Pure and Applied Functional Analysis
Volume 4, Number 4, 2019, 765–801

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

TRANSFINITE INTERPOLATIONS FOR FREE AND MOVING BOUNDARY PROBLEMS

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Abstract. In imaging and numerical analysis a *transfinite interpolation* is a special case of extension of a function defined on a closed subset E of the Euclidean n-dimensional space $\mathbb{R}^n$ to some larger subset of $\mathbb{R}^n$. Given an open domain Ω with a compact locally Lipschitzian boundary Γ, and a continuous function on Γ, we first generalize (k-TMI) the construction and relax the assumptions of the seminal paper of Dyken and Floater [*Transfinite mean value interpolation*, Computer Aided Geometric Design 26 (2009), 117–134] of its continuous extension to Ω. Secondly, we introduce the new family of k-*Transfinite Barycentric Interpolation* (k-TBI) for compact H^d-rectifiable subsets E of $\mathbb{R}^n$ of arbitrary dimension $0 \le d < n$. Modulo the specific requirements of the application at hand, it is both computationally simpler and mathematically more general than the k-TMI for which E is limited to the boundary of an open domain. Thirdly, the constructions are extended to new enhanced interpolations using derivatives that preserve polynomials of a specified degree. Finally, dynamical versions of the k-TMI and the k-TBI are introduced to iteratively construct the rate of change of the position of the points of $\mathbb{R}^n\backslash E$ from the rate of change of the points of E as if E was a moving/deforming body in a fluid medium. This paper is motivated by pressing numerical and theoretical issues in free/moving boundary problems, *Arbitrary Lagrangian-Eulerian* (ALE) methods, and iterative schemes in shape/topological optimization and control.

1. Introduction

Given a closed subset E of the n-dimensional Euclidean space $\mathbb{R}^n$ endowed with the *scalar product* and norm

$$x \cdot y \stackrel{\text{def}}{=} \Sigma_{i=1}^n x_i\, y_i, \quad \|x\| \stackrel{\text{def}}{=} \sqrt{x \cdot x}, \tag{1.1}$$

and a continuous function $f : E \to \mathbb{R}$, Tietze [29] in 1915 proved the existence of a continuous extension to $\mathbb{R}^n$ and Hausdorff [15] in 1919 gave the following explicit extension

$$F(x) \stackrel{\text{def}}{=} \begin{cases} f(x), & \text{if } x \in E, \\ \inf\limits_{a\in E} \left\{ f(a) + \dfrac{\|x-a\|}{d_E(x)} - 1 \right\}, & \text{if } x \in \mathbb{R}^n\backslash E, \end{cases}$$

where $d_E(x)$ is the distance function from x to E.

2010 *Mathematics Subject Classification.* 65D05, 32V25, 65D18, 65M60.
Key words and phrases. Transfinite Interpolation, continuous Extension, rectifiable sets, computational geometry, finite element meshes, imaging.

As is often the case a close form formula may not always be the best object to numerically compute an extension and other choices may turn out to be more efficient. In the literature on Imaging, extentions of a function defined at an infinite number of points are called *transfinite interpolations.* For instance, the *Transfinite Mean Value Interpolation* (TMI) in the seminal paper of Dyken and Floater [9] has important applications in Imaging and in the construction of adaptive finite element meshes. Roughly speaking, given an open subset Ω of $\mathbb{R}^n$ with a smooth compact boundary Γ, a continuous function $f : \Gamma \to \mathbb{R}$ is extended to Ω by introducing the function

$$\mathcal{M}(f)(y) \overset{\text{def}}{=} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^{n+1}} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^{n+1}} \cdot n_\Omega(\xi')\, Redd\Gamma(\xi')}\, Redd\Gamma(\xi), \quad y \in \Omega, \tag{1.2}$$

where $n_\Omega(\xi)$ is the unit exterior normal to Ω at the point $\xi \in \Gamma$. This interpolation and its generalization from $\mathbb{R}^2$ to $\mathbb{R}^n$ by Bruvoll and Floater [4] both used a parametrization of the boundary Γ. The proof that, for each $x \in \Gamma$, we have the *pointwise convergence* $\mathcal{M}(f)(y) \to f(x)$ *as* $y \to x$ is straightforward for a convex set Ω, but is non-trivial when Ω is not convex. In both papers they assume that Ω is a set of *positive reach* (see Definition A.3 (ii)) plus what they call a *mild assumption* that turns out to be the key assumption in the proof of *pointwise convergence* of $\mathcal{M}(f)$ to f at points of Γ. This positive reach assumption rules out non-convex polygons in $\mathbb{R}^2$ that were previously studied by Hormann and Floater [17]. For non-convex sets the difficulty is the presence of the normal to Ω that induces changes in the sign of the integrand of the numerator and the denominator.

The object of this paper is fourfold. We first extend the family of TMI to a larger parametrization-free family called *k-Transfinite Mean Value Interpolations* (k-TMI) from Γ to Ω, Ω^c, or $\mathbb{R}^n\backslash\Gamma$ where the exponent $n+1$ is replaced by a real number $k > n$ and the continuous function $f : \Gamma \to \mathbb{R}$ by an L^2-integrable function. We show that this extension is the minimizing solution of local variational problems. For a continuous f, we prove the *pointwise convergence* under a weaker *local boundedness condition* that is verified for convex sets and non-convex n-polytopes as in [17]. Two questions remain open. Is the interpolation property true for all open domains Ω with compact locally Lipschitzian boundary Γ? Is the positive reach property a pertinent condition?

Secondly, we introduce the new *k-Transfinite Barycentric Interpolation* (k-TBI) for a compact subset E of $\mathbb{R}^n$ with $H^d(E) < \infty$ (H^d, the d-dimensional Hausdorff measure in $\mathbb{R}^n$), a function $f \in L^2(E; H^d)$, and a real number $k > n$

$$\mathcal{B}(f)(y) \overset{\text{def}}{=} \int_E f(\xi)\, \frac{\frac{1}{\|\xi-y\|^k}}{\int_E \frac{1}{\|\xi'-y\|^k}\, dH^d(\xi')}\, dH^d(\xi), \quad y \in \mathbb{R}^n\backslash E. \tag{1.3}$$

E can be a *cloud* of a finite number of points ($d = 0$) as in the early work of Shepard [24] or a d-dimensional, $1 \le d < n$, submanifold of $\mathbb{R}^n$ such as a piece of curve ($d = 1$) or of surface ($d = 2$) in $\mathbb{R}^3$. One important technical advantage is that the weight in the numerator and denominator is now a positive function that does not involve the normal. Again, we show that this extension/interpolation is the minimizing solution of local variational problems. Moreover, it is shown that, for a continuous f, we have *pointwise convergence* not only for smooth d-dimensional submanifolds but also for arbitrary H^d-rectifiable sets of any dimension such as sets that are locally Lipschitz d-graphs. In the case of an open subset Ω (not necessarily connected) of $\mathbb{R}^n$ with compact locally Lipschitzian boundary, Γ is H^{n-1}-rectifiable and the k-TBI is a competitive alternative to the k-TMI.

Thirdly, the constructions are extended by using the function and its derivatives up to degree m. Knowing that the $(n+1)$-TMI of Dyken and Floater [9] preserve polynomials of degree one, we show that the *enhanced* $(n+m)$-*TMI* and (k,m)-*TBI*, $k > d$, preserve polynomials of degree $m \geq 0$.

Finally, a dynamical version of both families of transfinite interpolations is introduced for applications in the construction of finite element meshes for moving/deforming bodies in a surrounding medium. In this approach a *differential interpolation equation* is obtained by interpolating the *rate of change* at the boundary of the object. Solving that equation numerically effectively constructs the rate of change of the points of the surrounding medium from the rate of change of the boundary points of the body. For instance, it allows without re-meshing a complete rotation of an immersed solid body onto itself. The detailed numerical implementation of the k-TMI and the k-TBI with extensive experimentation and many comparisons are the object of the companion paper [13].

This paper is motivated by pressing numerical and theoretical applications. In the numerical analysis of free/moving boundary problems, one of the challenging issue is the re-meshing of the initial finite element grid as the boundary of a domain or the interface evolves. It is also at the hearth of *Arbitrary Lagrangian-Eulerian* (ALE) methods that try to alleviate the drawbacks of the traditional Lagrangian and Eulerian-based finite element simulations. Such issues also naturally arise during the successive iterations in shape/topological optimization and control.

Perspectives. In that context, the transfinite interpolations that originated in Imaging are a serious alternative to *pseudo-solid methods* that involve heavy computations of solutions of large systems of partial differential equations. In the pseudo-solid method initiated by Lynch and O'Neil [22] in 1980, the linear elasticity equations are solved to propagate the boundary deformations within the computational domain. The pseudo-solid method is certainly the most prevalent approach and has been successfully applied to monolithic and loosely coupled simulations of fluid-structure interaction problems.

There are many beautiful problems in Mechanics that could potentially benefit from transfinite interpolations. For instance, the pioneering work of Sokołowski and Zolésio [25, 26, 27, 28] for the sensitivity analysis of contact problems via variational inequalities, the advanced formulations of Plotikov and Sokołowski [23] for the moving obstacle in compressible gas, the drag minimization for the obstacle in compressible flow by Kaźmierczak, Sokołowski, and Żochowski [18], the shape and topology optimization for passive control of crack propagation by Leugering, Sokołowski, and Żochowski [21], and the numerical methods currently developped by Léger, Fortin, Tibirna, and Fortin [20] for very large deformation in elasticity problems in the design of tires for the automotive industry, the anisotropic mesh adaptation for scroll wave turbulence dynamics in reaction-diffusion systems by

Belhamadia, Fortin, and Bourgault [3], and the fluid-structure interaction problems by Hay, Etienne, Garon, and Pelletier [16].

Finally, there are genuine mathematical motivations that are not related to numerical methods. Looking at transfinite interpolations as extensions of a function defined on some subset of a *normal topological space* to some larger space as in the generalization of the work of Tietze [29], the solution of the non-homogeneous Dirichlet problem can be seen as an extension of a function specified on the boundary Γ to a function defined on Ω with a trace at H^{n-1} almost all points of Γ. As we shall see in this paper, interpolation H^{n-1} almost everywhere comes in naturally and we have to live with it if we want general results for domains with non-smooth boundaries. Another example is the *double layer potential.* So, there is a natural conceptual convergence of objectives and tools with Boundary Valued Problems and, even more interesting, with Control Theory of Partial Differential Equations as in the remarkable recent work of Lasiecka, Szulc, and Żochowski [19] on the boundary control of small solutions to fluid-structure interactions arising in coupling of elasticity with Navier-Stokes equation under mixed boundary condition.

Notation and Terminology. The *n-dimensional volume* $V_n(r)$ and the *surface area* $A_{n-1}(r)$ of the n-ball of radius r in $\mathbb{R}^n$ are given by the formulae

$$V_n(r) = \alpha_n\, r^n, \quad \alpha_n \stackrel{\text{def}}{=} \pi^{\frac{n}{2}}/\Gamma\,(n/2+1), \quad A_{n-1}(r) = \frac{d}{dr}V_n(r) = \alpha_n n\, r^{n-1},$$

where Γ is Euler's gamma function, α_n is the volume and $\beta_n = \alpha_n n$ is the surface area of the n-ball of radius one in $\mathbb{R}^n$. Denote by m_d, $1 \le d \le n$, the Lebesgue measure and by H^d, $0 \le d \le n$, the d-dimensional Hausdorff measure in $\mathbb{R}^n$. The *distance function* d_E from a point $x \in \mathbb{R}^n$ to a subset E of $\mathbb{R}^n$ and the *set of projections* $\Pi_E(x)$ of a point x onto $\overline{E}$ are defined as follows

$$d_E(x) \stackrel{\text{def}}{=} \inf_{y\in E} \|y-x\|, \quad \Pi_E(x) \stackrel{\text{def}}{=} \{p \in \overline{E} : \|p-x\| = d_E(x)\}. \tag{1.4}$$

$\Pi_E(x)$ is compact and never empty. When $\Pi_E(x)$ is a singleton, the *projection* is denoted $p_E(x)$. The *characteristic function* $\chi_E(x)$ of a subset E is the function equal to 1 if $x \in E$ and 0 if $x \notin E$. Other definitions and theorems are given in the Appendix.

2. k-Transfinite mean value interpolation (k-TMI)

Let Ω be an open subset of $\mathbb{R}^n$ with compact boundary Γ. Denote by $\Omega^c = \mathbb{R}^n\backslash\overline{\Omega}$ the open domain associated with the complement of Ω. Assume that Ω is *locally Lipschitzian*, that is, Ω is the hypergraph of a Lipschitzian function in the neighbourhood of each point $x \in \Gamma$. Such domains can have several connected components and/or holes. If, in addition, Γ is compact, the $(n-1)$-dimensional Hausdorff measure $H^{n-1}(\Gamma)$ of Γ is finite, the exterior unit normal $n_\Omega(\xi)$ to Ω and the exterior unit normal $n_{\Omega^c}(\xi)$ to Ω^c exist at H^{n-1}-almost all points $\xi \in \Gamma$, and the Gauss Divergence Theorem applies provided the integrand over the possibly unbounded domains Ω or Ω^c is integrable. This family of open domains includes all bounded open convex domains and n-polytopes.

We now extend the TMI of [9] to locally Lipschitzian domains and replace the exponent $n+1$ by a real number $k > n$. This creates two ranges: $(n,n+1)$ and $(n+1,\infty)$ on each side of $n+1$. The parameter k actually controls the width of the region in the neighbourhood of Γ where the interpolation is close to the value of the function on Γ. The width is larger when k is large. We also extend the definition of the TMI from continuous functions f on Γ to L^2 functions by introducing a new local variational problem.

2.1. **Least Squares Locally L^2-Interpolation.** In this paper the *transfinite mean-value interpolation* is introduced via a variational problem. Part (i) of the following theorem generalizes [9, Thms. 1 and 2] and [4, Thm. 2] given for $k = n+1$ and a connected bounded open Ω with positive reach. Here, connectedness and positive reach are both unnecessary properties.

Theorem 2.1. *Let $n \geq 1$ be an integer and $k > n$ a real number. Assume that Ω is an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary and that $f \in L^2(\Gamma; H^{n-1})$.*

(i) *The function*

$$y \mapsto \phi(y) \stackrel{\text{def}}{=} \int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1} : \mathbb{R}^n \backslash \Gamma \to \mathbb{R} \tag{2.1}$$

is well defined, continuously infinitely differentiable, and

$$\begin{aligned} &0 < \phi(y) \leq n\,\alpha_n / d_\Gamma(y)^{k-n}, \quad y \in \Omega, \\ &0 < -\phi(y) \leq n\,\alpha_n / d_\Gamma(y)^{k-n}, \quad y \in \Omega^c, \end{aligned} \tag{2.2}$$

(α_n, the n-dimensional volume of the ball of radius one in $\mathbb{R}^n$).

(ii) *The function*

$$\hat{F}(y) \stackrel{\text{def}}{=} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1}, \quad y \in \mathbb{R}^n \backslash \Gamma, \tag{2.3}$$

is well defined and infinitely continuously differentiable in $\mathbb{R}^n \backslash \Gamma$.

(iii) *For each compact $K \subset \mathbb{R}^n \backslash \Gamma$, the restriction of $\hat{F}$ to K is the unique solution in $L^2(K)$ of the following minimization problem*

$$\inf_{F \in L^2(K)} \int_K \int_\Gamma |F(y) - f(\xi)|^2 \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1}\, dy. \tag{2.4}$$

This theorem provides a new angle to tackle the interpolation problem and allows more general $k > n$, f, and Ω than the ones originally introduced in the seminal paper of Dyken and Floater [9].

