_1, ..., \mu_n)$ in $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, we write μ as (μ_i, μ_{-i}) where $\mu_{-i} := (\mu_1, ..., \mu_{i-1}, \mu_{i+1}, ..., \mu_n)$ is in

$$\mathbb{P}(A_1) \times \cdots \mathbb{P}(A_{i-1}) \times \mathbb{P}(A_{i+1}) \times \cdots \times \mathbb{P}(A_n).$$

For each species i we assign a payoff function $J_i : \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) \to \mathbb{R}$ defined as

$$J_i(\mu_1, ..., \mu_n) := \int_{A_1} \cdots \int_{A_n} U_i(a_1, ..., a_n)\mu_n(da_n)...\mu_1(da_1), \tag{3.2}$$

where $U : A_1 \times \cdots \times A_n \to \mathbb{R}$ is a given measurable function. If $\delta_{\{a_i\}}$ is a probability measure concentrated at $a_i \in A_i$, the vector $(\delta_{\{a_i\}}, \mu_{-i})$ is written as $(a?i, \mu_{-i})$, and then

$$\begin{aligned} J_i(\delta_{\{a_i\}}, \mu_{-i}) &= J_i(a_i, \mu_{-i}) \\ &= \int_{A_{-i}} U(a_i, a_{-i})\mu_{-i}(da_{-i}). \end{aligned}$$

In particular, (3.2) yields

$$J_i(\mu_i, \mu_{-i}) := \int_{A_i} J_i(a_i, \mu_{-i})\mu_i(da_i). \tag{3.3}$$

For each $i \in I$, $E_i \in \mathcal{B}(A_i)$ and time $t \in [0, \infty)$, let $N_i(t, E_i)$ and $\mu_i(t, E_i)$ be the mass-measure $N_i(t) \in \mathbb{M}(A_i)$ and the probability measure $\mu(t) \in \mathbb{P}(A_i)$ (as in (3.1)) evaluated at E_i. Let γ_1, γ_2 be the background per capita birth and death rates in the population. The background per capita net birth rate $\gamma := \gamma_1 - \gamma_2$ is modified by the payoff $J_i(a_i, \cdot)$ for using strategy $a_i \in A_i$. The rate of change of the number of individuals is

$$N_i'(t, E_i) = \gamma N_i(t, E_i) + N_i(t, A_i) \int_{E_i} J_i(a_i, \mu_{-i}(t))\mu_i(t, da_i) \tag{3.4}$$

for $E_i \in \mathcal{B}(A_i)$, with some initial positive measure $N_i(0)$ in $\mathbb{M}(A_i)$. The notation $N_i'(t, E_i)$ represents the strong derivative of $N_i(t)$ in the Banach space $\mathbb{M}(A_i)$ (see Definition 2.3) valued at $E_i \in \mathcal{B}(A_i)$ and $\mu_i(t)$ is a probability measure defined as in (3.1).

For each t in $[0, \infty)$ the term $\int_{E_i} J_i(a_i, \mu_{-i}(t))\mu_i(t, da_i)$ in (3.4) values the efficiency of the strategies in the set E_i when the other species have a distribution $\mu_{-i}(t)$. Note that if $J_i(\cdot, \cdot) \equiv 0$, the solution of (3.4) is $N_i(t, E_i) = N_i(0, E_i)e^{\gamma t}$ for all $E_i \in \mathcal{B}(A_i)$, and $t \geq 0$. Using (3.1) we have that

$$N_i'(t, E_i) = N_i(t, A_i)\mu_i'(t, E_i) + N_i'(t, A_i)\mu_i(t, E_i) \ \forall E_i \in \mathcal{B}(A_i), \ t \geq 0. \tag{3.5}$$

Hence, by (3.5), we may rewrite (3.4) as the differential equation

$$\mu_i'(t, E_i) = \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t))\Big]\mu_i(t, da_i) \tag{3.6}$$

for each E_i in $\mathcal{B}(A_i)$ and $t \geq 0$, whose solution lives in the space of probability measures. The equation (3.6) is known as the *replicator dynamics* for the asymmetric case.

3.2. **Asymmetric evolutionary games.** In an evolutionary game, the strategies' dynamics is determined by a differential equation of the form

$$\mu_i'(t) = F_i(\mu_1(t), ..., \mu_n(t)) \ \forall i \in I, \ t \geq 0, \tag{3.7}$$

with some initial condition $\mu_i(0) = \mu_{i,0}$ for each $i \in I$. The notation $\mu_i'(t)$ represents the strong derivative of $\mu_i(t)$ (see Definition 2.3), and $F_i(\cdot)$ is a mapping $F_i : \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) \to \mathbb{M}(A_i)$, which is associated with the payoff (3.2). Let $F : \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) \to \mathbb{M}(A_n) \times \cdots \times \mathbb{M}(A_n)$, where $F(\mu) := (F_1(\mu), ..., F_n(\mu))$, and consider the vector $\mu'(t) = (\mu_1'(t), ..., \mu_n'(t))$. Then the system (3.7) can be expressed as

$$\mu'(t) = F(\mu(t)) \tag{3.8}$$

which is defined on the Banach spaces $\mathbb{M}(A_1) \times \cdots \times \mathbb{M}(A_n)$ endowed with the norm (2.12).

More explicitly we write (3.7) as

$$\mu_i'(t, E_i) = F_i(\mu(t), E_i) \ \forall E_i \in \mathcal{B}(A_i), \tag{3.9}$$

where $\mu_i'(t, E_i)$ and $F_i(\mu(t), E_i)$ are the measures $\mu_i'(t)$ and $F_i(\mu(t))$ valued at $E_i \in \mathcal{B}(A_i)$.

We shall be working with a special class of so-called asymmetric evolutionary games which can be described as a quadruple

$$\Big[I, \ \Big\{\mathbb{P}(A_i)\Big\}_{i \in I}, \ \Big\{J_i(\cdot)\Big\}_{i \in I}, \ \mu'(t) = F(\mu(t))\Big], \tag{3.10}$$

where

i) $I = \{1, ..., n\}$ is the set of players;

ii) for each player $i \in I$ we have a set $\mathbb{P}(A_i)$ of mixed actions and a payoff function $J_i : \mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n) \to \mathbb{R}$ (as in (3.2)); and *iii)* the dynamics $\mu'(t) = F(\mu(t))$ (as in (3.8)) is described by the replicator equation (3.6), i.e., for $i \in I$ and each E_i in $\mathbb{B}(A_i)$,

$$F_i(\mu(t), E_i) := \int_{E_i} \Big[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t)) \Big] \mu_i(t, da_i). \tag{3.11}$$

3.3. **Technical issues on the replicator dynamics.** For future reference, in the remainder of this section we summarize conditions for the existence of a unique solution to the differential equation (3.8), and an important property of this solution (see Theorems 3.1 and 3.3, respectively) . These results can be traced back to Mendoza-Palacios and Hernández-Lerma [31]. See also Bomze [5], Oechssler and Riedel [35] for the symmetric case.

For each $i \in I$ and $t \geq 0$, let

$$\beta_i(a_i|\mu(t)) := J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t)), \tag{3.12}$$

which is the integrand of (3.6). Hence, by (3.12), $\beta(\cdot|\mu(t))$ is the Radon-Nikodym density of $F_i(\mu(t))$ with respect to $\mu(t)$, i.e.,

$$F_i(\mu(t), E_i) = \int_{E_i} \beta_i(a_i|\mu(t))\mu_i(t, da_i) \ \forall E_i \in \mathcal{B}(A_i).$$

Theorem 3.1. *Suppose that, for each $i \in I$, the function $\beta_i(\cdot|\mu)$ in (3.12) satisfies:*

i) there exists $C_i \geq 0$ such that

$$|\beta_i(a_i|\mu)| \leq C_i \ \forall a_i \in A_i \text{ and } \|\mu|_\infty \leq 2,$$

ii) there is a constant $D_i > 0$, such that

$$\sup_{a_i \in A_i} |\beta_i(a_i|\eta) - \beta_i(a_i|\nu)| \leq D_i \|\eta - \nu\|_\infty \ \forall \nu, \eta \text{ with } \|\eta\|_\infty, \|\nu\|_\infty \leq 2.$$

Then there exists a unique solution to the replicator dynamics (3.6)-(3.8).

