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A TURNPIKE RESULT FOR CONVEX HYPERBOLIC OPTIMAL BOUNDARY CONTROL PROBLEMS

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Abstract. In this paper the turnpike phenomenon is studied for problems of optimal boundary control. We consider systems that are governed by a linear 2×2 hyperbolic partial differential equation with a source term. Turnpike results are obtained for problems of optimal Dirichlet boundary control for such systems with a convex objective function that depends on the control and the boundary traces of the system states and is strongly convex with respect to the control. In the problem we also allow for an additional convex inequality constraint. We show that asymptotically for large T the influence of the initial state becomes smaller and smaller in the sense that the L^2-norm of the difference between the dynamic optimal control and the stationary control that solves the corresponding static optimal control problem remains uniformly bounded for arbitrarily large T. As an application, we consider gas pipeline flow.

1. Introduction

The turnpike property has been discussed by P. A. Samuelson in mathematical economics in 1949 (see [5]). Since this time, the turnpike phenomenon for optimization problems has been analyzed in various contexts, see for example [21] and [4]. In the area of infinite dimensional control problems the turnpike phenomenon has been investigated later, for example in [20]. For optimal control problems with partial differential equations see [18] or [15], where distributed control is considered for linear–quadratic optimal control problems. In this paper, we analyze the turnpike phenomenon for the case of problems of optimal boundary control governed by a 2×2 hyperbolic partial differential equation. We present turnpike results that state that the normalized L^2-norm of the difference between the optimal dynamic control and the optimal static control on the time interval $(0, T)$ converges to zero as the time-horizon T tends to infinity. Problems of optimal boundary control with hyperbolic systems have been considered in [9], where linear–quadratic optimal control problems with the wave equation are studied. In this paper, we consider a system that is governed by a 2×2 linear hyperbolic equation. In [7] we have already shown turnpike results for such a system for problems of optimal Dirichlet boundary control with quadratic objective functions and zero initial state. In this paper we show turnpike results for problems of optimal Dirichlet boundary control with more

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general convex objective functions and arbitrary initial states and allow for convex inequality constraints.

Our study is motivated by boundary control problems with hyperbolic systems as described in [1] with applications in the control of gas flow in pipelines, water flow in channels and traffic flow. In [17], both integral- and measure–turnpike properties are considered. In this paper, we consider integral–turnpike properties.

This paper has the following structure. First we introduce some notation and define the dynamic optimal control problem. Then we derive the corresponding necessary optimality conditions, using the adjoint operator that corresponds to the initial boundary value problem. We define the static optimal control problem and derive the corresponding necessary optimality conditions. Then we state the turnpike result. For the proof, we need several auxiliary results. We show that certain operator norms remain uniformly bounded with respect to time. Moreover, for constant (that is time-independent) inputs, certain operators generate outputs whose norms on the time interval remain uniformly bounded with respect to time. Our auxiliary results allow us to give a rather concise proof of the turnpike theorem. At the end of the paper, we discuss the control of gas pipeline flow as an application.

1.1. **Notation.** Let a length $L > 0$ and a time interval $[0, T]$ be given. Let functions d_- and $d_+ \in C^1([0, L])$ be given such that for all $x \in [0, L]$ the inequality $d_-(x) < 0 < d_+(x)$ holds. Define the (x-dependent) diagonal matrices

$$D = \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix}, \; D' = \begin{pmatrix} d_+' & 0 \\ 0 & d_-' \end{pmatrix} \text{ and } |D| = \begin{pmatrix} |d_+| & 0 \\ 0 & |d_-| \end{pmatrix}.$$

For all $x \in [0, L]$, let $M(x)$ denote a symmetric 2×2 matrix that depends continuously on x. Let η_0 be a real number such that $\eta_0 \leq 0$. Assume that for all $x \in [0, L]$ the matrices

$$|D|\, M + M\, |D| \tag{1.1}$$

and thus also $|D(x)|^{-1}\, M(x) + M(x)\, |D(x)|^{-1}$ are positive semi–definite.

Let an initial state $r^0 = (r_+^0, r_-^0) \in (L^2(0, L))^2$ be given. For $t \in (0, T)$ and $x \in (0, L)$ we consider a system that is governed by the initial boundary value problem

$$\begin{cases} r(0, x) = r^0, \\ r_t + D\, r_x = \eta_0\, M\, r, \\ r_+(t, 0) = u_+(t), \\ r_-(t, L) = u_-(t). \end{cases} \tag{1.2}$$

For $u_+,\ u_- \in L^2(0, T)$, system (1.2) has a solution $r \in C([0, T], L^2((0, L); \mathbb{R}^2))$. Moreover, for the boundary traces of the solution we have $r_+(\cdot, L),\ r_-(\cdot, 0) \in L^2(0, T)$. This follows with a Picard iteration along the characteristic curves similar as in [11], [12].

Remark 1.1. An example for a system of the form (1.2) are the linearized Saint-Venant equations that can be used as a model for the flow of water through channels, see [2].

For $x = (x_+, x_-) \in \mathbb{R}^2$, we use the notation $\|x\|_{\mathbb{R}^2} = \sqrt{x_+^2 + x_-^2}$. For $T > 0$ define the Hilbert space

$$H(T) = L^2(0, T) \times L^2(0, T)$$

with the scalar product $\langle u, v\rangle_{H(T)} = \int_0^T u^\top(\tau)\, v(\tau)\, d\tau$ and the corresponding norm

$$\|u\|_{H(T)} = \left(\int_0^T \|u(\tau)\|_{\mathbb{R}^2}^2\, d\tau\right)^{1/2}.$$

1.2. **A dynamic optimal control problem.** In this section we define our problem of dynamic optimal Dirichlet boundary control with the hyperbolic system (1.2). Let the function $f : \mathbb{R}^4 \to [0, \infty)$ be convex.

Assume that f is continuously differentiable. For the partial derivatives of f with respect to the first two components we use the notation $f_u = (f_{u_+}, f_{u_-})$. For the partial derivatives of f with respect to the last two components we use the notation $f_R = (f_{R_+}, f_{R_-})$.

We assume that f is strongly convex with respect to the first two components (these will later be the components of the control u) in the sense that there exists a number $\kappa > 0$ such that for all $u_1, u_2, r_1, r_2 \in \mathbb{R}^2$ we have

$$\begin{aligned} [f_u(u_1, r_1) - f_u(u_2, r_2)]^\top (u_1 - u_2) + [f_R(u_1, r_1) - f_R(u_2, r_2)]^\top (r_1 - r_2) \\ \geq \kappa\, \|u_1 - u_2\|_{\mathbb{R}^2}^2. \end{aligned} \tag{1.3}$$

Assume that for all $u, r \in H(T)$ we have $f(u(\cdot), r(\cdot)) \in L^2(0, T)$. For $u = (u_+, u_-) \in H(T)$ and the generated state $r = (r_+, r_-)$ that solves (1.2) define the objective function

$$J_T(u) = \int_0^T f(u_+(\tau), u_-(\tau), r_+(\tau, L), r_-(\tau, 0))\, d\tau. \tag{1.4}$$

Then J_T is strongly convex in the sense that there exists a constant $\kappa > 0$ that is independent of T such that for all $T > 0$ and all $u, v \in H(T)$ we have

$$\langle J_T'(u) - J_T'(v), u - v\rangle_{H(T)} \geq \kappa \|u - v\|_{H(T)}^2. \tag{1.5}$$

Note that (1.3) holds for example if f is strongly convex (see [13]). It also holds if $f(u, r) = g_1(u) + g_2(r)$ with a strongly convex function g_1 and a convex function g_2.