Definition 2.2. Let $n \geq 1$ be an integer and $k > n$ a real number. Assume that Ω is an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary. Given $f \in L^2(\Gamma; H^{n-1})$, the infinitely continuously differentiable function

$$\boxed{\mathcal{M}_k(f)(y) \stackrel{\text{def}}{=} \int_\Gamma f(\xi) \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1},\ y \in \mathbb{R}^n \backslash \Gamma.} \tag{2.5}$$

will called the *k-Transfinite Mean Value Interpolation* (k-TMI) of f. □

Remark 2.3. The exterior normal n_Ω to Ω can be replaced by the exterior normal n_{Ω^c} to Ω^c

$$\mathcal{M}_k(f)(y) = \int_\Gamma f(\xi) \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_{\Omega^c}(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_{\Omega^c}(\xi')\, dH^{n-1}} \, dH^{n-1}, \quad y \in \mathbb{R}^n \backslash \Gamma, \tag{2.6}$$

since the minus signs in the numerator and the denominator cancel out. □

Corollary 2.4. *For each compact $K \subset \mathbb{R}^n \backslash \Gamma$,*

$$\begin{aligned} \int_K \int_\Gamma |\hat{F}(y) - f(\xi)|^2 & \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}} \, dH^{n-1}\, dy \\ &= \int_K \left(\mathcal{M}_k(f^2) - \mathcal{M}_k(f)^2\right) dy. \end{aligned}$$

Proof of Theorem 2.1. (i) It is sufficient to prove the result for $y \in \Omega$. For $\xi \in \Omega^c$, the computation of the following divergence yields

$$\begin{aligned} \operatorname{div}_\xi \left(\frac{\xi - y}{\|\xi - y\|^k} \right) &= \frac{1}{\|\xi - y\|^k} \operatorname{div}_\xi (\xi - y) + (\xi - y) \cdot \nabla_\xi \frac{1}{\|\xi - y\|^k} \\ &= n \frac{1}{\|\xi - y\|^k} - k(\xi - y) \cdot \frac{\xi - y}{\|\xi - y\|^{k+2}} = (n-k) \frac{1}{\|\xi - y\|^k}. \end{aligned}$$

Since, as a function of ξ, it is integrable[1] over Ω^c for $y \in \Omega$ and $k > n$, we can use Gauss Divergence Theorem on the complement Ω^c to simplify the arguments in [9] and [4]

$$\begin{aligned} \int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1} &= - \int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_{\Omega^c}(\xi)\, dH^{n-1} \\ &= - \int_{\Omega^c} \operatorname{div}_\xi \left(\frac{\xi - y}{\|\xi - y\|^k} \right) d\xi. \end{aligned}$$

$$\Rightarrow \int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1} = (k-n) \int_{\Omega^c} \frac{1}{\|\xi - y\|^k}\, d\xi > 0 \tag{2.7}$$

for $k > n$. From identity (2.7) for $y \in \Omega$ and the fact that $B_{d_\Gamma(y)}(y) \subset \Omega$ and $\Omega^c \subset \mathbb{R}^n \backslash B_{d_\Gamma(y)}(y)$

$$\begin{aligned} \phi(y) &\le (k-n) \int_{\mathbb{R}^n \backslash B_{d_\Gamma(y)}(y)} \frac{1}{\|\xi - y\|^k}\, d\xi = (k-n) \int_{\mathbb{R}^n \backslash B_{d_\Gamma(y)}(0)} \frac{1}{\|\xi\|^k}\, d\xi \\ &= (k-n) \int_{d_\Gamma(y)}^\infty \beta_n \frac{r^{n-1}}{r^k}\, dr = \frac{\beta_n}{d_\Gamma(y)^{k-n}}, \end{aligned}$$

where $\beta_n = n\, \alpha_n$ is the surface area of the n-dimensional ball of radius one in $\mathbb{R}^n$. This proves the second inequality. So, $\phi(y)$ is well defined and continuously infinitely differentiable in Ω.

[1] If Ω is bounded then the domain Ω^c is infinite and the exponent k in the denominator of the integrand must be strictly greater than n since for $k \le n$ the integral over Ω^c explodes.

(ii) Given a compact $K \subset \mathbb{R}^n \backslash \Gamma$ and $f \in L^2(\Gamma; H^{n-1})$, consider the following quadratic function of $F \in L^2(K)$

$$\int_K \frac{\int_\Gamma |F(y) - f(\xi)|^2 \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy, \tag{2.8}$$

for which the Taylor's formula is exact: for all F, $\hat{F} \in L^2(K)$

$$\begin{aligned}
&\int_K \frac{\int_\Gamma |F(y) - f(\xi)|^2 \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy \\
&= \int_K \frac{\int_\Gamma |\hat{F}(y) - f(\xi)|^2 \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy \\
&\quad + 2 \int_K \frac{\int_\Gamma \left(\hat{F}(y) - f(\xi)\right) (F(y) - \hat{F}(y)) \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy \\
&\quad + \int_K \left|F(y) - \hat{F}(y)\right|^2 \underbrace{\frac{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}}_{=1}\, dy.
\end{aligned}$$

If $\hat{F}$ is a minimizer, then for all $F \in L^2(K)$

$$\int_K \int_\Gamma \left(\hat{F}(y) - f(\xi)\right) (F(y) - \hat{F}(y)) \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}\, dy = 0 \tag{2.9}$$

$$\forall F \in L^2(K), \quad \int_K \left|F(y) - \hat{F}(y)\right|^2 dy \geq 0. \tag{2.10}$$

The positivity condition (2.10) is always verified. As for condition (2.9),

$$\hat{F}(y) = \frac{\int_\Gamma f(\xi) \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}} \quad \text{for a.a. } y \in K \tag{2.11}$$

is the unique solution in $L^2(K)$. Therefore, $\hat{F} \in L^2(K)$ is the unique solution of the minimizing problem (2.4). Finally, for any bounded open set O such that $\overline{O} \subset \mathbb{R}^n \backslash \Gamma$, $\hat{F}$ is infinitely continuously differentiable and bounded in O and, hence, infinitely continuously differentiable in $\mathbb{R}^n \backslash \Gamma$. □

Proof of Corollary 2.4. Expand the quadratic term

$$\int_K \frac{\int_\Gamma |\hat{F}(y) - f(\xi)|^2 \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy$$

$$= \int_K |\hat{F}(y)|^2 \frac{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy + \int_K \frac{\int_\Gamma |f(\xi)|^2 \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy$$

$$- 2 \int_K \hat{F}(y) \frac{\int_\Gamma f(\xi) \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}\, dy.$$

This yields the final expression

$$\int_K |\hat{F}(y)|^2 + \mathcal{M}(f^2)(y)\, dy - 2|\hat{F}(y)|^2\, dy = \int_K \left(\mathcal{M}(f^2) - \mathcal{M}(f)^2\right)\, dy.$$

□

The property that $\mathcal{M}_{n+1}$ preserves first order polynomials as stated in [9] does not extend from $n+1$ to k, $n+1 \neq k > n$.

Theorem 2.5. *Assume that Ω is an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary. For polynomials $f \in P^1(\mathbb{R}^n)$, $\mathcal{M}_{n+1}(f) = f$ in $\mathbb{R}^n \backslash \Gamma$.*

Remark 2.6. Since the proof only requires the use of the divergence theorem, the theorem seems to be true for Caccioppoli sets Ω. □

Proof. For $f(\xi) = A^0 + A^1 \cdot \xi$, $A^0 \in \mathbb{R}$ and $A^1 \in \mathbb{R}^n$, and $y \in \mathbb{R}^n \backslash \Gamma$, $f(\xi) - f(y) = A^1 \cdot (\xi - y)$,

$$\mathcal{M}_k(f)(y) - f(y) = \frac{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_{\Omega^c}(\xi)\, A^1 \cdot (\xi - y)\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_{\Omega^c}(\xi)\, dH^{n-1}}.$$

Using the divergence theorem, the numerator is equal to

$$-\int_{\Omega^c} \operatorname{div}\left(\frac{\xi - y}{\|\xi - y\|^k}\, A^1 \cdot (\xi - y)\right) d\xi = -(n-k+1) \int_{\Omega^c} \frac{A^1 \cdot (\xi - y)}{\|\xi - y\|^k} dH^{n-1}.$$

For $k = n+1$ the integral is finite and $(n-k+1) = 0$. For $y \in \Omega^c$, we get the same result and $\mathcal{M}_{n+1}(f)(y) = f(y)$ on $\mathbb{R}^n \backslash \Gamma$. □

2.2. **Trace and Pointwise Convergence on Γ.** We have proved in Theorem 2.1 that for $k > n$ the denominator $\phi(y) \neq 0$ and that, for a continuous function f on Γ, the function $\hat{F}(y)$ is well defined and infinitely differentiable on $\mathbb{R}^n \backslash \Gamma$. Then, given $f \in C^0(\Gamma)$ and $x \in \Gamma$, the necessary and sufficient condition for $\hat{F}$ to be an interpolation of f at x is

$$\lim_{\substack{y \to x \\ y \in \Omega}} \int_\Gamma f(\xi) \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1} = f(x). \tag{2.12}$$

There are two variants: one for the interpolation in the complement $\Omega^c = \mathbb{R}^n \backslash \overline{\Omega})$ of Ω and one for the interpolation in all of $\mathbb{R}^n$:

$$\lim_{\substack{y\to x\\ y\in\Omega^c}} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k}\cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1} = f(x), \tag{2.13}$$

$$\lim_{\substack{y\to x\\ y\in\mathbb{R}^n\backslash\Gamma}} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k}\cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1} = f(x). \tag{2.14}$$

For the first case in Ω with $k = n+1$, Dyken and Floater [9] and Bruvoll and Floater [4] proved that it is true when Ω is convex and Hormann and Floater [17] when Ω is a non-necessarily convex n-polytope. But, we don't know if this is true for an arbitrary compact locally Lipschitzian boundary Γ. Dyken and Floater [9] and Bruvoll and Floater [4] working with a parametrization of the boundary Γ by the unit sphere in $\mathbb{R}^n$ invoked the positive reach property of Ω and a local boundedness condition. As we have seen in Theorem 2.1 and later in Theorem 2.7, the positive reach property of Ω can be dropped. Note the analogy of this interpolation problem with the *double-layer potential theory* ($k = n$) where there is a jump across Γ.

2.2.1. *Case $n = 1$.* In dimension $n = 1$ for an interval $\Omega = (a,b)$, the interpolation is well defined and continuous for all $y \in \mathbb{R}$ since

$$g(y) = \frac{-\frac{a-y}{|a-y|^2}\, f(a) + \frac{b-y}{|b-y|^2}\, f(b)}{-\frac{a-y}{|a-y|^2} + \frac{b-y}{|b-y|^2}} = \frac{1}{a-b}\left[(a-y)\, f(a) - (b-y)\, f(b)\right],$$

$g''(y) = 0$, and g is *harmonic.* But the cases $0 \le k$ are allowed since

$$g(y) = \frac{-(y-a)\,|y-b|^k\, f(a) + (y-b)\,|y-a|^k\, f(b)}{-(y-a)\,|y-b|^k + (y-b)\,|y-a|^k}$$

is well defined and continuous for all $y \in \mathbb{R}$. However, the Divergence Theorem cannot be applied since for $0 \le k \le 1$

$$\int_{\Omega^c} \frac{1}{|\xi-y|^k}\, d\xi = \int_{-\infty}^a \frac{1}{|\xi-y|^k}\, d\xi + \int_b^{+\infty} \frac{1}{|\xi-y|^k}\, d\xi = +\infty.$$

For $k = 0$, the boundary expression is linear and hence harmonic. However, for $0 < k \neq 2$, to be harmonic we would need linearity and, in particular, $g(0) = 0$. However, for $a \neq b$, $a \neq 0$, and $b \neq 0$,

$$g(0) = \frac{\frac{a}{|a|^k}\, f(a) - \frac{b}{|b|^k}\, f(b)}{\frac{a}{|a|^k} - \frac{b}{|b|^k}} = 0 \qquad \Rightarrow\ f(b) = \left|\frac{b}{a}\right|^k \frac{a}{b}\, f(a).$$

For functions f that do not satisfy the above identity, the corresponding g is not harmonic.

2.2.2. *Preliminaries.* In the following theorems the key property (and equivalent definition) of an open domain Ω with a locally Lipschizian boundary Γ is that it satisfies the so-called *uniform cone property* in each point of its boundary. Since $\partial\Omega^c = \Gamma$, the domain Ω^c also has a locally Lipschizian boundary and satisfies a *uniform cone property*[2] at each boundary point. For Γ compact this means that there exists $\rho > 0$, $h > 0$, and an angle $0 < \theta < \pi$ such that at each point $x \in \Gamma$ there exists a direction $\vec{d_x}$, $\|\vec{d_x}\| = 1$, such that

$$\forall x \in \Gamma, \forall \xi \in B_\rho(x) \cap \overline{\Omega^c}, \quad \xi + C(\vec{d_x}, h, \theta) \subset \Omega^c, \tag{2.15}$$

where $C(\vec{d_x}, h, \theta)$ is the open cone with vertex 0, axis $\vec{d_x}$, height $h > 0$, and angle $0 < \theta < \pi$.

The next theorem generalizes [9, Thm. 3, sec. 2.4] and [4, Thm. 3, sec. 2.4] given for $k = n + 1$ and a connected bounded open domain Ω with positive reach. That last property turns out to be an unnecessary restriction.

Theorem 2.7. *Let $n \geq 1$ be an integer and $k > n$ a real number. Let Ω be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary and let $\rho > 0$, $h > 0$, and $0 < \theta < \pi$ be the parameters associated with the uniform cone property* (2.15) *at $x \in \Gamma$. For $\overline{\rho} \stackrel{\text{def}}{=} \min\{\rho, h\}$ and $y \in B_{\overline{\rho}}(x) \cap \Omega$*

$$\phi(y) \geq \frac{(k-n)}{n\, 2^k} \frac{c(\theta)}{d_\Gamma(y)^{k-n}} \quad \text{and} \quad \lim_{\substack{y \to x \\ y \in \Omega}} \phi(y) = +\infty, \tag{2.16}$$

where $c(\theta)$ is the n-volume of the conical sector of angle θ and radius 1.

Proof. Since the domain Ω^c has a compact locally Lipschizian boundary, it satisfies a *uniform cone property*: there exists $\rho > 0$, $h > 0$, and $0 < \theta < \pi$, such that the uniform cone property (2.15) is satisfied at any $x \in \Gamma$. Choose $\overline{\rho} \stackrel{\text{def}}{=} \min\{\rho/2, h\}$. For $y \in B_{\overline{\rho}}(x)$ and a projection $p \in \Pi_\Gamma(y)$ of y onto Γ, $d_\Gamma(y) = \|y - p\| \leq \|y - x\| < \overline{\rho} < \min\{\rho/2, h\}$. The ball $B_{d_\Gamma(y)(p)}(x) \subset B_\rho(x)$ since for $z \in B_{d_\Gamma(y)(p)}(x)$

$$|z - x| \leq |z - p| + |p - x| < d_\Gamma(y) + d_\Gamma(y) < \rho.$$

So, the uniform cone property is verified at the point p: $p + C(\vec{d_x}, h, \theta) \subset \Omega^c$. Therefore,

$$\begin{aligned}\int_\Gamma \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1} &= (k-n) \int_{\Omega^c} \frac{1}{\|\xi - y\|^k}\, d\xi \\ &\geq (k-n) \int_{p + C(\vec{d_x}, h, \theta)} \frac{1}{\|\xi - y\|^k}\, d\xi.\end{aligned}$$

Since $\|\xi - y\| \leq \|p - y\| + \|\xi - p\| = d_\Gamma(y) + \|\xi - p\|$,

$$\begin{aligned}\int_{p + C(\vec{d_x}, h, \theta)} \frac{1}{\|\xi - y\|^k}\, d\xi &\geq \int_{p + C(\vec{d_x}, h, \theta)} \frac{1}{|d_\Gamma(y) + \|\xi - p\||^k}\, d\xi \\ &= \int_{C(\vec{d_x}, h, \theta)} \frac{1}{|d_\Gamma(y) + \|\xi\||^k}\, d\xi.\end{aligned}$$

[2]Cf, for instance, [7, Chap. 2, Dfn. 5.1, p. 33, Dfn. 6.1 and Thm. 6.3, pp. 49–58] or [8, Chap.r 2, pp. 114–116].