Proposition 3.2. *Let $i \in I$. If the payoff function $U_i(\cdot)$ in (3.2) is bounded, then $\beta_i(\cdot|\mu)$ satisfies the conditions i) and ii) of Theorem 3.1.*

Theorem 3.3. *Suppose that the conditions i) and ii) of Theorem 3.1 are satisfied. If $\mu(t)$ is a solution of the replicator dynamics (3.6)-(3.8) with initial condition $\mu(0)$ in $\mathbb{P}(A_1) \times \cdots \times \mathbb{P}(A_n)$, then:*

i) $\mu_i(0) << \mu_i(t)$ and $\mu_i(t) << \mu_i(0)$ for all $i \in I$ and $t > 0$, with Radon-Nikodym density

$$\frac{d\mu_i(t)}{d\mu_i(0)}(a_i) = e^{\int_0^t \beta_i(a_i|\mu(s))ds}. \tag{3.13}$$

ii) In particular, for every $i \in I$ and $t > 0$, if ν_i is a probability measure satisfying that $\nu_i << \mu_i(t)$ whenever $\nu_i << \mu_i(0)$, then

$$\log \frac{d\nu_i}{d\mu_i(t)}(a_i) = \log \frac{d\nu_i}{d\mu_i(0)}(a_i) - \int_0^t \beta_i(a_i|\mu(s))ds. \tag{3.14}$$

4. Stability for the asymmetric case

4.1. **The replicator dynamics, NE, SUP and SUbP.** In this section we consider asymmetric evolutionary games as in (3.10) and compare them with normal form games (4.1), below. We wish to study the relation between a Nash equilibrium of a normal form game and the replicator dynamics (Proposition 4.4) . We also define two important concepts *strong uninvadable profile* (Definition 4.2) and *strong unbeatable profile* (Definition 4.3), and analyze their relation to a Nash equilibrium (Proposition 4.5).

A normal form game Γ (also known as a game in strategic form) can be described as

$$\Gamma := \Big[I,\ \Big\{ \mathbb{P}(A_i) \Big\}_{i\in I},\ \Big\{ J_i(\cdot) \Big\}_{i\in I} \Big], \tag{4.1}$$

where

i) $I = \{1, 2, ..., n\}$ is the set of players,

ii) for each player $i \in I$ we specify a set of actions (or strategies) $\mathbb{P}(A_i)$ and a payoff function $J_i : \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) \to \mathbb{R}$ (as in (3.2)).

Definition 4.1. Let Γ be a normal form game. A vector μ^* in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$ is called an ϵ-equilibrium ($\epsilon > 0$) if, for all $i \in I$,

$$J_i(\mu_i^*, \mu_{-i}^*) \geq J_i(\mu_i, \mu_{-i}^*) - \epsilon \ \forall \mu_i \in \mathbb{P}(A_i).$$

If the inequality is true when $\epsilon = 0$, then μ^* is called a Nash equilibrium.

The following definition is an extended version of strongly uninvadable strategies of symmetric games (for details see Bomze [5]).

Definition 4.2. A vector $\mu^* \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n)$ is called a strong uninvadable profile (SUP) in a set $\mathcal{C}$ if μ^* is in $\mathcal{C}$ and the following holds. There exists $\epsilon > 0$ such that for any $\mu \in \mathcal{C}$ with $\|\mu - \mu^*\|_\infty < \epsilon$, and every $i \in I$, $J_i(\mu_i^*, \mu_{-i}) > J_i(\mu_i, \mu_{-i})$ *if* $\mu_i \neq \mu_i^*$. In particular, if

$$\mathcal{C} = \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n),$$

then μ^* is simply called a strong uninvadable profile (SUP). In both cases, we call ϵ the global invasion barrier.

The following definition is an extended version of strongly unbeatable strategies of symmetric games (for details see Hingu, Rao, and Shaiju [18]).

Definition 4.3. A vector $\mu^* \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n)$ is called a strong unbeatable profile (SUbP) if there exists $\epsilon > 0$ such that for any $\mu \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n)$ with $\|\mu - \mu^*\|_\infty < \epsilon$, $J_i(\mu_i^*, \mu_{-i}) \geq J_i(\mu_i, \mu_{-i})$ for every $i \in I$.

As usual, the open neighborhood with center μ^* and radius $\varepsilon > 0$ is defined as

$$V_\varepsilon(\mu^*) := \{\mu \in \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n) :\ \|\mu - \mu^*\|_\infty < \varepsilon\}. \tag{4.2}$$

The following proposition gives an important property, namely the relation between a Nash equilibrium of a normal form game and the replicator equation.

Proposition 4.4. *Suppose that $\mu^* = (\mu_1^*, ..., \mu_n^*)$ is a Nash equilibrium of Γ . Then μ^* is a critical point of the replicator dynamics (3.6)-(3.8), i.e., $F(\mu^*) = 0$.*

Proof. See Mendoza-Palacios and Hernández-Lerma [31], Theorem 5.4. □

The following proposition gives the relation between an ϵ-equilibrium (or a Nash equilibrium), strong uninvadable profiles, and strong unbeatable profiles.

Proposition 4.5. *i) Suppose that the payoff function $U_i(\cdot)$ in (3.2) is bounded for all $i \in I$. Let μ^* be a* SUP in a set $\mathcal{C}$ *with global invasion barrier $\epsilon_1 > 0$. If the set $\mathcal{C} \cap V_{\epsilon_1}(\mu^*)$ has a convex and nonempty interior, then μ^* is an ϵ_2-equilibrium of Γ, where $\epsilon_2 > 0$ depends on ϵ_1. Moreover, if μ^* is a* SUP, *then μ^* is a Nash equilibrium and the boundedness hypothesis is not required.*
ii) If μ^* a SUbP, *then μ^* is a Nash equilibrium.*
ii) If μ^* a SUP, *then μ^* is a* SUbP.

Proof. See Mendoza-Palacios and Hernández-Lerma [31] Theorem 5.7; and Narang and Shaiju [33]. □

4.2. **Stability.** In this section we are interested in the stability of the replicator dynamics (3.6)-(3.8) (see Definition 4.6). To this end, we establish that strong uninvadable profiles (Definition 4.2) and strong unbeatable profiles (Definition 4.3) have some type of stability.

Definition 4.6. Let μ^* be a critical point of the replicator dynamics (3.6)-(3.8), i.e., $F(\mu^*) = 0$.

i) μ^* is called Lyapunov stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\mu(0) - \mu^*\|_\infty < \delta$, then $\|\mu(t) - \mu^*\|_\infty < \epsilon$ for all $t > 0$.
ii) μ^* is called weakly attracting if it is Lyapunov stable and, in addition, there exists $\delta > 0$ such that if $\|\mu(0) - \mu^*\|_\infty < \delta$, then as $t \to \infty$, $\mu_i(t) \to \mu_i^*$ weakly for all $i \in I$.

The following theorem establishes that strong uninvadable profiles are stable for the replicator dynamics.

Theorem 4.7. *Suppose that the conditions i) and ii) of Theorem 3.3 hold. Let $\delta_{a^*} = (\delta_{a_1^*}, ..., \delta_{a_n^*})$ be a vector of Dirac measures, and $\mathcal{C}$ an invariant set for the replicator dynamics (3.6)-(3.8). If δ_{a^*} is a* SUP in the set $\mathcal{C}$, *then there exists $\epsilon > 0$ such that the set*

$$\mathcal{C} \cap V_\epsilon(\delta_{a^*}),$$

is invariant for (3.6)-(3.8). Moreover, suppose that for all i in I, the map $\mu \mapsto \beta_i(a_i^*|\mu)$ is weakly continuous and the set of strategies A_i is a compact set. If $\mathcal{C}$ is a closed set and $\mu(0)$ is in $\mathcal{C} \cap V_\epsilon(\delta_{a^*})$, then as $t \to \infty$, $\mu(t) \to \delta_{a^*}$ weakly.

Proof. See Mendoza-Palacios and Hernández-Lerma [31]. □

If the vector δ_{a^*} in Theorem 4.7 is a SUP, then we obtain the following corollary, taking $\mathcal{C} = \mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$.