The choice of the objective function is motivated by the situation in the operation of transportation networks such as gas pipeline networks where the customer's preferences depend on the input and the output at the boundary nodes. The objective function in this application is determined by the desired nodal profiles, see also [8]. The quadratic objective function for the optimal Dirichlet boundary control problem with the wave equation studied in [9] has a similar structure.

Let a convex continuously differentiable function $G_T : H(T) \to \mathbb{R}$ be given. Define the convex set $K_T = \{u \in H(T) : G_T(u) \leq 0\}$. Assume that there exists a Slater point $u_S \in \mathbb{R}^2$ such that for all $T > 0$ we have $G_T(u_S) < 0$. Assume that for all $T_1 > 0$, $T_2 > 0$ we have

$$K_{T_1} \cap \mathbb{R}^2 = K_{T_2} \cap \mathbb{R}^2. \tag{1.6}$$

Remark 1.2. For optimal control problems without additional constraints, define $G_T = -1$.

We consider the dynamic optimal control problem

$$\left\{ \begin{array}{l} \min_{u \in K_T} J_T(u) \\ \text{subject to (1.2).} \end{array} \right. \tag{1.7}$$

Due to the strong convexity assumption (1.3) the existence of an optimal control follows with the Direct Method of the Calculus of Variations.

1.3. **An adjoint operator.** For the analysis of the boundary control problem, the study of certain adjoint operators is essential. For a given time $T > 0$, we define the operator $F_T\,(u,\, r^0)$ that maps the initial state r^0 and the boundary control $u = (u_+(\cdot),\, u_-(\cdot)) \in H(T)$ to the boundary trace $(r_+(\cdot, L),\, r_-(\cdot,\, 0)) \in H(T)$ of the solution of the linear initial boundary value problem (1.2). Thus we have

$$F_T \begin{pmatrix} u_+(\cdot) \\ u_-(\cdot) \\ r_+^0(\cdot) \\ r_-^0(\cdot) \end{pmatrix} = \begin{pmatrix} r_+(\cdot, L) \\ r_-(\cdot,\, 0) \end{pmatrix}. \tag{1.8}$$

Lemma 1.3 states that the operator norm of F_T is uniformly bounded with respect to T.

Lemma 1.3. *The operator F_T is uniformly bounded with respect to T as an operator from the Hilbert space $H(T) \times (L^2(0, L)^2)$ to $H(T)$. For the corresponding operator norm of F_T for all $T > 0$ we have*

$$\|F_T\| \leq \max\left\{1,\ \sup_{x\in[0,\,L]} \{\tfrac{1}{\sqrt{d_+(x)}},\ \tfrac{1}{\sqrt{|d_-(x)|}}\}\right\}. \tag{1.9}$$

For the operator $\tilde{F}_T\,(u) = F_T\,(u,\, 0)$ we have $\|\tilde{F}_T\| \leq 1$. Moreover, we have

$$\|F_T(0,\, r^0)\|_{H(T)} \leq \sup_{x\in[0,\,L]} \left\{\frac{1}{\sqrt{d_+(x)}},\ \frac{1}{\sqrt{|d_-(x)|}}\right\} \|r^0\|_{(L^2(0,\,L))^2}. \tag{1.10}$$

Proof: Let $u_+,\, u_- \in L^2(0, T)$ be given. Let $r_+,\, r_-$ denote the generated solution of (1.2). Due to (1.2) we have

$$\begin{aligned} & \|u\|_{H(T)}^2 - \|F_T(u,\, r^0)\|_{H(T)}^2 \\ = & \int_0^T r_+(t,0)^2 + r_-(t,L)^2 - r_+(t,L)^2 - r_-(t,0)^2\, dt \\ = & -\int_0^T \int_0^L \left(r_+(t,x)^2\right)_x - \left(r_-(t,x)^2\right)_x\, dx\, dt \\ = & -\int_0^T \int_0^L 2\, r^\top\, |D|^{-1}\, D\, r_x\, dx\, dt \\ = & \int_0^T \int_0^L 2\, r^\top |D|^{-1}\, r_t + 2\, |\eta_0|\, r^\top\, |D|^{-1}\, M\, r\, dx\, dt. \end{aligned}$$

Hence due to (1.1) we have

$$
\begin{aligned}
& \|u\|^2_{H(T)} - \|F_T(u,\, r^0)\|^2_{H(T)} \\
= \; & \int_0^L \int_0^T \left(r^\top |D|^{-1}\, r\right)_t + 2\, |\eta_0|\, r^\top\, |D|^{-1} M\, r\, dt\, dx \\
= \; & \int_0^L \left(r^\top |D|^{-1}\, r\right)\Big|_{t=0}^{T} + \int_0^L \int_0^T 2\, |\eta_0|\, r^\top\, |D|^{-1} M\, r\, dt\, dx \\
= \; & \int_0^L r^\top(T,x) |D|^{-1}\, r(T,x)\, dx - \int_0^L (r^0)^\top(x) |D|^{-1}\, r^0(x)\, dx \\
+ \; & \int_0^L \int_0^T 2\, |\eta_0|\, r^\top\, |D|^{-1} M\, r\, dt\, dx \\
\geq \; & -\int_0^L (r^0)^\top(x) |D|^{-1}\, r^0(x)\, dx \\
+ \; & \int_0^L \int_0^T r^\top \left[\, |\eta_0|\, \left(|D|^{-1}\, M + M |D|^{-1}\right)\right] r\, dt\, dx \\
\geq \; & -\sup_{x \in [0,\, L]} \left\{\tfrac{1}{d_+(x)},\, \tfrac{1}{|d_-(x)|}\right\} \int_0^L (r^0)^\top(x)\, r^0(x)\, dx
\end{aligned}
$$

Hence (1.9) and (1.10) follow.

We are interested in the adjoint operator F_T^* that satisfies for all $z_T \in H(T) = \mathcal{D}(F_T^*)$ the equation

$$
\begin{aligned}
& \langle F_T\,(u,\, r^0),\, z_T\rangle_{H(T)} \qquad\qquad (1.11) \\
= & \int_0^T \langle u(t),\, \begin{pmatrix} F_T^*(z_T)(t)_1 \\ F_T^*(z_T)(t)_2 \end{pmatrix} \rangle_{\mathbb{R}^2}\, dt + \int_0^L \langle r^0(x),\, \begin{pmatrix} F_T^*(z_T)(x)_3 \\ F_T^*(z_T)(x)_4 \end{pmatrix} \rangle_{\mathbb{R}^2}\, dx
\end{aligned}
$$

where $\langle\, \cdot\,,\, \cdot\,\rangle_{\mathbb{R}^2}$ denotes the usual scalar product in $\mathbb{R}^2$. Due to (1.9) we have the inequality

$$
\|F_T^*\| \leq \max \left\{ 1,\, \sup_{x \in [0,\, L]} \left\{\tfrac{1}{\sqrt{d_+(x)}},\, \tfrac{1}{\sqrt{|d_-(x)|}}\right\} \right\}. \tag{1.12}
$$

Similar as in [3], we determine F_T^* in the following lemma.

