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TIME DISCRETIZATION OF A QUASI-VARIATIONAL INEQUALITY RELATED TO THE HUMID ATMOSPHERE

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Abstract. In this article, we study the numerical approximation of solutions to some quasi-variational inequalities introduced in [16] modeling the humid atmosphere with a multi-phase saturation. The humid atmosphere model studied here comprises three components, namely water vapor, rain water and cloud condensate, with respective mass ratios q_v, q_r and q_c. We also consider the saturation concentration q_{vs}, which is a diagnostic variable depending itself on the state. We construct a penalized and regularized implicit Euler method to overcome the difficulties caused by the nonlinear constraints on the water vapor mass ratio q_v ($q_v \leq q_{vs}$), and the discontinuity in the quasi-variational inequalities. By deriving delicate a priori estimates and applying compactness arguments, we manage to show that the approximation functions associated with the numerical scheme converge to the solutions of the quasi-variational inequality. A numerical simulation is included at the end to help illustrate the model.

1. Introduction

Clouds, with their continuous changing nature, have been the greatest source of uncertainty for the current numerical weather and climate predictions and have given rise to decades of challenges to climate modellers. Therefore, the investigation of the humid atmosphere plays a vital role in the better understanding of the weather forecasting in the short term and climate changes in the long term.

The Primitive Equations are the classical tool used in the study of the climate and weather predictions. They describe the dynamics of the atmosphere when the hydrostatic approximation is enforced (see e.g. [32],[34],[52]).The mathematical theory of the equations of the humid atmosphere appearing in [28], [52] has been initiated in [44] and more recently in [30, 31]. However, these references only accounted for the humidity through the air vapor concentration q_v, and the saturation of water vapor in the air was not considered, so that the equation for the concentration q_v of water vapor in the air was just a simple transport equation. The references [18], [17], and [10] are among the first articles to have accounted for the water saturation, with the two additional simplifying assumptions that the fluid velocity $\mathbf{u}$ is known and that the saturation concentration q_{vs} is constant. This is acceptable from the physical viewpoint since q_{vs} does not vary too much. In cloud microphysics parameterizations it is often assumed that the vapor-to-cloud water conversion is instantaneous,

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i.e. that *either* the air is saturated, that is, the water vapor content matches its saturation value, $q_v = q_{vs}(T,p)$, and the cloud water droplets can exist with $q_c > 0$, *or* the air is undersaturated , i.e. $q_v < q_{vs}$, in which case $q_c \equiv 0$; see [29]. This is also the viewpoint of the above mentioned references ([18, 17, 10]). Accordingly, the involvement of thresholds for the cloud water condensation and evaporation leads to the introduction of a Heaviside function, so that the equations for q_v and T (the temperature) appear as nonlinear, discontinuous and non-monotone. Nevertheless, the above mentioned references managed to establish results of existence, uniqueness and regularity of solutions. In [37], the authors provided another bulk microphysics description, where they did not assume this limiting vapor-to-cloud water conversion behavior from the outset and demonstrated how it may be derived in a consistent asymptotic framework given large but finite condensation rates. This model has recently been studied mathematically in [33], in which the authors proved the global existence and uniqueness of uniformly bounded solutions of the model in [37]. However, in the references [58, 60, 59, 16] cited below and in our current article, we still used the classical formulation, which involves a discontinuous Heaviside function, for the expression of cloud water condensation and evaporation rate. For other equations of geophysics associated with a discontinuous Heaviside function, we may refer the readers to, for instance, [20, 21, 25, 26, 28].

In trying to extend the study to the case where q_{vs} is not constant [58], it was found that the equations of the humid atmosphere in the classical references, e.g. [32, 34, 55], are inconsistent for the extreme cases $q_v = 0$ and $q_v = 1$. whether q_{vs} is constant or not. Here $q_v = 0$ corresponds to a totally dry atmosphere, $q_v = 1$ corresponds to a totally humid atmosphere. A physically satisfactory resolution of this difficulty is proposed in [58] and studied mathematically in [60], namely the equation for q_v is formulated as a variational inequality. For general results on variational inequalities and their utilization in mechanics and physics, see among a vast literature [12, 8, 9, 22, 24, 27, 36]. In [59], the authors investigateed the time-discretization scheme of the solution to the variational inequality introduced in [60].

After the references [60] and [58] resolved the inconsistency of humidity equations using the variational inequality under the simplifying assumption that q_{vs} is constant, a more recent article [16] generalized the model studied in [60] and [58] by considering the more realistic situation where the humid atmosphere comprises three components instead of only one, namely water vapor q_v, rain water q_r and cloud condensates q_c. Furthermore the saturation concentration is no longer assumed to be a constant as it was in [60], [58]. With the nonlinear constraint that the vapor mass ratio q_v is less than the saturation concentration q_{vs}, which depends itself on the temperature T, the authors introduce and handle a system of equations and inequations involving a so-called *quasi-variational inequality* for which they prove the existence of solutions. Quasi-variational inequalities have been introduced in [9], [4] by Bensoussan and Lions, motivated by the study of economical problems; see also [2], [3], [6], [7], [5] and [8]. For general results on quasi-variational inequalities and their applications in mechanics, physics and imagery, see for instance, [39, 35, 42, 45, 50, 49, 48].

The aim of the present article is to study the numerical approximation of the quasi variational inequalities introduced in [16]. In the current work, we shall follow the former works and assume, for simplicity, that the velocity $\mathbf{u}$ of the humid air is known. We believe that, despite these simplifications, the resulting system contains the essence of the nonlinearity that is present in the moist advection. See [15] for the study on a coupled system involving q_v, T and $\mathbf{u}$ where the velocity is no longer prescribed. From the mathematical point of view, this article combines the methods in [17, 18] with the methods for the 3-dimensional primitive equations (PEs) in [13], [38]. As mentioned before, the existence of change in phases leads to the introduction of a set-valued Heaviside function. The discontinuities due to the changes of phases and the quasi-variational inequality resulting from the extreme cases for the vapor concentration are the two distinct features of the model that we study and they bring significant mathematical difficulties towards the understanding of the model.

In this article, we propose an implicit Euler scheme to approach the solutions to the system. However, we can not simply proceed from this scheme directly as usual due to the difficulties induced by the discontinuities and the physical requirement for the vapor concentration q_v. We need to approximate the original scheme by a relatively standard nonlinear problem which can be treated by classical methods. For that purpose, we first introduce a regularized version of the scheme where we used a continuous function to approximate the Heaviside function. Meanwhile, the unknown function q_v should satisfy the range condition $q_v \le q_{vs}$ *a.e.* in the underlying domain which is denoted by $\mathcal{M}$ below. To guarantee that the functions which we recursively define in the Euler scheme obey this constraint, we have introduced a penalization term in the form of $\frac{1}{\varepsilon}((q_v - q_{vs})^+)^\alpha$ in the regularized Euler scheme to achieve this range condition in the limit through delicate energy estimates. Note that the use of the penalization method is a convenient mathematical tool and we do not try to give a physical meaning to the penalization term. Penalization has been introduced by R. Courant [19] and it is very common in Optimization Theory (see e.g. [14] and [53]). In summary, we discretize the quasi variational inequality using an implicit Euler scheme and we use penalization and regularization techniques to show the existence of solutions to the Euler scheme. Then to prove the convergence of the Euler scheme, we classically need some strong convergence results which follow from additional a priori estimates on the solutions of the discretized Euler scheme. The most challenging part in our estimates is to show that the discrete time derivative of q_v will remain in a bounded set of the space $L^\beta(0, t_1, V^*)$ for some $\beta > 1$, in view of using a suitable version of the Aubin-Lions compactness theorem [41]. Here $t_1 > 0$ is an arbitrary fixed time and the space V^* is the dual space of $V = H^1(\mathcal{M})$. The challenge comes from the large factor $\frac{1}{\varepsilon}$ in the penalization term in the q_v equation, where ε is a positive parameter that is aimed at converging to 0. The key step in the proof is to show that the saturation concentration q_{vs} in the time-discretized Euler scheme will remain in a bounded set of $L^\beta(0, t_1, H^2(\mathcal{M}))$. To achieve this goal, we need to use the relationship between q_{vs} and the temperature T and prove higher order regularity in the space $L^2(0, t_1; H^2(\mathcal{M})) \cap L^\infty(0, t_1; V)$ for T, compared with what was done in [59] when q_{vs} was assumed to be constant. Moreover, the above a priori estimates also require delicate choices of the value of α in the penalization term $\frac{1}{\varepsilon}((q_v - q_{vs})^+)^\alpha$ and of β in the space $L^\beta(0, t_1, V^*)$.

Later in Section 2 and 3 we will see that α is set to $\frac{3}{2}$ and β is set to $\frac{5}{3}$. From the mathematical view point, we systematically use the tools (e.g., integration by parts, interpolation inequalities, etc) found in [43, 56] and [57].

The rest of the article is organized as follows. In Section 2, we give a precise formulation of the problem. In Section 3, we introduce the Euler scheme and derive various uniform estimates for the functions associated with the penalized and regularized scheme. In Section 4, we investigate the convergence of the Euler scheme. Lastly, in Section 5, we illustrate the theory studied in the previous sections with some numerical simulations done in a slightly different setting where the viscosity terms are omitted as they are not significant for short term forecast, and the space dimension is 2 (with coordinates x and p, see below).

2. Formulation and setting of the exact problem

In this part, we shall introduce our system. We let $\mathcal{M} \subset \mathbb{R}^3$ be the spatial domain for our study in the x, y, p variables and a typical point in $\mathcal{M}$ is denoted by $\mathbf{x} = (x, y, p)$ where p is the pressure.The boundary $\partial\mathcal{M}$ of the domain $\mathcal{M}$ is decomposed as $\partial\mathcal{M} = \Gamma_i \cup \Gamma_u \cup \Gamma_l$ corresponding respectively to the bottom, top and lateral boundaries of $\mathcal{M}$. We use ρ, q, θ, T and e_{vs} to denote density, concentration, potential temperature, temperature, and saturation vapor pressure, respectively.

In our current study, we will consider the water vapor, cloud-condensate and rain water for the warm humid atmosphere (above freezing point $T = 273K$). For a specific quantity, we shall use the subindices v, c, and r to represent this quantity for the water vapor, cloud-condensate and rain water. For example, q_v represents the concentration of water vapor, q_c is the concentration of cloud-condensate, and q_r is the concentration of rain water, etc.

Assuming that the velocity $\mathbf{u} = (u, v, \omega)$ is known and sufficiently regular (see (2.16)), the unknowns for our current study are the potential temperature θ, the concentrations of the water vapor, cloud-condensate and rain water q_v, q_c, q_r and the saturation concentration q_{vs}. If T is the temperature then we classically have

$$\theta = T(\frac{p_0}{p})^{\varkappa} = \frac{T}{\Pi},\ \Pi = (\frac{p}{p_0})^{\varkappa}, \tag{2.1}$$

where $\varkappa = (\gamma - 1)/\gamma$ and $\gamma = c_p/c_v$ is the ratio of specific heats of dry air at constant pressure and at constant volume, p is the pressure and p_0 is a reference pressure. The usual range for p and p_0 in the atmosphere is $[200, 1000]$ (see e.g. in [11] and [23]).

Before going any further, we shall first make some simple observations. Of course, the quantities q_v, q_c, q_r, q_{vs} being relative mass fractions ratios take their values in the interval $[0, 1]$. Furthermore, following the common assumption that the vapor-to-cloud water conversion is instantaneous (see e.g., [29]), the air can not be supersaturated (in general). In other words, we have the constraint $0 \le q_v \le q_{vs}$.

The function q_{vs} is a diagnostic variable; it is explicitly given at each instant of time as a function of p and T (or θ), that is

$$q_{vs} = Q_{vs}(T, p). \tag{2.2}$$

The expression of q_{vs} as a function of T and p results from the application of the Clausius–Clapeyron equation and it can be expressed as a function of the saturation vapor pressure e_{vs}

$$q_{vs} = \frac{3.8}{p - 0.378\, e_{vs}} \frac{e_{vs}}{6.11} = 0.6219 \frac{e_{vs}}{p - 0.378\, e_{vs}}, \tag{2.3}$$

where by Tetens' formula,

$$e_{vs} = 6.11 \exp(a \frac{T - 273}{T - b}). \tag{2.4}$$

Here T is in Kelvin, $a = 17.27$, $b = 36$ for $T \geq 273K$ in [40]. Because we only consider the above freezing case in our model, $b \ll 273$, $b \ll T$, we may set $b = 0$ for simplicity. Considering the physical range of the temperature T found in the troposphere, we can replace T by $\varphi(T)$ in (2.4) where φ is a smooth (e.g. C^2) positive real function with $\varphi(T)$

$$\begin{cases} = T & for\ T_* \leq T \leq T_{**}, \\ \geq T_*/2 & for\ T \leq T_*, \\ = 0 & for\ T \geq 2T_{**}. \end{cases} \tag{2.5}$$

Here $T_* > 0$ is smaller than any temperature on earth (e.g. 100K) and T_{**} is larger than any temperature on earth (e.g. 355K). By the above modification of (2.4), we can avoid the singularity at $T = 0$, and thus e_{vs} and $q_{vs} = Q_{vs}(T, p)$ can be viewed as positive bounded smooth functions of p and T for all values of $p \geq 0$ and $T \in \mathbb{R}$.

With this expression for q_{vs}, we notice that the constraint $q_v \leq q_{vs}$ itself depends on the solution of the conservation equations; this leads us to formulate the q_v-equation in the form of a quasi-variational equation (for details, see [16]).

We then investigate the conservation equations for the relative mass densities q_v, q_c, q_r and for the temperature T (or more precisely the difference θ' between the potential temperature θ and a reference temperature θ_h, $\theta' = \theta - \theta_h$). For the original model equations, we may refer the readers to [40], [46], [47] and [51].

2.1. **Exact Problem.** We set $U = (q_v, q_c, q_r, \theta')$ and $\bar{U} = (q_c, q_r, \theta')$. Let $t_1 \geq 0$ be a fixed positive time and let $\mathcal{K}(U)$ be the non-empty closed convex set in $H^1(\mathcal{M})$ defined as $\mathcal{K}(U) = \{q_v \in H^1(\mathcal{M}); q_v \leq q_{vs}\ a.e.\}$; our problem is formulated as follows:

To find $\bar{U} : [0, t_1] \to H^1(\mathcal{M})^3$, $q_v : [0, t_1] \to \mathcal{K}(U)$ and $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$, such that for any $q_v^b \in L^2(0, t_1; \mathcal{K}(U))$ and $t \in [0, t_1]$, there hold

$$\begin{cases} \langle \partial_t q_v, q_v^b - q_v \rangle + (\mathcal{A}_v q_v + \mathbf{v} \cdot \nabla q_v + \omega \frac{\partial q_v}{\partial p}, q_v^b - q_v) \geq (f_{q_v}(U) - \frac{\omega^-}{p} F h_{q_v}, q_v^b - q_v), \\ \partial_t q_c + \mathcal{A}_c q_c + \mathbf{v} \cdot \nabla q_c + \omega \frac{\partial q_c}{\partial p} = f_{q_c}(U) - \frac{\omega^-}{p} F h_{q_v}, \\ \partial_t q_r + \mathcal{A}_r q_r + \mathbf{v} \cdot \nabla q_r + \omega \frac{\partial q_r}{\partial p} = f_{q_r}(U), \\ \partial_t \theta' + \mathcal{A}_\theta \theta' + \mathbf{v} \cdot \nabla \theta' + \omega \frac{\partial \theta'}{\partial p} = f_{\theta'}(U) + \frac{\mathcal{L}}{c_p \Pi} \frac{\omega^-}{p} F h_{q_v}, \end{cases} \tag{2.6}$$

with the following initial and boundary conditions:

$$U(x, y, p, 0) = U_0(x, y, p) = (q_{v0}, q_{c0}, q_{r0}, \theta'_0)^t(x, y, p), \tag{2.7}$$

$$\begin{cases} \partial_p q_v = \beta_v(q_{v*} - q_v) \text{ on } \Gamma_i, \quad \partial_p q_v = 0 \text{ on } \Gamma_u, \quad \partial_{n_v} q_v = 0 \text{ on } \Gamma_l, \\ \partial_p q_c = \beta_c(q_{c*} - q_c) \text{ on } \Gamma_i, \quad \partial_p q_c = 0 \text{ on } \Gamma_u, \quad \partial_{n_c} q_c = 0 \text{ on } \Gamma_l, \\ \partial_p q_r = \beta_r(q_{r*} - q_r) \text{ on } \Gamma_i, \quad \partial_p q_r = 0 \text{ on } \Gamma_u, \quad \partial_{n_r} q_r = 0 \text{ on } \Gamma_l, \\ \partial_p \theta' = \alpha(\theta'_* - \theta') \text{ on } \Gamma_i, \quad \partial_p \theta' = 0 \text{ on } \Gamma_u, \quad \partial_{n_{\theta'}} \theta' = 0 \text{ on } \Gamma_l. \end{cases} \tag{2.8}$$

The regularity for $\bar{U}_*$ and q_{v*} will be specified below. In the above expressions, $\mathcal{H}$ is the set-valued Heaviside function satisfying $\mathcal{H}(0) = [0,1]$. The heat and vapor diffusion operators $\mathcal{A}_\theta$ and $\mathcal{A}_q$ are described as

$$\mathcal{A}_\theta = -\mu_\theta \Delta - \nu_\theta \partial_p\big((\frac{gp}{R\bar{\theta}})^2 \partial_p\big), \ \mathcal{A}_q = -\mu_q \Delta - \nu_q \partial_p\big((\frac{gp}{R\bar{\theta}})^2 \partial_p\big), \tag{2.9}$$

where μ_q, ν_q $(q \in \{q_v, q_c, q_r\})$, $\mu_\theta, \nu_\theta, g, R, c_p$ are all positive constants and $\bar{\theta} = \bar{\theta}(p)$ is the average potential temperature over the isobar with pressure p. We assume that $\bar{\theta}$ satisfies:

(2.10)

$$\bar{\theta}_* \le \bar{\theta}(p) \le \bar{\theta}^*, \ |\partial_p \bar{\theta}(p)| \le M, \text{ for some positive constants } \bar{\theta}_*, \bar{\theta}^*, M \text{ and } p \in [p_0, p_1].$$

Throughout the paper, we shall assume that the boundary datum $U_* = (q_{v*}, q_{c*}, q_{r*}, \theta')$ satisfy

$$U_* \in L^2(0, t_1; L^2(\Gamma_i)^4). \tag{2.11}$$

Later, we will impose higher regularity assumptions on $\bar{U}_* = (q_{c*}, q_{r*}, \theta')$ for the homogenization of the Robin boundary conditions.