Recalling that $d_\Gamma(y) \le h$

$$\int_{C(\vec{d}_x,h,\theta)} \frac{1}{|d_\Gamma(y)+\|\xi\||^k}\,d\xi = c(\theta)\int_0^h \frac{r^{n-1}}{(d_\Gamma(y)+r)^k}\,dr \geq c(\theta)\int_0^{d_\Gamma(y)} \frac{r^{n-1}}{(d_\Gamma(y)+r)^k}\,dr$$

$$\Rightarrow\ c(\theta)\int_0^{d_\Gamma(y)} \frac{r^{n-1}}{(d_\Gamma(y)+r)^k}\,dr = \frac{c(\theta)}{d_\Gamma(y)^{k-n}}\int_0^1 \frac{r^{n-1}}{(1+r)^k}\,dr \geq \frac{c(\theta)}{d_\Gamma(y)^{k-n}}\,\frac{1}{n\,2^k}$$

$$\Rightarrow\ (k-n)\int_{\Omega^c} \frac{1}{\|\xi-y\|^k}\,d\xi \geq (k-n)\,\frac{c(\theta)}{d_\Gamma(y)^{k-n}}\,\frac{1}{n\,2^k},$$

where $c(\theta))$ is the n-volume of the conical sector of angle θ and radius 1. ☐

2.2.3. *Continuous Interpolation Theorems.* The next task is to prove that, for f continuous on Γ and $k > n$, $\mathcal{M}_k(f)(y) \to f(x)$ as $y \in \Omega \to x \in \Gamma$. In view of the fact that we have proved Theorems 2.1 and 2.8 without the positive reach property, it is the so-called *mild assumption* of [9, eq. (18)] and [4, eq, (10)] and not the positive reach property that makes the proof of interpolation work. Since the positive reach property used in their Theorem 4 rules out non-convex n-polytopes, is the positive reach property a sufficient condition to get their mild assumption? If not, it might be unnecessarily restrictive and possibly redundant to prove the interpolation.

We relax their *mild assumption* that is really a *global boundedness property on* Γ ([9, first paragraph at the top of page 125]) to a *local one at H^{n-1} almost all points of* Γ. This weaker property is verified for Ω convex, but it is also verified for non-necessarily convex n-polytopes where the H^{n-1} almost everywhere exception is important since the condition is verified everywhere except at the vertices in $\mathbb{R}^2$ and at the vertices and at the edges in $\mathbb{R}^3$.

Theorem 2.8. *Let $n \ge 1$ be an integer and $k > n$ a real number. Let Ω be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary and let $\rho > 0$, $h > 0$, and $0 < \theta < \pi$ be the parameters associated with the uniform cone property* (2.15) *at $x \in \Gamma$.*

(i) *(Local Boundedness Property for Ω) Assume that there exists a constant $C(x)$ and a radius $\delta(x) > 0$ at $x \in \Gamma$ such that*

$$\forall y \in B_{\delta(x)/2}(x)\cap\Omega,\quad \frac{\int_{\Gamma\cap B_{\delta(x)}(x)} \left|\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\right|\,dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\,dH^{n-1}} \le C(x). \tag{2.17}$$

Then

$$\lim_{\substack{y\to x\\ y\in\Omega}} \int_\Gamma f(\xi)\,\frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k}\cdot n_\Omega(\xi')\,dH^{n-1}}\,dH^{n-1} = f(x). \tag{2.18}$$

(ii) *(Local Boundedness Property for Ω^c) Assume that there exists a constant $C(x)$ and a radius $\delta(x) > 0$ at $x \in \Gamma$ such that*

$$\forall y \in B_{\delta(x)/2}(x) \cap \Omega^c, \quad \frac{\int_{\Gamma\cap B_{\delta(x)}(x)} \left|\frac{\xi-y}{\|\xi-y\|^k} \cdot n_{\Omega^c}(\xi)\right| dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_{\Omega^c}(\xi)\, dH^{n-1}} \leq C(x). \tag{2.19}$$

Then

$$\lim_{\substack{y\to x\\ y\in\Omega^c}} \int_\Gamma f(\xi) \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1} = f(x). \tag{2.20}$$

(iii) *(Bilateral Local Boundedness Property) Assume that there exists a constant $C(x)$ and a radius $\delta(x) > 0$ at $x \in \Gamma$ such that*

$$\forall y \in B_{\delta(x)/2}(x)\backslash\Gamma, \quad \frac{\int_{\Gamma\cap B_{\delta(x)}(x)} \left|\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\right| dH^{n-1}}{\left|\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}\right|} \leq C(x). \tag{2.21}$$

Then

$$\lim_{\substack{y\to x\\ y\in\mathbb{R}^n\backslash\Gamma}} \int_\Gamma f(\xi) \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1} = f(x). \tag{2.22}$$

(iv) *If the condition of part* (i), (ii), *or* (iii) *is verified at H^{n-1} almost all $x \in \Gamma$, then the convergence is true at all $x \in \Gamma$.*

Remark 2.9. Open question: is the bilateral local boundedness property verified at all points $x \in \Gamma$ where the normal exists. □

Proof. (i) Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \xi, \xi' \in \Gamma,\ \|\xi - \xi'\| < \delta, \quad |f(\xi) - f(\xi')| < \varepsilon, \quad M \stackrel{\text{def}}{=} \max_{\xi\in\Gamma} |f(\xi)| < \infty.$$

Given $y \in \Omega$ and the ball $B_\delta(x)$, split the integral into two parts

$$\begin{aligned}\mathcal{M}(f)(y) - f(x) =& \frac{\int_{\Gamma\cap B_\delta(x)} \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, [f(\xi) - f(x)]\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}} \\ &+ \frac{\int_{\Gamma\backslash B_\delta(x)} \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, [f(\xi) - f(x)]\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}.\end{aligned}$$

Denote the first term $I_1(y,\delta)$ and the second term $I_2(y,\delta)$. We get $|\mathcal{M}(f)(y) - f(x)| \leq |I_1(\delta,y)| + |I_2(\delta,y)|$ and

$$\begin{aligned}|I_1| &\leq \varepsilon \underbrace{\frac{\int_{\Gamma\cap B_\delta(x)} \left|\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\right| dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}}_{B_1(y,\delta)} \\ |I_2| &\leq 2M \underbrace{\frac{\int_{\Gamma\backslash B_\delta(x)} \left|\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\right| dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1}}}_{B_2(y,\delta)}.\end{aligned} \tag{2.23}$$

To get convergence we need to show that $B_1(y,\delta)$ is bounded and that $B_2(y,\delta) \to 0$, as $y \to x$.

By assumption, it is verified for B_1 by reducing the radius $\delta(x)$ to $\delta'(x) = \min\{\delta, \delta(x)\}$ so that for all $y \in B_{\delta'(x)/2}(x)$

$$|I_1(y,\delta'(x))| \leq \varepsilon \frac{\int_{\Gamma\cap B_{\delta'(x)}(x)} \left|\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\right| dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\, dH^{n-1}} \leq \varepsilon\, C(x).$$

Now consider $I_2(y,\delta'(x))$. For $\xi \in \Gamma\backslash B_{\delta'(x)}(x)$ and $y \in B_{\delta'(x)/2}(x)\cap\Omega$, $\|\xi - y\| \geq \|\xi - x\| - \|y-x\| \geq \delta'(x) - \delta'(x)/2 = \delta'(x)/2$ and

$$\begin{aligned}\int_{\Gamma\backslash B_{\delta'(x)}(x)} \left|\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\right| dH^{n-1} &\leq \int_{\Gamma\backslash B_{\delta'(x)}(x)} \frac{1}{\|\xi-y\|^{k-1}}\, dH^{n-1}\\ &\leq \frac{H^{n-1}(\Gamma)}{(\delta'(x)/2)^{k-1}}.\end{aligned}$$

By (2.2) in Theorem 2.7, for $\overline{\rho} \stackrel{\text{def}}{=} \min\{\rho, h\}$ and $y \in B_{\overline{\rho}}(x)\cap\Omega$

$$\phi(y) \geq \frac{(k-n)}{n\,2^k}\frac{c(\theta)}{d_\Gamma(y)^{k-n}}. \tag{2.24}$$

So, for $\delta'(x) \leq \bar\rho$ and $y \in B_{\delta'(x)/2}(x)$,

$$\frac{\int_{\Gamma\backslash B_{\delta'(x)}(x)} \frac{1}{\|\xi-y\|^{k-1}}\, dH^{n-1}}{\phi(y)} \leq \frac{H^{n-1}(\Gamma)}{c(\theta)}\,\frac{n\,2^{2k-1}}{(k-n)}\,\frac{d_\Gamma(y)^{k-n}}{(\delta'(x))^{k-1}}$$

that goes to zero as $y \to x$. Replacing δ by a $\delta'(x) \leq \bar\rho$ in the first part of the proof for $I_1(\delta'(x),y)$, $\mathcal{M}(f)(y) \to f(x)$ as $y \in \Omega\cap B_{\delta'(x)/2}(x)$ goes to x.

(ii) The arguments for Ω^c are the same as the ones for Ω in (i).

(iii) The arguments for $\mathbb{R}^n\backslash\Gamma$ are the same as the ones for Ω in (i).

(iv) We give the proof for the condition of part (iii). The proof for parts (i) and (ii) is similar. By uniform continuity of f on Γ, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall y, z \in \Gamma,\ \|y - z\| < \delta, \quad |f(y) - f(z)| < \varepsilon/2.$$

By assumption and part (iii), the function g=$\mathcal{M}(f)$ interpolates f on a subset E' of Γ such that $H^{n-1}(\Gamma\backslash E') = 0$. Let $x \in \Gamma\backslash E'$ and $\{y_n\}$ be an arbitrary sequence in $\mathbb{R}^n\backslash\Gamma$ that converges to x such that $\|y_n - x\| < \delta/2$.

Since Γ is locally Lipschitzian, it can be represented as a graph in a neighbourhood $N(x)$ of x. Therefore, there exists a sequence $\{x_i\} \subset E'\cap N(x)$ such that $x_i \to x$ and $\|x_i - x\| < \delta/2$. In view of condition (2.21) and part (ii), there exists $0 < \delta_i \leq \min\{\delta(x_i), \delta\}$ and $y_{n_i} \in B_{\delta_i/2}(x_i)$ such that $|g(y_{n_i}) - f(x_i)| < \varepsilon/2$. Hence

$$\begin{gathered}\|y_{n_i} - x\| \leq ||y_{n_i} - x_i\| + \|x_i - x\| < \delta,\\ |g(y_{n_i}) - f(x)| \leq |g(y_{n_i}) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon.\end{gathered}$$

Therefore, the sequence $\{g(y_{n_i})\}$ is bounded and there exist g_x and a subsequence, still denoted $\{g(y_{n_i})\}$, such that $g(y_{n_i}) \to g_x$. Letting n_i go to infinity, $g(y_i) \to g_x$ and $|g_x - f(x)| < \varepsilon$. Since this is true for all ε, $g_x = f(x)$. In conclusion, we have proved that for all sequences $\{y_n\} \subset \mathbb{R}^n \backslash \Gamma$ converging to $x \in E'$, the limit of $g(y_n)$ is $f(x)$. $\square$

It will be useful to preserve some of the properties used in the proof of Theorem 2.8 in the form of the following lemma.

Lemma 2.10. *Let $n \geq 1$ be an integer and $k > n$ a real number. Let Ω be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary and let $\rho > 0$, $h > 0$, and $0 < \theta < \pi$ be the parameters associated with the uniform cone property* (2.15) *at $x \in \Gamma$. Given $\overline{\rho} \overset{\text{def}}{=} \min\{\rho, h\}$, for all $0 < \delta \leq \overline{\rho}$ and $y \in B_{\delta/2}(x) \cap \Omega$*

$$\frac{\int_{\Gamma \backslash B_\delta(x)} \left| \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi) \right| dH^{n-1}}{\phi(y)} \leq \frac{H^{n-1}(\Gamma)}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)} \frac{d_\Gamma(y)^{k-n}}{\delta^{k-1}} \tag{2.25}$$

$$\frac{\int_{\partial B_\delta(x) \cap \Omega^c} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1}}{\phi(y)} \leq \frac{\beta_n}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)} \left(\frac{d_\Gamma(y)}{\delta} \right)^{k-n} \tag{2.26}$$

and they go to zero as $y \to x$.

Proof. (i) For $\xi \in \Gamma \backslash B_\delta(x)$ and $y \in B_{\delta/2}(x) \cap \Omega$, $\|\xi - y\| \geq \|\xi - x\| - \|y - x\| \geq \delta - \delta/2 = \delta/2$ and

$$\begin{aligned} \int_{\Gamma \backslash B_\delta(x)} \left| \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi) \right| dH^{n-1} &\leq \int_{\Gamma \backslash B_\delta(x)} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1} \\ &\leq \frac{1}{(\delta/2)^{k-1}}\, \beta_n\, \delta^{n-1}. \end{aligned}$$

By (2.16) in Theorem 2.7, for $\overline{\rho} \overset{\text{def}}{=} \min\{\rho, h\}$ and $y \in B_{\overline{\rho}}(x) \cap \Omega$

$$\phi(y) \geq \frac{(k-n)}{n\, 2^k} \frac{c(\theta)}{d_\Gamma(y)^{k-n}}. \tag{2.27}$$

So, for $\delta \leq \bar{\rho}$ and $y \in B_{\delta/2}(x)$,

$$\frac{\int_{\Gamma \backslash B_\delta(x)} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1}}{\phi(y)} \leq \frac{H^{n-1}(\Gamma)}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)} \frac{d_\Gamma(y)^{k-n}}{\delta^{k-1}}.$$

(ii) For $\xi \in \partial B_\delta(x) \cap \Omega^c$ and $y \in B_{\delta/2}(x) \cap \Omega$, $\|\xi - y\| \geq \xi - x\| - \|y - x\| \geq \delta - \delta/2 = \delta/2$ and

$$\int_{\partial B_\delta(x) \cap \Omega^c} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1} \leq \frac{1}{(\delta/2)^{k-1}}\, \beta_n\, \delta^{n-1}.$$

Again by (2.16) in Theorem 2.7, for $\delta \leq \bar{\rho}$ and $y \in B_{\delta/2}(x)$,

$$\frac{\int_{\partial B_\delta(x) \cap \Omega^c} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1}}{\phi(y)} \leq \frac{\beta_n}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)} \left(\frac{d_\Gamma(y)}{\delta} \right)^{k-n}.$$

$\square$

The next two theorems specialize Theorem 2.8.