Lemma 1.4. *For $z_T = (z_+^T,\, z_-^T) \in \mathcal{D}(F_T^*) = H(T)$, define $z = (z_+(\cdot),\, z_-(\cdot))$ as the solution of the adjoint system (where $(t,\, x) \in (0,\, T) \times (0,\, L)$)*

$$
\begin{cases}
z(T,\, x) = 0,\; x \in (0, L), \\
z_t(t,\, x) + D(x)\, z_x(t,\, x) = [|\eta_0|\, M(x) - D'(x)]\; z(t,\, x), \\
z_+(t,\, L) = \frac{1}{d_+(L)}\, z_+^T(t), \\
z_-(t,\, 0) = \frac{1}{|d_-(0)|}\, z_-^T(t).
\end{cases} \tag{1.13}
$$

Then we have

$$
F_T^* \begin{pmatrix} z_+^T(\cdot) \\ z_-^T(\cdot) \end{pmatrix} = \begin{pmatrix} d_+(0)\, z_+(\cdot,\, 0) \\ |d_-(L)|\, z_-(\cdot, L) \\ z_+(0, \cdot) \\ z_-(0,\, \cdot) \end{pmatrix}. \tag{1.14}
$$

Proof: Let r denote the solution of (1.2). Using integration by parts we obtain the equation

$$\begin{aligned}
0 &= \int_0^T \int_0^L \langle z,\, r_t + D\, r_x - \eta_0\, M\, r\rangle_{\mathbb{R}^2}\, dx\, dt \\
&= \int_0^T \int_0^L -r^\top z_t\, dx\, dt + \int_0^L \left[r(t,\, x)^\top z(t,\, x)\right]\Big|_{t=0}^T dx \\
&- \int_0^T \int_0^L r^\top D\, z_x\, dx\, dt - \int_0^T \int_0^L r^\top D'\, z\, dx\, dt \\
&+ \int_0^T \left[r(t,\, x)^\top D(x)\, z(t,\, x)\right]\Big|_{x=0}^L dt - \int_0^T \int_0^L \eta_0\, r^\top M\, z\, dx\, dt \\
&= \int_0^T \int_0^L -r^\top \left[z_t + D\, z_x + D'\, z + \eta_0\, M\, z\right]\, dx\, dt \\
&+ \int_0^T \left[r(t,\, x)^\top D(x)\, z(t,\, x)\right]\Big|_{x=0}^L dt - \int_0^L r^0(x)^\top z(0,\, x)\, dx.
\end{aligned}$$

Due to (1.13) this implies

$$\begin{aligned}
0 &= \int_0^T \left[r(t,\, x)^\top D\, z(t,\, x)\right]\Big|_{x=0}^L dt - \int_0^L r^0(x)^\top z(0,\, x)\, dx \\
&= \int_0^T d_+\, z_+(t,\, x)\, r_+(t,\, x)|_{x=0}^L\, dt + \int_0^T d_-\, z_-(t,\, x)\, r_-(t,\, x)|_{x=0}^L\, dt \\
&- \int_0^L r^0(x)^\top z(0,\, x)\, dx.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_0^T d_+(0)\, z_+(t,\, 0)\, r_+(t,\, 0) + d_-(0)\, z_-(t,\, 0)\, r_-(t,\, 0)\, dt \\
&= \int_0^T d_+(L)\, z_+(t,\, L)\, r_+(t,\, L) + d_-(L)\, z_-(t,\, L)\, r_-(t,\, L)\, dt \\
&\qquad - \int_0^L r^0(x)^\top z(0,\, x)\, dx.
\end{aligned} \tag{1.15}$$

Due to the definition of F_T and (1.13) this implies the equation

$$\begin{aligned}
&\int_0^T \langle F_T\,(u,\, r^0)(t),\, z_T(t)\rangle_{\mathbb{R}^2}\, dt \\
&= \int_0^T r_+(t,\, L)\, z_+^T(t) + r_-(t,\, 0)\, z_-^T(t)\, dt \\
&= \int_0^T d_+(L)\, r_+(t,\, L)\, z_+(t,\, L) + |d_-(0)|\, r_-(t,\, 0)\, z_-(t,\, 0)\, dt \\
&= \int_0^T d_+(0)\, r_+(t,\, 0)\, z_+(t,\, 0) + |d_-(L)|\, r_-(t,\, L)\, z_-(t,\, L)\, dt
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^L r^0(x)^\top z(0,\, x)\, dx \\
&= \int_0^T u_+(t)\, d_+(0)\, z_+(t,\, 0) + u_-(t)\, |d_-(L)|\, z_-(t,\, L)\, dt \\
&+ \int_0^L r^0(x)^\top z(0,\, x)\, dx \\
&= \left\langle \begin{pmatrix} u \\ r^0 \end{pmatrix},\, F_T{}^*(z_T) \right\rangle_{H(T)\times (L^2(0,L))^2}.
\end{aligned}
$$

Hence (1.11) holds.

1.4. **Necessary optimality conditions for the dynamic problem.** In order to determine the structure of the optimal control $u^{(\delta,\,T)}$ that solves the dynamic optimal control problem (1.7) we look at the necessary optimality conditions. Due to the definition of the convex admissible set K_T and the assumed Slater condition $u^{(\delta,\,T)}$ can only solve (1.7) if there exists a multiplier $\mu^{(\delta,\,T)} \geq 0$ such that the complementarity condition

$$\mu^{(\delta,\,T)}\, G_T(u^{(\delta,\,T)}) = 0 \tag{1.16}$$

holds and with the Fréchet derivatives J_T' and G_T' we have

$$J_T'(u^{(\delta,\,T)}) = -\mu^{(\delta,\,T)}\, G_T'(u^{(\delta,\,T)}). \tag{1.17}$$

For all $u \in H(T)$ and r that satisfy (1.2), we have

$$J_T(u) = \int_0^T f(u(t),\, F_T\,(u,\, r^0))\, dt. \tag{1.18}$$

Let $\tilde{u} = u + \delta^{(1)}$ with a control variation $\delta^{(1)} \in H(T)$. Let $\tilde{F}_T$ be as in Lemma 1.3. Then Lemma 1.4 implies $\tilde{F}_T^* \begin{pmatrix} z_+^T(\cdot) \\ z_-^T(\cdot) \end{pmatrix} = \begin{pmatrix} d_+(0)\, z_+(\cdot,\, 0) \\ |d_-(L)|\, z_-(\cdot, L) \end{pmatrix}$. Since J_T is convex, we have

$$
\begin{aligned}
J_T(\tilde{u}) \;\; &\geq \;\; J_T(u) + \int_0^T f_u(u,\, F_T(u,\, r^0))\, \delta^{(1)} + f_R(u,\, F_T(u,\, r^0))\, \tilde{F}_T(\delta^{(1)})\, dt \\
&= J_T(u) + \int_0^T \Big[f_u(u,\, F_T(u,\, r^0)) + \tilde{F}_T^* f_R(u,\, F_T(u,\, r^0)) \Big]\, \delta^{(1)}\, dt.
\end{aligned}
$$

This implies

$$J_T'(u) = f_u(u,\, F_T(u,\, r^0)) + \tilde{F}_T^* f_R(u,\, F_T(u,\, r^0)).$$

Hence (1.17) yields the optimality conditions that are stated in the following lemma. Due to the convexity of the problem, they are necessary and sufficient (see also the Lagrange multiplier rule as e.g. in [14]).