In line with what we did in (2.3)-(2.5), we assume that the source terms $f_{q_v}(U), f_{q_c}(U), f_{q_r}(U), f_{\theta'}(U)$ are all continuous bounded functions of U, compactly supported in the region of $\mathbb{R}^3$ corresponding to the domain of (q_v, q_c, q_r). The function F is defined as

$$F = F(T) = Q_{vs}(T,p)\varphi(T)\big(\frac{\mathcal{L}R - c_p R_v \varphi(T)}{c_p R_v \varphi(T)^2 + Q_{vs}(T,p)\mathcal{L}^2}\big), \tag{2.12}$$

and $\frac{-\omega^-}{p}F$ is the expression for dq_{vs}/dt. Here $\mathcal{L}$ is the latent heat of vaporization; R, R_v are the gas constants for dry air and water vapor respectively and c_p represents the specific heat of dry air at constant pressure (see [32] and [34]). From the mathematical point of view, we can treat $\mathcal{L}$, R, R_v and c_p as constants. With $\varphi(T)$ given as in (2.5), it can be easily verified that $F(T)$ is Lipschitz continuous and uniformly bounded in T.

Notice that in (2.6), the equations for q_c, q_r, θ' have the same form, so that we can rewrite them in the following compact form

$$\partial_t \bar{U} + \bar{A}\bar{U} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \cdot \bar{U} = \bar{f}(U) - \frac{\omega^-}{p}\bar{\mathcal{F}} h_{q_v}, \tag{2.13}$$

where $\bar{A} = \operatorname{diag}\{\mathcal{A}_c, \mathcal{A}_r, \mathcal{A}_{\theta'}\}$, $\bar{\mathcal{F}} = (-F, 0, -\frac{\mathcal{L}}{c_p \Pi}F)^t$. If we adopt the following notations for $\bar{U}_0 = U_0(x,y,p)$, $\bar{U}_* = U_*(x,y,p)$

$$\bar{U}_0 = (q_{c0}, q_{r0}, \theta'_0)^t, \ \bar{U}_* = (q_{c*}, q_{r*}, \theta'_*)^t,$$

and define the coefficient matrix $\bar{\mathcal{C}} = \mathrm{diag}\{\beta_c, \beta_r, \alpha\}$, then the initial and boundary conditions associated with the system (2.13) can be written as follows

$$\bar{U}(x, y, p, 0) = \bar{U}_0(x, y, p), \tag{2.14}$$

$$\partial_p \bar{U} = \bar{\mathcal{C}}(\bar{U}_* - \bar{U}) \text{ on } \Gamma_i, \quad \partial_{n_{\bar{\mathcal{A}}}} \bar{U} = 0 \text{ on } \Gamma_u \cup \Gamma_l. \tag{2.15}$$

For the weak formulation in the coming subsection, we will treat differently the equation for $\bar{U} = (q_c, q_r, \theta')$ and the equation for q_v which is subjected to the constraint $q_v \leq q_s$.

2.2. **Notations and weak formulation.** We denote as usual $H = L^2(\mathcal{M})$, $V = H^1(\mathcal{M})$ and we set $\mathbb{H} = H \times H \times H \times H$ and $\mathbb{V} = V \times V \times V \times V$. We use $(\cdot, \cdot)_{L^2}$ (regarded the same as $(\cdot, \cdot)_H$) and $|\cdot|_{L^2}$ to denote the usual scalar product and induced norm in H. In the space V, we will use $((\cdot, \cdot))$ to denote the scalar product adapted to the problem under investigation

$$((\varphi, \phi)) := (\nabla\varphi, \nabla\phi) + (\partial_p\varphi, \partial_p\phi) + \int_{\Gamma_i} \varphi\phi \, d\Gamma_i,$$

and the induced norm is denoted $\|\cdot\|$. The symbol $\langle \cdot, \cdot \rangle$ will denote the duality pair between a Banach space E and its dual space E^*. In relation with the Navier-Stokes equations, we also use the following standard notations:

$$\mathbf{H} = \{\mathbf{u} \in H \times H \times H \,\big|\, div\, \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot n = 0 \text{ on } \partial\mathcal{M}\},$$

$$\mathbf{V} = \{\mathbf{u} \in V \times V \times V \,\big|\, div\, \mathbf{u} = \mathbf{0} \text{ and } \mathbf{u} \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial\mathcal{M}\},$$

which will serve as the natural function spaces for the vector field $\mathbf{u}$. In fact we will assume that

$$\mathbf{u} \in L^\infty(0, t_1; \mathbf{V}) \cap L^\infty((0, t_1) \times \mathcal{M}). \tag{2.16}$$

In view of deriving the weak (variational) formulation of the boundary value problem, we multiply e.g. the expression $\mathcal{A}_{q_v} q_v$ by a test function q_v^b. Assuming smoothness and taking into account the boundary conditions (2.8) for q_v we find

$$\begin{aligned}\langle \mathcal{A}_{q_v} q_v, q_v^b \rangle &= \Big(-\mu_{q_v} \Delta q_v - \nu_{q_v} \partial_p\big((\frac{gp}{R\bar{\theta}})^2 \partial_p q_v, \big), q_v^b\Big) \\ &:= \mu_{q_v} (\nabla q_v, \nabla q_v^b)_H + \nu_{q_v} \int_{\mathcal{M}} \big(\frac{gp}{R\bar{\theta}}\big)^2 \partial_p q_v \partial_p q_v^b \, d\mathcal{M} \\ &\quad + \nu_{q_v} \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \beta_{q_v} (q_v - q_{v*}) q_v^b \, d\Gamma_i.\end{aligned} \tag{2.17}$$

We do the same for q_c, q_r and θ' and thus

$$\begin{aligned}\langle \mathcal{A}_{q_c} q_c, q_c^b \rangle &= \mu_{q_c} (\nabla q_c, \nabla q_c^b)_H + \nu_{q_c} \int_{\mathcal{M}} \big(\frac{gp}{R\bar{\theta}}\big)^2 \partial_p q_c \partial_p q_c^b \, d\mathcal{M} \\ &\quad + \nu_{q_c} \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \beta_{q_c} (q_c - q_{c*}) q_c^b \, d\Gamma_i,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\langle \mathcal{A}_{q_r} q_r, q_r^b \rangle &= \mu_{q_r} (\nabla q_r, \nabla q_r^b)_H + \nu_{q_r} \int_{\mathcal{M}} \big(\frac{gp}{R\bar{\theta}}\big)^2 \partial_p q_r \partial_p q_r^b \, d\mathcal{M} \\ &\quad + \nu_{q_r} \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \beta_{q_r} (q_r - q_{r*}) q_r^b \, d\Gamma_i,\end{aligned} \tag{2.19}$$

and

$$\langle \mathcal{A}_\theta \theta', \theta'^b\rangle = \mu_\theta(\nabla\theta', \nabla\theta'^b)_H + \nu_\theta \int_{\mathcal{M}} \big(\frac{gp}{R\bar{\theta}}\big)^2 \partial_p\theta'\partial_p\theta'^b \, d\mathcal{M}$$

$$+ \nu_\theta \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \alpha(\theta' - \theta'_*)\theta'^b \, d\Gamma_i. \tag{2.20}$$

Consequently, we define the following bilinear forms

$$a_\theta(\theta', \theta'^b) = \mu_\theta(\nabla\theta', \nabla\theta'^b)_H + \nu_\theta \int_{\mathcal{M}} \big(\frac{gp}{R\bar{\theta}}\big)^2 \partial_p\theta'\partial_p\theta'^b \, d\mathcal{M} + \nu_\theta\alpha \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \theta'\theta'^b \, d\Gamma_i, \tag{2.21}$$

$$a_q(q, q^b) = \mu_q(\nabla q, \nabla q^b)_H + \nu_q \int_{\mathcal{M}} \big(\frac{gp}{R\bar{\theta}}\big)^2 \partial_p q\partial_p q^b \, d\mathcal{M} + \nu_q\beta_q \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 q q^b \, d\Gamma_i. \tag{2.22}$$

Similarly, we define $b(\mathbf{u}, \psi, \psi^b)$ as follows:

$$b(\mathbf{u}, \psi, \psi^b) = \int_{\mathcal{M}} (\mathbf{v}\cdot\nabla\psi + \omega\partial_p\psi)\psi^b \, d\mathcal{M}, \tag{2.23}$$

which we will use with $(\psi, \psi^b) = (\theta', \theta'^b), (q_v, q_v^b), (q_r, q_r^b), (q_c, q_c^b)$. We recall here that $\mathbf{u} = (\mathbf{v}, \omega)$ is the three dimensional velocity, $\mathbf{v}$ is the horizontal velocity and ω is the vertical velocity of the air in the x, y, p system.

Analogously, we define the linear functionals:

$$l_\theta(\theta'^b) = \nu_\theta\alpha \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 \theta_*\theta'^b \, d\Gamma_i, \quad l_q(q^b) = \nu_q\beta_q \int_{\Gamma_i} \big(\frac{gp_1}{R\bar{\theta}}\big)^2 q_* q^b \, d\Gamma_i, \tag{2.24}$$

$$l(U^b) = l_{q_c}(q_c^b) + l_{q_v}(q_v^b) + l_{q_r}(q_r^b) + l_\theta(\theta'^b), \tag{2.25}$$

which correspond to the reference state terms on the boundary Γ_i in (2.17)-(2.20).

We introduce the multilinear forms for U and $U^b = (q_c^b, q_v^b, q_r^b, \theta'^b)$

$$a(U, U^b) = a_{q_c}(q_c, q_c^b) + a_{q_v}(q_v, q_c^b) + a_{q_r}(q_r, q_r^b) + a_\theta(\theta', \theta'^b), \tag{2.26}$$

$$b(\mathbf{u}, U, U^b) = \int_{\mathcal{M}} (\mathbf{u}\cdot\nabla_{x,y,p} U)\cdot U^b \, d\mathcal{M}. \tag{2.27}$$

It is easy to see that

$$b(\mathbf{u}, U, U^b) = b(\mathbf{u}, q_c, q_c^b) + b(\mathbf{u}, q_v, q_v^b) + b(\mathbf{u}, q_r, q_r^b) + b(\mathbf{u}, \theta', \theta'^b). \tag{2.28}$$

In view of $\nabla\cdot\mathbf{u} = 0$, we readily see by performing integration by parts that

$$b(\mathbf{u}, \psi, \psi) = 0, \ \forall\, \psi \in V. \tag{2.29}$$

Before we move further, we first give the following well-known estimates.

More precisely, we have the following lemma concerning the boundedness of the above functionals.

Lemma 2.1. *Assume $U = (q_v, q_c, q_r, \theta'), U^b = (q_v^b, q_c^b, q_r^b, \theta'^b) \in \mathbb{V}$ and $\mathbf{u} \in \mathbf{V}$. There exist universal positive constants C_a and C_b such that (q denotes here q_v, q_c or q_r):*

$$|a_\theta(\theta, \theta^b)| \le C_a\|\theta'\|\|\theta^b\|, \ a_\theta(\theta, \theta) \ge C_b\|\theta\|^2; \tag{2.30}$$

$$|a_q(q, q^b)| \leq C_a\|q\|\|q^b\|, \ a_q(q,q) \geq C_b\|q\|^2; \tag{2.31}$$

$$|b(\mathbf{u}, U, U^b)| \leq C_a\|\mathbf{u}\|_{\mathbf{V}}|U|_{L^2}^{\frac{1}{2}}\|U\|^{\frac{1}{2}}\|U'^b\|; \tag{2.32}$$

$$|l_\theta(\theta'^b)| \leq C_a\|\theta'^b\|, \ |l_q(q^b)| \leq C_a\|q^b\|. \tag{2.33}$$

The proof of Lemma 2.1 is based on a routine use of the Cauchy-Schwarz inequality and the trace theorem, see e.g. [54]. We shall omit the details here.

Weak Formulation

Now we can describe the weak formulation for the exact problem (2.6) as in [16]. Let $U_0 \in \mathbb{V}$ be such that $0 \leq q_{v0} \leq q_{vs}(t=0)$ and let $t_1 > 0$ be an arbitrary but fixed time. A vector $U = U(t) = (q_v, \bar{U}) \in L^2(0, t_1; \mathcal{K} \times V^3) \cap C([0, t_1]; \mathbb{V})$ with $\partial_t \bar{U} \in L^2(0, t_1; (V^3)^*)$, $\partial_t q_v \in L^{5/3}(0, t_1; V^*)$ is a solution to the initial-boundary value problem (2.6)-(2.8), if, for almost every $t \in [0, t_1]$ and for every $U^b \in \mathcal{K} \times V \times V \times V$, we have

$$\int_0^{t_1} \Big[\langle \partial_t \bar{U}, \bar{U}^b\rangle + \bar{a}(\bar{U}, \bar{U}^b) + \bar{b}(\mathbf{u}, \bar{U}, \bar{U}^b) - \bar{l}(\bar{U}^b)\Big]\,dt = \int_0^{t_1} (f(\bar{U}) - \frac{\omega^-}{p}\bar{\mathcal{F}}h_{q_v}, \bar{U}^b)dt, \tag{2.34}$$

for all $\bar{U}^b \in L^2(0, t_1; (H^1)^3)$ with initial condition

$$\bar{U}(t=0) = \bar{U}_0,$$

and

$$\int_0^{t_1} \Big[\langle \partial_t q_v, q_v^b - q_v\rangle + a_{q_v}(q_v, q_v^b - q_v) + b(\mathbf{u}, q_v, q_v^b - q_v) - l_{q_v}(q_v^b - q_v)\Big]\,dt$$

$$\geq \int_0^{t_1} (f_{q_v}(U) - \frac{\omega^-}{p}Fh_{q_v}, q_v^b - q_v)dt, \tag{2.35}$$

for all $q_v^b \in L^\infty(0, t_1; H^1)$ with $q_v^b \leq q_{vs} = Q_{vs}(p, T)$ and initial condition

$$q_v(t=0) = q_{v0}. \tag{2.36}$$

3. Time discretization: The Euler scheme

3.1. **Time-discretization.** Let N be an integer which will later go to ∞ and set $\Delta t := k = t_1/N$. We will define recursively a family of elements of $\mathcal{K} \times V \times V \times V$, say $(q_v^m, \bar{U}^m)$, $m = 0, 1, \ldots, N$ where $(q_v^m, \bar{U}^m)$ is intended to be an approximation of $(q_v, \bar{U})$ at time $m\Delta t$.

We begin by defining $\mathbf{u}^m$, ω^m for $m = 1, \ldots, N$:

$$\mathbf{u}^m = \frac{1}{k}\int_{(m-1)k}^{mk} \mathbf{u}(t)dt, \ \omega^m = \frac{1}{k}\int_{(m-1)k}^{mk} \omega(t)dt. \tag{3.1}$$

It is easy to observe that $\mathbf{u}^m$ inherits the divergence-free property of $\mathbf{u}^m$, and also

$$|\mathbf{u}^m|_{L^2} = |\frac{1}{k}\int_{(m-1)k}^{mk} \mathbf{u}(t)dt|_{L^2} \leq \frac{1}{k}\int_{(m-1)k}^{mk} |\mathbf{u}(t)|_{L^2}dt \leq |\mathbf{u}|_{L^\infty(0,t_1;H)}. \tag{3.2}$$

Now we discretize (2.34) and (2.35) in time using the semi-implicit Euler scheme. The initial datum $(\bar{U}^0, q_v^0)$ is given, and when $(\bar{U}^0, q_v^0), (\bar{U}^1, q_v^1), \ldots, (\bar{U}^m, q_v^m)$ are known, $\bar{U}^{m+1} \in V^3$ and $q_v^{m+1} \in \mathcal{K}(U^{m+1})$ are formally determined by:

$$\langle \frac{\bar{U}^{m+1}-\bar{U}^m}{k}, \bar{U}^b\rangle + \bar{a}(\bar{U}^{m+1}, \bar{U}^b) + \bar{b}(\mathbf{u}^{m+1}, \bar{U}^{m+1}, \bar{U}^b) - \bar{l}(\bar{U}^b) \tag{3.3}$$
$$= (\bar{f}(U^m) - \frac{[\omega^m]^-}{p}\bar{\mathcal{F}}(T^m)h_{q_v^m}, \bar{U}^b),$$

$$\langle \frac{q_v^{m+1}-q_v^m}{k}, q_v^b - q_v^{m+1}\rangle + a_{q_v}(q_v^{m+1}, q_v^b - q_v^{m+1}) + b(\mathbf{u}, q_v^{m+1}, q_v^b - q_v^{m+1}) \tag{3.4}$$
$$-l_{q_v}(q_v^b - q_v^{m+1}) \geq (f_{q_v}(U^m) - \frac{[\omega^m]^-}{p}F(T^m)h_{q_v^m}, q_v^b - q_v^{m+1}),$$

for any $\bar{U}^b \in V^3$, $q_v^b \in \mathcal{K}(U^{m+1})$.