Theorem 2.11 (Convex Case)**.** *If Ω is a convex open domain with compact boundary and $k > n$,*

$$\forall x \in \Gamma, \quad \lim_{\substack{y\to x\\ y\in\Omega}} \int_\Gamma f(\xi)\, \frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k}\cdot n_\Omega(\xi')\, dH^{n-1}}\, dH^{n-1} = f(x). \tag{2.28}$$

Proof. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \xi, \xi' \in \Gamma,\ \|\xi - \xi'\| < \delta, \quad |f(\xi) - f(\xi')| < \varepsilon, \quad M \overset{\text{def}}{=} \max_{\xi\in\Gamma} |f(\xi)| < \infty.$$

If Ω is an open convex set, for any $y \in \Omega$ and $\xi \in \Gamma$, the vector $y - \xi$ is contained in the tangent space to Ω at the point ξ and $(y - \xi)\cdot\nu \le 0$ for all ν in the dual cone $(T_\xi\Omega)^* = \{\nu \in \mathbb{R}^n : \nu\cdot\tau \ge 0,\ \forall\tau \in T_\xi\Omega\}$. Since the boundary Γ is locally Lipschitzian, the unit exterior normal $n_\Omega(\xi)$ exists at H^{n-1} almost all points ξ and $-n_\Omega(\xi) \in (T_\xi\Omega)^*$ and $(\xi - y)\cdot n_\Omega(\xi) \ge 0$. Therefore,

$$B_1(\delta,y) = \frac{\int_{\Gamma\cap B_\delta(x)} \left|\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\right| dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\, dH^{n-1}} = \frac{\int_{\Gamma\cap B_\delta(x)} \frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\, dH^{n-1}}{\int_\Gamma \frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\, dH^{n-1}} \le 1.$$

As for B_2, we use (2.25) in Lemma 2.10,

$$\frac{\int_{\Gamma\backslash B_\delta(x)} \left|\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\right| dH^{n-1}}{\phi(y)} \le \frac{H^{n-1}(\Gamma)}{c(\theta)}\, \frac{n\, 2^{2k-1}}{(k-n)}\, \frac{d_\Gamma(y)^{k-n}}{\delta^{k-1}}$$

and $|I_2(\delta,y)|$ goes to zero as $y \to x$ for $k > n$. □

Theorem 2.12. *Let $n \ge 1$ be an integer and $k > n$ a real number. Let Ω be an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary and let $\rho > 0$, $h > 0$, and $0 < \theta < \pi$ be the parameters associated with the uniform cone property* (2.15) *at $x \in \Gamma$.*

(i) *Assume that, for H^{n-1} almost each $x \in \Gamma$, there exists $\delta(x) > 0$, such that*

$$\forall \xi \in B_{\delta(x)}(x),\ \forall y \in B_{\delta(x)/2}(x)\cap\Omega, \quad (\xi - y)\cdot n_\Omega(\xi) \ge 0. \tag{2.29}$$

Then, given $\overline{\rho} \overset{\text{def}}{=} \min\{\rho, h\}$ and $\delta'(x) = \min\{\overline{\rho}, \delta(x)\}$, for all $y \in B_{\delta'(x)/2}(x)\cap\Omega$,

$$\frac{\int_{\Gamma\cap B_{\delta'(x)}(x)} \frac{1}{\|\xi-y\|^k}\, |(\xi - y)\cdot n_\Omega(\xi)|\, dH^{n-1}}{\phi(y)} \le 1 + \frac{\beta_n}{c(\theta)}\, \frac{n\, 2^{2k-1}}{(k-n)}.$$

and $\mathcal{M}(f)$ interpolates f in Ω. In particular, condition (2.29) *is verified for Ω convex.*

(ii) *Assume that, for H^{n-1} almost each $x \in \Gamma$, there exists $\delta(x) > 0$, such that*

$$\forall \xi \in B_{\delta(x)}(x),\ \begin{array}{ll} \forall y \in B_{\delta(x)/2}(x)\cap\Omega, & (\xi - y)\cdot n_\Omega(\xi) \ge 0, \\ \forall y \in B_{\delta(x)/2}(x)\cap\Omega^c, & (\xi - y)\cdot n_{\Omega^c}(\xi) \ge 0. \end{array} \tag{2.30}$$

Then, $\mathcal{M}(f)$ interpolates f in $\mathbb{R}^n\backslash\Gamma$. In particular, condition (2.30) *is verified for n-polytopes (non necessarily convex).*

Proof. (i) To simplify the notation let $\delta = \delta(x)$. By assumption, given $y \in B_{\delta/2}(x) \cap \Omega$ and $\xi \in B_\delta(x)$, using the Divergence Theorem

$$\begin{aligned}
&\int_{\Gamma \cap B_\delta(x)} \left| \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi) \right| dH^{n-1} = \int_{\Gamma \cap B_\delta(x)} \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi)\, dH^{n-1} \\
&= - \int_{\Omega^c \cap B_\delta(x)} \operatorname{div} \frac{\xi - y}{\|\xi - y\|^k}\, d\xi + \int_{\partial B_\delta(x) \cap \Omega^c} \frac{\xi - y}{\|\xi - y\|^k} \cdot n_{B_\delta(x)}(\xi)\, dH^{n-1} \\
&= (k-n) \int_{\Omega^c \cap B_\delta(x)} \frac{1}{\|\xi - y\|^k}\, d\xi + \int_{\partial B_\delta(x) \cap \Omega^c} \frac{\xi - y}{\|\xi - y\|^k} \cdot n_{B_\delta(x)}(\xi)\, dH^{n-1} \qquad (2.31) \\
&\leq (k-n) \int_{\Omega^c} \frac{1}{\|\xi - y\|^k}\, d\xi + \int_{\partial B_\delta(x) \cap \Omega^c} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1} \\
&\leq \phi(y) + \int_{\partial B_\delta(x) \cap \Omega^c} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1}.
\end{aligned}$$

From (2.26) in Lemma 2.10, given $\overline{\rho} \stackrel{\text{def}}{=} \min\{\rho, h\}$ and $\delta'(x) = \min\{\overline{\rho}, \delta(x)\}$, for all $y \in B_{\delta'(x)/2}(x) \cap \Omega$, $d_\Gamma(y) < \delta'(x)$ and

$$\begin{aligned}
&\frac{\int_{\partial B_\delta(x) \cap \Omega^c} \frac{1}{\|\xi - y\|^{k-1}}\, dH^{n-1}}{\phi(y)} \leq \frac{\beta_n}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)} \left(\frac{d_\Gamma(y)}{\delta'(x)} \right)^{k-n} \leq \frac{\beta_n}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)} \\
&\Rightarrow \frac{\int_{\Gamma \cap B_{\delta'(x)}(x)} \left| \frac{\xi - y}{\|\xi - y\|^k} \cdot n_\Omega(\xi) \right| dH^{n-1}}{\phi(y)} \leq 1 + \frac{\beta_n}{c(\theta)} \frac{n\, 2^{2k-1}}{(k-n)}
\end{aligned}$$

and this proves the local boundedness assumption (2.17) of Theorem 2.8.

(ii) For an n-polytope, all the pieces of the boundary Γ that are k-polytopes, of dimension $k < n-1$ have zero H^{n-1} measure and can be neglected. So we are left with the *faces* which are (flat) $(n-1)$-polytopes each contained in an affine subspace of dimension $(n-1)$ where the positivity condition (2.30) is satisfied. Hence, from part (i) the local boundedness assumption (2.17) of Theorem 2.8 (i) is verified almost everywhere for Ω. By a similar argument from part (i) using Ω^c in place of Ω, the local boundedness assumption (2.19) of Theorem 2.8 (ii) is verified almost everywhere for Ω^c. We get the final result by combining the two. □

3. k-Transfinite barycentric interpolation (k-TBI)

We proceed as in the case of the k-TMI and replace Γ by a compact set E that can be a smooth d-dimensional, $0 < d < n$, submanifold of $\mathbb{R}^n$ such as a rectifiable curve ($d = 1$) or surface ($d = 2$) in $\mathbb{R}^3$.

3.1. Least Squares Locally L^2 Interpolation.

Theorem 3.1. *Let $0 \leq d < n$ be two integers, $k > 0$ a real number, E a non-empty compact subset of $\mathbb{R}^n$ such that $H^d(E) < +\infty$, and $f \in L^2(E; H^d)$.*

(i) *The function*

$$\hat{F}(y) \stackrel{\text{def}}{=} \int_E f(\xi) \frac{\frac{1}{\|\xi - y\|^k}}{\int_E \frac{1}{\|\xi' - y\|^k}\, dH^d}\, dH^d, \quad y \in \mathbb{R}^n \backslash E, \tag{3.1}$$

is well defined and infinitely continuously differentiable in $\mathbb{R}^n \backslash E$.

(ii) *For each compact $K \subset \mathbb{R}^n \backslash E$, the restriction of $\hat{F}$ to K is the unique solution in $L^2(K)$ of the minimization problems*

$$\inf_{F \in L^2(K)} \int_K \int_E |F(y) - f(\xi)|^2 \frac{1}{\|\xi - y\|^k} \, dH^d \, dy \tag{3.2}$$

$$\inf_{F \in L^2(K)} \int_K \int_E |F(y) - f(\xi)|^2 \, \frac{\frac{1}{\|\xi - y\|^k}}{\int_E \frac{1}{\|\xi' - y\|^k} \, dH^d} \, dH^d \, dy. \tag{3.3}$$

(iii) *If, in addition, $f \in L^\infty(E; H^d)$, then $\hat{F}$ is continuous and bounded on the open set $\mathbb{R}^n \backslash E$.*

Proof. The proof is similar to the one of Theorem 2.1, but the problem is simpler since the weight function is non-negative. If fact, the solution of the minimization problem (3.2) is the same as the solution of the minimization problem (3.3). □

3.2. **Definition and Properties.** In view of Theorem 3.1, we introduce the following definition.

Definition 3.2. Let $0 \leq d < n$ be two integers, $k > 0$ a real number, E a non-empty compact subset of $\mathbb{R}^n$ such that $H^d(E) < +\infty$, and $f \in L^2(E; H^d)$. The *k-Transfinite Barycentric Interpolation* is defined by

$$\mathcal{B}(f)(y) \stackrel{\text{def}}{=} \int_E f(\xi) \, \frac{\frac{1}{\|\xi - y\|^k}}{\int_E \frac{1}{\|\xi' - y\|^k} \, dH^d} \, dH^d, \quad y \in \mathbb{R}^n \backslash E. \tag{3.4}$$

□

Theorem 3.3. *Let $0 \leq d < n$ be two integers, $k > 0$ a real number, and E a non-empty compact subset of $\mathbb{R}^n$ such that $H^d(E) < +\infty$.*

(i) *Given $f \in L^2(E; H^d)$,[3] the function $\mathcal{B}(f)$ is infinitely continuously differentiable on $\mathbb{R}^n \backslash E$ and*

$$\lim_{\|y\| \to +\infty} \mathcal{B}(f)(y) = \frac{\int_E f(\xi) \, dH^d}{\int_E dH^d}. \tag{3.5}$$

(ii) *Given $f \in L^\infty(E; H^d)$, $\mathcal{B}(f)$ is continuous and bounded on $\mathbb{R}^n \backslash E$,*

$$\sup_{y \in \mathbb{R}^n \backslash E} \|\mathcal{B}(f)(y)\| \leq M \stackrel{\text{def}}{=} \|f\|_{L^\infty(E[H^d)}. \tag{3.6}$$

(iii) *For each compact $K \subset \mathbb{R}^n \backslash E$,*

$$\begin{aligned} &\int_K \int_E |\hat{F}(y) - f(\xi)|^2 \frac{1}{\|\xi - y\|^k} \, dH^{n-1} \, dy \\ &= \int_K \left[\mathcal{B}(f^2) - \mathcal{B}(f)^2\right] \phi \, dy, \quad \phi(y) \stackrel{\text{def}}{=} \int_\Gamma \frac{1}{\|\xi - y\|^k} \, dH^{n-1}. \end{aligned} \tag{3.7}$$

Remark 3.4. If we had the uniform continuity of $\mathcal{B}(f)$ in $\mathbb{R}^n \backslash E$, its continuous trace on E would be well-defined. □

[3]In the case $d = 0$, E is a finite number of isolated points and any function defined on E is continuous and, a fortiori, $f \in L^2(E; H^0)$.

Proof. (i) Since E is compact, its diameter is finite and for any points $x \in \mathbb{R}^n$ and $\xi \in E$

$$d_E(x) \le \|\xi - x\| \le d_E(x) + \operatorname{diam}(E), \quad \operatorname{diam}(E) \overset{\text{def}}{=} \max_{\xi,\xi' \in E} \|\xi' - \xi\| < +\infty.$$

Then, for $y \notin E$

$$\frac{\int_E \frac{1}{|d_E(y)+\operatorname{diam}(E)|^k} f(\xi)\, dH^d}{\int_E \frac{1}{d_E(y)^k}\, dH^d} \le \frac{\int_E \frac{1}{\|y-\xi\|^k} f(\xi)\, dH^d}{\int_E \frac{1}{\|y-\xi\|^k}\, dH^d} \le \frac{\int_E \frac{1}{d_E(y)^k} f(\xi)\, dH^d}{\int_E \frac{1}{d_E(y)+\operatorname{diam}(E)^k}\, dH^d}.$$

From the above inequalities,

$$\begin{aligned} &\left(\frac{d_E(y)}{d_E(y)+\operatorname{diam}(E)}\right)^k \frac{\int_E f(\xi)\, dH^d}{\int_E dH^d} \\ &\le \mathcal{B}_\Omega(f)(y) \le \left(\frac{d_E(y)+\operatorname{diam}(E)}{d_E(y)}\right)^k \frac{\int_E f(\xi)\, dH^d}{\int_E dH^d}. \end{aligned}$$

As $|y| \to +\infty$, $d_E(y) \to \infty$ and, since $k > 0$ is fixed,

$$\left(\frac{d_E(y)}{d_E(y)+\operatorname{diam}(E)}\right)^k \to 1 \quad \text{and} \quad \left(\frac{d_E(y)+\operatorname{diam}(E)}{d_E(y)}\right)^k \to 1.$$

(ii) For $y \in \mathbb{R}^n \backslash E$

$$|\mathcal{B}(f)(y)| \le \frac{\int_E \frac{1}{\|y-\xi\|^k} |f(\xi)|\, dH^d}{\int_E \frac{1}{\|y-\xi\|^k}\, dH^d} \le \|f\|_{L^\infty(E)}$$

and the continuous function $\mathcal{B}(f)$ is bounded on the open set $\mathbb{R}^n \backslash E$.

(iii) Similar to the proof of Theorem 2.1. □

3.3. **Trace and Pointwise Convergence on E.**

3.3.1. *Case $d = 0$: Unstructured Cloud of Points E.* For $d = 0$, the Hausdorff measure H^0 is the counting measure and for a finite number of isolated points $E = \{x_1, \dots, x_M\}$ in $\mathbb{R}^n$ the formula (3.4) becomes

$$\mathcal{B}(f)(y) = \sum_{i=1}^{M} \frac{\frac{1}{\|y-x_i\|^k}}{\sum_{\ell=1}^{M} \frac{1}{\|y-x_\ell\|^k}} f(x_i), \quad y \in \mathbb{R}^n \backslash E, \tag{3.8}$$

and the interpolation property is immediate.

Theorem 3.5. *Let $k > d = 0$ be a real number and $E = \{x_1, \dots, x_M\}$ be a set of $M \ge 1$ isolated points*[4] *of $\mathbb{R}^n$. The function $\mathcal{B}(f)$ on $\mathbb{R}^n \backslash E$ continuously extends f from E to $\mathbb{R}^n$:*

$$\forall x \in E, \quad \lim_{\substack{y \to x \\ y \in \mathbb{R}^n \backslash E}} \mathcal{B}(f)(y) = f(x) \quad \text{or} \quad \forall x_i \in E, \quad \lim_{\substack{y \to x_i \\ y \in \mathbb{R}^n \backslash E}} \mathcal{B}(f)(y) = f(x_i).$$

[4]Hence, E is a compact subset of $\mathbb{R}^n$ with finite measure $H^0(E) < \infty$.

Proof. For $x_j \in E$ and $y \in \mathbb{R}^n \backslash E$,

$$\begin{aligned}\mathcal{B}(f)(y) - f(x_j) &= \frac{\sum_{i=1}^{M} \frac{1}{\|y-x_i\|^k} [f(x_i) - f(x_j)]}{\sum_{\ell=1}^{M} \frac{1}{\|y-x_\ell\|^k}} \\ &= \frac{\sum_{\substack{1\le i\le M\\ i\neq j}} \frac{1}{\|y-x_i\|^k} [f(x_i) - f(x_j)]}{\frac{1}{\|y-x_j\|^k} + \sum_{\substack{1\le \ell\le M\\ \ell\neq j}} \frac{1}{\|y-x_\ell\|^k}}\end{aligned}$$

and, for $k > 0$, this expression goes to zero since $1/\|y - x_j\|$ goes to infinity as $y \to x_j$. □

Remark 3.6. In 1968 Shepard [24] defines for the first time this interpolation method for unstructured data in dimension two. This method has become an essential tool for data analysis in meteorology, biology, imagery, and geoscience. It easily extends to higher dimensions of space and is in fact a generalization of the Lagrange approximation to multidimensional spaces. In recent years, Shepards method has become a very competitive method for updating meshes in fluid-structure interactions. We shall see in the following section how the generalization of this interpolation for $d \geq 1$ can handle structured meshes. □

3.3.2. *Cases $1 \le d < n$: Structured E.* In this section we use definitions and theorems from the Appendix.

