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SPACE OF INFINITESIMAL ISOMETRIES AND BENDING OF SHELLS

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Abstract. We discuss infinitesimal isometries of the middle surfaces and present some characteristic conditions for a function to be the normal component of an infinitesimal isometry. Our results show that those characteristic conditions depend on the Gaussian curvature of the middle surfaces: Normal components of infinitesimal isometries satisfy an elliptic problem, or a parabolic one, according to the middle surface being elliptic, or parabolic, respectively. A problem of determining an infinitesimal isometry is changed to that of 1-dimension. Then we apply those results to the energy functionals of bending of shells which have been obtained as two-dimensional problems by the limit theory of Γ-convergence from the three-dimensional nonlinear elasticity. Therefore the limit theory of Γ-convergence reduces to be a one-dimensional problem in the above two cases.

1. Introduction

Let $M \subset I\!\!R^3$ be a smooth surface and let $\Omega \subset M$ be a bounded, open set. A map $V : \Omega \to I\!\!R^3$ is said to be an infinitesimal isometry on Ω if

$$\left\langle \hat{D}_X V, X \right\rangle = 0 \quad \text{for} \quad X \in M_x, \; x \in \Omega,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric of $I\!\!R^3$ and $\hat{D}$ denotes the covariant differential of the Euclidean space $I\!\!R^3$. Let $m \geq 2$ be an integer. We denote by $\mathrm{IS}^m(\Omega, I\!\!R^3)$ all H^m infinitesimal isometries on Ω.

The study of infinitesimal isometries has been a long history, see [34] and references there. Their purposes were to establish "infinitesimal rigidity" for some closed surfaces and their interests were not on the structure of infinitesimal isometries themselves. For a detail survey along this direction, see [34] also.

Our interests in the space $\mathrm{IS}^m(\Omega, I\!\!R^3)$ of infinitesimal isometries are motivated by the recent lower dimensional models for thin structures through Γ-convergence, see [10, 11, 12, 13, 15, 16, 17, 20, 21, 22, 23, 24, 26, 27, 29, 32, 35], and many others. The minimization of quadratic integral functions over the space $\mathrm{IS}^2(\Omega, I\!\!R^3)$ of infinitesimal isometries arises in the linear bending theory. In addition, based on some quantitative rigidity estimate due to [16], [23] demonstrates that the first term in the expansion $u - R$, in terms of the thickness, belongs to the space $\mathrm{IS}^2(\Omega, I\!\!R^3)$ of infinitesimal isometries, where u is a admissible deformation and R is a rigid motion. Moreover, as shown in [22, 23, 24], the some limit energy functionals of shells by the

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Γ-convergence are also over the space $\mathrm{IS}^2(\Omega, I\!\!R^3)$ of infinitesimal isometries. Thus the space $\mathrm{IS}^2(\Omega, I\!\!R^3)$ naturally plays a crucial role in the analysis of shells. The aim of the present paper is to understand the space $\mathrm{IS}^m(\Omega, I\!\!R^3)$.

We now give heuristic overview of our results, whose precise formulations will be presented in the sections later. Let N be the unit normal field of surface M and let $\mathfrak{X}(\Omega)$ be all vector fields on Ω. For $V \in H^m(\Omega, I\!\!R^3)$, we decompose as

$$V = W + wN \quad \text{for} \quad W \in \mathfrak{X}(\Omega), \quad w \in H^m(\Omega).$$

We look for conditions on functions w such that there are H^m vector fields $W \in \mathfrak{X}(\Omega)$ to guarantee $V \in \mathrm{IS}^m(\Omega, I\!\!R^3)$.

Let $H^m_{\mathrm{is}}(\Omega)$ denote all functions $w \in H^m(\Omega)$ such that there are H^m vector fields $W \in \mathfrak{X}(\Omega)$, which are perpendicular to all Killing fields, to ensure that $V = W + wN \in \mathrm{IS}^m(\Omega, I\!\!R^3)$. Section 2 shows that if $w \in H^m_{\mathrm{is}}(\Omega)$, then w satisfies equation (2.25). Moreover, if

$$\bar{\Omega} \subset \exp_o \Sigma(o), \tag{1.1}$$

then equation (2.25) is also sufficient (Theorem 2.5), where $o \in \Omega$ is such that Ω is star-shaped with respect to o and $\exp_o \Sigma(o)$ is the interior of the cut locus of o.