To deal with the discontinuity of h_{q_v} and the constraint $q_v \leq q_{vs}$, we consider the associated regularized and penalized problem like in [16]:

$$\langle \frac{\bar{U}_\varepsilon^{m+1}-\bar{U}_\varepsilon^m}{k}, \bar{U}^b\rangle + \bar{a}(\bar{U}_\varepsilon^{m+1}, \bar{U}^b) + \bar{b}(\mathbf{u}^{m+1}, \bar{U}_\varepsilon^{m+1}, \bar{U}^b) - \bar{l}(\bar{U}^b) \tag{3.5}$$
$$= (\bar{f}(U_\varepsilon^m) - \frac{[\omega^m]^-}{p}\bar{\mathcal{F}}(T_\varepsilon^m)\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), \bar{U}^b),$$

$$\langle \frac{q_{v\varepsilon}^{m+1}-q_{v\varepsilon}^m}{k}, q_v^b\rangle + a_{q_v}(q_{v\varepsilon}^{m+1}, q_v^b) + b(\mathbf{u}, q_{v\varepsilon}^{m+1}, q_v^b) + (\frac{1}{\varepsilon_1}((q_{v\varepsilon}^{m+1} - q_{vs,\varepsilon}^{m+1})^+)^{3/2}, q_v^b) \tag{3.6}$$
$$-l_{q_v}(q_v^b) \geq (f_{q_v}(U_\varepsilon^m) - \frac{[\omega^m]^-}{p}F(T_\varepsilon^m)\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), q_v^b)$$

for all $\bar{U}^b \in V^3$, $q_v^b \in V$. Here, the function $\mathcal{H}_{\varepsilon_2}(r)$ is defined as

$$\mathcal{H}_{\varepsilon_2}(r) = \begin{cases} 0 & for\ r \leq 0, \\ r/\varepsilon_2 & for\ r \in (0, \varepsilon_2], \\ 1 & for\ r > \varepsilon_2, \end{cases} \tag{3.7}$$

and $q_{vs,\varepsilon}^{m+1} = q_{vs}(T_\varepsilon^{m+1}, p)$. When $(\bar{U}^0, q_v^0), (\bar{U}^1, q_v^1), \ldots, (\bar{U}^m, q_v^m)$ are known, the existence of solutions $\bar{U}_\varepsilon^{m+1}, q_{v\varepsilon}^{m+1}$ to (3.5) and (3.6) follows from Theorem I-1.2 of [57], using the Galerkin method. Due to the nonlinearity of the penalization term $\frac{1}{\varepsilon_1}((q_{v\varepsilon}^{m+1} - q_{vs,\varepsilon}^{m+1})^+)^{3/2}$, the semi-implicit Euler scheme becomes fully implicit.

Remark 3.1. In the above scheme , we can replace $h_{q_v^m}$ (resp. $\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m)$) by $h_{q_v^{m+1}}$ (resp. $\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^{m+1} - q_{vs,\varepsilon}^{m+1})$). The a priori estimates in the following section still hold.

3.2. **A Priori Estimates for** $(\bar{U}_\varepsilon^m, q_{v\varepsilon}^m)$**.** In this subsection, we aim to obtain some a priori estimates on the $(\bar{U}_\varepsilon^m, q_{v\varepsilon}^m)$ that are independent of k, ε_1 and ε_2. The dependence on $\varepsilon = (\varepsilon_1, \varepsilon_2)$ will be omitted when there is no confusion in the text. And we will use C and C_1 to denote some generic constants which may depend on the initial datum $(\bar{U}_0, q_{v0})$, the velocity field $\mathbf{u}$ and the time t_1 but are independent of k and ε.

Lemma 3.2. *The following estimates hold:*

$$\begin{aligned}
&|\bar{U}_\varepsilon^j|_{L^2}^2 \le C(\mathbf{u}, U_0, t_1),\ |q_{v\varepsilon}^j|_{L^2}^2 \le C(\mathbf{u}, U_0, t_1),\ 1 \le j \le N,\ N = \frac{t_1}{k},\\
&\sum_{m=0}^{N-1} |\bar{U}_\varepsilon^{m+1} - \bar{U}_\varepsilon^m|_{L^2}^2 \le C(\mathbf{u}, U_0, t_1),\ k\sum_{m=1}^{N} ||\bar{U}_\varepsilon^m||^2 \le C(\mathbf{u}, U_0, t_1),\\
&\sum_{m=0}^{N-1} |q_{v\varepsilon}^{m+1} - q_{v\varepsilon}^m|_{L^2}^2 \le C(\mathbf{u}, U_0, t_1),\ k\sum_{m=1}^{N} ||q_{v\varepsilon}^m||^2 \le C(\mathbf{u}, U_0, t_1).
\end{aligned} \tag{3.8}$$

Proof. We first prove that the above bounds hold for q_v^m; setting $q_v^b = 2kq_v^{m+1}$ in (3.6), we have

$$\begin{aligned}
&|q_v^{m+1}|^2 - |q_v^m|^2 + |q_v^{m+1} - q_v^m|^2 + 2ka_{q_v}(q_v^{m+1}, q_v^{m+1})\\
&+2kb(u^{m+1}, q_v^{m+1}, q_v^{m+1}) + \frac{2k}{\varepsilon_1}(((q_v^{m+1} - q_{vs}^{m+1})^+)^{3/2}, q_v^{m+1})\\
&=2k[l_v(q_v^{m+1}) + (f_{q_v}(U^m) - \frac{1}{p}[\omega^m]^- F(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m - q_{vs}^m), q_v^{m+1})]\\
&\le k(C_a||q_v^{m+1}||^2 + C(\mathbf{u}, U_0)).
\end{aligned} \tag{3.9}$$

Here we have used Lemma 2.1 and the fact that $f_{q_v}(U^m) - \frac{1}{p}[\omega^m]^- F(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m - q_{vs}^m)$ is bounded in $L^\infty(\mathcal{M})$ in (3.9).

Observing that the term $(((q_v^{m+1} - q_{vs}^{m+1})^+)^{3/2}, q_v^{m+1})$ is positive and using Lemma 2.1 again in the LHS of (3.9), we conclude that

$$|q_v^{m+1}|^2 - |q_v^m|^2 + |q_v^{m+1} - q_v^m|^2 + kC_a||q_v^{m+1}||^2 \le kC(\mathbf{u}, U_0). \tag{3.10}$$

Summing (3.10) in m from 0 to $N-1$ we obtain

$$|q_v^N|^2 - |q_v^0|^2 + \sum_{m=0}^{N-1} |q_v^{m+1} - q_v^m|^2 + \sum_{m=0}^{N-1} kC_a||q_v^{m+1}||^2 \le C(\mathbf{u}, U_0)t_1,$$

which implies

$$\sum_{m=0}^{N-1} |q_v^{m+1} - q_v^m|^2 \le C(\mathbf{u}, U_0, t_1),\ \sum_{m=1}^{N-1} k||q_v^m||^2 \le C(\mathbf{u}, U_0, t_1). \tag{3.11}$$

Summing (3.10) in m from 0 to $j-1$ for any j between 1 and N and droping some positive terms, we also have

$$|q_v^j|^2 \le |q_v^0|^2 + Ct_1 \le C(\mathbf{u}, U_0, t_1),\ 1 \le j \le N. \tag{3.12}$$

So we have the desired estimates on $q_{v\varepsilon}^m$. The estimates on $\bar{U}_\varepsilon^m$ can be derived in the same way and we omit the details here. $\square$

Remark 3.3. Equation (3.9) also implies

$$
\begin{aligned}
&\frac{2k}{\varepsilon_1}(((q_v^{m+1}-q_{vs}^{m+1})^+)^{3/2}, q_v^{m+1}-q_{vs}^{m+1})\\
&+\underbrace{\frac{2k}{\varepsilon_1}(((q_v^{m+1}-q_{vs}^{m+1})^+)^{3/2}, q_{vs}^{m+1})}_{\geq 0}+|q_v^{m+1}|^2-|q_v^m|^2\\
&\leq kC(\mathbf{u},U_0,t_1)
\end{aligned}
\tag{3.13}
$$

Summing (3.13) in m from $m=0$ to $m=N-1$ and dropping some positive terms in the LHS, we obtain the additional a priori estimate

$$
\frac{2k}{\varepsilon_1}\sum_{m=0}^{N-1}|(q_v^{m+1}-q_{vs}^{m+1})^+|^{5/2}\leq C(\mathbf{u},U_0,t_1). \tag{3.14}
$$

Next, we will seek a priori bounds for the approximate time derivatives of $\bar{U}$ and q_v, namely, we will show that $k\sum_{m=0}^{N-1}|\frac{\bar{U}_\varepsilon^{m+1}-\bar{U}_\varepsilon^m}{k}|_{L^2}^2$ and $k\sum_{m=0}^{N-1}|\frac{q_{v\varepsilon}^{m+1}-q_{v\varepsilon}^m}{k}|_{v^*}^{5/3}$ are bounded independently of k and ε. These estimates will be used later on in the compactness argument when we pass to the limit $k\to 0$. We have the following estimate for the $\bar{U}_\varepsilon^m$:

Lemma 3.4. *For any $\varepsilon_1>0$, $\varepsilon_2>0$, the inequality*

$$
k\sum_{m=0}^{N-1}\left|\frac{\bar{U}_\varepsilon^{m+1}-\bar{U}_\varepsilon^m}{k}\right|_{L^2}^2\leq C(\mathbf{u},U_0,t_1)<\infty \tag{3.15}
$$

holds for some constant $C(\mathbf{u},U_0,t_1)$ depending on $\mathbf{u},U_0,t_1$ but not on ε and k.

Proof. Before we start to derive these a priori estimates, we need to homogenize the boundary conditions on $\bar{U}$. We introduce $\bar{U}_s$, the solution of the stationary problem associated with (2.13). Namely,

$$
\bar{\mathcal{A}}\bar{U}_s^m=0,\ m=1,2,\dots,N \tag{3.16}
$$

$$
\bar{U}_s^0=\bar{U}_0(x,y,p), \tag{3.17}
$$

$$
\partial_p\bar{U}_s^m=\bar{\mathcal{C}}(\bar{U}_*^m-\bar{U}_s^m)\text{ on }\Gamma_i,\quad \partial_{n_{\bar{\mathcal{A}}}}\bar{U}_s^m=0\text{ on }\Gamma_u\cup\Gamma_l, \tag{3.18}
$$

where $\bar{U}_*^m=\frac{1}{k}\int_{(m-1)k}^{mk}\bar{U}_*(t)dt$. The regularity of the solutions $\bar{U}_s^m$ of the stationary problem (3.16)-(3.18) have been proved in Theorem 4.5 of [54]. To guarantee the validity of Theorem 4.5 in [54], we add the following assumptions on the boundary Γ_i and the boundary datum $\bar{U}_*^m$. We assume the boundary Γ_i is of class $\mathcal{C}^3$. For the boundary datum $\bar{U}_*$, we assume that it satisfies $\bar{U}_*\in L^2(0,t_1;H_0^1(\Gamma_i)^3)$, and

$$
k\sum_{m=1}^{N}\|\frac{\bar{U}_*^{m+1}-\bar{U}_*^m}{k}\|_{H^1(\Gamma_i)}^2\leq\kappa, \tag{3.19}
$$

where κ is some constant independent of k and depending only on $\mathcal{M}$. Then according to Theorem 4.5 of [54], there exists a unique solution $\bar{U}_s^m \in H^2(\mathcal{M})$ for (3.16)-(3.18), and it satisfies

$$\|\bar{U}_s^m\|_{H^2(\mathcal{M})} \leq C\|\bar{U}_*^m\|_{H^1(\Gamma_i)}. \tag{3.20}$$

Due to the linearity of the system (3.16)-(3.18) and using the assumption (3.19), we also have

$$k\sum_{m=1}^{N}\|\frac{\bar{U}_s^{m+1}-\bar{U}_s^m}{k}\|^2_{H^2(\mathcal{M})} \leq Ck\sum_{m=1}^{N}\|\frac{\bar{U}_*^{m+1}-\bar{U}_*^m}{k}\|^2_{H^1(\Gamma_i)} \leq C\kappa. \tag{3.21}$$

We then consider the function $\bar{U}_h^m = \bar{U}^m - \bar{U}_s^m$ which satisfies the following equation

$$\begin{aligned}&\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k} + \bar{\mathcal{A}}\bar{U}_h^{m+1} + \mathbf{u}^{m+1}\cdot\nabla_{\mathbf{x}}\cdot\bar{U}^{m+1}\\ =&\bar{f}(U^m) - \frac{1}{p}[\omega^m]^-\bar{\mathcal{F}}(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m - q_{vs}^m) - \frac{\bar{U}_s^{m+1}-\bar{U}_s^m}{k},\end{aligned} \tag{3.22}$$

with homogeneous boundary conditions

$$\partial_p\bar{U}_h^m + \bar{\mathcal{C}}\bar{U}_h^m = 0 \text{ on } \Gamma_i, \quad \partial_{n_{\bar{\mathcal{A}}}}\bar{U}_h^m = 0 \text{ on } \Gamma_u\cup\Gamma_l. \tag{3.23}$$

Multiplying (3.22) by $\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}$, we have

$$\begin{aligned}&\left|\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\right|^2_{L^2} + \langle\bar{\mathcal{A}}\bar{U}_h^{m+1}, \frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\rangle\\ =&(\bar{f}(\bar{U}^m) - \frac{[\omega^m]^-}{p}\bar{\mathcal{F}}^m\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m)\\ &- \mathbf{u}^{m+1}\cdot\nabla_3\bar{U}^{m+1} - \frac{\bar{U}_s^{m+1}-\bar{U}_s^m}{k}, \frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k})\\ \leq&J_1^m + \frac{1}{2}\left|\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\right|^2_{L^2},\end{aligned} \tag{3.24}$$

with

$$J_1^m = C\left|\bar{f}(\bar{U}^m) - \frac{[\omega^m]^-}{p}\bar{\mathcal{F}}^m\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m) - \mathbf{u}^{m+1}\cdot\nabla_3\bar{U}^{m+1} - \frac{\bar{U}_s^{m+1}-\bar{U}_s^m}{k}\right|^2_{L^2}. \tag{3.25}$$

Now we introduce the piecewise linear function $\tilde{\bar{U}}_{hk} : [0, t_1] \to V$ defined by

$$\tilde{\bar{U}}_{hk} = \bar{U}_h^{m+1} - (1 - \frac{t-mk}{k})(\bar{U}_h^{m+1} - \bar{U}_h^m), \tag{3.26}$$

for $t \in [mk, (m+1)k]$, $m = 0, 1, \ldots, N-1$. We see that

$$\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k} = \partial_t\tilde{\bar{U}}_{hk},\ k\sum_{m=0}^{N-1}\left|\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\right|^2_{L^2} = |\partial_t\tilde{\bar{U}}_{hk}|^2_{L^2(0,t_1;L^2)}. \tag{3.27}$$

Then in the LHS of (3.24), as $\bar{U}_h^m$ satisfies the homogeneous boundary conditions (3.23), we can use the symmetry of the operator $\bar{\mathcal{A}}$,

$$
\begin{aligned}
&\langle \bar{\mathcal{A}}\bar{U}_h^{m+1}, \frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\rangle \\
=&\langle \bar{\mathcal{A}}\big(\tilde{\bar{U}}_{hk} + (1-\frac{t-mk}{k})(\bar{U}_h^{m+1}-\bar{U}_h^m)\big), \frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\rangle \\
=&(\bar{\mathcal{A}}\tilde{\bar{U}}_{hk}, \partial_t\tilde{\bar{U}}_{hk}) + \frac{1}{k}(1-\frac{t-mk}{k})\langle \bar{\mathcal{A}}(\bar{U}_h^{m+1}-\bar{U}_h^m), \bar{U}_h^{m+1}-\bar{U}_h^m\rangle \\
\geq&\frac{1}{2}\frac{d}{dt}(\bar{\mathcal{A}}\tilde{\bar{U}}_{hk}, \tilde{\bar{U}}_{hk}).
\end{aligned}
\tag{3.28}
$$

In the RHS of (3.24), the functions $\bar{f}(\bar{U}^m)$ and $\bar{\mathcal{F}}(T^m)$ are uniformly continuous bounded functions of $\bar{U}^m$ and T^m respectively, so

$$
k\sum_{m=0}^{N-1}|\bar{f}(\bar{U}^m) - \frac{[\omega^m]^-}{p}\bar{\mathcal{F}}(T^m)\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m)|_{L^2}^2 \leq Ck\sum_{m=0}^{N-1}|\bar{U}^m|_{L^2}^2 + C_1.
$$

Then by Lemma 3.2, we have $k\sum_{m=0}^{N-1}|\nabla_3\bar{U}^{m+1}|_{L^2}^2 \leq C$ and the velocity $\mathbf{u}^{m+1}$ is assumed to be given in $L^\infty(V)$, $0 \leq m \leq N-1$. These estimates together with (3.21) imply that the J_1^m satisfy $k\sum_{m=0}^{N-1}J_1^m \leq C$, where C is some constant independent of k and ε. Hence (3.24) implies

$$
\begin{aligned}
&\left|\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\right|_{L^2}^2 + \frac{d}{dt}(\bar{A}\tilde{\bar{U}}_{hk}, \tilde{\bar{U}}_{hk}) \\
=&\left|\frac{d}{dt}\tilde{\bar{U}}_{hk}\right|_{L^2}^2 + \frac{d}{dt}(\bar{A}\tilde{\bar{U}}_{hk}, \tilde{\bar{U}}_{hk}) \\
\leq& 2J_1^m = \mathcal{G}(t),\ t \in [mk, (m+1)k].
\end{aligned}
\tag{3.29}
$$