Lemma 3.7. *Let d, $1 \le d < n$, be two integers and $k > d$ be a real number. Assume that E is a non-empty compact H^d-rectifiable subset of $\mathbb{R}^n$ with finite measure $H^d(E)$. Then*

$$\text{for } H^d \text{ a.a. } x \in E, \quad \lim_{\substack{y\to x\\ y\notin E}} \int_E \frac{1}{\|y-\xi\|^k}\, dH^d = +\infty.$$

Proof. Given some $\varepsilon > 0$ and $y \in B_\varepsilon(x)$,

$$\forall \xi \in B_\varepsilon(x) \cap E, \quad \|y - \xi\| \le \|y - x\| + \|\xi - x\| < 2\varepsilon$$

and, using the characteristic function χ_E of E,

$$\begin{aligned}&\int_E \frac{1}{\|y-\xi\|^k}\, dH^d \ge \int_{E\cap B_\varepsilon(x)} \frac{1}{\|y-\xi\|^k}\, dH^d \\ &\ge \frac{1}{(2\varepsilon)^k} \int_{E\cap B_\varepsilon(x)} dH^d \ge \frac{\alpha_d}{2^k} \frac{1}{\varepsilon^{k-d}} \frac{1}{\alpha_d \varepsilon^d} \int_{E\cap B_\varepsilon(x)} dH^d \\ &\ge \frac{\alpha_d}{2^d} \frac{1}{(2\varepsilon)^{k-d}} \underbrace{\frac{1}{\alpha_d \varepsilon^d} \int_{B_\varepsilon(x)} \chi_E(\xi)\, dH^d}_{\to \chi_E(x)=1\, H^d \text{ a.e.}}\end{aligned}$$

by Theorem A.7 in the Appendix. Therefore, for $k > d$ and for H^d almost all $x \in E$,

$$\lim_{y\to x} \int_E \frac{1}{\|y-\xi\|^k}\, dH^d \ge \lim_{\varepsilon\searrow 0} \frac{\alpha_d}{2^d} \frac{1}{(2\varepsilon)^{k-d}} = +\infty.$$

□

We need the notion of *Lipschitz d-graph* that is defined in Example A.1.

Theorem 3.8. *Let d, $1 \le d < n$, be an integer and $k > d$ a real number. Assume that E is a non-empty compact subset of $\mathbb{R}^n$ which is H^d-rectifiable with finite measure $H^d(E)$.*

(i) *The function $\mathcal{B}(f)$ on $\mathbb{R}^n \backslash E$ interpolates f at H^d almost all points $x \in E$, that is,*

$$\text{for } H^d \text{ a.a. } x \in E, \qquad \lim_{\substack{y \to x \\ y \in \mathbb{R}^n \backslash E}} \mathcal{B}(f)(y) = f(x).$$

(ii) *If, in addition, E is locally a Lipschitz d-graph, then the function $\mathcal{B}(f)$ on $\mathbb{R}^n \backslash E$ interpolates f at all points $x \in E$.*

Remark 3.9. Note that, under the assumption that E is locally a Lipschitz d-graph, E can have several connected components. In the case of an open subset Ω of $\mathbb{R}^n$ with compact locally Lipschitzian boundary, Γ is H^{n-1}-rectifiable (Proposition A.5), Also, from Theorem A.4, if E is a compact subset of $\mathbb{R}^n$ with positive reach, then E is $(n-1)$-rectifiable. $\square$

Proof. (i) Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall \xi, \xi', z \in E,\ \|\xi - \xi'\| < \delta, \quad |f(\xi) - f(\xi')| < \varepsilon, \quad M \stackrel{\text{def}}{=} \max_{\xi \in E} |f(\xi)| < \infty.$$

Using δ split the integral into two parts: for $x \in E$, and $y \in \mathbb{R}^n \backslash E$,

$$\begin{aligned} \mathcal{B}(f)(y) - f(x) =& \frac{\int_{E \cap B_\delta(x)} \frac{1}{\|y-\xi\|^k} [f(\xi) - f(x)]\, dH^d}{\int_E \frac{1}{\|y-\xi\|^k}\, dH^d} \\ &+ \frac{\int_{E \backslash B_\delta(x)} \frac{1}{\|y-\xi\|^k} [f(\xi) - f(x)]\, dH^d}{\int_E \frac{1}{\|y-\xi\|^k}\, dH^d} \end{aligned}$$

and denote by I_1 the first and by I_2 the second. For the first integral

$$\|\xi - x\| < \delta \quad \Rightarrow |I_1| \le \sup_{E \cap B_\delta(x)} |f(\xi) - f(x)| < \varepsilon.$$

For the second integral, if $\|y - x\| < \delta/2$

$$\forall \xi \in E \backslash B_\delta(x), \quad \|\xi - y\| \ge \|\xi - x\| - \|y - x\| \ge \delta - \|y - x\| > \delta/2 > 0$$

and

$$|I_2| = \frac{\int_{E \backslash B_\delta(x)} \frac{1}{\|y-\xi\|^k} [f(\xi) - f(x)]\, dH^d}{\int_E \frac{1}{\|y-\xi\|^k}\, dH^d} \le \frac{2M\, H^d(E)}{(\delta/2)^k} \frac{1}{\int_E \frac{1}{\|y-\xi\|^k}\, dH^d}.$$

By Lemma 3.7

$$\lim_{\substack{y \to x \\ y \notin E}} \int_E \frac{1}{\|y - \xi\|^k}\, dH^d = +\infty.$$

So, given $\varepsilon > 0$, there exists $0 < \delta_1 < \delta/2$ such that

$$\begin{gathered} \forall\, y \notin E,\ \|y - x\| < \delta_1, \quad \left| \int_E \frac{1}{\|y - \xi\|^k}\, dH^d \right| > \frac{2M\, H^d(E)}{(\delta/2)^k \varepsilon} \\ \Rightarrow |I_2| < \varepsilon \quad \Rightarrow |I_1| + |I_2| < 2\varepsilon. \end{gathered}$$

Finally, given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$\forall y \in B_{\delta_1}(x)\backslash E, \quad |\mathcal{B}(f)(y) - f(x)| < 2\varepsilon$$

and $\mathcal{B}(f)$ interpolates f.

(ii) From part (i), the function g=$\mathcal{B}(f)$ interpolates f on a subset E' of E such that $H^d(E\backslash E') = 0$. Let $x \in E\backslash E'$ and $\{y_n\}$ be an arbitrary sequence in $\mathbb{R}^n\backslash E$ that converges to x. Since g is bounded on $\mathbb{R}^n\backslash E$, there exist g_x and a subsequence of $\{y_n\}$, still denoted $\{y_n\}$ such that $g(y_n) \to g_x$. Since E is locally a Lipschitz d-graph, it can be represented as a graph in a neighbourhood $N(x)$ of x. Therefore, there exists a sequence $\{x_i\} \subset E' \cap N(x)$ such that $x_i \to x$. By uniform continuity of f on E, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall y, z \in E, \, \|y - z\| < \delta, \quad |f(y) - f(z)| < \varepsilon/2.$$

For each i, there exists δ_i, $0 < \delta_i < \delta$, such that

$$\forall y \in \mathbb{R}^n\backslash E, \, \|y - x_i\| < \delta_i, \quad |g(y) - f(x_i)| < \varepsilon/2.$$

Since we have convergence of $g(y)$ to $f(x_i)$ at each x_i, there exists $x_i \in E'$ such that $\|x_i - x\| < \delta_i/2$ and there exists $y_{n_i} \in \mathbb{R}^n\backslash E$ such that $\|y_{n_i} - x\| < \delta_i/2$

$$\begin{aligned} &\Rightarrow \|y_{n_i} - x_i\| \le \|y_{n_i} - x\| + \|x_i - x\| < \delta_i \quad \text{and} \quad |g(y_{n_i}) - f(x_i)| < \varepsilon/2 \\ &\Rightarrow |g(y_{n_i}) - f(x)| \le |g(y_{n_i}) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon, \end{aligned}$$

since $\|x_i - x\| < \delta_i/2 < \delta$. Letting n_i go to infinity, $g(y_{n_i}) \to g_x$ and $|g_x - f(x)| < \varepsilon$. Since this is true for all ε, $g_x = f(x)$. In conclusion, we have proved that for all sequences $\{y_n\} \subset \mathbb{R}^n\backslash E$ converging to $x \in E'$, the limit of $g(y_n)$ is $f(x)$. □

4. Enhanced interpolations from the function f and its derivatives

In the previous sections the k-TMI (resp. k-TBI) was constructed using only the function f on Γ (resp. E). The $(n+1)$-TMI is remarkable in the sense that it preserves $P^1(\mathbb{R}^n)$. The issue of constructing interpolations that preserve higher order polynomials within the TMI framework was raised in 2008 by Floater and Schulz [12] who proposed a construction using an Hermite interpolation where Γ is parametrized by the unit sphere in $\mathbb{R}^n$. Their construction requires the knowledge of f and its higher derivatives and involves $\hat{F}$ and its corresponding derivatives of the same degree.

In this section we show that, when derivatives of f are available, relatively simple constructions based on Taylor's formula extend the constructions of the previous sections to new interpolations $\hat{F}$ that preserve higher order polynomials. To do that we need some notation and Taylor's formula for an m-times continuously differentiable function f. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a *multi-index* and for $x \in \mathbb{R}^n$, $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $\alpha! = \alpha_1!\,\alpha_2! \ldots \alpha_n!$. The Taylor's formula of degree $m \ge 1$ is

$$f(x) \cong f(\xi) + \sum_{\ell=1}^{m} \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(\xi)\, (x - \xi)^\alpha \tag{4.1}$$

$$\partial^\alpha f \stackrel{\text{def}}{=} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|} f}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2} \ldots \partial \xi_n^{\alpha_n}}. \tag{4.2}$$

4.1. **(k,m)-TBI Interpolation.** Given a real $k > 0$, a H^d-rectifiable compact $E \subset \mathbb{R}^n$, and an m-times continuously differentiable function f, consider the minimization of the following quadratic functionals

$$\int_K \int_E \left| F(y) - \left[f(\xi) + \sum_{\ell=1}^{m} \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha \right] \right|^2 \frac{1}{\|\xi - y\|^k}\, dH^d\, dy \tag{4.3}$$

$$\int_K \int_E \left| F(y) - \left[f(\xi) + \sum_{\ell=1}^{m} \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha \right] \right|^2 \frac{\frac{1}{\|\xi-y\|^k}}{\int_E \frac{1}{\|\xi'-y\|^k}\, dH^d}\, dH^d\, dy \tag{4.4}$$

with respect to $F \in L^2(K)$ for a compact subset $K \subset \mathbb{R}^n \backslash E$. The unique solution of both problems is the restriction to K of the function

$$\boxed{\hat{F}(y) = \int_E \left[f(\xi) + \sum_{\ell=1}^{m} \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha \right] \frac{\frac{1}{\|\xi-y\|^k}}{\int_E \frac{1}{\|\xi'-y\|^k}\, dH^d}\, dH^d,} \tag{4.5}$$

which is well-defined and infinitely differentiable on $\mathbb{R}^n \backslash E$.

We introduce the notation $\mathcal{B}_{k,m}(f)$ for the function $\hat{F}$ and call it the *m-th order k-TBI interpolation* or simply (k,m)-TBI. So, the *0-th order k-TBI interpolation* is the interpolation of Definition 3.2 in section 3.

Theorem 4.1. *Let d, $0 \le d < n$, and $m \ge 0$ be integers and $k > d$ a real number. Assume that E is a non-empty compact subset of $\mathbb{R}^n$ which is H^d-rectifiable with finite measure $H^d(E)$.*

(i) *For all $f \in P^m(\mathbb{R}^n)$, $\mathcal{B}_{k,m}(f) = f$.*

(ii) *The function $\mathcal{B}_{k,m}(f)$ on $\mathbb{R}^n \backslash E$ interpolates f at H^d almost all points $x \in E$, that is,*

$$\text{for } H^d \text{ a.a. } x \in E, \qquad \lim_{\substack{y \to x \\ y \in \mathbb{R}^n \backslash E}} \mathcal{B}_{k,m}(f)(y) = f(x).$$

(iii) *If, in addition, E is locally a Lipschitz d-graph, then the function $\mathcal{B}_{k,m}(f)$ on $\mathbb{R}^n \backslash E$ interpolates f at all points $x \in E$.*

Remark 4.2. For $m = 1$, $\mathcal{B}_{k,1}$ preserves $P^1(\mathbb{R}^n)$ as the $(n+1)$-TMI does, but it requires the knowledge of f and its first order derivatives. □

Proof. (i) Let $f \in P^m(\mathbb{R}^n)$ and observe that at $y \in \mathbb{R}^n \backslash E$ the difference

$$\begin{aligned} &\hat{F}(y) - f(y) \\ &= \int_E \left[f(\xi) + \sum_{\ell=1}^{m} \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha - f(y) \right] \frac{\frac{1}{\|\xi-y\|^k}\, dH^d}{\int_E \frac{1}{\|\xi'-y\|^k}\, dH^d} = 0 \end{aligned}$$

is zero since the term in the square bracket is zero for $f \in P^m(\mathbb{R}^n)$, $m \ge 0$.

(ii) It is sufficient to consider the case $m \ge 1$. Let $x \in E$ and $y \to x$. The right-hand side of (4.5) is the sum of a first term involving only $f(\xi)$ and a term containing all its derivatives. By applying Theorem 3.5 and 3.8 (i), the first term converges to $f(x)$ for $d=0$ (resp. H^d a.e. for $0<d<n$). As for the second term

$$\int_E \left[\sum_{\ell=1}^{m}\sum_{\alpha,\,|\alpha|=\ell}\frac{1}{\alpha\,!}\,\partial^\alpha f(\xi)\,(y-\xi)^\alpha\right]\frac{\frac{1}{\|\xi-y\|^k}}{\int_E \frac{1}{\|\xi'-y\|^k}\,dH^d}\,dH^d, \tag{4.6}$$

it converges for $d=0$ (resp. H^d a.e. for $0<d<n$) to

$$\sum_{\ell=1}^{m}\sum_{\alpha,\,|\alpha|=\ell}\frac{1}{\alpha\,!}\,\partial^\alpha f(x)\,(\underbrace{x-x}_{=0})^\alpha=0. \tag{4.7}$$

□

4.2. **(k,m)-TMI Interpolation.** Let $k>n$ be a real number, Ω an open set with compact locally Lipschitzian boundary Γ, and f an m-times continuously differentiable function, $m\geq 0$. Given $K\subset\mathbb{R}^n\backslash\Gamma$ compact, consider the minimization of the following quadratic functionals

$$\int_K\int_\Gamma\left|F(y)-\left[f(\xi)+\sum_{\ell=1}^{m}\sum_{\alpha,\,|\alpha|=\ell}\frac{1}{\alpha\,!}\,\partial^\alpha f(\xi)\,(y-\xi)^\alpha\right]\right|^2\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)\,d\Gamma\,dy \tag{4.8}$$

$$\begin{aligned}&\int_K\int_\Gamma\left|F(y)-\left[f(\xi)+\sum_{\ell=1}^{m}\sum_{\alpha,\,|\alpha|=\ell}\frac{1}{\alpha\,!}\,\partial^\alpha f(\xi)\,(y-\xi)^\alpha\right]\right|^2\\ &\qquad\qquad\frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma\frac{\xi'-y}{\|\xi'-y\|^k}\cdot n_\Omega(\xi')\,d\Gamma}\,d\Gamma\,dy\end{aligned} \tag{4.9}$$

with respect to $F\in L^2(K)$. They both have the same unique solution that is the restriction to K of the infinitely differentiable function in $\mathbb{R}^n\backslash\Gamma$

$$\boxed{\hat{F}(y)\overset{\text{def}}{=}\int_\Gamma\left[f(\xi)+\sum_{\ell=1}^{m}\sum_{\alpha,\,|\alpha|=\ell}\frac{1}{\alpha\,!}\,\partial^\alpha f(\xi)\,(y-\xi)^\alpha\right]\frac{\frac{\xi-y}{\|\xi-y\|^k}\cdot n_\Omega(\xi)}{\int_\Gamma\frac{\xi'-y}{\|\xi'-y\|^k}\cdot n_\Omega(\xi')\,d\Gamma}\,d\Gamma} \tag{4.10}$$

We denote it $\mathcal{M}_{k,m}(f)$ and call it the *m-th order k-TMI interpolation* or simply (k,m)-TMI. So, the *0-th order k-TMI interpolation* is the interpolation of Definition 2.2 in section 2.