The type of equation (2.25) is subject to the Gaussian curvature function: It is elliptic, or parabolic according to ellipticity, or parabolicity of the middle surface Ω, respectively. The two cases are studied, respectively, in Sections 3 and 4. We have shown that without assumption (1.1) equation (2.30) is still sufficient for a closed spherical shell (Theorem 3.11) but it is not for a cylinder (Theorem 4.2) or a conical shell (Theorem 4.4).

Our results show that the problem to determine whether $w \in H^m_{\mathrm{is}}(\Omega)$ is actually that of 1-dimension in the above two types, respectively. As a consequence of the above reults, we show that $H^m_{\mathrm{is}}(\Omega) \cap C^\infty(\Omega)$ is dense in $H^m_{\mathrm{is}}(\Omega)$ in the norm of $H^m(\Omega)$ if assumption (1.1) holds (Theorems 3.6 and 4.3): Such an issue is actually not trivial. In general, even though Ω is elliptic, an element $V \in \mathrm{IS}^2(\Omega, I\!\!R^3)$ may not be approximated by smooth infinitesimal isometries. An interesting example, discovered in [9] (also see [34]), is a closed smooth surface of non-negative curvature for which the infinitesimal rigidity holds true: All C^∞ infinitesimal isometries are trivial. But there is a C^2 non-trivial infinitesimal isometry. Therefore $H^2_{\mathrm{is}}(\Omega) \cap C^\infty(\Omega)$ is not dense in $H^2_{\mathrm{is}}(\Omega)$ for this surface.

In Section 5 we apply the above reults to some limit energy of Γ-convergence. Then the limit energy functional is changed to a one-dimensional formula over a function space with one variable (Theorem 5.3). In particular, we present the explicit formulas of the limit energy functionals for a spherical shell (Theorem 5.5) and a cylinder shell (Theorem 5.6), respectively, for nonlinear isotropic materials.

Furthermore, we motion that the strain tensor equation and it's applications in elasticity for the hyperbolic surface have also been studied in [39, 40, 41, 42].

Here we do not use the traditional methods where everything is done in a coordinate. We view the middle surface Ω as a 2-dimensional Riemannian manifold with the induced metric to make everything coordinate free as far as possible. When necessary, some special coordinates are chosen to simplify computations as in modelling and control for the classical thin shells, see [1, 2, 3, 4, 5, 6, 7, 14, 19, 25, 36, 37, 38] and many others.

2. Infinitesimal Isometries

Let M be a surface with the induced metric g from $I\!R^3$. Let N be the unit normal field of M. Let $\Omega \subset M$ be an open set. We shall give some characteristic conditions on a function w for which there exists a vector field W such that $V = W + wN$ is to be an infinitesimal isometry (Theorem 2.5).

Let Π be the second fundamental form of M. Let $m \geq 2$ be an integer. Let $\mathrm{IS}^m(\Omega, I\!R^3)$ be all H^m infinitesimal isometries on Ω. It is easy to check that $V = W + wN \in \mathrm{IS}^m(\Omega, I\!R^3)$ is an infinitesimal isometry if and only if, W, w are H^m, and

$$DW(X,X) + w\Pi(X,X) = 0 \quad \text{for} \quad X \in M_x,\ x \in \Omega, \tag{2.1}$$

where D is the Levi-Civita connection of the induced metric g. In particular, if $w = 0$ and $V = W$ is an infinitesimal isometry, then W is said to be a Killing field. Let

$$\mathrm{KF}\,(\Omega, I\!R^3) = \{\, \text{all } C^\infty \text{ Killing fields on } \Omega \,\}. \tag{2.2}$$

Then [30]

$$\dim \mathrm{KF}\,(M, I\!R^3) \leq 3.$$

Let $H^m_{\mathrm{is}}(\Omega)$ denote all functions $w \in H^m(\Omega)$ such that there is a H^m vector field $W \in \mathfrak{X}(\Omega)$, which is perpendicular to all Killing fields in $\mathrm{KF}\,(\Omega, I\!R^3)$, to ensure that $V = W + wN \in \mathrm{IS}^m(\Omega, I\!R^3)$.