Corollary 4.8. *Suppose that the conditions i) and ii) of Theorem 3.3 hold. Let $\delta_{a^*} = (\delta_{a_1^*}, ..., \delta_{a_n^*})$ be a vector of Dirac measures, and suppose that it is a SUP. Then δ_{a^*} is Lyapunov stable for the replicator dynamics (3.6)-(3.8). Moreover, if the map $\mu \mapsto \beta_i(a_i^*|\mu)$ is weakly continuous and the set of strategies A_i is compact for all $i \in I$, then δ_{a^*} is weakly attracting.*

Note that if for each i in I the payoff function $U_i(\cdot)$ in (3.2) is continuous, then the map $\mu \mapsto \beta_i(a_i^*|\mu)$ is weakly continuous. This fact is of relevance because many games satisfy that $U_i(\cdot)$ in (3.2) is continuous.

The following definition was introduced by Narang and Shaiju [33].

Definition 4.9. A vector $\mu^* \in \mathbb{P}(A_1) \times \mathbb{P}(A_2) \times ... \times \mathbb{P}(A_n)$ is called a polymorphic profile if for every $i \in I$ there exist many finite distinct $a_i^1, ..., a_i^{k_i} \in A_i$ and numbers $h_i^1, ..., h_i^{k_i} \in (0, 1]$ such that $\mu_i^* = \sum_{j=1}^{k_i} h_i^j \delta_{a_i^j}$.

The following theorem establishes conditions under which the strong unbeatable profiles are stable for the replicator dynamics.

Theorem 4.10. *Suppose that the payoff function $U_i(\cdot)$ in (3.2) is bounded for all $i \in I$. Let $\mu^* = (\mu_1^*, ..., \mu_n^*)$ be a polymorphic profile, which is also a SUbP. Then μ^* is Lyapunov stable for the replicator dynamics.*

Proof. See Narang and Shaiju [33]. □

Narang and Shaiju [33] obtained the following results for polymorphic profiles:

i) if a polymorphic profile is strong uninvadable, then it is a vector of Dirac measures;

ii) if a set of polymorphic profiles is a strong uninvadable set, then it is asymptotically stable for the replicator dynamics.

5. The replicator dynamics: symmetric games

Is this section we consider symmetric evolutionary games as in (5.3), below, and compare them with normal-form games (5.4). We define the important concept of strongly uninvadable strategy (Definition 5.2) and its relation to a Nash equilibrium (Proposition 5.4).

We can obtain from (3.10) a symmetric evolutionary game when $I := \{1, 2\}$ and the sets of actions and payoff functions are the same for both players, i.e., $A = A_1 = A_2$ and $U(a, b) = U_1(a, b) = U_2(b, a)$, for all $a, b \in A$. As a consequence, the sets of mixed actions and the expected payoff functions are the same for both players , i.e., $\mathbb{P}(A) = \mathbb{P}(A_1) = \mathbb{P}(A_2)$ and $J(\mu, \nu) = J_1(\mu, \nu) = J_2(\nu, \mu)$ for all $\mu, \nu \in \mathbb{P}(A)$. This kind of model determines the dynamic interaction of strategies of a unique species through the replicator dynamics

$$\mu'(t) = F(\mu(t)), \tag{5.1}$$

where $F : \mathbb{P}(A) \to \mathbb{M}(A)$ is given by

$$F(\nu(t), E) := \int_E \Big[J(a, \nu(t)) - J(\nu(t), \nu(t))\Big]\nu(t, da) \ \forall E \in \mathcal{B}(A). \tag{5.2}$$

Finally, as in (3.10), we can describe a symmetric evolutionary games as

$$\left[I = \{1, 2\},\ \mathbb{P}(A),\ J(\cdot),\ \nu'(t) = F(\nu(t))\right]. \tag{5.3}$$

Similarly, we can obtain from (4.1) a *two-player symmetric normal-form game* described as

$$\Gamma_s := \Big[I = \{1, 2\},\ \mathbb{P}(A),\ J(\cdot)\Big]. \tag{5.4}$$

For symmetric normal-form games Γ_s we can express a symmetric Nash equilibrium (μ^*, μ^*) in terms of the strategy $\mu^* \in \mathbb{P}(A)$, as follows.

Definition 5.1. We say that $\mu^* \in \mathbb{P}(A)$ is a Nash equilibrium strategy (NES) if the pair (μ^*, μ^*) is a Nash equilibrium for Γ_s. That is,

$$J(\mu^*, \mu^*) \geq J(\mu, \mu^*) \ \forall \mu \in \mathbb{P}(A).$$

By Proposition 4.4 if μ^* is a NES for Γ_s, then μ^* is a critical point of (5.1) when $F(\cdot)$ is described by the replicator dynamics (5.2).

The following definition is a slightly modified version of the strongly uninvadable strategies used in Bomze [5].

Definition 5.2. Let r be a metric on $\mathbb{P}(A)$. A measure $\mu^* \in \mathbb{P}(A)$ is called an r-strongly uninvadable strategy (r-SUS) if there exists $\epsilon > 0$ such that for any μ with $r(\mu, \mu^*) < \epsilon$, it follows that $J(\mu^*, \mu) > J(\mu, \mu)$. We call ϵ the global invasion barrier.

When r is the Prokhorov metric r_p, Oechssler and Riedel [36] name a r_p-SUS as an *evolutionary robust strategy.* If r_{w^*} is any metric that metrizes the weak topology (recall Remark 2.1), Cressman and Hofbauer [11] call a r_{w^*}-SUS a *locally superior strategy.* Hingu, Roa and Shaiju [19] call a r_{w^*}-SUS a *superior strategy* if the Definition 5.2 is satisfie with the Kullback-Leibler distance (6.1), below.

We use the notation $\|\cdot\|$-SUS when the metric on $\mathbb{P}(A)$ is given by the total variation norm (2.1). Note that $\|\cdot\|$-SUS is a modified version of the SUP (see Definition 4.2) for the symmetric case.

Proposition 5.3. *Let r_{w^*} be a distance that metrizes the weak convergence on $\mathbb{P}(A)$. If a measure $\mu^* \in \mathbb{P}(A)$ is r_{w^*}-SUS, then it is $\|\cdot\|$-SUS.*

Proof. See Mendoza-Palacios and Hernández-Lerma [32] Proposition 3. □

The following proposition shows that a strongly uninvadable strategy is also a Nash equilibrum strategy. Compare with Proposition 4.5.

Proposition 5.4. *Let r be a metric on $\mathbb{P}(A)$. If μ^* is a r-SUS, then μ^* is a NES of Γ_s.*

Proof. See Mendoza-Palacios and Hernández-Lerma [32], Proposition 4. □

Summarizing, in this section we have that

$$r - \mathcal{SUS} \ \subset \ \mathcal{N} \ \subset \ \mathcal{C} \tag{5.5}$$

where $\mathcal{C}$ is the set of critical points of the replicator dynamics, $\mathcal{N}$ is the family of Nash equilibrium strategies for the replicator dynamics, $r - \mathcal{SUS}$ is the subfamily of r-strongly uninvadable strategies for any metric r. This result will be complemented in Corollary 7.3.

6. Stability of the replicator dynamics: symmetric games

6.1. **Stability of SUSs.** By Propositions 5.4 and 4.4, a SUS is a critical point of the replicator dynamics. In this section we present a review of results on the stability of a SUS in the replicator dynamics. These results include different stability criteria with respect to various metrics and topologies in the space of probability measures.

Assume that $\nu << \mu$. We define the cross entropy or Kullback-Leibler distance of ν with respect to μ as

$$K(\mu,\nu) := \int_A \log\left[\frac{d\nu}{d\mu}(a)\right]\nu(da). \tag{6.1}$$

From Jensen's inequality it follows that $0 \le K(\mu,\nu) \le \infty$ and $K(\mu,\nu) = 0$ if and only if $\mu = \nu$. The Kullback-Leibler distance is not a metric, since it is not symmetric, i.e., $K(\mu,\nu) \neq K(\nu,\mu)$.