Lemma 1.5. *The control $u^{(\delta,\,T)}$ is a solution of the dynamic optimal control problem (1.7) if and only if there exist a multiplier*

$$p^{(\delta,\,T)} = \tilde{F}_T^*(f_R(u^{(\delta,\,T)},\, F_T\,(u^{(\delta,\,T)},\, r^0))) \in H(T) \tag{1.19}$$

and a multiplier $\mu^{(\delta,T)} \geq 0$ such that the optimality system

$$f_u(u^{(\delta,T)},\, F_T\,(u^{(\delta,T)},\, r^0)) + p^{(\delta,T)} = -\mu^{(\delta,T)}\, G'_T(u^{(\delta,T)}) \tag{1.20}$$

holds, that is for $(t,\, x) \in (0,\, T) \times (0,\, L)$ almost everywhere we have

$$\begin{cases} R^{(\delta,T)}(0,\, x) = r^0, \\ R_t^{(\delta,T)} + D\, R_x^{(\delta,T)} = \eta_0\, M\, R^{(\delta,T)}, \\ R_+^{(\delta,T)}(t,\, 0) = u_+^{(\delta,T)}(t), \\ R_-^{(\delta,T)}(t,\, L) = u_-^{(\delta,T)}(t), \\ p^{(\delta,T)}(T,\, x) = 0, \\ p_t^{(\delta,T)} + D\, p_x^{(\delta,T)} = [-\eta_0\, M - D']\; p^{(\delta,T)}, \\ p_+^{(\delta,T)}(t,\, L) = \frac{1}{d_+(L)}\, f_{R_+}(u^{(\delta,T)}(t),\, R^{(\delta,T)}(t)), \\ p_-^{(\delta,T)}(t,\, 0) = \frac{1}{|d_-(0)|}\, f_{R_-}(u^{(\delta,T)}(t),\, R^{(\delta,T)}(t)) \end{cases} \tag{1.21}$$

and

$$\begin{cases} f_{u_+}(u^{(\delta,T)}(t),\, R^{(\delta,T)}(t)) + d_+(0)\, p_+^{(\delta,T)}(t,\, 0) = -\mu^{(\delta,T)}\, G'_T(u^{(\delta,T)})_+, \\ f_{u_-}(u^{(\delta,T)}(t),\, R^{(\delta,T)}(t)) + |d_-(L)|\, p_-^{(\delta,T)}(t,\, L) = -\mu^{(\delta,T)}\, G'_T(u^{(\delta,T)})_-. \end{cases} \tag{1.22}$$

1.5. **A static optimal control problem.** In the static optimal control problem the initial boundary value problem (1.2) is replaced by the boundary value problem with an ordinary differential equation

$$\begin{cases} D\, R_x^{(\sigma)}(x) = \eta_0\, M\, R^{(\sigma)}(x), \\ R_+^{(\sigma)}(0) = u_+^{(\sigma)}, \\ R_-^{(\sigma)}(L) = u_-^{(\sigma)} \end{cases} \tag{1.23}$$

with $x \in (0,\, L)$ and $u^{(\sigma)} = \begin{pmatrix} u_+^{(\sigma)} \\ u_-^{(\sigma)} \end{pmatrix} \in \mathbb{R}^2$. The static optimization problem that corresponds to the dynamic optimal control problem (1.7) from Section 1.2 is

$$\begin{cases} \min_{u^{(\sigma)} \in \mathbb{R}^2 \cap K_1} f(u^{(\sigma)},\, R_+^{(\sigma)}(L),\, R_-^{(\sigma)}(0)) \\ \text{subject to (1.23).} \end{cases} \tag{1.24}$$

Due to assumption (1.6) for all $T > 0$ problem (1.24) is equivalent to

$$\begin{cases} \min_{u^{(\sigma)} \in \mathbb{R}^2 \cap K_T} f(u^{(\sigma)},\, R_+^{(\sigma)}(L),\, R_-^{(\sigma)}(0)) \\ \text{subject to (1.23).} \end{cases} \tag{1.25}$$

1.6. **An adjoint operator for the static problem.** We define the static operator $F_{(\sigma)}\,(u^{(\sigma)})$ that maps the boundary control $u^{(\sigma)} = (u_+^{(\sigma)},\, u_-^{(\sigma)}) \in \mathbb{R}^2$ to the point $(r_+^{(\sigma)}(L),\, r_-^{(\sigma)}(0))$, where $r^{(\sigma)}$ solves the linear boundary value problem (for $x \in (0, L)$)

$$\begin{cases} r_x^{(\sigma)} = \eta_0\, D^{-1}\, M\, r^{(\sigma)}, \\ r_+^{(\sigma)}(0) = u_+^{(\sigma)}, \\ r_-^{(\sigma)}(L) = u_-^{(\sigma)}. \end{cases} \tag{1.26}$$

Thus we have

$$F_{(\sigma)} \begin{pmatrix} u_+^{(\sigma)} \\ u_-^{(\sigma)} \end{pmatrix} = \begin{pmatrix} r_+^{(\sigma)}(L) \\ r_-^{(\sigma)}(0) \end{pmatrix}. \tag{1.27}$$

In Lemma 1.6 an explicit representation of the adjoint operator F_T^* is given that satisfies for all $z \in \mathbb{R}^2$ the equation $\langle F_{(\sigma)}\,(u^{(\sigma)}),\, z\rangle_{\mathbb{R}^2} = \langle u^{(\sigma)},\, F^*_{(\sigma)}(z)\rangle_{\mathbb{R}^2}$.