Here $\mathcal{G}(t) : [0, t_1] \to V$ is the function defined by $\mathcal{G}(t) = 2J_1^m$ for $t \in [mk, (m+1)k]$, $m = 0, ..., N-1$. We see that $\mathcal{G}(t)$ is a function bounded independently of k and ε in $L^1(0, t_1)$. We integrate (3.29) from 0 to t_1, drop the positive term $(A\bar{U}_h^N, \bar{U}_h^N)$ in the LHS of (3.29) and use the fact that $\bar{U}_{h0} = \bar{U}_0 - \bar{U}_s(t=0) = 0$ to deduce that $\partial_t\tilde{\bar{U}}_{hk}$ is bounded independently of k and ε in $L^1(0, t_1; L^2)$. In other words,

$$
k\sum_{m=0}^{N-1}\left|\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\right|_{L^2}^2 \leq C(\mathbf{u}, U_0, t_1),
$$

where $C(\mathbf{u}, U_0, t_1)$ is some constant independent of k and ε. Therefore,

$$
k\sum_{m=0}^{N-1}\left|\frac{\bar{U}^{m+1}-\bar{U}^m}{k}\right|_{L^2}^2 \leq k\sum_{m=0}^{N-1}\Big(\left|\frac{\bar{U}_h^{m+1}-\bar{U}_h^m}{k}\right|_{L^2}^2 + \left|\frac{\bar{U}_s^{m+1}-\bar{U}_s^m}{k}\right|_{L^2}^2\Big) \leq C(\mathbf{u}, U_0, t_1).
$$

□

Remark 3.5. By an argument similar to the proof of Lemma 3.4, we can also prove higher-order uniform estimates for $\bar{U}_h^m$. First, with $\bar{U}_0 \in V$, integrating (3.29) from

0 to t for any $t \in [0, t_1]$, we can also infer that

$$\tilde{\tilde{U}}_{kh} \text{ is bounded independently of k and } \varepsilon \text{ in } L^\infty(0, t_1; V^3). \tag{3.30}$$

As $\tilde{\tilde{U}}_{kh} = \bar{U}_h^{m+1} - (1 - \frac{t-mk}{k})(\bar{U}_h^{m+1} - \bar{U}_h^m)$ is a linear combination of $\bar{U}_h^{m+1}$ and $\bar{U}_h^m$ on the interval $t \in [mk, (m+1)k]$, (3.30) also implies

$$||\bar{U}_h^m||_V \leq C(\mathbf{u}, U_0, t_1) < +\infty \text{ for any } m, 1 \leq m \leq N. \tag{3.31}$$

Moreover, if we multiply (3.22) by $\bar{A}\bar{U}_h^{m+1}$, following the similar steps as what we did in (3.24)-(3.29) for $\frac{d}{dt}\tilde{\tilde{U}}_{kh} = \frac{\bar{U}_h^{m+1} - \bar{U}_h^m}{k}$, we obtain

$$k \sum_{m=0}^{N-1} |\bar{\mathcal{A}}\bar{U}_h^{m+1}|_{L^2}^2 \leq C(\mathbf{u}, U_0, t_1) < +\infty, \tag{3.32}$$

where $C(\mathbf{u}, U_0, t_1)$ is independent of k and ε. As $U_{h0} = 0$, we infer that from (3.32),

$$\bar{\mathcal{A}}\tilde{\tilde{U}}_{kh} \text{ is bounded independently of k and } \varepsilon \text{ in } L^2(0, t_1; L^2(\mathcal{M})^3). \tag{3.33}$$

On another note, in (3.28), if we keep the positive term $\frac{1}{k}(1 - \frac{t-mk}{k})\langle \bar{\mathcal{A}}(\bar{U}_h^{m+1} - \bar{U}_h^m), \bar{U}_h^{m+1} - \bar{U}_h^m \rangle$ which is equal to $\frac{1}{k}(1 - \frac{t-mk}{k})\|\bar{U}_h^{m+1} - \bar{U}_h^m\|_V^2$, then (3.24) also implies

$$\begin{aligned} &\left| \frac{\bar{U}_h^{m+1} - \bar{U}_h^m}{k} \right|_{L^2}^2 + \frac{d}{dt}(\bar{A}\tilde{\tilde{U}}_{hk}, \tilde{\tilde{U}}_{hk}) + \frac{1}{k}(1 - \frac{t-mk}{k})\|\bar{U}_h^{m+1} - \bar{U}_h^m\|_V^2 \\ &\leq 2J_1^m = \mathcal{G}(t), \text{ for } t \in [mk, (m+1)k]. \end{aligned} \tag{3.34}$$

Integrating (3.34) from $t = mk$ to $t = (m+1)k$ and summing for m from 0 to $N-1$, we then find

$$\sum_{m=0}^{N-1} \|\bar{U}_h^{m+1} - \bar{U}_h^m\|_V^2 \leq C(\mathbf{u}, U_0, t_1). \tag{3.35}$$

Because $\bar{U}^m = \bar{U}_h^m + \bar{U}_s^m$, thanks to (3.20) and (3.21), the above estimates (3.30)-(3.35) also hold with $\bar{U}_h^m$ being replaced by $\bar{U}^m$.

Lemma 3.4 and (3.30)-(3.35) will be useful in the estimation of $k \sum_{m=0}^{N-1} ||\frac{q_{v\varepsilon}^{m+1} - q_{v\varepsilon}^m}{k}||_{v^*}^{5/3}$ and in the passage to limit $k \to 0^+$.

Estimate of $\mathbf{dq_v/dt}$.

To estimate the approximate time derivative for q_v, we first need to control the penalization term which contains the "large" factor $\frac{1}{\varepsilon_1}$. For that purpose, we establish the following technical lemma.

Lemma 3.6. *For any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, the inequality*

$$k\sum_{m=0}^{N-1}\frac{1}{\varepsilon_1^{5/3}}|(q_{v\varepsilon}^{m+1}-q_{vs,\varepsilon}^{m+1})^+|_{L^{5/2}}^{5/2}\le C(\mathbf{u},U_0,t_1)<\infty,$$

holds for some constant $C(\mathbf{u},U_0,t_1)$ depending on $\mathbf{u},U_0,t_1$ but not on ε and k

Proof. Replacing q_v^b by $(q_v^{m+1}-q_{vs}^{m+1})^+$ in (3.6), we see that

$$\begin{aligned}&\langle\frac{q_v^{m+1}-q_v^m}{k},(q_v^{m+1}-q_{vs}^{m+1})^+\rangle+a_{q_v}(q_v^{m+1},(q_v^{m+1}-q_{vs}^{m+1})^+)-l_{q_v}((q_v^{m+1}-q_{vs}^{m+1})^+)\\&\quad+b(\mathbf{u}^{m+1},q_v^{m+1},(q_v^{m+1}-q_{vs}^{m+1})^+)+\frac{1}{\varepsilon_1}|(q_v^{m+1}-q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2}\\&\quad=\big(f_{q_v}(U^m)-\frac{[\omega^m]^-}{p}F(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m-q_{vs}^m),(q_v^{m+1}-q_{vs}^{m+1})^+\big).\end{aligned}\tag{3.36}$$

In the RHS of (3.36)

$$\begin{aligned}&\langle\frac{q_v^{m+1}-q_v^m}{k},(q_v^{m+1}-q_{vs}^{m+1})^+\rangle\\&=\langle\frac{(q_v^{m+1}-q_{vs}^{m+1})-(q_v^m-q_{vs}^m)}{k},(q_v^{m+1}-q_{vs}^{m+1})^+\rangle\\&+\langle\frac{q_{vs}^{m+1}-q_{vs}^m}{k},(q_v^{m+1}-q_{vs}^{m+1})^+\rangle.\end{aligned}\tag{3.37}$$

We denote $q_v^{m+1}-q_{vs}^{m+1}$ by d^{m+1} and write d^{m+1} as $(d^{m+1})^+-(d^{m+1})^-$; then the first term in the RHS of (3.37) becomes

$$\begin{aligned}&\langle\frac{(d^{m+1})^+-(d^m)^+}{k},(d^{m+1})^+\rangle+\underbrace{\langle\frac{-(d^{m+1})^-+(d^m)^-}{k},(d^{m+1})^+\rangle}_{\ge 0}\\&\ge\frac{1}{2k}\big(|(d^{m+1})^+|^2-|(d^m)^+|^2\big).\end{aligned}\tag{3.38}$$

For the a_{q_v}-term in (3.36), we have

$$\begin{aligned}&a_{q_v}(q_v^{m+1},(q_v^{m+1}-q_{vs}^{m+1})^+)\\&=a_{q_v}((q_v^{m+1}-q_{vs}^{m+1})^+,(q_v^{m+1}-q_{vs}^{m+1})^+)+a_{q_v}(q_{vs}^{m+1},(q_v^{m+1}-q_{vs}^{m+1})^+)\\&\ge a_{q_v}(q_{vs}^{m+1},(q_v^{m+1}-q_{vs}^{m+1})^+).\end{aligned}\tag{3.39}$$

We move the a_{q_v}-term, b-term and l_{q_v}-term to the RHS of (3.36) and recall the definitions (2.21)-(2.23). Then it follows from (3.37)-(3.39) that

$$\frac{1}{2k}\big(|(d^{m+1})^+|^2-|(d^m)^+|^2\big)+\frac{1}{\varepsilon_1}|(q_v^{m+1}-q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2}$$

$$
\begin{aligned}
(3.40)\qquad &\leq \Big|\langle \frac{q_{vs}^{m+1}-q_{vs}^{m}}{k}, (q_v^{m+1}-q_{vs}^{m+1})^+\rangle + \big(\mathcal{A}_v q_{vs}^{m+1}, (q_v^{m+1}-q_{vs}^{m+1})^+\big)\\
&+\big(\mathbf{u}^{m+1}\cdot\nabla_3 q_v^{m+1}, (q_v^{m+1}-q_{vs}^{m+1})^+\big)\\
&-\big(f_{q_v}(U^m) - \frac{[\omega^m]^-}{p}F(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m - q_{vs}^m), (q_v^{m+1}-q_{vs}^{m+1})^+\big)\Big|.
\end{aligned}
$$

Using the Hölder and Young inequalities, the first term in the RHS of (3.40) can be estimated in the following way

$$
\begin{aligned}
&\big|\langle \frac{q_{vs}^{m+1}-q_{vs}^{m}}{k}, (q_v^{m+1}-q_{vs}^{m+1})^+\rangle\big|\\
(3.41)\qquad &= \Big|\int_{\mathcal{M}} \varepsilon_1^{2/5}\frac{(q_{vs}^{m+1}-q_{vs}^{m})}{k}\cdot\frac{(q_v^{m+1}-q_{vs}^{m+1})^+}{\varepsilon_1^{2/5}}d\mathcal{M}\Big|\\
&\leq C\varepsilon_1^{2/3}\Big|\frac{(q_{vs}^{m+1}-q_{vs}^{m})}{k}\Big|_{L^{5/3}}^{5/3} + \frac{1}{8\varepsilon_1}\big|(q_v^{m+1}-q_{vs}^{m+1})^+\big|_{L^{5/2}}^{5/2}.
\end{aligned}
$$

Treating the other terms in the RHS of (3.40) as in (3.41), we infer that

$$
\begin{aligned}
&\frac{1}{2k}\big(|(d^{m+1})^+|^2 - |(d^m)^+|^2\big) + \frac{1}{\varepsilon_1}|(q_v^{m+1}-q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2}\\
(3.42)\qquad &\leq C\varepsilon_1^{2/3}\big(\Big|\frac{q_{vs}^{m+1}-q_{vs}^{m}}{k}\Big|_{L^{5/3}}^{5/3} + |\mathcal{A}_v q_{vs}^{m+1}|_{L^{5/3}}^{5/3} + |\nabla_3 q_v^m|_{L^{5/3}}^{5/3} + |q_v^m|_{L^{5/3}}^{5/3} + C_1\big)\\
&+\frac{1}{2\varepsilon_1}|(q_v^{m+1}-q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2},
\end{aligned}
$$

where we have bounded $\big|f_{q_v}(U^m) - \frac{[\omega^m]^-}{p}F(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m - q_{vs}^m)\big|_{L^{5/3}}^{5/3}$ by $|q_v^m|_{L^{5/3}}^{5/3} + C_1$.

Multiplying (3.42) by k and summing in m from $m=0$ to $N-1$, we obtain

(3.43)

$$
\begin{aligned}
&\frac{1}{2}|(q_v^N - q_{vs}^N)^+|_{L^2}^2 - \frac{1}{2}|(q_v^0 - q_{vs}^0)^+|_{L^2}^2 + k\sum_{m=0}^{N-1}\frac{1}{2\varepsilon_1}|(q_v^{m+1}-q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2}\\
&\leq C\varepsilon_1^{2/3}k\sum_{m=0}^{N-1}\big(\Big|\frac{q_{vs}^{m+1}-q_{vs}^{m}}{k}\Big|_{L^{5/3}}^{5/3} + |\mathcal{A}_v q_{vs}^{m+1}|_{L^{5/3}}^{5/3} + |\nabla_3 q_v^m|_{L^{5/3}}^{5/3} + |q_v^m|_{L^{5/3}}^{5/3} + C_1\big).
\end{aligned}
$$

In the LHS of (3.43), The first term is positive and the second term is 0 because of the constraint on the initial value $q_v^0 \leq q_{vs}^0$. We now estimate the RHS of (3.43) term by term.

Firstly, for $k\sum_{m=0}^{N-1}|\frac{q_{vs}^{m+1}-q_{vs}^{m}}{k}|_{L^{5/3}}^{5/3}$, we recall that $q_{vs}^m = Q_{vs}(T^m, p)$ and the function $Q_{vs}(T,p)$ is Lipschitz continuous in the variable T. Hence $|q_{vs}^{m+1} - q_{vs}^m| \leq C|T^{m+1} - T^m|$ pointwise for some constant C independent of k and ε. We have shown in Lemma 3.4 that $k\sum_{m=0}^{N-1}\Big|\frac{\bar{U}^{m+1}-\bar{U}^m}{k}\Big|_{L^2}^2$ is uniformly bounded in terms of k and ε; thus

$$
k\sum_{m=0}^{N-1}\Big|\frac{q_{vs}^{m+1}-q_{vs}^{m}}{k}\Big|_{L^{5/3}}^{5/3} \leq Ck\sum_{m=0}^{N-1}\Big|\frac{T^{m+1}-T^m}{k}\Big|_{L^{5/3}}^{5/3}
$$

$$\leq \sum_{m=0}^{N-1} \left| \frac{\bar{U}^{m+1} - \bar{U}^m}{k} \right|_{L^2}^2 \tag{3.44}$$
$$\leq C(\mathbf{u}, U_0, t_1).$$

Secondly, for $|\mathcal{A}_v q_{vs}^{m+1}|_{L^{5/3}}^{5/3}$, because of the expression of $Q_{vs}(T,p)$ in (2.2)-(2.4), the following inequalities hold pointwise (see also Lemma 4.2 in [16])

$$\left|\frac{\partial^2 q_{vs}^{m+1}}{\partial x^2}\right| \leq C\left(\left|\frac{\partial T^{m+1}}{\partial x}\right|^2 + \left|\frac{\partial^2 T^{m+1}}{\partial x^2}\right|\right), \quad \left|\frac{\partial^2 q_{vs}^{m+1}}{\partial y^2}\right| \leq C\left(\left|\frac{\partial T^{m+1}}{\partial y}\right|^2 + \left|\frac{\partial^2 T^{m+1}}{\partial y^2}\right|\right),$$

$$\left|\frac{\partial^2 q_{vs}^{m+1}}{\partial p^2}\right| \leq C\left(\left|\frac{\partial T^{m+1}}{\partial p}\right| + \left|\frac{\partial T^{m+1}}{\partial p}\right|^2 + \left|\frac{\partial^2 T^{m+1}}{\partial p^2}\right| + C_1\right).$$

It follows that

$$|\mathcal{A}_v q_{vs}^{m+1}|_{L^{5/3}}^{5/3} \leq C\left(|\Delta_3 T^{m+1}|_{L^{5/3}}^{5/3} + |\nabla_3 T^{m+1}|_{L^{10/3}}^{10/3} + C_1\right). \tag{3.45}$$

Then by the Gagliardo-Nirenberg interpolation inequality, we have

$$|\nabla_3 T^{m+1}|_{L^{10/3}}^{10/3} \leq C\left(|\nabla_3 T^{m+1}|_{L^2}^{10/3} + |\nabla_3 T^{m+1}|_{L^2}^{4/3} |\Delta_3 T^{m+1}|_{L^2}^2\right). \tag{3.46}$$

With the help of Remark 3.5 and (3.31)-(3.32), we see that $|\nabla_3 T^{m+1}|_{L^2} \leq C(\mathbf{u}, U_0, t_1)$ for any m, $0 \leq m \leq N-1$; thus

$$k \sum_{m=0}^{N-1} |\Delta_3 T^{m+1}|_{L^2}^2 \leq C(\mathbf{u}, U_0, t_1).$$

In view of (3.46) and the inclusion $L^2(\mathcal{M}) \subset L^{5/3}(\mathcal{M})$, after we multiply (3.45) by k and sum from $m = 0$ to $N-1$, we obtain

$$\begin{aligned} k \sum_{m=0}^{N-1} |\mathcal{A}_v q_{vs}^{m+1}|_{L^{5/3}}^{5/3} &\leq C\big(k \sum_{m=0}^{N-1} |\Delta_3 T^{m+1}|_{L^{5/3}}^{5/3} + \sup_{0\leq m\leq N-1} |\nabla_3 T^{m+1}|_{L^2}^{10/3} \cdot t_1 \\ &\quad + \sup_{0\leq m\leq N-1} |\nabla_3 T^{m+1}|_{L^2}^{4/3} \cdot k \sum_{m=0}^{N-1} |\Delta_3 T^{m+1}|_{L^2}^2 + C_1 t_1\big) \\ &\leq C(\mathbf{u}, U_0, t_1). \end{aligned} \tag{3.47}$$

The other terms in the RHS of (3.43) can be easily bounded by Lemma 3.2 due to the inclusion $L^2(\mathcal{M}) \subset L^{5/3}(\mathcal{M})$, so that

$$k \sum_{m=0}^{N-1} \left(|\nabla_3 q_v^{m+1}|_{L^{5/3}}^{5/3} + |q_v^m|_{L^{5/3}}^{5/3} + C_1\right) \leq C(\mathbf{u}, U_0, t_1). \tag{3.48}$$

Now the RHS of (3.43) can be bounded by $C\varepsilon_1^{2/3} C(\mathbf{u}, U_0, t_1)$. Dividing by $\varepsilon_1^{2/3}$ on both sides of (3.43), we find the desired estimate

$$k \sum_{m=0}^{N-1} \frac{1}{\varepsilon_1^{5/3}} |(q_v^{m+1} - q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2} \leq C(\mathbf{u}, U_0, t_1).$$

□

With the help of Lemma 3.6, we are ready to derive the needed estimate for the term $k\sum_{m=0}^{N-1}||\frac{q_{v\varepsilon}^{m+1}-q_{v\varepsilon}^{m}}{k}||_{v^*}^{5/3}$. We have the following lemma.