Theorem 4.3. *Assume that Ω is an open subset of $\mathbb{R}^n$ with compact locally Lipschitzian boundary.*

(i) *For an integer $m\geq 0$, the $\mathcal{M}_{n+m+1,m}$ interpolation in $\mathbb{R}^n$ preserves polynomials in $P^{m+1}(\mathbb{R}^n)$.*
(ii) *Under any one of the assumptions of Theorem 2.8, $k>n$, and $m\geq 0$, $\mathcal{M}_{k,m}$ interpolates m-times continuously differentiable functions.*

Remark 4.4. Part (i) with $m=0$ generalizes Theorem 2.5 in section 2. □

Proof. (i) For $f \in P^{m+1}(\mathbb{R}^n)$, consider the difference $\hat{F}(y) - f(y)$

$$\int_\Gamma \left[f(\xi) + \sum_{\ell=1}^m \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha - f(y) \right] \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, d\Gamma}\, d\Gamma$$

at $y \in \mathbb{R}^n \backslash \Gamma$. But, since Taylor's formula is exact for $f \in P^{m+1}(\mathbb{R}^n)$,

$$f(\xi) + \sum_{\ell=1}^m \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha - f(y) = -\sum_{\alpha,\, |\alpha|=m+1} \frac{1}{\alpha!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha,$$

where the partial derivatives $\partial^\alpha f(\xi)$ of order $|\alpha| = m+1$ are all independent of ξ. Using the notation $\tau_\alpha = \partial^\alpha f(\xi)/\alpha!$ we have

$$\hat{F}(y) - f(y) = -\int_\Gamma \left[\sum_{\alpha,\, |\alpha|=m+1} \tau_\alpha\, (y-\xi)^\alpha \right] \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_\Gamma \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, d\Gamma}\, d\Gamma.$$

where the expression between square brackets is an $(m+1)$-linear form. Using the divergence theorem, rewrite the boundary integral of the numerator as an integral over the complement Ω^c of Ω as we did in the proof of Theorem 2.5 and notice that

$$\begin{aligned} & -\int_\Gamma \left[\sum_{\alpha,\, |\alpha|=m+1} \tau_\alpha\, (y-\xi)^\alpha \right] \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, d\Gamma \\ & = \int_{\Omega^c} \operatorname{div}_\xi \left[\sum_{\alpha,\, |\alpha|=m+1} \tau_\alpha\, (y-\xi)^\alpha\, \frac{\xi-y}{\|\xi-y\|^k} \right] d\xi \\ & = (m+1+n-k) \int_{\Omega^c} \left[\sum_{\alpha,\, |\alpha|=m+1} \tau_\alpha\, (y-\xi)^\alpha\, \frac{1}{\|\xi-y\|^k} \right] d\xi. \end{aligned} \tag{4.11}$$

We have proved it for $m=0$. For $m=1$, $f(\xi)$ is of the form $f(\xi) = a + b \cdot \xi + (Q\xi) \cdot \xi$ for $a \in \mathbb{R}$, $b \in \mathbb{R}^n$, and Q a symmetric $n \times n$ matrix or 2-tensor. It is easy to check that

$$\begin{aligned} & \int_{\Omega^c} \operatorname{div}_y \left[\frac{1}{2} Q(\xi-y) \cdot (\xi-y)\, \frac{\xi-y}{\|\xi-y\|^k} \right] d\xi \\ & = (2+n-k) \int_{\Omega^c} \frac{1}{2} Q(\xi-y) \cdot (\xi-y)\, \frac{1}{\|\xi-y\|^k}\, d\xi. \end{aligned}$$

For $m > 1$, Q is replaced by a symmetric $(m+1)$-tensor τ constructed from the τ_α's. The last term in (4.11) is zero if k is chosen equal to $m+1+n$ in which case the interpolation preserves $P^{m+1}(\mathbb{R}^n)$.

(ii) For $x \in \Gamma$, let $y \to x$. Isolate the first term in $f(\xi)$ from the sum of terms involving derivatives

$$\begin{aligned} \hat{F}(y) = & \frac{\int_E f(\xi) \frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)\, d\Gamma}{\int_E \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, d\Gamma} \\ & + \int_E \left[\sum_{\ell=1}^m \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(\xi)\, (y-\xi)^\alpha \right] \frac{\frac{\xi-y}{\|\xi-y\|^k} \cdot n_\Omega(\xi)}{\int_E \frac{\xi'-y}{\|\xi'-y\|^k} \cdot n_\Omega(\xi')\, d\Gamma}\, d\Gamma. \end{aligned}$$

By Theorem 2.8, the first term converges to $f(x)$ for $k > n$. As for the second term containing the derivatives, it converges to

$$\sum_{\ell=1}^{m} \sum_{\alpha,\, |\alpha|=\ell} \frac{1}{\alpha\,!}\, \partial^\alpha f(x)\,(x-x)^\alpha = 0.$$

□

5. Dynamical k-transfinite interpolations

5.1. **Dynamics of the Parametrized Varying Body.** Whether it is the boundary Γ of an open subset Ω or a closed subset E of $\mathbb{R}^n$, the following set-up will be used to generate a *dynamical interpolation* where Ω or E are evolving according to some known dynamics. Let μ, $0 \le \mu \le 1$, be a parameter that can be viewed as an *artificial time* associated with the intermediary open sets Ω_μ (resp. closed sets E_μ) evolving from an initial state $\Omega_0 = \Omega$ (resp. $E_0 = E$) at time 0 to a final state Ω_1 (resp. E_1) at time 1. Assume that those sets are characterized by the solutions of the ordinary differential equation

$$\frac{dx}{d\mu}(\mu) = V(\mu, x(\mu)), \quad x(0) = x_0, \tag{5.1}$$

for some *velocity field* $(\mu, \xi) \mapsto V(\mu, \xi) : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that (5.1) has a unique solution $x(\mu) = x(\mu; y_0)$ for all $x_0 \in \Omega$ (resp. $x_0 \in E$). Assume, for simplicity, that the solution is unique for all $x_0 \in \mathbb{R}^n$ so that the solutions generate a family of transformations of $\mathbb{R}^n$

$$x_0 \mapsto T_\mu(x_0) \overset{\text{def}}{=} x(\mu; x_0) : \mathbb{R}^n \to \mathbb{R}^n$$

that are bijective and bi-continuous. Further assume that the μ partial derivative exists and that the function

$$x \mapsto \frac{\partial T_\mu}{\partial \mu}(x) : \mathbb{R}^n \to \mathbb{R}^n \tag{5.2}$$

is continuous. As a result

$$V(\mu,x) = \left(\frac{\partial T_\mu}{\partial \mu} \circ T_\mu^{-1}\right)(x) \tag{5.3}$$

is the the *velocity of a point* $x \in \Omega_\mu$ (resp. $x \in E_\mu$) at time μ and, in view of the previous assumptions, the function $x \mapsto V(\mu,x) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Specific conditions[5] can be used to make T_μ a bi-Lipschitzian continuous transformation.

Define the intermediary sets

$$\Omega_\mu \overset{\text{def}}{=} T_\mu(\Omega) \quad \Big(\text{resp. } E_\mu \overset{\text{def}}{=} T_\mu(E)\Big), \quad 0 \le \mu \le 1, \tag{5.4}$$

and recall that for a bi-continuous bijection T_μ and an open subset Ω

$$\Gamma_\mu = T_\mu(\Gamma) \quad \text{and} \quad \mathbb{R}^{\backslash}\overline{\Omega_\mu} = T_\mu(\mathbb{R}^n \backslash \overline{\Omega}). \tag{5.5}$$

The additional bi-Lipschitzian continuity is used to transport the set Γ of zero measure onto sets Γ_μ of zero measure.

It is important to understand that the transformations $\{T_\mu\}$ and the velocity V are not unique since it is always possible to use other transformations $\{T'_\mu\}$ and a

[5]See, for instance, [8, Chapter 4, sec. 4, pp. 180–193].

velocity V' such that $T_\mu(\Gamma) = T'_\mu(\Gamma)$. This will be illustrated later in the example of section 5.4.

5.2. **k-Transfinite Mean Value Interpolation.** Assume that, under appropriate conditions on the velocity field V, the image $\Omega_\mu = T_\mu(\Omega)$ of a locally Lipschitzian domain Ω that satisfies the conditions of Theorem 2.12 is also locally Lipschitzian and satisfies the conditions of Theorem 2.12. Recall that the k-TMI on Γ_μ of a function $f_\mu : \Gamma_\mu \to \mathbb{R}$ is defined as

$$\mathcal{M}(f_\mu)(y) \stackrel{\text{def}}{=} \frac{\displaystyle\int_{\Gamma_\mu} \frac{(\xi - y)\cdot n_{\Omega_\mu}(\xi)}{\|\xi - y\|^k}\, f_\mu(\xi)\, dH^{n-1}}{\displaystyle\int_{\Gamma_\mu} \frac{(\xi - y)\cdot n_{\Omega_\mu}(\xi)}{\|\xi - y\|^k}\, dH^{n-1}}. \tag{5.6}$$

Consider at each μ the interpolation of the velocity field $V(\mu, \cdot)$:

$$\mathcal{V}(\mu,y) \stackrel{\text{def}}{=} \frac{\displaystyle\int_{\Gamma_\mu} \frac{(\xi - y)\cdot n_{\Omega_\mu}(\xi)}{\|\xi - y\|^k}\, V(\mu,\xi)\, dH^{n-1}}{\displaystyle\int_{\Gamma_\mu} \frac{(\xi - y)\cdot n_{\Omega_\mu}(\xi)}{\|\xi - y\|^k}\, dH^{n-1}}, \quad y \in \mathbb{R}^n\backslash\Gamma_\mu. \tag{5.7}$$

Since $\xi \mapsto V(\mu,\xi)$ is continuous, then $y \mapsto \mathcal{V}(\mu,y)$ is continuous and interpolates $\xi \mapsto V(\mu,\xi)$ from Γ_μ to $\mathbb{R}^n$.

Assume that the differential equation

$$\frac{dy}{d\mu}(\mu) = \mathcal{V}(\mu,y(\mu)), \quad y(\mu) = y_0 \in \Omega_0, \tag{5.8}$$

has a solution such that $y(\mu; y_0) \in \Omega_\mu$ at each μ. Then, by using the expression of $\mathcal{V}$, the function $y(\mu)$ is a solution of the following *differential interpolation equation*

$$\frac{dy}{d\mu}(\mu) = \frac{\displaystyle\int_{\Gamma_\mu} \frac{(\xi - y(\mu))\cdot n_{\Omega_\mu}(\xi)}{\|\xi - y(\mu)\|^k}\, V(\mu,\xi)\, dH^{n-1}}{\displaystyle\int_{\Gamma_\mu} \frac{(\xi - y(\mu))\cdot n_{\Omega_\mu}(\xi)}{\|\xi - y(\mu)\|^k}\, dH^{n-1}}, \quad y(0) = y_0, \tag{5.9}$$

for each $y_0 \in \mathbb{R}^n\backslash\Gamma_0$. Therefore, for $x_0 \in \Gamma_0$ and $y_0 \in \mathbb{R}^n\backslash\Gamma_0$ such that $y_0 \to x_0$,

$$y(\mu; y_0) \to x(\mu; x_0) \quad \text{and} \quad \frac{dy}{d\mu}(\mu; y_0) \to \frac{dx}{d\mu}(\mu; x_0), \quad 0 \le \mu \le 1,$$

and we get an interpolation of the dynamics of Ω_μ to $\mathbb{R}^n$ at each μ.

5.3. **k-Transfinite Barycentric Interpolation.** The family $\{T_\mu : \mu \ge 0\}$ of transformations of $\mathbb{R}^n$ transforms the set E into the family of sets $\{E_\mu = T_\mu(E) : \mu \ge 0\}$. If for each μ the transformation T_μ is Lipschitzian, then the H^d-rectifiable set E is transformed into an H^d-rectifiable[6] set E_μ and the k-TBI interpolation

[6]For instance, a d-rectifiable set E is the image $L(K)$ of a compact subset of $K \subset \mathbb{R}^d$ by a Lipschitzian map $L : \mathbb{R}^d \to \mathbb{R}^n$. So if the transformation $T_\mu : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitzian, then the composition $T_\mu \circ L$ is Lipschitzian and $T_\mu(L(K)) = T_\mu(E)$ is the image of the compact $K \subset \mathbb{R}^d$ by the Lipschitzian map $T_\mu \circ L$. Hence, by definition, $T_\mu(E) = T_\mu(L(K))$ is d-rectifiable.

can be applied with E_μ in place of E. Given $f_\mu : E_\mu \to \mathbb{R}$ a family of continuous functions parametrized by μ their k-TBI is

$$\mathcal{B}(f_\mu)(y) \stackrel{\text{def}}{=} \frac{\int_{E_\mu} \frac{1}{\|\xi-y\|^k} f_\mu(\xi)\, dH^d}{\int_{E_\mu} \frac{1}{\|\xi-y\|^k}\, dH^d}. \tag{5.10}$$

For each μ consider the interpolation of the velocity field $V(\mu,\cdot)$:

$$\boxed{\mathcal{V}(\mu,y) \stackrel{\text{def}}{=} \frac{\int_{E_\mu} \frac{1}{\|\xi-y\|^k} V(\mu,\xi)\, dH^d}{\int_{E_\mu} \frac{1}{\|\xi-y\|^k}\, dH^d}, \quad y \in \mathbb{R}^n \backslash E_\mu.} \tag{5.11}$$

If $\xi \mapsto V(\mu,\xi)$ is continuous, then $y \mapsto \mathcal{V}(\mu,y)$ is continuous and interpolates $\xi \mapsto V(\mu,\xi)$ from E_μ to $\mathbb{R}^n$:

$$\forall x \in E_\mu, \quad \lim_{\substack{y\to x\\ y\notin E_\mu}} \mathcal{V}(\mu,y) = \lim_{y\to x} \frac{\int_{E_\mu} \frac{1}{\|\xi-y\|^k} V(\mu,\xi)\, dH^d}{\int_{E_\mu} \frac{1}{\|\xi-y\|^k}\, dH^d} = V(\mu,x)$$

$$\forall x \in E, \quad \lim_{\substack{y\to T_\mu(x)\\ y\notin E_\mu}} \frac{\int_{E_\mu} \frac{1}{\|\xi-y\|^k} V(\mu,\xi)\, dH^d}{\int_{E_\mu} \frac{1}{\|\xi-y\|^k}\, dH^d} = V(\mu,T_\mu(x)) = \frac{\partial T_\mu}{\partial \mu}(x).$$

Assume that the differential equation

$$\frac{dy}{d\mu}(\mu) = \mathcal{V}(\mu,y(\mu)), \quad y(\mu) = y_0 \in \mathbb{R}^n \backslash E_0, \tag{5.12}$$

has a solution $y(\mu) = y(\mu; y_0) \in \mathbb{R}^n \backslash E_\mu$ for each μ. Then, by using the expression of $\mathcal{V}$, the function $y(\mu)$ is a solution of the following *differential interpolation equation*

$$\boxed{\frac{dy}{d\mu}(\mu) = \frac{\displaystyle\int_{E_\mu} \frac{1}{\|\xi - y(\mu)\|^k} V(\mu,\xi)\, dH^d}{\displaystyle\int_{E_\mu} \frac{1}{\|\xi - y(\mu)\|^k}\, dH^d}, \quad y(0) = y_0,} \tag{5.13}$$

for each $y_0 \in \mathbb{R}^n \backslash \mathbb{E}_0$. Therefore, for $x_0 \in E_0$ and $y_0 \in \mathbb{R}^n \backslash E_0$ such that $y_0 \to x_0$,

$$y(\mu; y_0) \to x(\mu; x_0) \quad \text{and} \quad \frac{dy}{d\mu}(\mu; y_0) \to \frac{dx}{d\mu}(\mu; x_0), \quad 0 \le \mu \le 1,$$

and we get an interpolation of the dynamics of E_μ to $\mathbb{R}^n$ at each μ.