Lemma 1.6. *For $z = (z_+,\, z_-)^T \in \mathbb{R}^2$, define $(z_+^{(\sigma)}(\cdot),\, z_-^{(\sigma)}(\cdot)) \in (C^{(1)}(0,\, L))^2$ as the solution of the adjoint system*

$$\begin{cases} z_x^{(\sigma)} = -D^{-1}\left[\eta_0\, M + D'\right]\, z^{(\sigma)}, \\ z_+^{(\sigma)}(L) = \frac{1}{d_+(L)}\, z_+, \\ z_-^{(\sigma)}(0) = \frac{1}{|d_-(0)|}\, z_-. \end{cases} \tag{1.28}$$

Then we have

$$F^*_{(\sigma)} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} d_+(0)\, z_+^{(\sigma)}(0) \\ |d_-(L)|\, z_-^{(\sigma)}(L) \end{pmatrix}. \tag{1.29}$$

Proof: Due to (1.26) we have the equation

$$\begin{aligned} 0 \;&=\; \int_0^L \left(z^{(\sigma)}\right)^\top \left[D\, r_x^{(\sigma)} - \eta_0\, M\, r^{(\sigma)}\right]\, dx \\ &=\; -\int_0^L \left(r^{(\sigma)}\right)^\top\, D\, z_x^{(\sigma)}\, dx - \int_0^L \left(r^{(\sigma)}\right)^\top\, D'\, z^{(\sigma)}\, dx \\ &+\; \left[\left(z^{(\sigma)}(x)\right)^\top\, D\, r^{(\sigma)}(x)\right]\Big|_{x=0}^{L} - \int_0^L \left(r^{(\sigma)}\right)^\top\, \eta_0\, M\, z^{(\sigma)}\, dx \\ &=\; -\int_0^L \left(r^{(\sigma)}\right)^\top \left(D\, z_x^{(\sigma)} + \left[D' + \eta_0\, M\right]\, z^{(\sigma)}\right)\, dx + \left[\left(z^{(\sigma)}(x)\right)^\top\, D\, r^{(\sigma)}(x)\right]\Big|_{x=0}^{L}. \end{aligned}$$

Due to (1.28) this implies

$$\begin{aligned} 0 \;&=\; \left[\left(z^{(\sigma)}(x)\right)^\top\, D\, r^{(\sigma)}(x)\right]\Big|_{x=0}^{L} \\ &=\; d_+(x)\, z_+^{(\sigma)}(x)\, r_+^{(\sigma)}(x)|_{x=0}^{L} + d_-(x)\, z_-^{(\sigma)}(x)\, r_-^{(\sigma)}(x)|_{x=0}^{L}. \end{aligned}$$

Thus we have

$$\begin{aligned} &d_+(0)\, z_+^{(\sigma)}(0)\, r_+^{(\sigma)}(0) + d_-(0)\, z_-^{(\sigma)}(0)\, r_-^{(\sigma)}(0) \\ =\;& d_+(L)\, z_+^{(\sigma)}(L)\, r_+^{(\sigma)}(L) + d_-(L)\, z_-^{(\sigma)}(L)\, r_-^{(\sigma)}(L). \end{aligned}$$

This implies the equation

$$\begin{aligned} \langle F_{(\sigma)}\,(u),\, z\rangle_{\mathbb{R}^2} \;&=\; r_+^{(\sigma)}(L)\, z_+ + r_-^{(\sigma)}(0)\, z_- \\ &=\; d_+(L)\, r_+^{(\sigma)}(L)\, z_+^{(\sigma)}(L) + |d_-(0)|\, r_-^{(\sigma)}(0)\, z_-^{(\sigma)}(0) \\ &=\; d_+(0)\, r_+^{(\sigma)}(0)\, z_+^{(\sigma)}(0) + |d_-(L)|\, r_-^{(\sigma)}(L)\, z_-^{(\sigma)}(L) \\ &=\; u_+^{(\sigma)}\, d_+(0)\, z_+^{(\sigma)}(0) + u_-^{(\sigma)}\, |d_-(L)|\, z_-^{(\sigma)}(L) \end{aligned}$$

$$= \quad \langle u^{(\sigma)},\, F^*_{(\sigma)}(z)\rangle_{\mathbb{R}^2}$$

with $F^*_{(\sigma)}$ as defined in (1.29). Hence the adjoint operator $F^*_{(\sigma)}$ is indeed given by (1.29).

1.7. **Necessary optimality conditions for the static problem.** Let $u^{(\sigma)}$ denote the optimal control that solves (1.24) and $R^{(\sigma)}$ the state generated by $u^{(\sigma)}$ as a solution of (1.23). Due to the definition of the convex admissible set K_T and the assumed Slater condition $u^{(\sigma)}$ can only solve (1.25) if there exists a multiplier $\mu^{(\sigma,T)} \geq 0$ such that the complementarity condition

$$\mu^{(\sigma,T)}\, G_T(u^{(\sigma)}) = 0 \tag{1.30}$$

holds and we have

$$f_u(u^{(\sigma)},\, F_{(\sigma)}(u^{(\sigma)})) + F^*_{(\sigma)} f_R(u^{(\sigma)},\, F_{(\sigma)}(u^{(\sigma)})) = -\mu^{(\sigma,T)}\, G'_T(u^{(\sigma)}). \tag{1.31}$$

This is equivalent to the optimality system

$$\begin{cases} D\, R_x^{(\sigma)} = \eta_0\, M\, R^{(\sigma)}, \\ R_+^{(\sigma)}(0) = u_+^{(\sigma)}, \\ R_-^{(\sigma)}(L) = u_-^{(\sigma)}, \\ D\, P_x^{(\sigma)} = -\left[D' + \eta_0\, M\right]\, P^{(\sigma)}, \\ P_+^{(\sigma)}(L) = \frac{1}{d_+(L)}\, f_{R_+}(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})), \\ P_-^{(\sigma)}(0) = \frac{1}{|d_-(0)|}\, f_{R_-}(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \end{cases} \tag{1.32}$$

and

$$\begin{cases} f_{u_+}(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) + d_+(0)\, P_+^{(\sigma)}(0) = -\mu^{(\sigma,T)}\, G'_T(u^{(\sigma)})_+, \\ f_{u_-}(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) + |d_-(L)|\, P_-^{(\sigma)}(L) = -\mu^{(\sigma,T)}\, G'_T(u^{(\sigma)})_-. \end{cases} \tag{1.33}$$

2. A turnpike result

In this section we show that the optimal controls that solve the dynamic optimal control problem (1.7) and the corresponding static optimal control problem (1.24) satisfy a turnpike inequality in the sense that there exists a constant $\zeta_0 > 0$ such that for all $T > 0$ we have the inequality

$$\frac{1}{T}\int_0^T \left\| u^{(\delta,T)}(\tau) - u^{(\sigma)} \right\|^2_{\mathbb{R}^2}\, d\tau \leq \frac{\zeta_0}{T}. \tag{2.1}$$

The norm on the left–hand side is divided by T in order to normalize it, in the sense that for a constant function c we have $\frac{1}{T}\int_0^T c^2\, dt = c^2$ for all $T > 0$. Note that the turnpike inequality (2.1) implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u^{(\delta,T)}(\tau)\, d\tau = u^{(\sigma)}. \tag{2.2}$$

This means that asymptotically for $T \to \infty$ the average value of the dynamic optimal control converges to the optimal static control. In fact the convergence is of the order $\frac{1}{\sqrt{T}}$. Now we state our main turnpike result.

Theorem 2.1. *Assume that the gradients f_R are Lipschitz continuous in the sense that there exists a Lipschitz constant L_i such that for all $T > 0$ for all $u_1,\, u_2 \in K_T$, $R_1,\, R_1 \in H(T)$ we have*

$$\|f_R(u_1,\, R_1) - f_R(u_1,\, R_2)\|^2_{H(T)} \le L_i^2 \left(\|u_1 - u_2\|^2_{H(T)} + \|R_1 - R_2\|^2_{H(T)}\right). \tag{2.3}$$

Then there exists a constant $\zeta_0 > 0$ that is independent of T such that for all $T > 0$ the turnpike inequality (2.1) holds.