Let $o \in M$ be fixed and let $\exp_o : M_o \to M$ be the exponential map in the metric g. For any $v \in M_o$ with $|v| = 1$, there is a unique $t_0(v) > 0$ (or $t_0(v) = \infty$) such that the normal geodesic $\gamma(t) = \exp_o tv$ is the shortest on the interval $[0, t_0]$. Let

$$C(o) = \{\ t_0(v)v \mid v \in M_o,\ |v| = 1\ \},$$

$$\Sigma(o) = \{\ tv \mid v \in M_o,\ |v| = 1,\ 0 \leq t < t_0(v)\ \}.$$

The set $\exp_o C(o) \subset M$ is said to be the cut locus of o and the set $\exp_o \Sigma(o) \subset M$ is called the interior of the cut locus of o. Then

$$M = \exp_o \Sigma(o) \cup \exp_o C(o).$$

Furthermore, $\exp_o : \Sigma(o) \to \exp_o \Sigma(o)$ is a diffeomorphism and $C(o)$ is a zero measure set on M_o. Then $\exp_o C(o)$ is a zero measure set on M since it is the image of the zero measure set $C(o)$, that is, $\exp_o \Sigma(o)$ is M excluding a zero measure set.

We introduce the polar coordinate system at $o \in M$ as follows. Let e_1, e_2 be an orthonormal basis of M_o. Set

$$\sigma(\theta) = \cos\theta e_1 + \sin\theta e_2 \quad \text{for} \quad \theta \in [0, 2\pi). \tag{2.3}$$

Consider a family of two-parameter curves on M given by

$$\mathfrak{F}(t,\theta) = \exp_o t\sigma(\theta) \quad \text{for} \quad t\sigma(\theta) \in \Sigma(o).$$

Then

$$\partial t = \frac{\partial}{\partial t}\mathfrak{F}(t,\theta) = \exp_{o*} \sigma(\theta), \quad \partial\theta = \frac{\partial}{\partial\theta}\mathfrak{F}(\,t\,,\theta) = \ t\ \exp_{o*} \dot\sigma(\theta). \tag{2.4}$$

In particular, the classical Gauss-Jacobi theorem yields

$$g = dt^2 + f^2(t, \theta)d\theta^2 \quad \text{for} \quad x = \exp_o t\sigma(\theta) \in \exp_o \Sigma(o),$$

where $f(t, \theta)$ is the solution to the problem

$$\begin{cases} f_{tt}(t,\theta) + \kappa(t,\theta) f(t,\theta) = 0, \\ f(0,\theta) = 0, \quad f_t(0,\theta) = 1, \end{cases} \tag{2.5}$$

where κ is the Gaussian curvature function on M and $\kappa(t, \theta) = \kappa(\mathfrak{F}(t, \theta))$ [30].

Let

$$T = \partial t, \quad E = \frac{1}{f}\partial\theta \quad \text{for} \quad x \in \exp_o \Sigma(o) - \{o\}. \tag{2.6}$$

Then T, E is a frame field on $\exp_o \Sigma(o) - \{o\}$. We have

$$D_T T = 0, \quad D_T E = 0, \quad D_E T = \frac{f_t}{f}E, \quad D_E E = -\frac{f_t}{f}T \tag{2.7}$$

for $x \in \exp_o \Sigma(o) - \{o\}$.

We suppose that $o \in \Omega$ is given such that Ω is star-shaped with respect to o. Let the frame field T, E be given in (2.6). Let

$$W = \varphi(t, \theta)T + \phi(t, \theta)E \quad \text{for} \quad x = \mathfrak{F}(t, \theta) \in \Omega \cap \exp_o \Sigma(o).$$

In the sequel all our computations are made on the region $\Omega \cap \exp_o \Sigma(o)$. A simple computation shows that the relation (2.1) is equivalent to

$$\begin{cases} \varphi_t + w\Pi(T,T) = 0, \\ f\phi_t - f_t\phi + \varphi_\theta + 2fw\Pi(T,E) = 0, \\ \phi_\theta + f_t\varphi + fw\Pi(E,E) = 0, \\ \varphi(0) = \langle W, \sigma(\theta)\rangle, \quad \phi(0) = \langle W, \dot\sigma(\theta)\rangle. \end{cases} \tag{2.8}$$