Lemma 3.7. *For any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, the inequality*

$$k\sum_{m=0}^{N-1}\|\frac{q_{v\varepsilon}^{m+1}-q_{v\varepsilon}^{m}}{k}\|_{V^*}^{5/3} \leq C(\mathbf{u}, U_0, t_1) < \infty,$$

holds for some constant $C(\mathbf{u}, U_0, t_1)$ depending on $\mathbf{u}, U_0, t_1$ but not on ε and k

Proof. We estimate the duality pair $\langle\frac{q_v^{m+1}-q_v^m}{k}, q_v^b\rangle$ for an arbitrary $q_v^b \in V$. Rearranging (3.6), we find

$$\begin{aligned}
|\langle\frac{q_v^{m+1}-q_v^m}{k}, q_v^b\rangle| = \Big| &- a_{q_v}(q_v^{m+1}, q_v^b) - b(\mathbf{u}, q_v^{m+1}, q_v^b) \\
&- (\frac{1}{\varepsilon_1}((q_v^{m+1} - q_{vs}^{m+1})^+)^{3/2}, q_v^b) + l_{q_v}(q_v^b) \\
&+ (f_{q_v}(U^m - \frac{[\omega^m]^-}{p}F(T^m)\mathcal{H}_{\varepsilon_2}(q_v^m - q_{vs}^m), q_v^b)\Big| \\
\leq C\Big(&\|q_v^{m+1}\|_V + \|\mathbf{u}^{m+1}\|_{\mathbf{V}}\|q_v^{m+1}\|_V \\
&+ \frac{1}{\varepsilon_1}|(q_v^{m+1} - q_{vs}^{m+1})^+|_{L^{5/2}}^{3/2} \\
&\quad + |q_v^m|_{L^2} + C_1\Big)\|q_v^b\|_V.
\end{aligned} \tag{3.49}$$

For the penalization term in (3.49), we have used the following fact

$$\begin{aligned}
\frac{1}{\varepsilon_1}\int_{\mathcal{M}}((q_v^{m+1} - q_{vs}^{m+1})^+)^{3/2} q_v^b d\mathcal{M} &\leq \frac{1}{\varepsilon_1}|((q_v^{m+1} - q_{vs}^{m+1})^+)^{3/2}|_{L^{5/3}}|q_v^b|_{L^{5/2}} \\
&\leq (V \subset L^{5/2}(\mathcal{M}) \text{ in } \mathbb{R}^3) \\
&\leq \frac{1}{\varepsilon_1}|(q_v^{m+1} - q_{vs}^{m+1})^+|_{L^{5/2}}^{3/2}\|q_v^b\|_V.
\end{aligned} \tag{3.50}$$

It then follows from (3.49) that

$$\begin{aligned}
\|\frac{q_v^{m+1}-q_v^m}{k}\|_{V^*} \leq C(&\|q_v^{m+1}\|_V + \|\mathbf{u}^{m+1}\|_{\mathbf{V}}\|q_v^{m+1}\|_V \\
&+ \frac{1}{\varepsilon_1}|(q_v^{m+1} - q_{vs}^{m+1})^+|_{L^{5/2}}^{3/2} + |q_v^m|_{L^2} + C_1),
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
\|\frac{q_v^{m+1}-q_v^m}{k}\|_{V^*}^{5/3} \leq C(&\|q_v^{m+1}\|_V^{5/3} + \|\mathbf{u}^{m+1}\|_{\mathbf{V}}^{5/3}\|q_v^{m+1}\|_V^{5/3} \\
&+ \frac{1}{\varepsilon_1^{5/3}}|(q_v^{m+1} - q_{vs}^{m+1})^+|_{L^{5/2}}^{5/2} + |q_v^m|_{L^2}^{5/3} + C_1).
\end{aligned} \tag{3.52}$$

Multiplying (3.52) by k and summing in m from $m = 0$ to $N - 1$, we end up with

$$k\sum_{m=0}^{N-1}\|\frac{q_v^{m+1}-q_v^m}{k}\|_{V^*}^{5/3} \leq C(\mathbf{u}, U_0, t_1),$$

thanks to Lemma 3.2 and Lemma 3.6. □

Orientation

Now we have all the a priori estimates we need to pass to the limit $\varepsilon = (\varepsilon_1, \varepsilon_2) \to (0^+, 0^+)$ in (3.5) and (3.6) for fixed k; we will pass to the limit $k \to 0^+$ in a second step.

Recalling Lemma 3.2 and (3.31)-(3.32), since the inclusion $H^2 \subset V \subset H$ is compact, we have, for $m = 1, 2, \ldots, N$, there exist functions $U^m = (\bar{U}^m, q_v^m) \in V^4 = V \times V \times V \times V$, such that,

$$\bar{U}_\varepsilon^m \to \bar{U}^m \text{ strongly in } V^3 \text{ and weakly in } H^2(\mathcal{M})^3, \tag{3.53}$$

$$q_{v\varepsilon}^m \to q_v^m \text{ strongly in } H \text{ and weakly in } V \tag{3.54}$$

as $\varepsilon \to (0^+, 0^+)$.

By an additional extraction of subsequences, we also have

$$\bar{U}_\varepsilon^m(x) \to \bar{U}^m(x) \;\; a.e \text{ in } \mathcal{M}, \tag{3.55}$$

$$q_{v\varepsilon}^m(x) \to q_v^m(x) \;\; a.e \text{ in } \mathcal{M}. \tag{3.56}$$

Since $q_{vs}^m = Q_{vs}(T_\varepsilon^m, p)$ is a smooth function of T_ε^m, by Lemma 4.5 of [16], we have $q_{vs}(T_\varepsilon^m, p) \to q_{vs}(T^m, p)$ strongly in V, and by extraction of a further subsequence

$$q_{vs}(T_\varepsilon^m, p) \to q_{vs}(T^m, p) \;\; a.e \text{ in } \mathcal{M}. \tag{3.57}$$

Meanwhile, for the source terms, thanks to the continuity and boundedness of the functions $f(U)$ and $\mathcal{F}(T)$, we have

$$f(\bar{U}_\varepsilon^m) \to f(\bar{U}^m), \; \mathcal{F}(T_\varepsilon^m) \to \mathcal{F}(T^m) \text{ strongly in } L^2(\mathcal{M}), \tag{3.58}$$

$$\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs}(T_\varepsilon^m, p)) \rightharpoonup h_{q_v^m} \text{ weak* in } L^\infty(\mathcal{M}), \tag{3.59}$$

and

$$\mathcal{F}(T_\varepsilon^m)\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs}(T_\varepsilon^m, p)) \to \mathcal{F}(T^m) h_{q_v}^m \text{ weakly in } L^2(\mathcal{M}) \tag{3.60}$$

for $m = 1, 2, \cdots, N$.

For $q_v^b \in \mathcal{K}^{m+1} = \{q \in V; q \le q_{vs}^{m+1}, \; a.e.\}$, we define $q_{v\varepsilon}^b = q_v^b - (q_v^b - q_{vs,\varepsilon}^{m+1})^+ = min(q_v^b, \, q_{vs,\varepsilon}^{m+1})$. Then we have

$$q_{v\varepsilon}^b \to q_v^b \;\; \text{strongly in } V, \tag{3.61}$$

see Lemma 4.6 of [16].

Now we pass to the limit $\varepsilon \to (0^+, 0^+)$ in (3.5) and (3.6). As the proof for the $\bar{U}_\varepsilon^m$ equation (3.5) is easier than for the $q_{v\varepsilon}^m$ equation (3.6) , we only show the details of the passage to limit in (3.6). We replace q_v^b in (3.6) by $q_{v\varepsilon}^b - q_{v\varepsilon}^{m+1}$, with

$q^b_{v\varepsilon} = min(q^b_v, q^{m+1}_{vs,\varepsilon})$, $q^b_v \in \mathcal{K}^{m+1}$. We arrive at

(3.62)

$$
\begin{aligned}
&\langle \frac{q^{m+1}_{v\varepsilon} - q^m_{v\varepsilon}}{k}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}\rangle + a_{q_v}(q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}) + b(\mathbf{u}^{m+1}, q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon})\\
&- l_{q_v}(q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}) + (\frac{1}{\varepsilon_1}[(q^{m+1}_{v\varepsilon} - q_{vs}(T^{m+1}_\varepsilon, p))^+]^{3/2}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon})\\
=&(f_{q_v}(U^m_\varepsilon) - \frac{[\omega^m]^-}{p} F(T^{m+1}_\varepsilon)\mathcal{H}_{\varepsilon_2}(q^{m+1}_{v\varepsilon} - q_{vs}(T^{m+1}_\varepsilon, p)), q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}).
\end{aligned}
$$

We first look at the a_{q_v}-term in (3.62)

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} a_{q_v}(q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}) &= \lim_{\varepsilon\to 0} a_{q_v}(q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon}) - \liminf_{\varepsilon\to 0} a_{q_v}(q^{m+1}_{v\varepsilon}, q^{m+1}_{v\varepsilon})\\
\text{(by weak l.s.c of norm)} &\le a_{q_v}(q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon}) - a_{q_v}(q^{m+1}_{v\varepsilon}, q^{m+1}_{v\varepsilon})\\
&= a_{q_v}(q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}).
\end{aligned}
$$

Then for the trilinear b-term in (3.62) , we have

$$
\begin{aligned}
&b(\mathbf{u}^{m+1}, q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}) - b(\mathbf{u}^{m+1}, q^{m+1}_v, q^b_v - q^{m+1}_v)\\
=&b(\mathbf{u}^{m+1}, q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon}) - b(\mathbf{u}^{m+1}, q^{m+1}_v, q^b_{v\varepsilon}) + b(\mathbf{u}^{m+1}, q^{m+1}_v, q^b_{v\varepsilon}) - b(\mathbf{u}^{m+1}, q^{m+1}_v, q^b_v)\\
\le&|b(\mathbf{u}^{m+1}, q^{m+1}_{v\varepsilon} - q^{m+1}_v, q^b_{v\varepsilon})| + |b(\mathbf{u}^{m+1}, q^{m+1}_v, q^b_{v\varepsilon} - q^b_v)|\\
\le&C\|\mathbf{u}^{m+1}\|(|q^{m+1}_{v\varepsilon} - q^{m+1}_v|^{1/2}_{L^2}\|q^{m+1}_{v\varepsilon} - q^{m+1}_v\|^{1/2}_V\|q^b_{v\varepsilon}\|_V + |q^{m+1}_v|^{1/2}_{L^2}|q^{m+1}_v|^{1/2}_V\|q^b_{v\varepsilon} - q^b_v\|_V).
\end{aligned}
$$

Since $q^{m+1}_{v\varepsilon} \to q^{m+1}_v$ strongly in H, $q^b_{v\varepsilon} \to q^b_v$ strongly in V as $\varepsilon \to (0^+, 0^+)$ and the other terms are bounded, we have

$$\lim_{\varepsilon\to 0} b(\mathbf{u}^{m+1}, q^{m+1}_{v\varepsilon}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}) = b(\mathbf{u}^{m+1}, q^{m+1}_v, q^b_v - q^{m+1}_v).$$

Next, for the penalization term we find

$$
\begin{aligned}
&\big(\frac{1}{\varepsilon_1}[(q^{m+1}_{v\varepsilon} - q_{vs}(T^{m+1}_\varepsilon, p))^+]^{3/2}, q^b_{v\varepsilon} - q^{m+1}_{v\varepsilon}\big)\\
=&\big(\frac{1}{\varepsilon_1}[(q^{m+1}_{v\varepsilon} - q_{vs}(T^{m+1}_\varepsilon, p))^+]^{3/2}, q_{vs}(T^{m+1}_\varepsilon, p) - q^{m+1}_{vs}\big)\\
+&\big(\frac{1}{\varepsilon_1}[(q^{m+1}_{v\varepsilon} - q_{vs}(T^{m+1}_\varepsilon, p))^+]^{3/2}, \underbrace{q^b_{v\varepsilon} - q_{vs}(T^{m+1}_\varepsilon, p)}_{\le 0}\big)\\
\le&0
\end{aligned}
$$

The passage to limit $\varepsilon \to (0^+, 0^+)$ in the other terms in (3.62) can be carried out in a straightforward manner thanks to (3.53)-(3.60). Then dropping the negative term in the LHS of (3.62), we can retrieve (3.4) from (3.62) after passing to the limit $\varepsilon \to (0^+, 0^+)$.

To see that the limit function q^m_v lies in the convex set $\mathcal{K}^m = \{q \in V, q \le q_{vs}(T^m, p) \;\; a.e.\}$, we infer from (3.14) in Remark 3.3 , that as $\varepsilon_1 \to 0^+$, with fixed

k,

$$2k\sum_{m=0}^{N-1}|(q_v^{m+1}-q_{vs}^{m+1})^+|^{5/2}\le C\varepsilon_1\to 0 \ \text{ as } \ \varepsilon_1\to 0^+.$$

Together with (3.55),(3.57) we see that

$$q_v^m\le q_{vs}(T^m,p) \ \ \textit{a.e.} \text{ in } \mathcal{M}, \ m=0,1,\dots,N.$$

It remains to show that $h_{q_v^m}\in\mathcal{H}(q_v^m-q_{vs}^m)$. As in [60], $h_{q_v^m}\in\mathcal{H}(q_v^m-q_{vs}^m)$ is characterized by

$$([q_v^b-q_{vs}^m]^+,1)-([q_v^m-q_{vs}^m]^+,1)\ge\langle h_{q_v^m},q_v^b-q_v^m\rangle,\quad 1\le m\le N,\ \forall q_v^b\in V. \tag{3.63}$$

Consider the following antiderivative $\mathcal{K}_{\varepsilon_2}$ of the function $\mathcal{H}_{\varepsilon_2}$

$$\mathcal{K}_{\varepsilon_2}(r)=\begin{cases}0 & for\ r\le 0,\\ r^2/2\varepsilon_2 & for\ r\in(0,\varepsilon_2],\\ r-\varepsilon_2/2 & for\ r>\varepsilon_2.\end{cases} \tag{3.64}$$

Both $\mathcal{H}_{\varepsilon_2}$ and $\mathcal{K}_{\varepsilon_2}$ are Lipschitz functions and the following inequalities hold for any $r_1,r_2\in\mathbb{R}$.

$$|\mathcal{H}_{\varepsilon_2}(r_1)-\mathcal{H}_{\varepsilon_2}(r_2)|\le\frac{1}{\varepsilon_2}|r_1-r_2|, \tag{3.65}$$

$$|\mathcal{K}_{\varepsilon_2}(r_1)-\mathcal{K}_{\varepsilon_2}(r_2)|\le|r_1-r_2|. \tag{3.66}$$

Moreover,

$$|\mathcal{K}_{\varepsilon_2}(r)-r|\le\frac{\varepsilon_2}{2},\ \forall r\ge 0. \tag{3.67}$$

We now consider the functional $q\to(\mathcal{K}_{\varepsilon_2}(q),1)$ from V to $\mathbb{R}$. Observing that the function $\mathcal{K}_{\varepsilon_2}$ is a convex function on $\mathbb{R}$, we know that the functional $q\to(\mathcal{K}_{\varepsilon_2}(q),1)$ is convex on V . Since the function $\mathcal{K}_{\varepsilon_2}$ is continuously differentiable, the functional $q\to(\mathcal{K}_{\varepsilon_2}(q),1)$ is actually Fréchet differentiable with Fréchet derivative at q equal to $\mathcal{H}_{\varepsilon_2}(q)\in V^*$. Considering its Gâteaux derivative at the point $q_{v\varepsilon}^m-q_{vs,\varepsilon}^m$ along the direction $q_v^b-q_{v\varepsilon}^m$, we end up with the following inequality

$$\begin{aligned}&\big(\mathcal{K}_{\varepsilon_2}(q_v^b-q_{vs,\varepsilon}^m),1\big)-\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m-q_{vs,\varepsilon}^m),1\big)\\ =&\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m-q_{vs,\varepsilon}^m+q_v^b-q_{v\varepsilon}^m),1\big)-\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m-q_{vs,\varepsilon}^m),1\big)\\ \ge&\langle\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m-q_{vs,\varepsilon}^m),q_v^b-q_{v\varepsilon}^m\rangle.\end{aligned} \tag{3.68}$$