Remark 2.2. To be more precise, we give an explicit representation of ζ_0. Define

$$c_0 = \sup_{x\in[0,\,L]} \{\tfrac{1}{\sqrt{d_+(x)}},\, \tfrac{1}{\sqrt{|d_-(x)|}}\}\, \|r^0\|_{(L^2(0,\,L))^2}, \tag{2.4}$$

$$c_1 = \sup_{x\in[0,\,L]} \{\tfrac{1}{\sqrt{d_+(x)}},\, \tfrac{1}{\sqrt{|d_-(x)|}}\}\, \|F_{(\sigma)} u^{(\sigma)})\|_{(L^2(0,\,L))^2}, \tag{2.5}$$

$$c_2 = \max\{d_+(0),\, |d_-(L)|\}\, \|F^*_{(\sigma)} f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\|^2_{(L^2(0,\,L))^2}. \tag{2.6}$$

Then

$$\sqrt{\zeta_0} = \frac{1}{2\,\kappa}\left(\sqrt{(c_2 + 2\,L_i(c_0 + c_1))^2 + 4\kappa L_i (c_0 + c_1)^2} - 2\,L_i\,(c_0 + c_1) - c_2\right).$$

This can be seen in the proof of Theorem 2.1. Note that if $\|r^0\|_{(L^2(0,\,L))^2}$ tends towards infinity, also c_0 becomes arbitrarily large which implies that also $\sqrt{\zeta_0}$ becomes arbitrarily large.

Example 2.3. The assumptions of Theorem 2.1 are valid for the linear quadratic case where $f(u,R) = (u_+, u_-, R_+, R_-)\, A\, (u_+, u_-, R_+, R_-)^\top + v^\top\, (u_+, u_-, R_+, R_-)^\top$ with a positive definite matrix A and a vector $v \in \mathbb{R}^4$.

For the proof of Theorem 2.1 we need an auxiliary result that we present in the next section.

3. On the difference of the dynamic and the static adjoint operator

In this section we show that the application of the difference of the dynamic and the static adjoint operator to a static input yields an output that decays exponentially backwards in time from T to 0.

Lemma 3.1. *For $h \in \mathbb{R}^2$ define the number*

$$C_*(h) = \max\{d_+(0),\, |d_-(L)|\}\, \|F^*_{(\sigma)} h\|^2_{(L^2(0,\,L))^2}. \tag{3.1}$$

Then for all $T > 0$ we have the inequality

$$\int_0^T \left\|\left(\left(\tilde F^*_T - F^*_{(\sigma)}\right) h\right)(\tau)\right\|^2_{\mathbb{R}^2} d\,\tau \le C_*(h). \tag{3.2}$$

Proof: For $t \in [0,\, T]$, let z be defined as the solution of

$$\begin{cases} z(T,\, x) = -F^*_{(\sigma)} h,\; x \in (0, L), \\ z_t(t,\, x) + D\, z_x(t,\, x) = [-D'(x) - \eta_0\, M(x)]\; z(t,\, x), \\ z_+(t,\, L) = 0, \\ z_-(t,\, 0) = 0. \end{cases} \tag{3.3}$$

We have

$$\left\| \left(\left(\tilde{F}_T^* - F_{(\sigma)}^* \right) h \right)(t) \right\|_{\mathbb{R}^2}^2 = \| (d_+(0)\, z_+(t,0),\, |d_-(L)|\, z_-(t,L)) \|_{\mathbb{R}^2}^2 .$$

Similar as in the proof of Lemma 1.3 we have

$$\begin{aligned}
& \left\| \begin{pmatrix} d_+(L)\, z_+(t,L) \\ d_-(0)\, z_-(t,\,0) \end{pmatrix} \right\|_{H(T)}^2 - \left\| \begin{pmatrix} d_+(0)\, z_+(t,\,0) \\ d_-(L)\, z_-(t,\,L) \end{pmatrix} \right\|_{H(T)}^2 \\
= & \int_0^T d_+(L)^2\, z_+(t,\,L)^2 + d_-(0)^2\, z_-(t,\,0)^2 - d_+(0)^2\, z_+(t,\,0)^2 - d_-(L)^2\, z_-(t,\,L)^2\, dt \\
= & \int_0^T \int_0^L \left(z^\top |D|\, D\, z \right)_x dx\, dt \\
= & \int_0^T \int_0^L 2\, z^\top |D|\, D\, z_x + 2\, z^\top |D|\, D'\, z\, dx\, dt.
\end{aligned}$$

Due to (3.3) this yields

$$\begin{aligned}
& \left\| \begin{pmatrix} d_+(L)\, z_+(t,L) \\ d_-(0)\, z_-(t,\,0) \end{pmatrix} \right\|_{H(T)}^2 - \left\| \begin{pmatrix} d_+(0)\, z_+(t,\,0) \\ d_-(L)\, z_-(t,\,L) \end{pmatrix} \right\|_{H(T)}^2 \\
= & \int_0^T \int_0^L -2\, z^\top |D|\, z_t + 2\, z^\top\, |D|\, \left[|\eta_0|\, M - D' \right] z + 2\, z^\top\, |D|\, D'\, z\, dx\, dt \\
= & \int_0^L - \left(z^\top |D|\, z \right)\Big|_{\tau=0}^T dx + 2 \int_0^T \int_0^L z^\top\, |\eta_0|\, |D|\, M\, z\, dx\, dt \\
= & \int_0^L \left(z^\top |D|\, z \right)(0,\,x)\, dx - \int_0^L \left(z^\top |D|\, z \right)(T,\,x)\, dx \\
+ & \int_0^T \int_0^L 2\, |\eta_0|\, z^\top\, |D|\, M z\, dx\, dt \\
= & \int_0^L \left(z^\top |D|\, z \right)(0,\,x)\, dx - \int_0^L \left(z^\top |D|\, z \right)(T,\,x)\, dx \\
+ & \int_0^T \int_0^L z^\top\, \left[|\eta_0|\, (|D|\, M + M\, |D|) \right] z\, dx\, dt \\
\geq & -\int_0^L \left(z^\top |D|\, z \right)(T,\,x)\, dx
\end{aligned}$$

where the last inequality follows with (1.1). Since $z_+(t,\,L) = 0 = z_-(t,\,0)$, this implies

$$\int_0^T \left\| \left(\left(\tilde{F}_T^* - F_{(\sigma)}^* \right) h \right)(\tau) \right\|_{\mathbb{R}^2}^2 d\tau \leq \max\{d_+,\, |d_-|\} \int_0^L \| z(T,\,x) \|_{\mathbb{R}^2}^2\, dx.$$

Thus (3.2) follows.

4. Proof of the main result

Our auxiliary results allow us to prove Theorem 2.1.