Given $\mu^* \in \mathbb{P}(A)$, $\epsilon > 0$, and a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$, we define the set

$$\mathcal{W}_{\varphi(\epsilon)}(\mu^*) := \Big\{\mu \in \mathbb{P}(A) : K(\mu,\mu^*) < \varphi(\epsilon)\Big\}. \tag{6.2}$$

Theorem 6.1. *Suppose that A is a separable metric space, and that the conditions $i)$ and $ii)$ of Theorem 3.1 hold. Let μ^* be a $\|\cdot\|$-SUS with global invasion barrier $\epsilon > 0$, and $\mu(\cdot)$ the solution of the replicator dynamics (5.1)-(5.2). If $\mu(0) \in \mathcal{W}_{\varphi(\epsilon)}(\mu^*)$, with $\varphi(\epsilon) = \left[\frac{\epsilon}{2}\right]^2$, then:*

i) $\mu(t) \in \mathcal{W}_{\varphi(\epsilon)}(\mu^*)$ for all $t \ge 0$;
ii) $\|\mu(t) - \mu^*\| < \epsilon$ for all $t \ge 0$;
iii) $\mu(t)$ is in some open ball $\mathcal{V}_\alpha^{r_{w^*}}(\mu^*)$ as (2.10) for all $t \ge 0$, where r_{w^*} is some distance that metrizes the weak topology.
iv) Moreover if A is compact and the map $\mu \to J(\mu^*,\mu) - J(\mu,\mu)$ is continuous in the weak topology, then $r_{w^*}(\mu(t),\mu^*) \to 0$,.
v) Furthermore, parts $i)$ to $iv)$ are also true with the hypothesis that μ^* is r_{w^*}-SUS.

Proof. Parts $i), ii)$ and $iv)$ are proved in Bomze [4] [1]. Part $iii)$ is a consequence of $ii)$ and Remark 2.2. Finally, $v)$ follows from Proposition 5.3. □

The following theorem characterizes the stability of the replicator dynamics with respect to the L_1-Wasserstein metric r_w (2.9). This distance metrizes the weak topology and has important relationships with other distances that also metrize the weak topology (see Proposition A.2). Furthermore, the L_1-Wasserstein metric is closely related to the variation norm (2.1) and the Kullback-Leibler distance (6.1); see Proposition A.1. The following two propositions give better approximations to parts $iii)$ and $iv)$ of Theorem 6.1.

Theorem 6.2. *Suppose that A is a compact Polish space (with diameter $C > 0$), and the conditions $i)$ and $ii)$ of Theorem 3.1 hold. Let μ^* be a r_w*-SUS *with global invasion barrier $\epsilon > 0$, and $\mu(\cdot)$ the solution of the replicator dynamics (5.1)-(5.2). If $\mu(0) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$, with $\varphi'(\epsilon) = \left[\frac{\epsilon}{2C}\right]^2$, then*

i) $\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \ge 0$;
ii) $\|\mu(t) - \mu^*\| < \frac{\epsilon}{C}$ for all $t \ge 0$;

[1] Bomze proves a more general case for point $iv)$ of Theorem 6.1, where any topology τ in $\mathbb{P}(A)$ is included. Bomze only requires that $\mathbb{P}(A)$ be a τ-compact set and the map $\mu \to J(\mu^*,\mu) - J(\mu,\mu)$ be τ-continuous.

iii) $r_w(\mu(t), \mu^*) < \epsilon$ *for all* $t \geq 0$.

iv) Moreover, if the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ *is continuous in the weak topology, then* $r_w(\mu(t), \mu^*) \to 0$.

v) Furthermore, parts i) to iv) are also true with the hypothesis that μ^* is $\|\cdot\|$-SUS, with barrier $\frac{\epsilon}{C}$.

Proof. See Mendoza-Palacios and Hernández-Lerma [32] Theorem 5.1. □

The next theorem characterizes the stability of the replicator dynamics of a SUS that is also a Dirac measure.

Theorem 6.3. *Let* A *be a separable metric space and suppose that the conditions i) and ii) of Theorem 3.1 hold. Let* δ_{a^*} *be a Dirac measure and* r *any metric on* $\mathbb{P}(A)$. *If* δ_{a^*} *is* r*-SUS ,* $\mu(\cdot)$ *is a solution of (5.1), with* $F(\cdot)$ *as (5.2), and* $\|\mu_0 - \delta_{a^*}\| < \epsilon$ *for some small* $\epsilon > 0$, *then*

i) $\|\mu(t) - \delta_{a^*}\| < \epsilon$ *for all* $t \geq 0$;

ii) $\mu(t)$ *is in some open ball* $\mathcal{V}_\alpha^{r_{w^*}}(\mu^*)$ *as in (2.10) for all* $t \geq 0$, *where* r_{w^*} *is some distance that metrizes the weak topology;*

iii) if A *is a compact Polish space (with diameter* $C > 0$*), then, for all* $t \geq 0$, $r_w(\mu(t), \delta_{a^*}) < C\epsilon$;

iv) if A *is compact (not necessary a Polish space) and the map* $\mu \to J(\delta_{a^*}, \mu) - J(\mu, \mu)$ *is continuous in the weak topology, then* $r_{w^*}(\mu(t), \mu^*) \to 0$, *where* r_{w^*} *is any distance that metrizes the weak topology.*

Proof. Parts *i)*, *ii)* and *iv)* follow from Proposition 5.3 and Corollary 4.8. Finally, part *iii)* follows from Proposition A.2. □

This theorem is also proved by Oechssler and Riedel [35] with slight changes in the definition of SUS.

6.2. **Other stability results.** The following conjecture was proposed by Oechssler and Riedel in [36], when r_{w^*} is a distance that metrizes the weak topology.

Conjecture 6.4. Let r be any metric on $\mathbb{P}(A)$ and r_{w^*} any distance that metrizes the weak topology. Suppose that A is a separable metric space, and that the conditions *i)* and *ii)* of Theorem 3.1 hold. Let μ^* be a r-SUS and $\mu(\cdot)$ the solution of the replicator dynamics (5.1)-(5.2). Then

i) for $\epsilon > 0$ there exist $\delta > 0$ such that if $r(\mu(0), \mu^*) < \delta$, we have that $r(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$;

ii) moreover, if part *i)* is satisfied, and the map $\mu \to J(\mu^*, \mu) - J(\mu, \mu)$ is continuous in the weak topology and $\mu^* << \mu(0)$, then $r_{w^*}(\mu(t), \mu^*) \to 0$.

Remark 6.5. A double symmetric game (named a potential game by Cressman and Hofbauer [11]) is a game where $J(\mu, \nu) = J(\nu, \mu)$ for any $\mu, \nu \in \mathbb{P}(A)$. Let r_{w^*} be any distance that metrizes the weak topology. Oechssler and Riedel [36] prove that if A is a compact set and μ^* is r_{w^*}-SUS, then for double symmetric games, μ^* satisfies part *i)* of the Conjecture 6.4. Cressman and Hofbauer [11] prove that if part *i)* is satisfied, then *ii)* follows for any symmetric game.

Oechssler and Riedel [36] prove that a r_{w^*}-SUS satisfies other static evolutionary concepts such as *evolutionary stable strategy* (ESS), *continuously stable strategy* (CSS), and *neighborhood invader strategy* (NIS), which are sufficient to guarantee dynamic stability in the weak topology for the replicator dynamics. Eshel and Sansone [13], Cressman [10], Cressman, Hofbauer and Riedel [12], use these evolutionary concepts and different hypotheses in the payoff function $U(\cdot)$ in (3.2) to guarantee dynamic stability. Norman [34] and Hingu [17] establish the dynamic stability in terms of strategy sets.

Finally, Hingu, Rao and Shaiju [18], establish two important concepts: strong unbeatable state (SUbS) which is a modified version of SUbP (see Definition 4.3) for the symmetric case; and polymorphic population state (PPS) which is a modified version of polymorphic profile (see Definition 4.9). Hingu, Rao and Shaiju [18] show the following results:

i) every asymptotically stable critical point of the replicator dynamics, in the norm (2.1), is finitely supported. If a polymorphic profile is strong uninvadable, then it is a vector of Dirac measures;

ii) if a PPS is SUbS, then it is Lyapunov stable; moreover, it is asymptotically stable in the norm (2.1).