Because the duality pair $\langle\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m-q_{vs,\varepsilon}^m),q_v^b-q_{v\varepsilon}^m\rangle$ can be realized by an L^2 inner product, and we have the convergences that $\mathcal{H}_{\varepsilon_2}(q_v^m-q_{vs}^m)\rightharpoonup h_{q_v^m}$ weak* in $L^\infty(\mathcal{M})$ and $q_{v\varepsilon}^m\to q_v^m$, $q_{vs,\varepsilon}^m\to q_{vs}^m$ strongly in H, we can then pass to the limit on ε in the RHS of (3.68),

$$\langle\mathcal{H}_{\varepsilon_2}(q_{v\varepsilon}^m-q_{vs,\varepsilon}^m),q_v^b-q_{v\varepsilon}^m\rangle\to\langle h_{q_v^m},q_v^b-q_v^m\rangle,\quad \text{for } q^b\in V. \tag{3.69}$$

As for the LHS of (3.68), we notice that $\mathcal{K}_{\varepsilon_2}(r)=0$ for $r<0$, by (3.67) and

$$\big|\big(\mathcal{K}_{\varepsilon_2}(q_v^b-q_{vs,\varepsilon}^m),1\big)-\big([q_v^b-q_{vs}^m]^+,1\big)\big|$$

$$
\begin{aligned}
&\le \left| \big(\mathcal{K}_{\varepsilon_2}([q_v^b - q_{vs,\varepsilon}^m]^+) - [q_v^b - q_{vs,\varepsilon}^m]^+, 1\big) \right| + \left| \big([q_v^b - q_{vs,\varepsilon}^m]^+, 1\big) \right| - \big([q_v^b - q_{vs}^m]^+, 1\big) \Big| \\
&\le \frac{\varepsilon_2}{2}|\mathcal{M}| + \sqrt{|\mathcal{M}|}(|q_{vs,\varepsilon}^m - q_{vs}^m|_{L^2} \to 0, \quad \text{as } \varepsilon \to (0^+, 0^+),
\end{aligned} \tag{3.70}
$$

which implies

$$
\big(\mathcal{K}_{\varepsilon_2}(q_v^b - q_{vs,\varepsilon}^m), 1\big) \to \big([q_v^b - q_{vs}^m]^+, 1\big) \quad \text{as } \varepsilon \to (0^+, 0^+). \tag{3.71}
$$

To show the convergence of $\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), 1\big)$ to $([q_v^m - q_{vs}^m]^+, 1)$, we split the difference $\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), 1\big) - ([q_v^m - q_{vs}^m]^+, 1)$ into the following sum

$$
\Big(\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), 1\big) - \big(\mathcal{K}_{\varepsilon_2}(q_v^m - q_{vs}^m), 1\big)\Big) + \Big(\big(\mathcal{K}_{\varepsilon_2}(q_v^m - q_{vs}^m), 1\big) - ([q_v^m - q_{vs}^m]^+, 1)\Big). \tag{3.72}
$$

The second term of (3.72) can be dealt with similarly as in (3.70). For the first term, using (3.65) and the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
&\left| \big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), 1\big) - \big(\mathcal{K}_{\varepsilon_2}(q_v^m - q_{vs}^m), 1\big) \right| \\
\le& \big(|(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m) - (q_v^m - q_{vs}^m)|, 1\big) \\
\le& \sqrt{|\mathcal{M}|}\big(|q_{v\varepsilon}^m - q_v^m|_{L^2} + |q_{vs,\varepsilon}^m - q_{vs}^m|_{L^2}\big) \to 0, \quad \text{as } \varepsilon \to (0^+, 0^+),
\end{aligned}
$$

in view of the strong convergence in $L^2(\mathcal{M})$ of $q_{v\varepsilon}^m$ and $q_{vs,\varepsilon}^m$. So we have

$$
\big(\mathcal{K}_{\varepsilon_2}(q_{v\varepsilon}^m - q_{vs,\varepsilon}^m), 1\big) \to ([q_v^m - q_{vs}^m]^+, 1) \quad \text{as } \varepsilon \to (0^+, 0^+). \tag{3.73}
$$

From (3.68),(3.69),(3.71) and (3.73), we can conclude that

$$
([q_v^b - q_{vs}^m]^+, 1) - ([q_v^m - q_{vs}^m]^+, 1) \ge \langle h_{q_v^m}, q_v^b - q_v^m \rangle, \quad 1 \le m \le N, \ \forall q_v^b \in V,
$$

which tells that $h_{q_v^m} \in \mathcal{H}(q_v^m - q_{vs}^m)$ as desired.

4. Convergence of the Euler Scheme

In this section, we want to prove that the solutions of the Euler scheme converge to the solutions of the system (2.34)-(2.35). As in the last section, we shall use the same conventions on subsequences and indices in the limit process $k \to 0^+$ in this part and up to subsequences.

Due to the weak lower semi-continuity property of the norms, we know that for the limit functions $U^m = (\bar{U}^m, q_v^m)$ which now have no dependence on ε, the bounds in Lemma 3.2-Lemma 3.7 are now valid with U_ε^m replaced by the limit functions U^m. Now we introduce some approximate functions associated with the elements $U^1, \dots, U^N$. Consider the piecewise constant functions $\bar{U}_k^{(i)}$, $q_{vk}^{(i)} : [0, t_1] \to V$, for $i \in 1, 2$, defined by

$$
\begin{aligned}
\bar{U}_k^{(1)}(t) = \bar{U}^m, \ \ \bar{U}_k^{(2)}(t) = \bar{U}^{m+1}, \\
q_{vk}^{(1)}(t) = q_v^m, \ \ q_{vk}^{(2)}(t) = q_v^{m+1},
\end{aligned}
$$

on the time interval $t \in [mk, (m+1)k)$, $m = 0, \dots, N-1$. We also define $\tilde{\bar{U}}_k, \tilde{q}_{vk}$ as the continuous functions from $[0, t_1]$ into V, which are linear on each interval $[mk, (m+1)k]$, and satisfy $\tilde{\bar{U}}_k(mk) = \bar{U}^m$, $\tilde{q}_{vk}(mk) = q_v^m$ for $m = 0, \dots, N$. We also denote $(\bar{U}_k^{(i)}, q_{vk}^{(i)})$ by $U_k^{(i)}$ $(i = 1, 2)$ and $(\tilde{\bar{U}}_k, \tilde{q}_{vk})$ by $\tilde{U}_k$.

Collecting the a priori estimates in Section 3, we can restate them in terms of $\bar{U}_k^{(i)}, q_{vk}^{(i)}, i = 1, 2$ and $\tilde{\bar{U}}_k, \tilde{q}_{vk}$.

Lemma 4.1. *As $k \to 0$, the functions $\bar{U}_k^{(i)}, i = 1, 2$ and $\tilde{\bar{U}}_k$ remain in a bounded set of $L^2(0, t_1; H^2(\mathcal{M})^3) \cap L^\infty(0, t_1; V^3)$; the functions $q_{vk}^{(i)}, i = 1, 2$ and $\tilde{q}_{vk}$ remain in a bounded set of $L^2(0, t_1; V) \cap L^\infty(0, t_1; H)$. And the functions $\partial_t \tilde{\bar{U}}_k$ form a bounded set in $L^2(0, t_1; L^2(\mathcal{M})^3)$; the functions $\partial_t \tilde{q}_{vk}$ form a bounded set in $L^{5/3}(0, t_1; V^*)$. Moreover, as $k \to 0$, $|\bar{U}_k^{(2)} - \bar{U}_k^{(1)}| \to 0$ strongly in $L^2(0, t_1; V^3)$; $|q_{vk}^{(2)} - q_{vk}^{(1)}| \to 0$ strongly in $L^2(0, t_1; H)$.*

We also have the following estimates.

Lemma 4.2. *For the $\bar{U}_k^{(i)}, q_{vk}^{(i)}, i = 1, 2$ and $\tilde{\bar{U}}_k, \tilde{q}_{vk}$ defined above, there holds*

$$|\bar{U}_k^{(2)} - \tilde{\bar{U}}_k|_{L^2(0,t_1;V^3)} \leq C(\mathbf{u}, U_0, t_1)\sqrt{k}, \tag{4.1}$$

$$|q_{vk}^{(2)} - \tilde{q}_{vk}|_{L^2(0,t_1;H)} \leq C(\mathbf{u}, U_0, t_1)\sqrt{k}. \tag{4.2}$$

Proof. We recall the definition of $\tilde{\bar{U}}_k$, for $t \in [mk, (m+1)k]$, $m = 0, 1, \ldots, N-1$

$$\tilde{\bar{U}}_k = \bar{U}_k^{(2)} - (1 - \frac{t - mk}{k})(\bar{U}_k^{(2)} - \bar{U}_k^{(1)}).$$

Then

$$\|\bar{U}_k^{(2)}(t) - \tilde{\bar{U}}_k(t)\|_V = \frac{(m+1)k - t}{k}\|\bar{U}_k^{(2)}(t) - \bar{U}_k^{(1)}(t)\|_V \text{ for } t \in [mk, (m+1)k), \tag{4.3}$$

$$\int_{mk}^{(m+1)k} \|\bar{U}_k^{(2)}(t) - \tilde{\bar{U}}_k(t)\|_V^2 dt = \frac{k}{3}\|\bar{U}_k^{(2)}(t) - \bar{U}_k^{(1)}(t)\|_V^2. \tag{4.4}$$

By (3.35), we obtain (4.1) by summation.

The proof for the estimate (4.2) is similar and can be found in e.g. [57] (see Lemma 7.3 in Chapter 3). □

Now we define the functions $\mathbf{u}_k : [0, t_1] \to V \times V \times V$ as follows:

$$\mathbf{u}_k(t) = \mathbf{u}^{m+1} = \frac{1}{k}\int_{mk}^{(m+1)k} \mathbf{u}(t)dt \text{ for } t \in [mk, (m+1)k), m = 0, \ldots, N-1.$$

With $\mathbf{u} \in C^1([0, t_1]; V^3)$, by Lemma 4.3 of [59], we have, for any $r \geq 1$

$$\mathbf{u}_k \to \mathbf{u} \text{ in } L^r(0, t_1; V^3) \text{ as } k \to 0. \tag{4.5}$$

We then define $q_{vs}^k : [0, t_1] \to V$ by $q_{vs}^k = Q_{vs}(T_k^{(2)}, p)$. We observe that $q_{vs}^k = q_{vs}^{m+1}$ on the time interval $[mk, (m+1)k)$ for $m = 0, \ldots, N-1$.

For later use, we also define the linear averaging map for the test functions $U^b = (\bar{U}^b, q_v^b)$ with $\bar{U}^b \in L^2(0, t_1; V^3)$, $q_v^b \in L^\infty(0, t_1; \mathcal{K}(U))$. Namely, we define $\bar{U}_k^b : [0, t_1] \to V^3$, $q_{vk}^b : [0, t_1] :\to V$ piecewise by

$$\bar{U}_k^b(t) = \frac{1}{k}\int_{mk}^{(m+1)k} \bar{U}^b(t)dt, \tag{4.6}$$

$$
\begin{aligned}
q^b_{vk} &= \frac{1}{k}\int_{mk}^{(m+1)k} q^b_v(t)dt - \big(\frac{1}{k}\int_{mk}^{(m+1)k} q^b_v(t)dt - q^k_{vs}(t)\big)^+ \\
&= min\big(\frac{1}{k}\int_{mk}^{(m+1)k} q^b_v(t)dt, q^k_{vs}(t)\big),
\end{aligned} \tag{4.7}
$$

for $t \in [mk,(m+1)k]$, $m = 0,\dots,N-1$. When $q^b_v \in \mathcal{K}$, we have $q^b_{vk} \le min(q_{vs}, q^k_{vs})$.

To proceed, we interpret as follow the scheme (3.3)-(3.4) in terms of the functions $\bar{U}^{(i)}_k, q^{(i)}_{vk}, i = 1,2$ and $\tilde{\bar{U}}_k, \tilde{q}_{vk}$:

$$
\langle \partial_t \tilde{\bar{U}}_k, \bar{U}^b_k\rangle + \bar{a}(\bar{U}^{(2)}_k, \bar{U}^b_k) + \bar{b}(\mathbf{u}_k, \bar{U}^{(2)}_k, \bar{U}^b_k) - \bar{l}(\bar{U}^b_k) = (\bar{f}(U^{(1)}_k) - \frac{[\omega_k]^-}{p}\bar{\mathcal{F}}(T^{(1)}_k)h_{q^{(1)}_{vk}}, \bar{U}^b_k), \tag{4.8}
$$

$$
\begin{aligned}
&\langle \partial_t \tilde{q}_{vk}, q^b_{vk} - q^{(2)}_{vk}\rangle + a_{q_v}(q^{(2)}_{vk}, q^b_{vk} - q^{(2)}_{vk}) + b(\mathbf{u}_k, q^{(2)}_{vk}, q^b_{vk} - q^{(2)}_{vk}) - l_{q_v}(q^b_{vk} - q^{(2)}_{vk}) \\
&\qquad \ge (f_{q_v}(U^{(1)}_k) - \frac{[\omega_k]^-}{p}F(T^{(1)}_k)h_{q^{(1)}_{vk}}, q^b_{vk} - q^{(2)}_{vk}).
\end{aligned} \tag{4.9}
$$

In the above equations $h_{q^{(1)}_{vk}} = h_{q^m_v}$ for $t \in [mk,(m+1)k)$, and accordingly, $h_{q^{(2)}_{vk}} = h_{q^{m+1}_v}$ for $t \in [mk,(m+1)k)$.

We are in a position to pass to the limit $k \to 0^+$ in (4.8)-(4.9). First, by Lemma 4.1 and Aubin-Lions compactness theorem (see e.g, [41]), we deduce the existence of a subsequence, still denoted by the subscript k, and functions $U = (\bar{U}, q_v)$, $\tilde{U} = (\tilde{\bar{U}}, \tilde{q_v})$ verifying (4.10)-(4.12) below

$$U, \tilde{U} \in L^\infty(0,t_1; L^2(\mathcal{M})^4) \cap L^2(0,t_1;V^4), \tag{4.10}$$

$$\bar{U}, \tilde{\bar{U}} \in L^\infty(0,t_1;V^3) \cap L^2(0,t_1;H^2(\mathcal{M})^3), \tag{4.11}$$

$$\partial_t \tilde{\bar{U}} \in L^2(0,t_1;H), \quad \partial_t \tilde{q_v} \in L^{5/3}(0,t_1;V^*), \tag{4.12}$$

such that, as $k \to 0^+$

(i) $\bar{U}^{(i)}_k \rightharpoonup \bar{U}$ weakly in $L^2(0,t_1;H^2(\mathcal{M})^3)$ and weak* in $L^\infty(0,t_1;V^3)$, i=1,2;
(ii) $q^{(i)}_{vk} \rightharpoonup q_v$ weakly in $L^2(0,t_1;V)$ and weak* in $L^\infty(0,t_1;H)$, i=1,2;
(iii) $\tilde{\bar{U}}_k \rightharpoonup \tilde{\bar{U}}$ weakly in $L^2(0,t_1;H^2(\mathcal{M})^3)$ and weak* in $L^\infty(0,t_1;V^3)$;
(iv) $\tilde{q}_{vk} \rightharpoonup \tilde{q_v}$ weakly in $L^2(0,t_1;V)$ and weak* in $L^\infty(0,t_1;H)$;
(v) $\partial_t\tilde{\bar{U}}_k \rightharpoonup \partial_t\tilde{\bar{U}}$ weakly in $L^2(0,t_1;L^2(\mathcal{M})^3)$;
(vi) $\partial_t\tilde{q}_{vk} \rightharpoonup \partial_t\tilde{q_v}$ weakly in $L^{5/3}(0,t_1;V^*)$,
(vii) $\tilde{\bar{U}}_k \rightharpoonup \tilde{\bar{U}}$ strongly in $L^2(0,t_1;V^3)$;
(viii) $q^\varepsilon_v \rightharpoonup \tilde{q_v}$ strongly in $L^2(0,t_1;H)$.