Proof of Theorem 2.1. Due to the optimality conditions (1.20) for the dynamic problem (1.7) we have

$$f_u(u^{(\delta,T)},\, F_T\,(u^{(\delta,T)},\, r^0)) + \tilde F_T^* f_R(u^{(\delta,T)},\, F_T(u^{(\delta,T)},\, r^0)) = -\mu^{(\delta,T)}\, G_T'(u^{(\delta,T)}). \tag{4.1}$$

Due to the optimality conditions (1.31) for the static problem (1.24) we have

$$f_u(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) + F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) = -\mu^{(\sigma,T)}\, G_T'(u^{(\sigma)}). \tag{4.2}$$

Due to (1.16), (1.30), the convexity of G_T and since $\mu^{(\delta,T)} \geq 0$ and $\mu^{(\sigma,T)} \geq 0$ we have

$$\mu^{(\delta,T)}\langle G_T'(u^{(\delta,T)}), u^{(\sigma)} - u^{(\delta,T)}\rangle_{H(T)} \leq 0, \tag{4.3}$$

$$\mu^{(\sigma,T)}\langle G_T'(u^{(\sigma)}), u^{(\delta,T)} - u^{(\sigma)}\rangle_{H(T)} \leq 0. \tag{4.4}$$

The difference of (4.1) and (4.2) yields the equation

$$\begin{aligned} & f_u(u^{(\delta,T)},\, F_T\,(u^{(\delta,T)},\, r^0)) - f_u(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \\ = {} & F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde F_T^* f_R(u^{(\delta,T)},\, F_T(u^{(\delta,T)},\, r^0)) \\ & + \mu^{(\sigma,T)}\, G_T'(u^{(\sigma)}) - \mu^{(\delta,T)}\, G_T'(u^{(\delta,T)}). \end{aligned} \tag{4.5}$$

Define the numbers

$$\begin{aligned} L_0 &= \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)}\right)^\top \left(f_u(u^{(\delta,T)},\, F_T\,(u^{(\delta,T)},\, r^0)) - f_u(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\right) d\tau, \\ L_1 &= \int_0^T \left(F_T\,(u^{(\delta,T)},\, r^0) - F_{(\sigma)}\,(u^{(\sigma)})\right)^\top \\ & \quad \left(f_R(u^{(\delta,T)},\, F_T\,(u^{(\delta,T)},\, r^0)) - f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\right) d\tau. \end{aligned}$$

The strong convexity assumption (1.3) implies the inequality

$$L_0 + L_1 \geq \kappa \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)}\right)^2 d\tau. \tag{4.6}$$

Equation (4.5), (4.3) and (4.4) imply

$$L_0 \leq \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)}\right)^\top$$
$$\left(F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde F_T^* f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\right) d\tau.$$

Hence we have

$$\begin{aligned} L_0 \leq {} & \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)}\right)^\top \\ & \left(F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde F_T^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\right. \\ + {} & \left.\tilde F_T^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde F_T^* f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\right) d\tau \\ = {} & \int_0^T \left(\tilde F_T\left(u^{(\delta,T)}(\tau) - u^{(\sigma)}\right)\right)^\top \end{aligned}$$

$$\Big(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\Big)\, d\tau$$
$$+\quad \int_0^T \Big(u^{(\delta,T)}(\tau) - u^{(\sigma)}\Big)^\top \Big(F^*_{(\sigma)} - \tilde F^*_T\Big)\, f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\, d\tau.$$

With the definition of L_1 this implies

$$\begin{aligned} L_0 + L_1 \quad &\le \quad \int_0^T \Big(\tilde F_T\, \Big(u^{(\delta,T)}(\tau) - u^{(\sigma)}\Big) - F_T\,(u^{(\delta,T)},\, r^0) + F_{(\sigma)}\,(u^{(\sigma)})\Big)^\top \\ &\qquad \Big(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\Big)\, d\tau \\ &+ \quad \int_0^T \Big(u^{(\delta,T)}(\tau) - u^{(\sigma)}\Big)^\top \Big(F^*_{(\sigma)} - \tilde F^*_T\Big)\, f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\, d\tau \end{aligned}$$

With the definition of the linear operator $\tilde F_T = F_T(\cdot, 0)$ this yields

$$\begin{aligned} L_0 + L_1 \quad &\le \quad \int_0^T \Big(F_{(\sigma)}\,(u^{(\sigma)}) - F_T(u^{(\sigma)}, 0) - F_T(0,\, r_0)\Big)^\top \\ &\qquad \Big(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\Big)\, d\tau \\ &+ \quad \int_0^T \Big(u^{(\delta,T)}(\tau) - u^{(\sigma)}\Big)^\top \Big(F^*_{(\sigma)} - \tilde F^*_T\Big)\, f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\, d\tau \\ &= \quad R_0 + R_1 + R_2 \end{aligned}$$

where

$$R_0 = \int_0^T -\left(F_T(0, r^0)\right)^\top \Big(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\Big)\, d\tau,$$
$$R_1 = \int_0^T \Big(\Big[F_{(\sigma)} - \tilde F_T\Big]\,(u^{(\sigma)})\Big)^\top \Big(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\Big)\, d\tau,$$
$$R_2 = \int_0^T \Big(u^{(\delta,T)}(\tau) - u^{(\sigma)}\Big)^\top \Big[F^*_{(\sigma)} - \tilde F^*_T\Big]\, f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\, d\tau.$$

Hence (4.6) yields the inequality

$$\kappa \int_0^T \Big(u^{(\delta,T)}(\tau) - u^{(\sigma)}\Big)^2\, d\tau \le L_0 + L_1 \le R_0 + R_1 + R_2. \tag{4.7}$$

The assumed Lipschitz condition (2.3) implies that

$$\Big\| f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0)) \Big\|^2_{H(T)}$$
$$\le L_i^2\, \Big(\|u^{(\delta,T)} - u^{(\sigma)}\|^2_{H(T)} + \|F_T(\, u^{(\delta,T)}),\, r^0) - F_{(\sigma)}\,(u^{(\sigma)})\|^2_{H(T)}\Big).$$

Hence we have

$$\begin{aligned} &\Big\| f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0)) \Big\|_{H(T)} \\ &\le L_i\, \Big(\|u^{(\delta,T)} - u^{(\sigma)}\|_{H(T)} + \|\tilde F_T(u^{(\delta,T)} - u^{(\sigma)}) \\ &\qquad + F_T(0,\, r^0) + \Big[\tilde F_T - F_{(\sigma)}\Big]\, u^{(\sigma)}\|_{H(T)}\Big) \end{aligned}$$

$$\leq L_i \left(1 + \|\tilde{F}_T\|\right) \|u^{(\delta,T)} - u^{(\sigma)}\|_{H(T)}$$
$$+ L_i \left(\|F_T(0,\, r^0)\|_{H(T)} + \| \left[\tilde{F}_T - F_{(\sigma)}\right] u^{(\sigma)}\|_{H(T)}\right)$$

Due to (1.10), for

$$c_0 := \sup_{x\in[0,\,L]} \{\tfrac{1}{\sqrt{d_+(x)}},\ \tfrac{1}{\sqrt{|d_-(x)|}}\} \,\|r^0\|_{(L^2(0,\,L))^2}$$

we have $\|F_T(0,\, r^0)\|_{H(T)} \leq c_0$. Define

$$c_1 := \sup_{x\in[0,\,L]} \{\tfrac{1}{\sqrt{d_+(x)}},\ \tfrac{1}{\sqrt{|d_-(x)|}}\} \,\|F_{(\sigma)}u^{(\sigma)})\|_{(L^2(0,\,L))^2}.$$

We have

$$\left[\tilde{F}_T - F_{(\sigma)}\right] u^{(\sigma)} = F_T(0,\, -F_{(\sigma)}u^{(\sigma)}).$$

Hence (1.10) implies $\left\|\left[\tilde{F}_T - F_{(\sigma)}\right] u^{(\sigma)}\right\|_{H(T)} \leq c_1$.