7. NESs and stability for symmetric games

In this section we introduce a general definition of dynamic stability for the replicator dynamics (see Definition 7.1), and prove that any stable critical point of the replicator dynamics is a NES of Γ_s (see Proposition 7.2). Finally, in Corollary 7.3 and Remarks 7.4 and 7.5 we establish relations between the stability of the replicator dynamics (5.1)-(5.2), and the static evolutionary concepts of a strategy, NES and SUS.

Consider $\mu, \nu \in \mathbb{P}(A)$. By Propositions A.2 and A.3 we know that if μ and ν are close with respect to the Kullback-Leibler distance K, then they are close in the total variation norm $\|\cdot\|$, and consequently they are close in the weak topology. This argument is not true in the opposite direction. Hence we say that the Kullback-Leibler distance is "stronger than" the total variation norm, and, similarly, the total variation norm is "stronger than" any distance that metrizes the weak topology.

Definition 7.1. Let A be a separable metric space, and r_1 and r_2 the Kullback-Leibler distance or some metric in $\mathbb{P}(A)$ where r_1 is equal to or "stronger than" r_2. A critical point μ^* of the replicator dynamics (5.1)-(5.2) is said to be

i) $[r_1, r_2]$-stable (*in symbols* : $[r_1, r_2]$-S) if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $r_1(\mu(0), \mu^*) < \delta$, then $r_2(\mu(t), \mu^*) < \epsilon$ for all $t > 0$. If $r_1 = r_2 = r^*$ then we only say that μ^* is r^*-stable (*in symbols* : r^*-S).

ii) $[r_1, r_2]$-asymptotically weakly stable if it is $[r_1, r_2]$-stable and $\lim_{t\to\infty} \mu(t) = \mu^*$ in the weak topology.

Consider the Kullback-Leibler distance K, the total variation norm $\|\cdot\|$, and any distance r_{w^*} that metrizes the weak topology. The following diagram gives the natural implications between the different concepts of stability. (For details see Mendoza-Palacios and Hernández-Lerma [32].)

$$
\begin{array}{ccccc}
K - \mathrm{S} & \Rightarrow & [K, \|\cdot\|] - \mathrm{S} & \Rightarrow & [K, r_{w^*}] - \mathrm{S} \\
 & & \Uparrow & & \Uparrow \\
 & & \|\cdot\| - \mathrm{S} & \Rightarrow & [\|\cdot\|, r_{w^*}] - \mathrm{S} \\
 & & & & \Uparrow \\
 & & & & r_{w^*} - \mathrm{S}
\end{array}
\tag{7.1}
$$

Van Veelen and Spreij [45] study other relationships among the different concepts of stability in diagram (7.1). They also study relationships between static evolutionary concepts and asymptotic evolutionary stability.

Proposition 7.2. *Let A be a separable metric space, and r_1, r_2 the Kullback-Leibler distance or some metric in $\mathbb{P}(A)$ where r_1 is equal to or "stronger than" r_2. Suppose that the conditions $i)$ and $ii)$ of Theorem 3.1 are satisfied, and let μ^* be a critical point of the replicator dynamics (5.1)-(5.2). If μ^* is $[r_1, r_2]$-stable, then μ^* is a Nash equilibrium strategy (NES) of Γ_s.*

Proof. See Mendoza-Palacios and Hernández-Lerma [32], Proposition 6. □

Now, we define the following sets:

i) $\mathcal{N} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is a NES of } \Gamma_s\}$,
$\mathcal{C} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is a critical point of } (5.1) - (5.2)\}$.

ii) If r is any metric in $\mathbb{P}(A)$,

$$r - \mathcal{SUS} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is } \; r - \mathrm{SUS}\ \}.$$

iii) Let r_1 and r_2 be the Kullback-Leibler distance or some metric in $\mathbb{P}(A)$, where r_1 is equal to or "stronger than" r_2,

$$[r_1, r_2] - \mathcal{S} := \{\mu^* \in \mathbb{P}(A) : \mu^* \text{ is } \; [r_1, r_2] - \mathrm{S}\}.$$

Corollary 7.3. *Let A be a separable metric space, and consider the conditions $i)$ and $ii)$ of Theorem 3.1. Let r_1 be a metric on $\mathbb{P}(A)$, and let r_2 be the Kullback-Leibler distance or some metric on $\mathbb{P}(A)$ equal to or "stronger than" r_1. Then we have:*

$$r_1 - \mathcal{SUS} \;\subset\; [K, r_2] - \mathcal{S} \;\subset\; \mathcal{N} \;\subset\; \mathcal{C}.$$

Proof. This is consequence of Theorem 6.1, and Propositions 4.4, 5.4 and 7.2. □

Compare Corollary 7.3 with (5.5).

Remark 7.4. Suppose the hypotheses of Corollary 7.3 and let A be a compact Polish space. Then by Theorem 6.2 and Propositions A.2, A.3, we can obtain the implications in Theorem 7.3 with a specific value for the barrier $\epsilon > 0$, for the metrics $\|\cdot\|$, r_p, r_{bl}, r_w, and r_{kr}. See Mendoza-Palacios and Hernández-Lerma [32], Corollary 1.

Remark 7.5. Let r_1 be a metric on $\mathbb{P}(A)$, and let r_2 be the total variation norm in $\mathbb{P}(A)$ or some metric that metrizes the weak topology. By Theorem 6.3 and Propositions 4.4, 5.3,5.4 and 7.2, we have the following implications if a Dirac measure δ_{a^*} is a r_1-SUS.

$$\delta_{a^*} \in r_1 - \mathcal{SUS} \;\Rightarrow\; \delta_{a^*} \in [\|\;\|, r_2] - \mathcal{S} \;\Rightarrow\; \delta_{a^*} \in \mathcal{N} \;\Rightarrow\; \delta_{a^*} \in \mathcal{C}.$$

8. Finite dimensional approximations

An infinite-dimensional dynamical system, as the replicator dynamics (3.6)-(3.8), is not a computable model. Mendoza-Palacios and Hernández-Lerma [30] introduce some approximation schemes and propose two approximation theorems that extend the results in [36]. They establish the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximation model) with different initial conditions. These approximations are studied in the weak topology using the Kantorovich-Rubinstein metric (2.8), and also in the strong topology using the norm of total variation (2.1).

8.1. **Discrete approximations to the replicator dynamics.** To obtain a finite-dimensional approximation of the replicator dynamics (3.6)-(3.8) for an asymmetric game (3.10) (or (5.1)-(5.2) for the symmetric game (5.3)), we can apply the following Theorems 8.1 and 8.4 to a discrete approximation of the payoff functions U_i in (3.2) and the initial probability measures $\mu_{i,0}$, for i in I. For some approximation techniques for the payoff function in games, see Bishop and Cannings [2], Simon [44].

Oechssler and Riedel [36] propose a finite approximation for a symmetric game. Following [36], consider an asymmetric game (4.1) where, for every i in I, $A_i = [c_{i,1}, c_{i,2}]$ (for some real numbers $c_{i,1} < c_{i,2}$), and U_i is a real-valued bounded function. For every i in I, consider the partition $P_{k_i} := \{\xi_{m_i}\}_{m_i=0}^{2^{k_i}-1}$ over A_i, where

$$\xi_{m_i} := [a_{m_i}, a_{m_i+1}), \; a_{m_i} = c_{i,1} + \frac{m_i[c_{i,2} - c_{i,1}]}{2^{k_i}},$$

for $m_i = 0, 1, ..., 2^{k_i} - 2$ and $\xi_{2^{k_i}-1} := [a_{2^{k_i}-1}, c_{i,2}]$. For every i in I, the discrete approximation to U_i is given by the function

$$U_{k_i}(x_1, ..., x_i, ..., x_n) := U_i(a_{m_1}, ..a_{m_i}, ..., a_{m_n}),$$

if $(x_1, ..., x_i, ..., x_n)$ is in $\xi_{m_1}\times\cdots\times\xi_{m_i}\times\cdots\times\xi_{m_n}$. Also, for each i in I we approximate a probability measure $\mu_i \in \mathbb{P}(A_i)$ by a discrete probability distribution μ_{k_i} on the partition set P_{k_i}. Then we can write the approximation to the payoff function (3.2) as

$$J_{k_i}(\mu_{k_1}, ..., \mu_{k_n}) := \sum_{\xi_{m_1}\in P_{k_1}} ... \sum_{\xi_{m_n}\in P_{k_n}} U_i(a_{m_1}, ..., a_{m_n})\mu_{k_n}(\xi_{m_n})\cdots\mu_{k_1}(\xi_{m_1}). \tag{8.1}$$