The more delicate issue is to study the properties of the interpolation of the velocity as a function of k. This last aspect has been numerically investigated in [13].

5.4. **A Simple Illustrative Example: Re-meshing a Finite Element Grid around a Moving Object.** Let D be a sufficiently large open set with a smooth boundary ∂D that will serve as a fixed *hold-all*. For instance, D could be a *control volume* in dimension three. Let A be a 2×2 invertible symmetric matrix. For instance, given constants $a > 0$ and $b > 0$

$$A \stackrel{\text{def}}{=} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad A^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot A^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2.$$

Using this function define the *open ellipse* E (resp. *ellipse* ∂E)

$$E \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^2 : A^{-1}x \cdot A^{-1}x < 1\right\}, \tag{5.14}$$

$$\partial E = \left\{x \in \mathbb{R}^2 : A^{-1}x \cdot A^{-1}x = 1\right\}. \tag{5.15}$$

The *rotation matrix* associated with a counterclockwise angle ϕ is

$$B_\phi \stackrel{\text{def}}{=} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{5.16}$$

Think of E as a rigid body in a surrounding fluid $D\backslash\overline{E}$. We want to continuously rotate the ellipse E from an angle 0 to an angle β, $0 < \beta < 2\pi$, within the kold-all D. Let μ, $0 \leq \mu \leq 1$ be an *artificial time* and let $B_{\mu\beta}$ be the rotation matrix corresponding to a counterclockwise rotation by an angle $\mu\beta$ and let $T_\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be the bijection

$$x \mapsto T_\mu(x) = B_{\mu\beta}x : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_\mu^{-1} = T_{-\mu}, \tag{5.17}$$

$$\frac{\partial T_\mu}{\partial \mu}(x) = \beta \begin{bmatrix} -\sin\mu\beta & -\cos\mu\beta \\ \cos\mu\beta & -\sin\mu\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5.18}$$

The corresponding velocity is given by the expression

$$V(\mu, x) \stackrel{\text{def}}{=} \left(\frac{\partial T_\mu}{\partial \mu} \cdot T_\mu^{-1}\right)(x) \quad = \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{5.19}$$

Denote by E_μ the rotated ellipses by an angle $\mu\beta$:

$$\begin{aligned} E_\mu \stackrel{\text{def}}{=} T_\mu(E) = \{B_{\mu\beta}\xi : \xi \in E\} &= \left\{x \in \mathbb{R}^2 : A^{-1}B_{-\mu\beta}x \cdot A^{-1}B_{-\mu\beta}x < 1\right\} \\ &= \left\{x \in \mathbb{R}^2 : B_{\mu\beta}A^{-2}B_{-\mu\beta}x \cdot x < 1\right\} \\ \partial E_\mu &= \left\{x \in \mathbb{R}^2 : B_{\mu\beta}A^{-2}B_{-\mu\beta}x \cdot x = 1\right\}. \end{aligned}$$

Given a finite element meshing in the open domain

$$\Omega_0 \stackrel{\text{def}}{=} D\backslash\overline{E}, \quad \Gamma_0 = \partial E \cup \partial D, \tag{5.20}$$

we want to move the nodes to remesh in the domains

$$\Omega_\mu \stackrel{\text{def}}{=} D\backslash\overline{E_\mu}, \quad \Gamma_\mu = \partial E_\mu \cup \partial D \tag{5.21}$$

by using the transfinite interpolations.

5.4.1. *First Scenario.* As a first scenario, choose the k-TBI with $V(\mu,x)$ on ∂E_μ and 0 on ∂D. For each μ the interpolated velocity in Ω_μ is given by

$$\mathcal{V}_1(\mu,y) \stackrel{\text{def}}{=} \frac{\displaystyle\int_{\partial E_\mu} \frac{1}{\|\xi - y\|^k} V(\mu,\xi)\, dH^1}{\displaystyle\int_{\partial E_\mu \cup \partial D} \frac{1}{\|\xi - y\|^k}\, dH^1} = \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\displaystyle\int_{\partial E_\mu} \frac{1}{\|\xi - y\|^k}\, \xi\, dH^1}{\displaystyle\int_{\partial E_\mu \cup \partial D} \frac{1}{\|\xi - y\|^k}\, dH^1},$$

since the velocity is zero on the component ∂D of Γ_μ. As seen in Figure 2 the nodes in Ω_μ near ∂E_μ follow the rotation of the object.

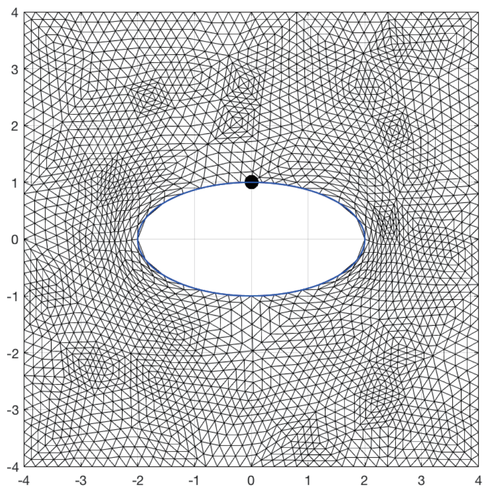


FIGURE 1. Initial finite element grid in $\Omega_0 = D\backslash E$.

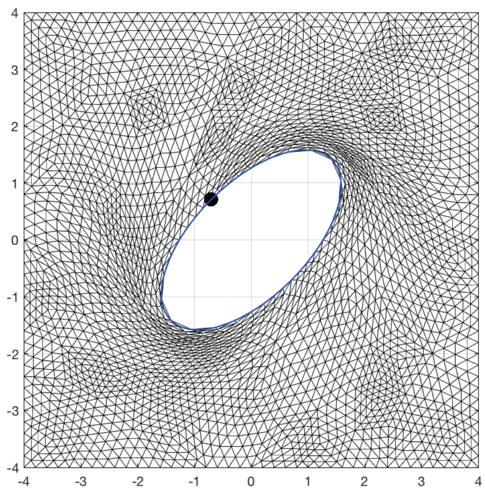
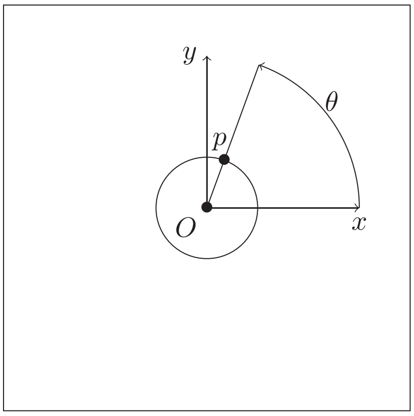


FIGURE 2. First scenario: finite element grid in Ω_1 for $\beta = \pi/4$.

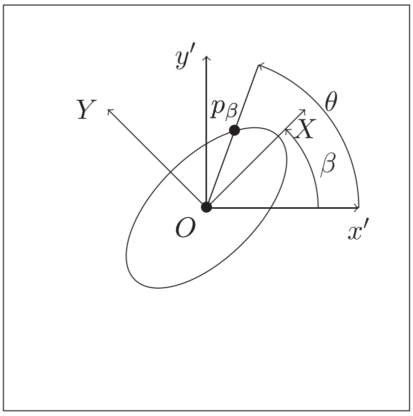
5.4.2. *Second Scenario.* In the first scenario the rotation β can move some points of Γ quite far from its initial position as seen by comparing their positions in Figures 1 and 2. In some applications the objective might be to generate a finite element grid of Ω_β from the initial finite element grid of Ω that does move the points on Γ too far. Intuitively, this can be done by having a new T_μ moving the points $T_\mu(x)$ closer to the initial points $x \in \Gamma$ along Γ_μ. There are many ways to do that. The best reorganization of the correspondance between the points $x \in \partial E$ and and those of $\partial E_\mu = B_\mu(\partial E)$ would be to minimize the distance from x to ∂E_μ, that is, to project each point of $x \in \partial E$ onto ∂E_μ:

$$T'_\mu(x) \stackrel{\text{def}}{=} p_{\partial E_\mu}(x), \quad x \in \partial E, \quad E_\mu = B_\mu(E), \tag{5.22}$$

where $p_{\partial E_\mu}(x)$ denotes the projection of x onto ∂E_μ. This would guarantee that the distance is minimum at each point. Yet, the projection might not be unique as μ increases and $T'_\mu(x)$ would be multivalued making it difficult to reproduce the constructions of the previous section.



(A) Point p on the reference unit circle C.



(B) Image p_β of p on the rotated ellipse ∂E_β.

FIGURE 3. Transformation of the circle C into the rotated ellipse ∂E_β that preserves the initial angle θ.

Consider now the construction given in [14]. Start with the circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, rotate the points backward by an angle $-\beta$, parametrize the ellipse as follows, and rotate the points by an angle β,

$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \mapsto \begin{bmatrix}\cos(\theta-\beta)\\ \sin(\theta-\beta)\end{bmatrix} \mapsto \begin{bmatrix}a\cos(\widehat{\theta-\beta})\\ b\sin(\widehat{\theta-\beta})\end{bmatrix} \mapsto \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}a\cos(\widehat{\theta-\beta})\\ b\sin(\widehat{\theta-\beta})\end{bmatrix},$$

where $\widehat{\theta-\beta}$ is the *pseudo angle* associated with $\theta-\beta$ and the pseudo angle $\hat\phi$ associated with ϕ is defined through the identity

$$\begin{bmatrix}a\cos(\widehat{\phi})\\ b\sin(\widehat{\phi})\end{bmatrix} = R(\phi)\begin{bmatrix}\cos(\phi)\\ \sin(\phi)\end{bmatrix} \quad \Rightarrow \tan\widehat{\phi} = \frac{a}{b}\tan\phi \tag{5.23}$$

$$\Rightarrow R(\phi)^2 = a^2\cos^2(\widehat{\phi}) + b^2\sin^2(\widehat{\phi}) = \frac{1+\tan^2(\phi)}{\frac{1}{a^2}+\frac{1}{b^2}\tan^2(\phi)}. \tag{5.24}$$

The transformation from C to ∂E_μ reduces to

$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \mapsto B_\beta R(\theta-\beta)B_{-\beta}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = R(\theta-\beta)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} : C \to \partial E_\beta. \tag{5.25}$$

What makes this transformation interesting is that an initial point $p \in C$ with an angle θ is moved to a new point p_β of ∂E_β along the line defined by the angle θ that is not affected by the rotation (see Figure 3) and this property is independent of β. In particular for $\beta = 0$ (the horizontal ellipse)

$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \mapsto R(\theta)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} : C \to \partial E.$$

From this the transformation T_β from the ∂E to ∂E_β is

$$\underbrace{R(\theta)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}}_{\xi} \mapsto T_\beta\underbrace{\left(R(\theta)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\right)}_{\xi} = \frac{R(\theta-\beta)}{R(\theta)}\underbrace{\left(R(\theta)\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\right)}_{\xi} : \partial E \to \partial E_\beta.$$

T_β extends to a non-linear bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $T_\beta(\partial E) = \partial E_\beta$:

$$\xi \mapsto T_\beta(\xi) = \frac{R(\theta_\xi - \beta)}{R(\theta_\xi)}\,\xi : \mathbb{R}^2 \mapsto \mathbb{R}^2, \tag{5.26}$$

$$\zeta \mapsto T_\beta^{-1}(\zeta) = \frac{R(\theta_\zeta)}{R(\theta_\zeta - \beta)}\,\zeta : \mathbb{R}^2 \mapsto \mathbb{R}^2, \tag{5.27}$$

where θ_ξ and θ_ζ are the respective angles associated with the vectors ξ and ζ in $\mathbb{R}^2$. The maximum distance between corresponding points is

$$\sup_{\theta\in[0,2\pi)} |R(\theta-\beta) - R(\theta)|. \tag{5.28}$$

Introducing a parameter μ, $0 \le \mu \le 1$, and replacing β by $\mu\beta$ we get

$$T_\mu(\zeta) = \frac{R(\theta_\zeta - \mu\beta)}{R(\theta_\zeta)}\,\zeta. \tag{5.29}$$

From this we can compute

$$\frac{\partial T_\mu}{\partial \mu}(\zeta) = -\beta\,\frac{R'(\theta_\zeta - \mu\beta)}{R(\theta_\zeta)}\,\zeta, \tag{5.30}$$

where $R'(\phi)$ is the derivative of $R(\phi)$:

$$\begin{aligned} R'(\phi) \overset{\text{def}}{=} \frac{dR}{d\phi}(\phi) &= \left(\frac{1}{a^2} - \frac{1}{b^2}\right) R^3(\phi)\,\sin(\phi)\,\cos(\phi) \\ &= \left(\frac{1}{a^2} - \frac{1}{b^2}\right)\left[\frac{1+\tan^2(\phi)}{\frac{1}{a^2} + \frac{1}{b^2}\tan^2(\phi)}\right]^{3/2} \sin(\phi)\,\cos(\phi). \end{aligned}$$

The inverse of T_μ and the velocity are given by the expressions

$$T_\mu^{-1}(\xi) = \frac{R(\theta_\xi)}{R(\theta_\xi - \mu\beta)}\,\xi, \quad V(\mu,\xi) = -\beta\,\frac{R'(\theta_\xi - \mu\beta)}{R(\theta_\xi - \mu\beta)}\,\xi. \tag{5.31}$$

After substitution

$$V(\mu,\xi) = -\beta\left(\frac{1}{a^2} - \frac{1}{b^2}\right) R^2(\theta_\xi - \mu\beta)\,\sin(\theta_\xi - \mu\beta)\,\cos(\theta_\xi - \mu\beta)\,\xi.$$

This yields the *differential interpolation equation*

$$\frac{dy}{d\mu}(\mu; y_0) = \frac{\displaystyle\int_{\partial E_\mu} \frac{1}{\|\xi - y(\mu; y_0)\|^k} V(\mu,\xi)\, dH^1}{\displaystyle\int_{\partial E_\mu \cup \partial D} \frac{1}{\|\xi - y(\mu; y_0)\|^k}\, dH^1}, \quad y(0,y_0) = y_0 \tag{5.32}$$

and its implementation gives excellent results as seen in Figure 4. More complex examples can be found in [14].

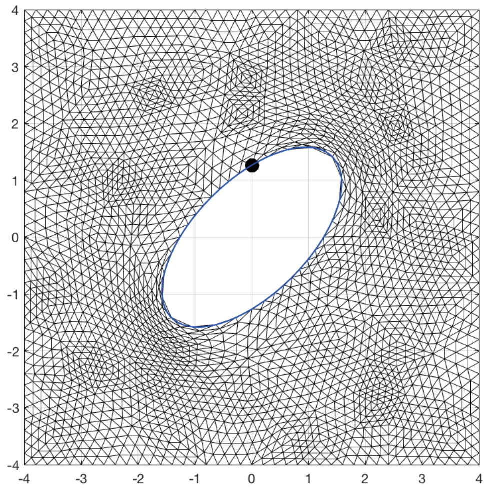


FIGURE 4. Finite element grid in Ω_1 for $\beta = \pi/4$ using V.