Since $\|\tilde{F}_T\| \leq 1$ this yields the inequality

$$\left\|f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,T)},\, F_T(\, u^{(\delta,T)},\, r^0))\right\|_{H(T)}$$
$$\leq L_i \left(2\,\|u^{(\delta,T)} - u^{(\sigma)}\|_{H(T)} + c_0 + c_1\right)$$

Now we derive an upper bound for R_0. We have

$$R_0 \leq L_i\, c_0 \left(2\, \left\|u^{(\delta,T)} - u^{(\sigma)}\right\|_{H(T)} + c_0 + c_1\right).$$

Similarly, for R_1 we obtain the upper bound

$$R_1 \leq L_i\, c_1 \left(2\, \left\|u^{(\delta,T)} - u^{(\sigma)}\right\|_{H(T)} + c_0 + c_1\right).$$

Now we derive an upper bound for R_2. Define $c_2 = C_*(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})))$. Due to (3.2) in Lemma 3.1 we have

$$\left\|\left(F^*_{(\sigma)} - \tilde{F}^*_T\right) f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\right\|^2_H \quad\leq\quad c_2.$$

Then

$$R_2 \leq c_2 \left\|u^{(\delta,T)} - u^{(\sigma)}\right\|_{H(T)}.$$

Now (4.7) yields the inequality

$$\kappa \left\|u^{(\delta,T)} - u^{(\sigma)}\right\|^2_{H(T)} \tag{4.8}$$

$$\leq L_i\, (c_0 + c_1) \left(2\, \left\|u^{(\delta,T)} - u^{(\sigma)}\right\|_{H(T)} + c_0 + c_1\right) + c_2 \left\|u^{(\delta,T)} - u^{(\sigma)}\right\|_{H(T)}.$$

Define the polynomial

$$P(x) = \kappa\, x^2 - (c_2 + 2\, L_i\, (c_0 + c_1))\, x - L_i\, (c_0 + c_1)^2\,.$$

Then the set $\mathcal{M} = \{x \in [0, \infty) : P(x) \leq 0\}$ is bounded and there exists a number $\zeta_0 > 0$ such that $\mathcal{M} = [0, \sqrt{\zeta_0}]$. Since (4.8) implies $\left\|u^{(\delta, T)} - u^{(\sigma)}\right\|_{H(T)} \in \mathcal{M}$, this implies the inequality

$$\left\|u^{(\delta, T)} - u^{(\sigma)}\right\|_{H(T)} \leq \sqrt{\zeta_0}$$

and inequality (2.1) follows. Thus we have proved Theorem 2.1.

5. Application: Gas pipeline flow

In [10] we have stated a quasilinear model for isothermal gas pipeline flow and derived the diagonal form with the corresponding Riemann invariants. The eigenvalues of the system matrix are $c + v$ and $-c + v$, where $c > 0$ is the sound speed in the gas which is constant for ideal gas and v is the speed of the gas flow. Note that in gas pipeline operations, $|v|$ is much smaller than c in order to a avoid strong fluid structure interactions. Hence as a simplified approximation to our model we can consider the system in diagonal form with the constant eigenvalues $d_+ = c$ and $d_- = -c$. Then in terms of the Riemann invariants the simplified system has the semilinear form $R_t + D\,R_x = F(R)$ with a nonlinear function F. If we linearize the system, that is the source term F around a stationary state $\bar{R}$ with $D\,\bar{R}_x = F(\bar{R})$, for $r = R - \bar{R}$ we obtain the system $r_t + D\,r_x = F'(\bar{R})\,r$. For the model from [10] we obtain

$$F'(\bar{R}) = -2\,\theta\,|\bar{R}_+(x) - \bar{R}_-(x)| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

where $\theta > 0$ is a constant. With the choice $\eta_0 = -2\,\theta$ and the matrix

$$M(x) = |\bar{R}_+(x) - \bar{R}_-(x)| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

we have $F'(\bar{R}) = \eta_0\,M$ and M is positive semidefinite. Since $|D| = c\,I$, assumption (1.1) holds.

The pressure in the gas is given by $\exp\left(\frac{1}{2}\left(r_+ + r_- + \bar{R}_+ + \bar{R}_-\right)\right)$ and the gas velocity is proportional to $r_+ - r_- + \bar{R}_+ - \bar{R}_-$. A possible strictly convex objective function for the entry at $x = 0$ that describes desired values for the pressure (with a transformed value p_{desi}) and gas velocity (given by v_{desi}) is is

$$f_0(r_+, r_-) = \left(\tfrac{1}{2}\left(r_+ + r_-\right) - p_{desi}\right)^2 + \left(\tfrac{1}{2}\left(r_+ - r_-\right) - v_{desi}\right)^2.$$

A possible strictly convex objective function for the exit at $x = L$ that describes a desired state is

$$f_L(r_+, r_-) = \left(r_+ - \bar{R}_+(L)\right)^2 + \left(r_- - \bar{R}_-(L)\right)^2.$$

For the optimal control problem, we combine the two functions f_0 and f_L and obtain with $\lambda \in (0, 1)$

$$f(u, r) = \lambda\,f_0(u_+, r_-) + (1 - \lambda)\,f_L(r_+, u_-).$$

The convex inequality constraint allows to take into account upper bounds for the gas pressure. For example, with a given value $q \geq 1$, consider the convex constraint

$$\frac{1}{T}\int_0^T \frac{1}{L}\int_0^L \left[\exp\left(\tfrac{1}{2}\left(r_+(t, x) + r_-(t, x) + \bar{R}_+(x) + \bar{R}_-(x)\right)\right)\right]^q\,dx\,dt \leq \bar{p}^q$$

with a given upper bound $\bar{p} \in \mathbb{R}$. If $\bar{p}$ is sufficiently large, a Slater point u_S exists. Then Theorem 2.1 is applicable. This allows to approximate the dynamic optimal controls with static optimal controls. The solution of static optimal control problems in gas pipeline networks has been studied for example in [16]. The transient system can be controlled by the stabilization of the state to the static optimal state, see [6].

6. Conclusion

We have shown a turnpike theorem for a problem of optimal boundary control for a system that is governed by a linear 2×2 hyperbolic system with negative and positive eigenvalues. The optimal control problem can also include a convex inequality constraint. The turnpike result shows that regardless of the initial state, the dynamic optimal control approaches the corresponding static optimal control with increasing time horizon. This result can be generalized in several directions. The first generalization concerns the dimension of the system. Instead of a 2×2 system, we can also consider a system with n positive and m negative eigenvalues. For the applications, also a generalization for to the case of coupled networked systems that are defined on a graph is important. Moreover, also the consideration of nonlinear models is interesting, however, this is out of the scope of the current work.

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