For every $i \in I$ and every vector $\mu_k := (\mu_{k_1}, ..., \mu_{k_n})$ in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_n})$, we write μ_k as (μ_{k_i}, μ_{-k_i}), where $\mu_{-k_i} := (\mu_{k_1}, ..., \mu_{k_{i-1}}, \mu_{k_{i+1}}, ..., \mu_{k_n})$ is in $\mathbb{P}(P_{k_1})\times...\times\mathbb{P}(P_{k_{i-1}}) \times \mathbb{P}(P_{k_{i+1}}) \times ... \times \mathbb{P}(P_{k_n})$. If $\delta_{\{\xi_{m_i}\}}$ is a probability measure concentrated at $\xi_{m_i} \in P_{k_i}$, then the vector $(\delta_{\{\xi_{m_i}\}}, \mu_{-i})$ is written as (a_{m_i}, μ_{-i}), and so

$$J_{k_i}(\delta_{\{\xi_{m_i}\}}, \mu_{-k_i}) = J_{k_i}(a_{m_{-k_i}}, \mu_{-k_i}). \tag{8.2}$$

In particular, (8.1) yields

$$J_{k_i}(\mu_{k_i}, \mu_{-k_i}) := \sum_{\xi_{m_i}\in P_{k_i}} J_{k_i}(a_{m_i}, \mu_{-k_i})\mu_{k_i}(\xi_{m_i}). \tag{8.3}$$

Note that $\mu_k := (\mu_{k_1}, ..., \mu_{k_n})$ in $\mathbb{P}(P_{k_1}) \times ... \times \mathbb{P}(P_{k_n})$ is a vector of measures in $\mathbb{P}(A_1) \times ... \times \mathbb{P}(A_n)$. Then for any $i \in I$ and $E_i \in \mathcal{B}(A_i) \cap P_{k_i}$, the replicator induced by $\{U_{k_i}\}_{i \in I}$ has the form

$$\mu'_{k_i}(t, E_i) = \sum_{\xi_{m_i} \in E_i \cap P_{k_i}} \left[J_{k_i}(a_{m_{k_i}}, \mu_{-k_i}(t)) - J_{k_i}(\mu_{k_i}(t), \mu_{-k_i}(t)) \right] \mu_{k_i}(t, \xi_{m_i}),$$

which is equivalent to the system of differential equations in $\mathbb{R}^{2^{k_1}+...+2^{k_n}}$ of the form

$$\mu'_{k_i}(t, \xi_{m_i}) = \left[J_{k_i}(a_{m_i}, \mu_{-k_i}(t)) - J_{k_i}(\mu_{k_i}(t), \mu_{-k_i}(t)) \right] \mu_{k_i}(t, \xi_{m_i}), \tag{8.4}$$

for $i = 1, 2, ..., n$ and $m_i = 0, 1, ..., k_i$, with initial condition $\{\mu_{k_i,0}(\xi_{m_i})\}_{m_i=0}^{2^k_i - 1}$.

Hence, using Theorem 8.1 or Theorem 8.4, we can approximate (3.6)-(3.8) by a system of differential equations in $\mathbb{R}^{2^{k_1}+...+2^{k_n}}$ of the form (8.4).

8.2. **Finite dimensional approximation: strong form.** In this section we provide an approximation theorem that gives conditions under which we can approximate (3.6)-(3.8) by a finite-dimensional dynamical system of the form (8.4) under the total variation norm (2.1).

Theorem 8.1. *For each i in I, let A_i be a separable metric space and let $U_i, U_i^\epsilon : A_1 \times ... \times A_n \to \mathbb{R}$ be bounded functions such that $\max_{i \in I} \|U_i - U_i^\epsilon\| < \epsilon$, where $\|\cdot\|$ is the sup norm in (e3.2). Consider the replicator dynamics induced by $\{U_i\}_{i=1}^n$ and $\{U_i^\epsilon\}_{i=1}^n$, i.e.,*

$$\mu'_i(t, E_i) = \int_{E_i} \left[J_i(a_i, \mu_{-i}(t)) - J_i(\mu_i(t), \mu_{-i}(t)) \right] \mu_i(t, da_i), \tag{8.5}$$

$$\nu'_i(t, E_i) = \int_{E_i} \left[J_i^\epsilon(a_i, \nu_{-i}(t)) - J_i^\epsilon(\nu_i(t), \nu_{-i}(t)) \right] \nu_i(t, da_i), \tag{8.6}$$

for each $i \in I$, $E \in \mathcal{B}(A_i)$, and $t \geq 0$. If $\mu(\cdot)$ and $\nu(\cdot)$ are solutions of (8.5) and (8.6), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu(0) = \nu_0$, then for $T < \infty$

$$\sup_{t \in [0,T]} \|\mu(t) - \nu(t)\|_\infty < \|\mu_0 - \nu_0\|_\infty e^{QT} + 2\epsilon \left(e^{QT} - \frac{1}{Q} \right). \tag{8.7}$$

where $Q := (2n+1)H$ and $H := \max_{i \in I} \|U_i\|$.

Corollary 8.2. *Let us assume the hypotheses of Theorem 8.1. Suppose that for each i in I, there exists a sequence of functions $\{U_i^{\epsilon_n}\}_{n=1}^\infty$ and probability measure vectors $\{\nu^n\}_{n=1}^\infty$ such that $\max_{i \in I} \|U_i - U_i^{\epsilon_n}\| \to 0$ and $\|\mu_0 - \nu_0^n\|_\infty \to 0$. If $\mu(\cdot)$ and $\nu^n(\cdot)$ are solutions of (8.5) and (8.6), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu^n(0) = \nu_0^n$, then for $T < \infty$,*

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|\mu(t) - \nu_n(t)\|_\infty = 0.$$

8.3. **Finite dimensional approximations: weak form.** The next approximation result, Theorem 8.4, establishes the proximity of two paths generated by two different dynamical systems (the original model and a discrete approximating model) with different initial conditions, under the weak topology. To this end we use the Kantorovich-Rubinstein norm $\|\cdot\|_{kr}$ on $\mathbb{M}(A)$, which metrizes the weak topology.

Remark 8.3. Let A be a separable metric space. We say that a mapping $\mu : [0,\infty) \to \mathbb{M}(A)$ is weakly differentiable if there exists $\mu'(t) \in \mathbb{M}(A)$ such that, for every $t > 0$ and $g \in \mathbb{C}_B(A)$

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\int_A g(a)\mu(t+\epsilon, da) - \int_A g(a)\mu(t, da)\right] = \int_A g(a)\mu'(t, da). \tag{8.8}$$

If $\|\cdot\|_{k,r}$ is the Kantorovich-Rubinstein metric in (2.8), then (8.8) is equivalent to

$$\lim_{\epsilon\to 0}\left\|\frac{\mu(t+\epsilon)-\mu(t)}{\epsilon} - \mu'(t)\right\|_{kr} = 0. \tag{8.9}$$

Theorem 8.4. *For each i in I, let (A_i, ϑ_i) be a bounded separable metric space (with diameter $C_i > 0$), and $U_i, U_i^\epsilon : A_1 \times ... \times A_n \to \mathbb{R}$ be two bounded functions such that $\max_{i\in I}\|U_i - U_i^\epsilon\| < \epsilon$. For each i in I, suppose that $\|U_i\|_L < \infty$ and consider the replicator dynamics induced by $\{U_i\}_{i=1}^n$ and $\{U_i^\epsilon\}_{i=1}^n$, as in (8.5) and (8.6). If $\mu(\cdot)$ and $\nu(\cdot)$ are solutions of (8.5) and (8.6), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu(0) = \nu_0$, then for $T < \infty$*

$$\sup_{t\in[0,T]} \|\mu(t) - \nu(t)\|_\infty^{kr} < \|\mu_0 - \nu_0\|_\infty^{kr} e^{QT} + 2\epsilon\left(e^{QT} - \frac{1}{Q}\right). \tag{8.10}$$

where $Q := [2H + (2n-1)CH_L]$, $H := \max_{i\in I}\|U_i\|$, $H_L := \max_{i\in I}\|U_i\|_L$, and $C := \max_{i\in I} C_i$.