5.4.3. *Third Scenario.* The choice of T_μ is not unique. We can slightly modify the parametrization of the ellipses of the second scenario. Start with the circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, rotate the points back by an angle $-\beta$ on the circle, change the circle into an ellipse, and rotate the ellipse by an angle β

$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \mapsto \begin{bmatrix}\cos(\theta-\beta)\\ \sin(\theta-\beta)\end{bmatrix} \mapsto A\begin{bmatrix}\cos(\theta-\beta)\\ \sin(\theta-\beta)\end{bmatrix} \mapsto \underbrace{B_\beta A\begin{bmatrix}\cos(\theta-\beta)\\ \sin(\theta-\beta)\end{bmatrix}}_{=B_\beta A B_{-\beta}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}},$$

and in term of the initial ellipse

$$A\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \mapsto B_\beta A B_{-\beta} A^{-1}\left(A\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}\right) : E \to E_\mu. \tag{5.33}$$

The distance between the initial and final points is now

$$\sup_{\theta\in[0,2\pi)} \left\| [B_\beta A B_{-\beta} - A]\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}\right\| \tag{5.34}$$

to be compared with the distance of the second scenario.

Introducing a parameter μ, $0 \le \mu \le 1$, and replacing β by $\mu\beta$ we get

$$\xi \mapsto T_\mu(\xi) = B_{\mu\beta} A B_{-\mu\beta} A^{-1} \xi : E \to E_{\mu\beta} \tag{5.35}$$

that naturally extends to a linear bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$. Therefore,

$$T_\mu^{-1}(\zeta) = A B_{\mu\beta} A^{-1} B_{-\mu\beta}\, \zeta,$$

$$\frac{\partial T_\mu}{\partial \mu}(\xi) = \left[(B_{\mu\beta})' A B_{-\mu\beta} A^{-1} + B_{\mu\beta} A (B_{-\mu\beta})' A^{-1}\right] \xi,$$

where

$$(B_{\mu\beta})' = \beta \begin{bmatrix} -\sin(\mu\beta) & -\cos(\mu\beta) \\ \cos(\mu\beta) & -\sin(\mu\beta) \end{bmatrix} = \beta\, J\, B_{\mu\beta}, \quad J \overset{\text{def}}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$(B_{-\mu\beta})' = -\beta\, J\, B_{-\mu\beta},$$

$$\frac{\partial T_\mu}{\partial \mu}(\xi) = \beta\, \left[J\, B_{\mu\beta} A B_{-\mu\beta} A^{-1} - B_{\mu\beta} A J\, B_{-\mu\beta} A^{-1}\right] \xi. \tag{5.36}$$

Hence,

$$V(\mu,\zeta) = \beta\, \left[J - B_{\mu\beta} A J\, A^{-1} B_{-\mu\beta}\right] \zeta. \tag{5.37}$$

This yields the *differential interpolation equation*

$$\frac{dy}{d\mu}(\mu; y_0) = \frac{\displaystyle\int_{\partial E_\mu} \frac{V(\mu,\xi)}{\|\xi - y(\mu; y_0)\|^k}\, dH^1}{\displaystyle\int_{\partial E_\mu \cup \partial D} \frac{1}{\|\xi - y(\mu; y_0)\|^k}\, dH^1}, \quad y(0; y_0) = y_0. \tag{5.38}$$

The rotated ellipse $T_\mu(\partial E)$ is given by

$$\begin{aligned} T_\mu(\partial E) &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : B_{\mu\beta} A^{-1} B_{-\mu\beta} \begin{bmatrix} x \\ y \end{bmatrix} \cdot B_{\mu\beta} A^{-1} B_{-\mu\beta} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : B_{\mu\beta} A^{-2} B_{-\mu\beta} \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 1 \right\}. \end{aligned} \tag{5.39}$$

Appendix A. Rectifiable sets and sets of positive reach

We recall some mathematical notions and results.

Definition A.1 ([11, pp. 251–252])**.** Let E be a subset of a metric space X. $E \subset X$ is *d-rectifiable*[7] if it is the image of a compact subset K of $\mathbb{R}^d$ by a Lipschitz continuous function $f : \mathbb{R}^d \to X$. □

There are several extensions of the d-rectifiability to non-compact sets.

Definition A.2 ([1, Dfn. 2.57, p. 80])**.** Let $E \subset \mathbb{R}^n$ be H^d-measurable.

(i) E is *countably d-rectifiable* if there exist countably many Lipschitzian functions $f_i : \mathbb{R}^d \to \mathbb{R}^n$ such that

$$E \subset \cup_{i=0}^{\infty} f_i(\mathbb{R}^d). \tag{A.1}$$

[7] In [10, Dfn. 2.7, p. 422] *bounded* is used in place of *compact*.

(ii) E is *countably H^d-rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^d \to \mathbb{R}^n$ such that $E \backslash \cup_i f_i(\mathbb{R}^d)$ is H^d-negligible:

$$H^d \left(E \backslash \cup_{i=0}^{\infty} f_i(\mathbb{R}^d) \right) = 0. \tag{A.2}$$

(iii) E is *H^d-rectifiable* if it is *countably H^d-rectifiable* and $H^d(E)<\infty$. □

We recall the definition of *positive reach* introduced by [10, p. 419].

Definition A.3 (Federer [10, p. 419]). Let E be a closed subset of $\mathbb{R}^n$.

(i) The set of points in $\mathbb{R}^n$ having a unique projection onto E:

$$\mathrm{Unp}\,(E) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \exists \text{ a unique } x \in E \text{ such that } d_E(y) = \|y - x\|\},$$

where $d_E(y)$ is the distance function from a point y to E.

(ii) The *reach* of a point $x \in E$ and the reach of E are defined as

$$\mathrm{reach}\,(E,x) \stackrel{\text{def}}{=} \sup\{r > 0 : B_r(x) \subset \mathrm{Unp}\,(E)\}$$
$$\mathrm{reach}\,(E) \stackrel{\text{def}}{=} \inf_{x \in E} \mathrm{reach}\,(E,x).$$

E is said to be a *set with positive reach* if $\mathrm{reach}\,(E) > 0$. □

The definition of $\mathrm{Unp}\,(E)$ implies the existence of a function $p_E : \mathrm{Unp}\,(E) \to E$ which assigns to $y \in \mathrm{Unp}\,(E)$ the unique point $p_E(y) \in E$ such that such that $d_E(y) = \|p_E(y) - x\|$.

Theorem A.4 ([1, Prop. 3, p. 743, and Corollary 3, p. 744]). *If E is a compact subset of $\mathbb{R}^n$ with positive reach, then E is $(n-1)$-rectifiable.*

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function of Lipschitz constant $\mathrm{Lip}\,(f)$, then $H^d(f(E)) \leq [\mathrm{Lip}\,(f)]^d\, H^d(E)$ for all Borel sets $E \subset \mathbb{R}^n$. An immediate consequence of [1, Prop. 2.49 (iv), p. 80] is the fact that rectifiable sets are stable under Lipschitz transformations. An important example of countably d-rectifiable set is the graph of a Lipschitz function of d variables in $\mathbb{R}^n$ (briefly, a Lipschitz d-graph).

Example A.1 (Lipschitz d-graphs [2, Example 2.58, p. 80]). Let $\pi \subset \mathbb{R}^n$ be a d-plane, $1 \leq d < n$, and $\phi : \pi \to \pi^\perp$ be a Lipschitz function. Let

$$\Gamma \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \phi(\pi x) = \pi^\perp x\} \tag{A.3}$$

be the graph of ϕ. Then, choosing an orthonormal basis $e_1, \ldots, e_d$ of π and setting

$$y \mapsto f(y) \stackrel{\text{def}}{=} \sum_{i=1}^{d} y_i e_i + \phi\,(y_i e_i) : \mathbb{R}^d \to \Gamma \tag{A.4}$$

we obtain that Γ is countably d-rectifiable. By [2, Proposition 2.49 (iv)], we conclude that any compact subset of Γ is H^d-rectifiable. □

Example A.2. A *locally Lipschtzian domain* Ω in $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ such that its boundary Γ is a Lipschitz $(n-1)$-graph in a neighbourhood of each point $x \in \Gamma$. So the boundary of a locally Lipschitzian domain is an example of H^{n-1}-rectifiable set. □

Proposition A.5 ([1, Prop. 1, p. 732]). *If $E \subset \mathbb{R}^n$ is a compact set with Lipschitz boundary, then $H^{n-1}(\partial E) < +\infty$.*

For a Lebesgue measurable set E we have the Lebesgue-Besicovitch Differentiation Theorem which involves the *n-dimensional density* of E in $\mathbb{R}^n$

$$\Theta_n(E,x) \stackrel{\text{def}}{=} \lim_{r\searrow 0} \frac{\mathrm{m}_n(B_r(x)\cap E)}{\mathrm{m}_n(B_r(x))} = \chi_E(x), \quad \mathrm{m}_n\text{-a.e. in } \mathbb{R}^n, \tag{A.5}$$

where χ_E is the characteristic function of E and m_n is the n-dimensional Lebegue measure. There is also a notion of *d-dimensional density* and a similar result for a H^d-rectifiable set.

Definition A.6 ([2, Dfn. 2.55, p. 78]). Given a Borel subset E of $\mathbb{R}^n$ such that $H^d(E) < +\infty$, the *upper* and *lower d-dimensional densities* of H^d at x are respectively defined as

$$\Theta^*_d(E,x) \stackrel{\text{def}}{=} \limsup_{r\searrow 0} \frac{H^d(B_r(x)\cap E)}{\alpha_d\, r^d}, \quad \Theta_{*d}(E,x) \stackrel{\text{def}}{=} \liminf_{r\searrow 0} \frac{H^d(B_r(x)\cap E)}{\alpha_d\, r^d}.$$

If they agree, we denote the common value of these densities by $\Theta_d(E,x)$. □

We shall need the Besicovitch-Marstrand-Mattila Theorem.

Theorem A.7 ([2, Thm. 2.63, p. 83]). *Let E be a Borel set in $\mathbb{R}^n$ with $H^d(E) < \infty$. Then, E is H^d-rectifiable if and only if $\Theta_d(E,x) = 1$ for H^d-a.e. $x \in E$ (that is, $\Theta_d(E,x) = \chi_E(x)$ for H^d-a.e. $x \in \mathbb{R}^n$).*

Acknowledgment

This research was supported by the Natural Sciences and Engineering research Council of Canada through Discovery Grants and a Grant from the Collaborative research and Training Experience (CREATE) program in Simulation-based Engineering Science.

Note added in the proofs

It is shown in the new paper [6] that the additional assumptions on Γ in Theorem 2.8 (iii) can be dropped for the k-TMI from Γ to $\mathbb{R}^n$ if f is assumed to be a Lipschitz continuous function on Γ. Similarly, the additional assumptions on Γ in Theorem 4.3 (ii) can be dropped for the (k,m)-TMI from Γ to $\mathbb{R}^n$ if f and its partial derivatives are Lipschitz continuous up to order m in a tubular neighbourhood of Γ. It is also shown that $\mathcal{M}_{k,m}(f)$ and its partial derivatives interpolate f and its partial derivatives up to order m from Γ to $\mathbb{R}^n$ solving a problem raised in [12].

References

[1] L. Ambrosio, A. Colesanti and E. Villa, *Outer Minkowski content for some classes of closed sets*, Math. Ann. **342** (2008), 727–748.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, The Clarendon Press, Oxford University Press, New York, 2000.

[3] Y. Belhamadia, A. Fortin and Y. Bourgault, *On the performance of anisotropic mesh adaptation for scroll wave turbulence dynamics in reaction-diffusion systems*, J. Comput. Appl. Math. **271** (2014), 233–246.

[4] S. Bruvoll and M. S. Floater, *Transfinite mean value interpolation in general dimension*, J. of Computational and Applied Mathematics **233** (2010), 1631–1639.

[5] G. Chen and J. Zhou, *Boundary Element Method*, Computational Mathematics and Applications, Academic Press, London 1992. Atlantis Press / World Scientific, Second edition 2010.

[6] M. C. Delfour and A. Garon, *On the transfinite mean value interpolation of Dyken and Floater*, accepted for publication in Pure and Applied Functional Analysis, November 6, 2019.

[7] M. C. Delfour and J.-P. Zolésio, *Shapes and geometries: Analysis, Differential Calculus and Optimization*, SIAM series on Advances in Design and Control, Society for Industrial and Applied Mathematics, Philadelphia, USA, 2001.

[8] M. C. Delfour and J.-P. Zolésio, *Shapes and geometries: Metrics, Analysis, Differential Calculus and Optimization*, SIAM series on Advances in Design and Control, Society for Industrial and Applied Mathematics, Philadelphia, USA, 2011.

[9] C. Dyken and M. S. Floater, *Transfinite mean value interpolation*, Computer Aided Geometric Design **26** (2009), 117–134.

[10] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–419.

[11] H. Federer, *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, 153. Springer, New York, 1969.

[12] M. S. Floater and C. Schulz, *Pointwise radial minimization: Hermite interpolation on arbitrary domains*, Comput. Graph. Forum **27** (2008), 1505–1512.

[13] A. Garon and M. C. Delfour, *Mesh Adaptation Based on Transfinite Mean Value Interpolation*, submitted to *Journal of Computational Physics*, 2019.

[14] A. Garon and M. C. Delfour, *Mesh adaptation based on transfinite mean value interpolation*, submitted to *Journal of Computational Physics*, 2019.

[15] F. Hausdorff, *Über halbstetige Funktionen und deren Verallgemeinerung*, Math. Zeitschr. **5** (1919), 292–309.

[16] A. Hay, S. Etienne, A. Garon, and D. Pelletier, *Time-integration for ALE simulations of fluid-structure interaction problems: stepsize and order selection based on the BDF*, Comput. Methods Appl. Mech. Engrg. **295** (2015), 172–195.

[17] K. Hormann and M. S. Floater, *Mean value coordinates for arbitrary planar polygons*, ACM Trans. Graph. **25** (2006), 1424–1441.

[18] A. Kaźmierczak, J. Sokołowski and A. Żochowski, *Drag minimization for the obstacle in compressible flow using shape derivatives and finite volumes*, Math. Control Relat. Fields **8** (2018), 89–115.

[19] I. Lasiecka, K. Szulc and A. Żochowski, *Boundary control of small solutions to fluid-structure interactions arising in coupling of elasticity with Navier-Stokes equation under mixed boundary conditions*, Nonlinear Anal. Real World Appl. **44** (2018), 54–85.

[20] S. Léger, A. Fortin, C. Tibirna and M. Fortin, *An updated Lagrangian method with error estimation and adaptive remeshing for very large deformation elasticity problems*. Internat. J. Numer. Methods Engrg. **100** (2014), 1006–1030.

[21] G. Leugering, J. Sokołowski and A. Żochowski, *Shape and topology optimization for passive control of crack propagation*, in New trends in shape optimization, pp. 141–197, Internat. Ser. Numer. Math., 166, Birkhäuser/Springer, Cham, 2015.

[22] D. R. Lynch and K. O'Neil, *Finite elements in water resources*, in: Proceedings of the third international conference on finite elements in water resources, 1980. Conference held at the University of Mississippi in May, 1980.

[23] P. Plotnikov and J. Sokołowski, *Compressible Navier-Stokes equations. Theory and shape optimization*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], 73. Birkhäuser/Springer Basel AG, Basel, 2012.

[24] D. Shepard, *A two-dimensional interpolation function for irregularly-spaced data*, in: Proceedings of the 1968 23rd ACM National Conference, ACM 68, New York, NY, USA, 1968, pp. 517–524.

[25] J. Sokołowski and J.-P. Zolésio, *Dérivée par rapport au domaine de la solution d'un problème unilatéral* (French) [Shape derivative for the solutions of variational inequalities], C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), 103–106.

[26] J. Sokołowski and J.-P. Zolésio, *Shape sensitivity analysis of unilateral problems*, SIAM J. Math. Anal. **18** (1987), 1416–1437.

[27] J. Sokołowski and J.-P. Zolésio, *Shape design sensitivity analysis of plates and plane elastic solids under unilateral constraints*, J. Optim. Theory Appl. **54** (1987), 361–382.

[28] J. Sokołowski and J.-P. Zolésio, *Shape sensitivity analysis of contact problem with prescribed friction*, Nonlinear Anal. (12) **12** (1988), 1399–1411.

[29] H. Tietze, *Über Funktionen die auf einer abgeschlossenen Menge stetig sind*, Journal für die reine und angewandte Mathematik **145** (1915), 9–14.

Manuscript received September 2 2018
revised February 12 2019

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