In view of (iv) and (vi), we also have

$$\tilde{q}_{vk}(t_1) \to \tilde{q_v}(t_1) \text{ strongly in } L^2(\mathcal{M}). \tag{4.13}$$

Thanks to Lemma 4.2, we can conclude that $U = \tilde{U}$ and

$$U_k^{(i)}(i=1,2), \tilde{U}_k \rightharpoonup U \text{ strongly in } L^2(0,t_1;L^2(\mathcal{M})^4). \tag{4.14}$$

In addition,

$$\bar{U}_k^{(i)}(i=1,2), \tilde{\bar{U}}_k \to \bar{U} \text{ strongly in } L^2(0,t_1;V^3). \tag{4.15}$$

By (4.15) and the fact that $q_{vs} = Q_{vs}(p,T)$ depends smoothly on T, we can follow the same argument as in Lemma 4.5 of [16] and deduce that

$$q_{vs}^k = Q_{vs}(p,T_k^{(2)}) \to Q_{vs}(p,T) = q_{vs} \text{ strongly in } L^2(0,t_1;V). \tag{4.16}$$

Accordingly, recalling the definition of q_{vk}^b in (4.7), we also have

$$q_{vk}^b \to q_v^b \text{ strongly in } L^2(0,t_1;V). \tag{4.17}$$

Remark 4.3. The relationship between q_{vs}^k and $T_k^{(2)}$ in (4.16) will allow us to deduce that $|\nabla_3 q_{vs}^k| \leq C|\nabla_3 T_k^{(2)}| + C_1$. Then noting that $\nabla_3 T_k^{(2)} \in L^\infty(0,t_1;H)$, q_{vs}^k actually lies in a bounded set of $L^\infty(0,t_1;V)$. And by our assumption, q_v^b belongs to the space $L^\infty(0,t_1;V)$. Hence, $q_{vk}^b = \min\left(\frac{1}{k}\int_{mk}^{(m+1)k} q_v^b(t)dt, q_{vs}^k\right)$ lies in a bounded set of $L^\infty(0,t_1;V)$ as well. Also q_{vk}^b converges to q_v^b almost everywhere in V for $t \in [0,t_1]$; (4.17) together with Lebesgue's dominated convergence theorem yields

$$q_{vk}^b \to q_v^b \text{ strongly in } L^r(0,t_1;V) \text{ for any } r > 1. \tag{4.18}$$

In particular, we will use the result with $r = \frac{5}{2}$ for passing to limit in the term $\langle \partial_t \tilde{q}_{vk}, q_{vk}^b \rangle$ in the q_v-equation, as we only have weak convergence in the space $L^{5/3}(0,t_1;V^*)$ for the term $\partial_t \tilde{q}_{vk}$.

For the forcing term $f(U_k^{(1)}) - \frac{1}{p}[\omega_k]^- \mathcal{F}(T_k^{(1)}) h_{q_{vk}^{(1)}}$, we have

$$f(U_k^{(1)}) \to f(U) \text{ strongly in } L^2(0,t_1;L^2(\mathcal{M})^4), \tag{4.19}$$

because of the continuity and boundedness of $f(U)$; and

$$\mathcal{F}(T_k^{(1)}) h_{q_{vk}^{(1)}} \rightharpoonup \mathcal{F}(T) h_{q_v} \text{ weakly in } L^2(0,t_1;L^2(\mathcal{M})), \tag{4.20}$$

as $\mathcal{F}$ is Lipschitz continuous and $h_{q_{vk}^{(i)}} \rightharpoonup h_{q_v}$ weak* in $L^\infty([0,t_1]\times\mathcal{M})$ for $i = 1,2$.

Thanks to (4.14) and (4.16), by an additional extraction of subsequences, we have

$$U_k^{(i)}(x) \to U(x) \quad a.e. \text{ in } [0,t_1]\times\mathcal{M}, i = 1,2, \tag{4.21}$$

$$q_{vs}^k(x) \to q_{vs}(x) \quad a.e. \text{ in } [0,t_1]\times\mathcal{M}. \tag{4.22}$$

We are now in position to pass to the limit in (4.9). First we observe in advance that the limit function q_v lies in $L^2(0,t_1;\mathcal{K})$ where $\mathcal{K} = \{q \in V; q_v \leq q_{vs} \ a.e.\}$. Indeed, by (3.14) and Lemma 3.6,

$$q_{vk}^{(2)} \leq q_{vs}^k = Q_{vs}(T_k^{(2)},p) \quad a.e. \text{ in } [0,t_1]\times\mathcal{M}.$$

Using (4.21) and (4.22), we pass to the limit $k \to 0^+$ in the inequality above , we see that $q_v \leq q_{vs}$ $a.e.$ in $[0,t_1]\times\mathcal{M}$.

Next we pass to the limit term by term in (4.9) after we integrate (4.9) form $t=0$ to t_1. The first term $\int_0^{t_1}\langle \partial_t\tilde{q}_{vk}, q_{vk}^b - q_{vk}^{(2)}\rangle dt$ is the the sum of $\int_0^{t_1}\langle \partial_t\tilde{q}_{vk}, q_{vk}^b - \tilde{q}_{vk}\rangle\, dt$ and $\int_0^{t_1}\langle \partial_t\tilde{q}_{vk}, \tilde{q}_{vk} - q_{vk}^{(2)}\rangle\, dt$. For the first part, we integrate by parts and write

$$\begin{aligned}
&\limsup \int_0^{t_1} \langle \partial_t\tilde{q}_{vk}, q_{vk}^b - \tilde{q}_{vk}\rangle\, dt\\
&= -\liminf \int_0^{t_1} \langle \partial_t\tilde{q}_{vk}, \tilde{q}_{vk}\rangle\, dt + \lim \int_0^{t_1} \langle \partial_t\tilde{q}_{vk}, q_{vk}^b\rangle\, dt\\
&= -\liminf \frac{1}{2}|\tilde{q}_{vk}(t_1)|_{L^2}^2 + \frac{1}{2}|q_{v0}|_{L^2}^2 + \int_0^{t_1} \langle \partial_t q_v, q_v^b\rangle\, dt\\
&\le (\text{ by (4.13) and the lower semi-continuity of the norm }) \qquad (4.23)\\
&\le -\frac{1}{2}|q_v(t_1)|_{L^2}^2 + \frac{1}{2}|q_{v0}|_{L^2}^2 + \int_0^{t_1} \langle \partial_t q_v, q_v^b\rangle\, dt\\
&= -\int_0^{t_1} \langle \partial_t q_v, q_v\rangle\, dt + \int_0^{t_1} \langle \partial_t q_v, q_v^b\rangle\, dt\\
&= \int_0^{t_1} \langle \partial_t q_v, q_v^b - q_v\rangle\, dt.
\end{aligned}$$

For the second part $\int_0^{t_1}\langle \partial_t\tilde{q}_{vk}, \tilde{q}_{vk} - q_{vk}^{(2)}\rangle\, dt$, we observe that $\partial_t\tilde{q}_{vk} = \frac{q_v^{m+1}-q_v^m}{k}$ and

$$\tilde{q}_{vk} - q_{vk}^{(2)} = (t-(m+1)k)\frac{q_v^{m+1}-q_v^m}{k} \quad \text{on} \quad [mk,(m+1)k);$$

then we have

(4.24)
$$\begin{aligned}
\int_0^{t_1} \langle \partial_t\tilde{q}_{vk}, \tilde{q}_{vk} - q_{vk}^{(2)}\rangle dt &= \sum_{m=0}^{N-1}\int_{(m+1)k}^{mk} \langle \frac{q_v^{m+1}-q_v^m}{k}, (t-(m+1)k)\frac{q_v^{m+1}-q_v^m}{k}\rangle\, dt\\
&= \sum_{m=0}^{N-1}\int_{mk}^{(m+1)k} \frac{|q_v^{m+1}-q_v^m|_{L^2}^2}{k^2}(t-(m+1)k)\, dt \le 0.
\end{aligned}$$

Then we can conclude that

$$\limsup \int_0^{t_1} \langle(\partial_t\tilde{q}_{vk}, q_{vk}^b - q_{vk}^{(2)}\rangle\, dt \le \int_0^{t_1} \langle \partial_t q_v, q_v^b - q_v\rangle\, dt. \tag{4.25}$$

For the a_{q_v}-term, due to the lower semi-continuity of the norm, we have

$$\begin{aligned}
\limsup \int_0^{t_1} a_{q_v}(q_{vk}^{(2)}, q_{vk}^b - q_{vk}^{(2)})\, dt &= \lim \int_0^{t_1} a_{q_v}(q_{vk}^{(2)}, q_{vk}^b)\, dt \qquad (4.26)\\
&\quad - \liminf \int_0^{t_1} a_{q_v}(q_{vk}^{(2)}, q_{vk}^{(2)})\, dt\\
&\le \int_0^{t_1} a_{q_v}(q_v, q_v^b)\, dt - \int_0^{t_1} a_{q_v}(q_v, q_v)\, dt \qquad (4.27)\\
&= \int_0^{t_1} a_{q_v}(q_v, q_v^b - q_v)\, dt.
\end{aligned}$$

By passing to the limit $k \to 0^+$ in the above two terms, we obtain the inequality in the desired direction.

Using the divergence free property of $\mathbf{u}$ and $\mathbf{u_k}$,

$$
\begin{aligned}
&\int_0^{t_1} b(\mathbf{u}_k, q_{vk}^{(2)}, q_{vk}^b - q_{vk}^{(2)})dt - \int_0^{t_1} b(\mathbf{u}, q_v, q_v^b - q_v)dt \\
=&\int_0^{t_1} b(\mathbf{u}_k, q_{vk}^{(2)}, q_{vk}^b) - b(\mathbf{u}, q_v, q_v^b)dt \\
=&\int_0^{t_1} b(\mathbf{u}_k - \mathbf{u}, q_{vk}^{(2)}, q_{vk}^b)dt + \int_0^{t_1} b(\mathbf{u}, q_{vk}^{(2)} - q_v, q_{vk}^b)dt + \int_0^{t_1} b(\mathbf{u}, q_v, q_{vk}^b - q_v^b)dt \\
=&\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

By Lemma 2.1 and Hölder's inequality, $\mathcal{I}_1 + \mathcal{I}_2$ can be controlled as follows,

$$
\begin{aligned}
|\mathcal{I}_1| + |\mathcal{I}_2| \le\, & C\int_0^{t_1} \|\mathbf{u}_k - \mathbf{u}\|_V |q_{vk}^{(2)}|_{L^2}^{\frac12} \|q_{vk}^{(2)}\|_V^{\frac12} \|q_{vk}^b\|_V dt \\
&+ C\int_0^{t_1} \|\mathbf{u}\|_V |q_{vk}^{(2)} - q_v|_{L^2}^{\frac12} \|q_{vk}^{(2)} - q_v\|_V^{\frac12} \|q_{vk}^b\|_V dt \\
\le\, & C\|\mathbf{u}_k - \mathbf{u}\|_{L^r(0,t_1;V)} |q_{vk}^{(2)}|_{L^2(0,t_1;H)}^{\frac12} \|q_{vk}^{(2)}\|_{L^2(0,t_1;V)}^{\frac12} \|q_{vk}^b\|_{L^{r^*}(0,t_1;V)} \\
&+ C\|\mathbf{u}\|_{L^r(0,t_1;V)} |q_{vk}^{(2)} - q_v|_{L^2(0,t_1;H)}^{\frac12} \|q_{vk}^{(2)} - q_v\|_{L^2(0,t_1;V)}^{\frac12} \|q_{vk}^b\|_{L^{r^*}(0,t_1;V)},
\end{aligned}
$$

where $\frac{1}{r} + \frac{1}{r^*} = \frac12$. By (4.5) and (4.14), we see that $|\mathcal{I}_1| + |\mathcal{I}_2| \to 0$ as $k \to 0^+$. In view of the strong convergence of q_{vk}^b to q_v^b in $L^2(0, t_1; V)$, we also have $|\mathcal{I}_3| \to 0$ as $k \to 0^+$.

Then as $q_{vk}^{(2)} \rightharpoonup q_v$ weakly in $L^2(0, t_1; V)$ and $q_{vk}^b \to q_v^b$ strongly in $L^2(0, t_1; V)$, we know by the Trace theorem that $\gamma_0(q_{vk}^{(2)}) \to \gamma_0(q_v)$ and $\gamma_0(q_{vk}^b) \to \gamma_0(q_v^b)$ weakly in $L^2(0, t_1; L^2(\partial\mathcal{M}))$ where γ_0 is the trace operator on $\partial\mathcal{M}$. Hence,

$$
\int_0^{t_1} l_{q_v}(q_{vk}^b - q_{vk}^{(2)})dt \to \int_0^{t_1} l_{q_v}(q_v^b - q_v)dt \text{ as } k \to 0^+.
$$

For the convergence of the forcing terms, we first consider the discontinuous part. We write the difference

$$
\int_0^{t_1} \big(-\frac{1}{p}[\omega_k]^- F(T_k^{(1)}) h_{q_{vk}^{(1)}}, q_{vk}^b - q_{vk}^{(2)}\big)dt - \int_0^{t_1} \big(-\frac{1}{p}[\omega]^- F(T) h_{q_v}, q_v^b - q_v\big)dt
$$

as the sum of the following two terms:

$$
\int_0^{t_1} \big(-\frac{1}{p}([\omega_k]^- - [\omega]^-) F(T_k^{(1)}) h_{q_{vk}^{(1)}}, q_{vk}^b - q_{vk}^{(2)}\big)dt,
$$

$$
\int_0^{t_1} \big(-\frac{1}{p}[\omega]^- F(T_k^{(1)}) h_{q_{vk}^{(1)}}, q_{vk}^b - q_{vk}^{(2)}\big) - \big(-\frac{1}{p}[\omega]^- F(T) h_{q_v}, q_v^b - q_v\big)dt.
$$

The first term above can be bounded by

$$\int_0^{t_1} \big(-\frac{1}{p}([\omega_k]^- - [\omega]^-)F(T_k^{(1)})h_{q_{vk}^{(1)}}, q_{vk}^b - q_{vk}^{(2)}\big)dt \\ \leq C\|\omega_k - \omega\|_{L^2(0,t_1;V)}|q_{vk}^b - q_{vk}^{(2)}|_{L^2(0,t_1;H)},$$

hence it tends to zero as $k \to 0^+$.

The second term above also tends to zero in view of (4.14),(4.17) and (4.20).

For the continuous part of the forcing terms we can pass to the limit $k \to 0^+$ using (4.19). Altogether we have

$$\lim_{k\to 0^+}\int_0^{t_1} \big(f_{q_v}(U_k^{(1)}) - \frac{1}{p}[\omega_k]^- F(T_k^{(1)})h_{q_{vk}^{(1)}}, q_{vk}^b - q_{vk}^{(2)}\big)dt \\ = \int_0^{t_1} \big(f_{q_v}(U) - \frac{1}{p}[\omega]^- F(T)h_{q_v}, q_v^b - q_v\big)dt.$$

So we can obtain (3.4) after we pass to the limit $k \to 0^+$ in (4.9).

Finally we need to verify that $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$. Rewriting (3.63) in terms of $q_{vk}^{(2)}, q_{vk}^b, q_{vs}^k$ and integrating on $[0, t_1]$ we have, for every $q^b \in L^2(0, t_1; V)$ and $q_{vk}^b = \frac{1}{k}\int_{mk}^{(m+1)k} q_v^b(t)dt$,

$$\int_0^{t_1} \big([q_{vk}^b - q_{vs}^k]^+, 1\big)\,dt - \int_0^{t_1} ([q_{vk}^{(2)} - q_{vs}^k]^+, 1)\,dt \geq \int_0^{t_1} \langle h_{q_{vk}^{(2)}}, q_{vk}^b - q_v\rangle\,dt. \tag{4.28}$$

Up to a subsequence, $h_{q_{vk}^{(2)}}$ converges to some limit h_{q_v} weak* in $L^\infty([0, t_1] \times \mathcal{M})$ and $0 \leq h_{q_v} \leq 1$ a.e.; then in view of (4.17) and (4.14), we infer that

$$\int_0^{t_1} \langle h_{q_{vk}^{(2)}}, q_{vk}^b - q_{vk}^{(2)}\rangle\,dt \to \int_0^{t_1} \langle h_{q_v}, q_v^b - q_v\rangle\,dt \text{ as } k \to 0^+. \tag{4.29}$$

For the second term in the LHS of (4.28), we have

$$\begin{aligned} &\Big|\int_0^{t_1} ([q_{vk}^{(2)} - q_{vs}^k]^+ - [q_v - q_{vs}]^+, 1)\,dt\Big| \\ \leq &\int_0^{t_1} \big(|q_{vk}^{(2)} - q_v|_{L^2} + |q_{vs}^k - q_{vs}|_{L^2}\big)|1|_{L^2}\,dt \\ \leq &\big(|q_{vk}^{(2)} - q_v|_{L^2(0,t_1;H)} + |q_{vs}^k - q_{vs}|_{L^2(0,t_1;H)}\big)\sqrt{|\mathcal{M}|t_1} \to 0. \end{aligned} \tag{4.30}$$

Hence

$$\int_0^{t_1} ([q_{vk}^{(2)} - q_{vs}^k]^+, 1)dt \to \int_0^{t_1} ([q_v - q_{vs}]^+, 1)dt.$$

Similarly as in (4.30), with the strong convergence of q_{vk}^b to q_v^b in $L^2(0, t_1; V)$, we have:

$$\int_0^{t_1} \big([q_{vk}^b - q_{vs}^k]^+, 1\big)\,dt \to \int_0^{t_1} \big([q_v^b - q_{vs}]^+, 1\big)\,dt. \tag{4.31}$$

In view of (4.29)-(4.31), we conclude that, for every $q_v^b \in L^2(0,t_1;V)$, the following inequality holds after passing to the limit in (4.28):

$$\int_0^{t_1} \big([q_v^b - q_{vs}]^+, 1\big)\,dt - \int_0^{t_1} ([q_v - q_{vs}]^+, 1)\,dt \geq \int_0^{t_1} \langle h_{q_v}, q_v^b - q_v\rangle\,dt. \tag{4.32}$$

The above inequality implies that, for every $q^b \in L^2(0,t_1;V)$, we have

$$\big([q_v^b - q_{vs}]^+, 1\big) - ([q_v - q_{vs}]^+, 1) \geq \langle h_{q_v}, q_v^b - q_v\rangle, \quad \text{for a.e. } t \in [0,t_1], \tag{4.33}$$

which tells that $h_{q_v} \in \mathcal{H}(q_v - q_{vs})$.