Corollary 8.5. *Let us assume the hypotheses of Theorem 8.4. Suppose that for each i in I, there exist sequences of functions $\{U_i^{\epsilon_n}\}_{n=1}^\infty$ and of vectors of probability measures $\{\nu^n\}_{n=1}^\infty$ such that $\max_{i\in I}\|U_i - U_i^{\epsilon_n}\| \to 0$ and $\|\mu_0 - \nu_0^n\|_\infty^{kr} \to 0$. If $\mu(\cdot)$ and $\nu^n(\cdot)$ are solutions of (8.5) and (8.6), respectively, with initial conditions $\mu(0) = \mu_0$ and $\nu^n(0) = \nu_0^n$, then, for $T < \infty$,*

$$\lim_{n\to\infty}\sup_{t\in[0,T]} \|\mu(t) - \nu_n(t)\|_\infty^{kr} = 0.$$

9. Examples

9.1. **A quadratic linear model: asymmetric case.** Consider games in which we have two players with the following payoff functions:

$$U_1(x,y) = -a_1x^2 - b_1xy + c_1x + d_1y, \tag{9.1}$$

$$U_2(x,y) = -a_2y^2 - b_2yx + c_2y + d_2x, \tag{9.2}$$

with $a_1, a_2, b_1, b_2, c_1, c_2 > 0$ and d_1, d_2 any real numbers. Let $A_1 = [0, M_1]$ and $A_2 = [0, M_2]$ for $M_1, M_2 > 0$ and large enough, be the strategy sets.

This class of games could represent a Cournot duopoly or models of international trade with linear demand and linear cost (see Mas-Colell, Whinston and Green [29]). It can also represent some models of public good games.

If

$$(2a_2c_1 - b_1c_2),\ (2a_1c_2 - b_2c_1),\ (4a_1a_2 - b_1b_2)$$

are all positive, then we have an interior Nash equilibrium

$$(x^*, y^*) = \left(\frac{2a_2c_1 - b_1c_2}{4a_1a_2 - b_1b_2}, \frac{2a_1c_2 - b_2c_1}{4a_1a_2 - b_1b_2}\right).$$

Let

$$\mathcal{C}_1 := \{(\mu,\nu) \in \mathbb{P}(A_1)\times\mathbb{P}(A_2) : \mu(x^*, M_1] = \nu(y^*, M_2] = 0\},$$
$$\mathcal{C}_2 := \{(\mu,\nu) \in \mathbb{P}(A_1)\times\mathbb{P}(A_2) : \mu[0, x^*) = \nu[0, y^*) = 0\},$$

and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. The set $\mathcal{C}$ is invariant for the replicator dynamics (3.6)-(3.8) and $(\delta_{x^*}, \delta_{y^*})$ is in $\mathcal{C}$. On the other hand, let

$$\bar{x}^\mu := \int_{A_1} x\mu(dx),\ \bar{y}^\mu := \int_{A_2} y\mu(dy).$$

If (μ,ν) is in $\mathcal{C}_1$, then by Jensen's inequality

$$J_1(\delta_{x^*},\nu) = \int_{A_2} U_1(x^*, y)\nu(dy) = U_1(x^*, \bar{y}^\nu) > U_1(\bar{x}^\mu, \bar{y}^\nu) \geq J_1(\mu,\nu)$$

$$J_2(\mu,\delta_{y^*}) = \int_{A_1} U_2(x, y^*)\mu(dx) = U_2(\bar{x}^\mu, y^*) > U_2(\bar{x}^\mu, \bar{y}^\nu) \geq J_2(\mu,\nu).$$

This is also true if (μ,ν) is in $\mathcal{C}_2$. Hence, for any $\epsilon > 0$, the vector $(\delta_{x^*}, \delta_{y^*})$ is a SUP in the set $\mathcal{C}$. Therefore, by Theorem 4.7, for $\epsilon > 0$ the set $\mathcal{C} \cap V_\epsilon(\delta_{a^*})$ is invariant for (3.6)-(3.8). Moreover, since for every i in I, the payoff functions $U_i(\cdot)$ are continuous and the sets of strategies A_i are compact sets, we conclude by Theorem 4.7 that if $\mu(0) \in \mathcal{C} \cap V_\epsilon(\delta_{a^*})$, then $\mu(t) \to \delta_{a^*}$ weakly.

9.2. **A quadratic linear model: symmetric case.** We now consider the symmetric form of the game in Section 9.1 Thus, we can rewrite the payoff functions (9.1) and (9.2) as

$$U(x,y) = -ax^2 - bxy + cx + dy, \tag{9.3}$$

with $a, b, c > 0$ and d any real number. Let $A = [0, M]$ for $M > 0$ and large enough, be the strategy set. If $2c(a-b) > 0$ and $4a^2 - b^2 > 0$, then we have an interior Nash equilibrium strategy (NES)

$$x^* = \frac{2c(a-b)}{4a^2 - b^2}.$$

For a fixed y the function $U(x,y)$ is concave in x and has the partial derivative $U_x(x,y) = -2ax - by + c$. Let

$$x(y) := \text{argmax} U(x,y) = \frac{(c - by)}{2a}$$

and note that $x'(y) = -(b/2a) < 0$. Then if $y < x^*$ or $x^* < y$, we have

$$U(x(y), y) > U(x^*, y) \geq U(y, y).$$

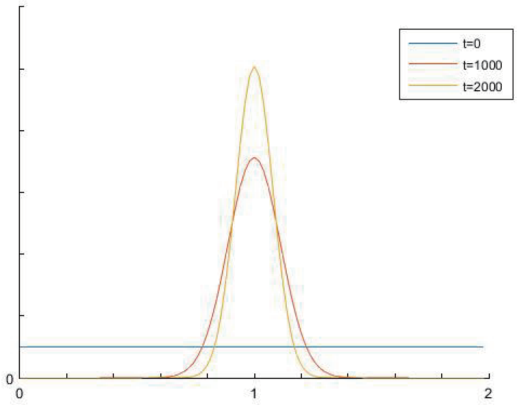

FIGURE 9.1

On the other hand, let $\bar{y}^\mu := \int_A y\mu(dy)$. If μ is such that $\bar{y}^\mu < x^*$, then by Jensen's inequality

$$J(\delta_{x^*}, \mu) = \int_A U(x^*, y)\mu(dy) = U(x^*, \bar{y}^\mu) > U(\bar{y}^\mu, \bar{y}^\mu) \geq J(\mu, \mu),$$

This is also true if $\bar{y}^\mu > x^*$. Hence, for any metric r on $\mathbb{P}(A)$, the strategy δ_{x^*} is r-SUS. Therefore, by Theorem 6.3, if $\|\mu_0 - \delta_{x^*}\| = 2(1 - \mu_0(\{x^*\})) < \epsilon$, then

$$\|\mu(t) - \delta_{x^*}\| = 2(1 - \mu(t, \{x^*\})) < \epsilon, \; r_w(\mu(t), \delta_{x^*}) < M\epsilon \; \forall t \geq 0.$$

Moreover, since the payoff function $U(\cdot)$ is continuous and the set A of strategies is compact, we conclude that $\mu(t) \to \delta_{x^*}$ weakly.

Consider a game where $a = 2, \; b = 1, \; c = 5, \; d = 1, \; M = 2$. For this game the payoff function (9.3) is bounded Lipschitz and by Theorem 8.4 we can approximate the replicator dynamics by a finite-dimensional dynamical system of the form (5.1)-(5.2) under the Kantorovich-Rubinstein norm. The Figure 9.1 shows a numerical approximation for this game where the Nash equilibrium is $x^* = 1$. For this numerical approximation we consider a partition with 100 elements with the same size, and use the forward Euler method for solving ordinary differential equations. We consider the uniform distribution as initial condition. We show the distribution for the times 0, 1000 and 2000.

Note that under the strong norm the Nash equilibrium $x^* = 1$ cannot be approximated by a probability measure with a continuous density function.

9.3. **A graduated risk game.** A graduated risk game is a symmetric game, where two players compete for a resource of value $v > 0$. Each player selects her "level of aggression" for the game. This "level of aggression" is captured by a probability distribution on $A := [0, 1]$. In this case, $x \in A$ can be interpreted as the probability that neither player is injured, and $\frac{1}{2}(1 - x)$ is the probability that player one (or player two) is injured. If the player is injured, its payoff is $v - c$ (with $c > 0$), and

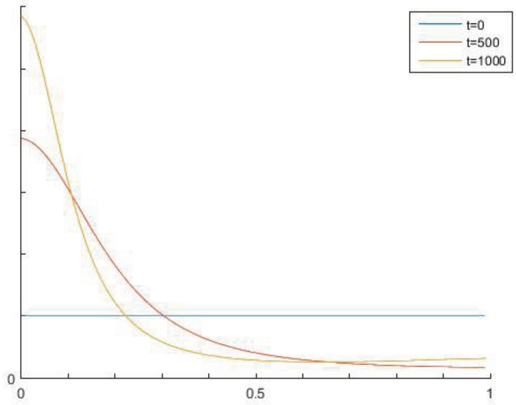

FIGURE 9.2

hence the expected payoff for the player is

$$U(x,y) = \begin{cases} vy + \frac{v-c}{2}(1-y) & \text{if } y > x \\ \frac{v-c}{2}(1-x) & \text{if } y \leq x \end{cases} \tag{9.4}$$

where x and y are the "levels of aggression" selected by the player and her opponent, respectively.

If $v < c$, this game has a NES with density function

$$\frac{d\mu^*(x)}{dx} = \frac{\alpha - 1}{2} x^{\frac{\alpha-3}{2}},$$

where $\alpha = \frac{c}{v}$. Bishop and Cannings [2] show that if $v < c$, then the NES satisfies that

$$J(\mu^*, \mu) - J(\mu, \mu) > 0 \ \ \forall \mu \in \mathbb{P}(A),$$

that is, μ^* is a r-SUS for any metric r in $\mathbb{P}(A)$, with $A = [0, 1]$.

Hence, by Theorem 6.2, if $K(\mu_0, \mu^*) < \varphi'(\epsilon) = \left(\frac{\epsilon}{2}\right)^2$, then

i) $\mu(t) \in \mathcal{W}_{\varphi'(\epsilon)}(\mu^*)$ for all $t \geq 0$;
ii) $\|\mu(t) - \mu^*\| < \epsilon$ for all $t \geq 0$;
iii) $r_w(\mu(t), \mu^*) < \epsilon$ for all $t \geq 0$.

Consider a game where $c = 10,\ v = 6.5$. For this game the payoff function (9.4) is bounded, and by theorem 8.1 we can approximate the replicator dynamics by a finite-dimensional dynamical system of the form (5.1)-(5.2) under the strong norm (2.1). The Figure 9.2 shows a numerical approximation for this game. For this numerical approximation we consider a partition with 100 elements with the same size, and use the forward Euler method for solving ordinary differential equations. We consider the uniform distribution as initial condition. We show the distribution for the times 0, 500 and 1000.

In the same way, Figure 9.3 shows a numerical approximation for a game where $c = 10,\ v = 0.5$. For this numerical approximation we consider a partition with

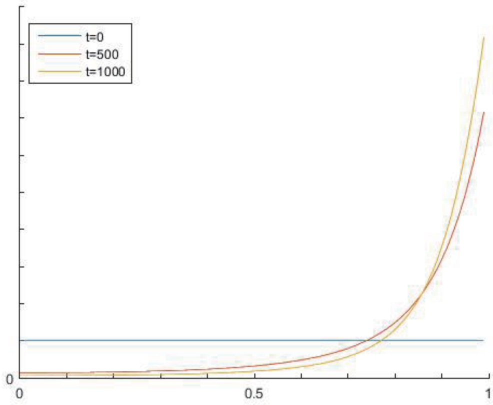

FIGURE 9.3

100 elements with the same size, and use again the forward Euler method for solving ordinary differential equations. We consider the uniform distribution as initial condition. We show the distribution for the times 0, 500 and 1000.

10. COMMENTS

In this survey, we introduced a model of evolutionary games with strategies in metric spaces. The model can be reduced, of course, to the particular case of evolutionary games with finite strategy sets. We provide a general framework to the replicator dynamics that allows us to analyze different stability criteria, and establish conditions to approximate the replicator dynamics in a metric space by a sequence of dynamical systems on finite-dimensional spaces. We also presented three examples. The first two models may be applicable to oligopoly models, theory of international trade, and public good models. The third example deals with a graduated risk game.

There are many questions, however, that remain open. For instance, when the set of pure strategies is finite, Cressman [9] shows that under some conditions the stability of monotone selection dynamics is locally determined by the replicator dynamics. Is this true for games with strategies in the space $\mathbb{P}(A)$ of probability measures? The study of the stability for other game dynamics with strategies in metric spaces has few theoretical results. A detailed study might require new theoretical developments in the stability analysis of dynamic systems in general spaces.

Another important issue would be to obtain stability results for evolutionary games with continuous strategies similar to the result by Hofbauer and Sigmund [22] (Theorem 14) for games with a finite strategy set A.

The replicator dynamics has been studied in other general spaces without direct applications to game theory. For instance, Kravvaritis et al. [27], [24], [25] [26], and Papanicolaou and Smyrlis [37] studied conditions for stability and examples for these general cases. These extensions may be applicable in areas such as migration, regional sciences, and spatial economics (see Fujita, Krugman, and Venables [14] chapters 5 and 6). However, these extensions have not been made for asymmetric models, that is, for multipopulation games.

Appendix A: Metrics on $\mathbb{P}(A)$

Proposition A.1. *Let (A, r) be a separable metric space. Then the Prokhorov metric r_p and the bounded Lipschitz metric r_{bl} metrize the weak convergence, i.e., for any sequence $\{\mu_n\} \subset \mathbb{P}(A)$, the following statements are equivalent:*

i) μ_n *converges in the weak topology,*

ii) $r_p(\mu_n, \mu) \to 0$,

iii) $r_{bl}(\mu_n, \mu) \to 0$.

Moreover, for any μ and ν in $\mathbb{P}(A)$,

$$\frac{1}{3}[r_p(\mu,\nu)]^2 \leq r_{bl}(\mu,\nu) \leq 2r_p(\mu,\nu) \tag{A.1}$$

Proof. See Shiryaev [43] chapter 3. □

Proposition A.2. *Let (A, r) be a Polish space and $1 \leq p < \infty$. The L_p-Wasserstein metric r_{w_p} metrizes the weak convergence on $\mathbb{P}_p(A)$, i.e., for any sequence $\{\mu_n\} \subset \mathbb{P}_p(A)$ and $\{\mu\} \subset \mathbb{P}(A)$, the following conditions are equivalent:*

i) μ_n *converges in the weak topology,*

ii) $r_{w_p}(\mu_n, \mu) \to 0$.

Moreover, if A is bounded, then the L_p-Wasserstein metric r_{w_p}, the Prokhorov metric r_p, the bounded Lipschitz metric r_{bl} and the Kantorovich-Rubinstein metric r_{kr} metrize the weak convergence of probability measures in $\mathbb{P}(A)$. In addition, if $p = 1$ then

$$\frac{1}{3}[r_p(\mu,\nu)]^2 \leq r_{bl}(\mu,\nu) \leq r_{kr}(\mu,\nu) = r_w(\mu,\nu). \tag{A.2}$$

Proof. See Shiryaev [43] chapter 3, and Givens and Shortt [15]. □

Proposition A.3. *Let A be a separable metric space. Let μ and ν be in $\mathbb{P}(A)$, with $\nu << \mu$. Then*

$$\|\mu - \nu\| \leq 2[K(\mu,\nu)]^{\frac{1}{2}}. \tag{A.3}$$

Moreover, if A is a bounded (with diameter $C > 0$) Polish space, then

$$r_w(\mu,\nu) \leq C\|\mu - \nu\| \leq 2C[K(\mu,\nu)]^{\frac{1}{2}}. \tag{A.4}$$

Proof. See Reiss [39] chapter 3, and Villani [46] chapter 6. □

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