The passage to limit $k \to 0^+$ in the $\bar{U}$-equation (4.8) is similar and easier, we omit the details here.

To conclude, we have the following theorem.

Theorem 4.4. *Given $U_0 = (\bar{U}_0, q_{v0}) = (q_{c0}, q_{r0}, \theta_0', q_{v0}) \in V \times V \times V \times V$, with $q_{v0} \leq q_{vs}(t=0)$ a.e. in $\mathcal{M}$, the functions $U_k^{(1)}, U_k^{(2)}, \tilde{U}_k$ associated with the Euler scheme* (3.3) *and* (3.4) *contain a subsequence $k \to 0^+$ which converges to a solution U of the system* (2.34)-(2.35) *in the sense of (i)-(viii) and* (4.14)-(4.15)*, and U satisfies* (4.10)-(4.12).

5. Numerical simulations

In this section, we illustrate the theory above with some numerical simulations done in a slightly different setting, easier in some sense, and more challenging in some sense.

We consider a two-dimensional problem with directions x (west-east) and p (the pressure), and the domain is not rectangular, corresponding to the geometry above one or two mountains. Finally, we omit the viscosity terms as the viscosity is not significant for short term forecast (up to one week). As we shall see the mountains produce a rain shadow on the leeward side of the mountains area, that is, away from the wind. Hence, in the simulations, the following system of equations is used.

$$\begin{cases} \dfrac{\partial T}{\partial t} + u\dfrac{\partial T}{\partial x} + \omega\dfrac{\partial T}{\partial p} = \dfrac{\omega}{p}\left(\dfrac{RT}{C_p} - \delta\dfrac{\mathcal{L}F}{C_p}\right), \\ \dfrac{\partial q}{\partial t} + u\dfrac{\partial q}{\partial x} + \omega\dfrac{\partial q}{\partial p} = \delta\dfrac{F}{p}\omega, \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \omega\dfrac{\partial u}{\partial p} + \phi_x = 0, \\ \dfrac{\partial \omega}{\partial p} + \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial \phi}{\partial p} = -\dfrac{RT}{p}, \\ \phi = zg, \quad z = z(x,p,t) \end{cases} \tag{5.1}$$

The unknown functions are

- $T = T(x,p,t)$: local temperature
- $q = q(x,p,t)$: specific humidity (called q_v in Sections 1 to 4 of this article)
- $u = u(x,p,t)$: the velocity in the x-direction

- $\omega = \omega(x, p, t)$: the vertical velocity in the x, p system

The variable ω is a diagnostic variable which will be computed using the prognostic variable u. We treat the geopotential $\phi = \phi(x, p, t)$ separately using $(5.1)_5$. All quantities are expressed in the metric system. The pressures are expressed in millibars. Note that $(5.1)_1$, $(5.1)_3$, and $(5.1)_4$ express the conservation of energy, momentum in the x-direction, and mass, respectively.

Also, for the numerical simulations, the following values and functions are used. See the details in [11]:

- $\delta = H(-\omega)H(q - q_{vs})$, where H is the Heaviside function $H(x) = \frac{1}{2}(1 + sign(x))$.
- $\mathcal{L}(T) = 2.5008 \times 10^6 - 2.3 \times 10^3(T - 275)\ J\ kg^{-1}$ is the latent heat of vaporization.
- $R = 287\ J\ K^{-1}\ kg^{-1}$ is the gas constant for dry air.
- $R_v = 461.50\ J\ K^{-1}\ kg^{-1}$ is the gas constant for water vapor.
- $p_A \in [0, 200]$, usually $\simeq 200$ (chosen).
- $p_0 = 1000$ (chosen).
- $C_p = 1004\ J\ K^{-1}\ kg^{-1}$ is the specific heat of dry air at constant pressure.
- $F(T, p)$ is given by

$$F(T,p) = q_{vs}(T,p)T\left(\frac{\mathcal{L}R - C_pR_vT}{C_pR_vT^2 + q_{vs}(T,p)\mathcal{L}^2(T)}\right). \tag{5.2}$$

- $q_{vs}(T, p)$ is the saturation specific humidity, defined by

$$q_{vs}(T,p) = \frac{0.622e_{vs}(T)}{p}, \tag{5.3}$$

  where $e_{vs}(T)$ is the saturation vapor pressure. We approximate its value with equation (2.17) in [55](compared with (2.4)):

$$e_{vs}(T) = 6.112\exp\left(\frac{17.67(T - 273.15)}{T - 29.65}\right). \tag{5.4}$$

  Compare (5.3) with (2.3), in which the term $0.378e_{vs}$ in the denominator is neglected by comparison with p.

We performed simulations with three different domains. All simulations were made with a west-east prevailing wind. (See the definition of $u(x, p, t = 0)$ in $(5.5)_3$ below.)

- In the first simulation, the domain represents the atmosphere above one mountain. We set the domain as $[x_0, x_f] \times [p_A, p_B(x)]$, where $x_0 = 0$, $x_f = 75000\,m$, and $p_A = 250\,mb$. The function $p_B(x)$, which defines the topography along the mountain, is set to be

$$p_B(x) = 1000 - 250\exp\left(-\frac{(x - 37500)^2}{6000^2}\right).$$

- In the second simulation, the domain represents the atmosphere above two mountains, with the left mountain lower than the right one. In this case, we choose the parameter values $x_0 = 0$, $x_f = 150000\,m$, and $p_A = 250\,mb$ for the domain $[x_0, x_f] \times [p_A, p_B(x)]$. The topography is defined to be

$$p_B(x) = \begin{cases} 1000 - 200\exp\left(-\dfrac{(x-37500)^2}{6000^2}\right), & x \le 75000 \\ 1000 - 250\exp\left(-\dfrac{(x-112500)^2}{6000^2}\right), & x > 75000 \end{cases}.$$

- In the third simulation, the domain also represents two mountains, but in this case, the left mountain is set to be higher than the right mountain, thus blocking the wind coming from the west. The values of x_0, x_f, and p_A are the same as in the second simulation, while p_B is defined as

$$p_B(x) = \begin{cases} 1000 - 250\exp\left(-\dfrac{(x-37500)^2}{6000^2}\right), & x \le 75000 \\ 1000 - 200\exp\left(-\dfrac{(x-112500)^2}{6000^2}\right), & x > 75000 \end{cases}.$$

We use the following initial conditions for all the simulations.

$$\begin{cases} T(x,p,t=0) = \bar{T}(p) = T_0 - \left(1 - \dfrac{p}{p_0}\right)\Delta T, \\ q(x,p,t=0) = q_{vs} - 0.0052, \\ u(x,p,t=0) = 7.5 + 2\cos\left(\dfrac{p\pi}{p_0}\right)\cos\left(\dfrac{2\pi x}{x_f}\right), \end{cases} \tag{5.5}$$

where $p_0 = 1000$, $T_0 = 300\,K$, and $\Delta T = 50\,K$.

The boundary conditions are

$$\begin{cases} \dfrac{\partial p_B}{\partial x} = 0, & \text{at } x = 0, x_f, \\ \omega = u\dfrac{\partial p_B}{\partial x}, & \text{at } p = p_B, \\ \omega = \phi_x = 0, & \text{at } p = p_A, \\ \mathbf{u} = \mathbf{G}(p), & \text{at } x = 0, \\ \dfrac{\partial \mathbf{u}}{\partial n} = 0, & \text{at } x = x_f, \\ \dfrac{\partial \omega}{\partial n} = 0, & \text{at } x = \{0, x_f\}, \end{cases}$$

where $\mathbf{u} = (T, q, u)$ is the solution, and $\mathbf{G} = (g_T, g_q, g_u)$ defines the boundary values of the solution on the left boundary (at $x = 0$). The following definition of $\mathbf{G}$ is used in all the simulations.

$$g_T(p) = \bar{T}(p), \quad g_q(p) = q_{vs}(\bar{T}(p), p), \quad g_u(p) = 7.5 + 2\cos\left(\frac{p\pi}{p_0}\right).$$

In the numerical simulations, we used the upwind Godunov scheme with an $n \times n$ mesh in the spatial domain and the 4th order Runge-Kutta method for the time discretization. The details of the numerical scheme are given in [11] and in [1] where one single mountain was considered. Now, we give the results of the numerical simulations.

Simulation 1

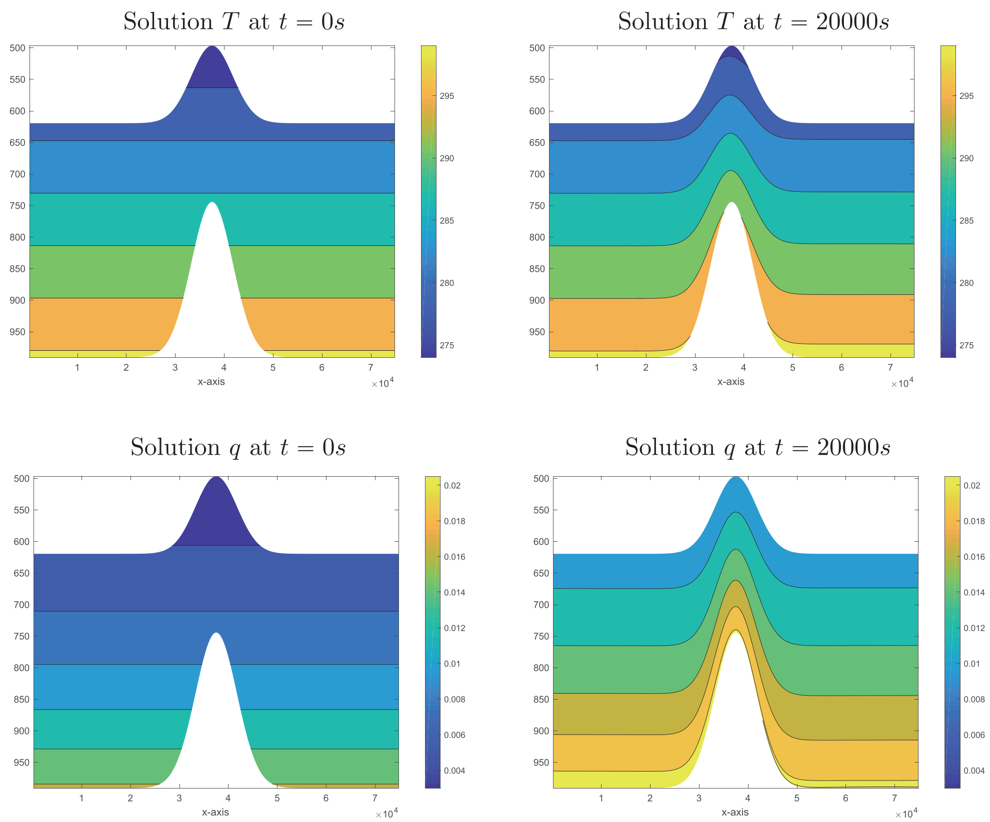

FIGURE 1. Solutions T and q are shown at initial time $t = 0$ and at $t = 20000s$ in the first simulation. Results were computed with a spatial mesh of size 200×200 and a time step of $\Delta t = 0.5s$. Note that since u is positive, the flow moves from left to right. At the end of the simulation, the temperature T is higher on the right side of the mountain than on the left side. On the other hand, the humidity q is higher in value on the left side of the mountain. These results are coherent with the physical context.

Simulation 2

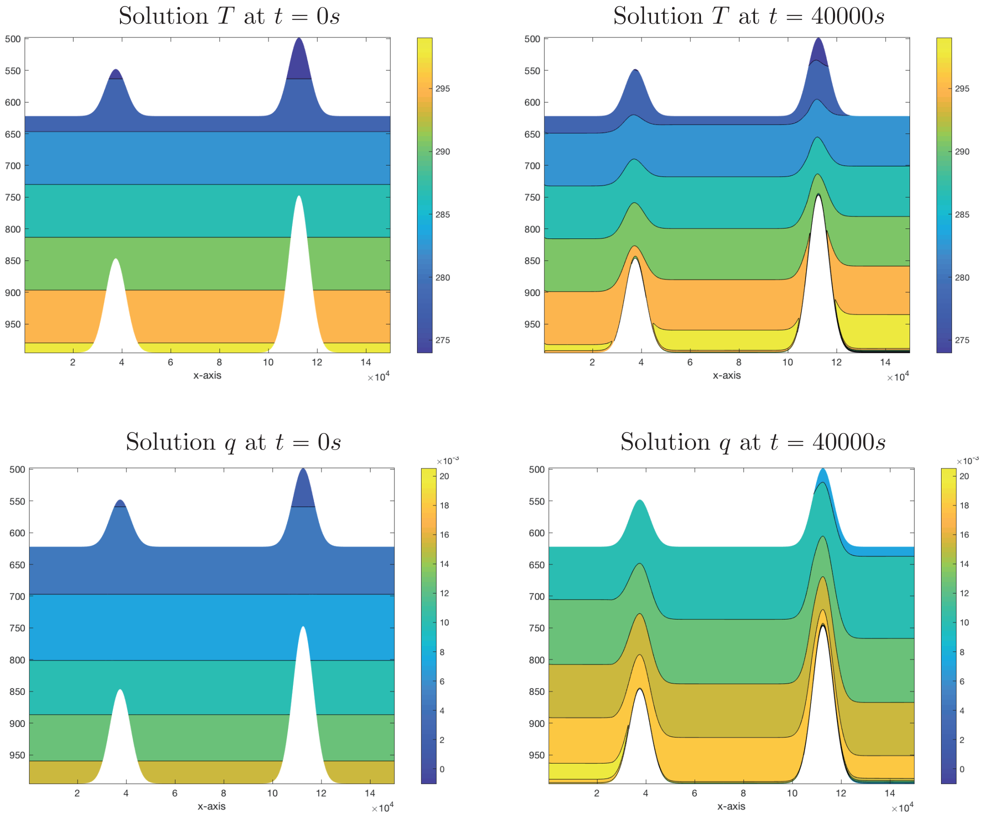

FIGURE 2. Solutions T and q are shown at initial time and at $t = 40000s$ in the second simulation. In this simulation, a spatial mesh of size 400×400 and a time step of $\Delta t = 0.5s$ were used. The flow moves from left to right, as in the first simulation. At the end of the simulation, the temperature T has the lowest value on the left side of the left mountain, and becomes higher as we cross the mountains towards the right side. The humidity q is higher in value on the left side of the left mountain and decreases in value going towards the right. These results are also coherent with the physical context.

Simulation 3

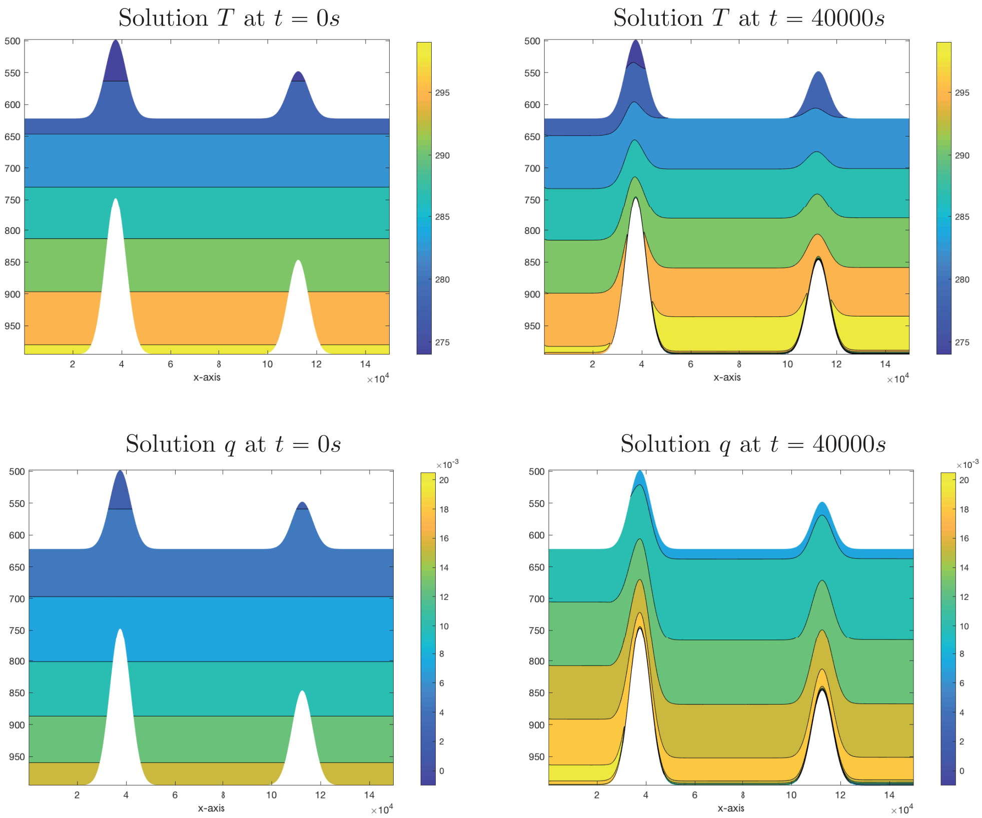

FIGURE 3. Solutions T and q are shown at initial time and at $t = 40000s$. In this simulation, a spatial mesh of size 400×400 and a time step of $\Delta t = 0.5s$ were used. The flow moves from left to right, as in the previous simulations. At the end of the simulation, the temperature T has lower values on the left side of the left mountain and assumes higher values in the area to the right of that mountain. The humidity q is higher on the left side of the left mountain and is lower in value on the right side of the left mountain. This indicates the higher mountain blocks the passing of moist from the west side and the air flow crossing the mountain is mostly dry. These results are coherent with the physical context.

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