Let φ solve the first equation in (2.8) with initial data $\varphi(0) = \langle W, \sigma(\theta)\rangle$. Then

$$\varphi = \langle W_0, \sigma(\theta)\rangle - \int_0^t w\Pi_{11} ds, \tag{2.9}$$

and ϕ solves the second equation in (2.8) if and only if it satisfies

$$\begin{cases} \phi_{tt} + \kappa\phi = P(w), \\ \phi(0) = \langle W, \dot\sigma(\theta)\rangle, \end{cases} \tag{2.10}$$

where

$$P(w) = -2w_1\Pi_{12} + w_2\Pi_{11} - w\Pi_{121} \quad \text{for} \quad x \in \Omega, \tag{2.11}$$

$$w_1 = \langle Dw, T\rangle, \quad w_2 = \langle DW, E\rangle, \quad \Pi_{12} = \Pi(T, E),$$
$$\Pi_{11} = \Pi(T,T), \quad \Pi_{121} = D\Pi(T, E, T),$$

etc. In the computation of (2.10) the formula (2.5) and the one below have been used

$$[\Pi(T, E)]_t = D\Pi(T, E, T) + \Pi(D_T T, E) + \Pi(T, D_T E) = \Pi_{121}.$$

Furthermore, differentiating the third equation in (2.8) with respect to the variable t and using the first equation of (2.8) yield

$$0 = \phi_{t\theta} + f_{tt}\varphi + f_t[w\Pi(E,E) - w\Pi(T,T)] + f[w\Pi(E,E)]_t \quad \text{for} \quad t > 0. \tag{2.12}$$

Letting $t \to 0$ in (2.12), we obtain another initial data for problem (2.10)

$$\phi_t(0) = -w(o)\Pi(\sigma(\theta), \dot{\sigma}(\theta)) + c_0, \tag{2.13}$$

where c_0 is constant.

Let k be an integer. Let $T^k(M)$ be all tensor fields of rank k on M. Let

$$\mathbf{R}_{XY}: \ T^k(M) \to T^k(M)$$

be the curvature operator where X, $Y \in \mathfrak{X}(M)$ are vector fields. For $K \in T^k(M)$, we have the following formulas, called the Ricci identities,

$$D^2K(\cdots, X, Y) = D^2K(\cdots, Y, X) + (\mathbf{R}_{XY}K)(\cdots). \tag{2.14}$$

The above formulas are very useful when we need to exchange the order of the covariant derivatives of a tensor field.

Let M be orientable. Let X be a vector field on M. We define a vector field QX on M by

$$QX = \langle X, e_2\rangle\, e_1 - \langle X, e_1\rangle\, e_2 \quad \text{for} \quad x \in \Omega, \tag{2.15}$$

where e_1, e_2 is an orthonormal basis of M_x with positive orientation. It is easy to check that the right hand side of (2.15) is independent of the choice of the positively oriented orthonormal basis e_1, e_2. The operator Q plays an important role in strain tensors for the hyperbolic surface, see [39, 42].

We seek some conditions on w such that problem (2.1) has a vector field solution W.

Lemma 2.1. *Let M be orientable. Let $V = W + wN$ be an infinitesimal isometry of Ω. Then*

$$\left\langle D^2w, Q^*\Pi\right\rangle + w\kappa \operatorname{tr}\Pi = \langle \nabla\kappa, W\rangle \quad \textit{for} \quad x \in \Omega, \tag{2.16}$$

where Q is defined by (2.15)*, κ is the Gaussian curvature function, and ∇,* tr *are the gradient, the trace of the induced metric of M, respectively.*

Proof. Let $o \in \Omega$ be any point. Then there is $\varepsilon > 0$ such that the geodesic ball centered at o with the radius ε is contained in Ω. Therefore, the systems (2.8) and (2.10) make sense for $(t, \theta) \in [0, \varepsilon) \times [0, 2\pi)$.

From (2.10) and using the symmetry of $D\Pi$, we have

$$\begin{aligned}
&\phi_{tt\theta} + \kappa_\theta\phi + \kappa\phi_\theta = -2(w_1\Pi_{12})_\theta + (w_2\Pi_{11})_\theta - (w\Pi_{121})_\theta \\
&= -2f(w_{12}\Pi_{12} + w_1\Pi_{122}) - 2f_t[w_2\Pi_{12} + w_1(\Pi_{22} - \Pi_{11})] \\
&\quad + f(w_{22}\Pi_{11} + w_2\Pi_{112}) + f_t(2w_2\Pi_{12} - w_1\Pi_{11}) \\
&- f(w_2\Pi_{121} + w\Pi_{1212}) - f_t w(2\Pi_{221} - \Pi_{111}) \\
&= f(w_{22}\Pi_{11} - 2w_{12}\Pi_{12} - 2w_1\Pi_{122} - w\Pi_{1212}) \\
&\quad + f_t[w_1(\Pi_{11} - 2\Pi_{22}) + w(\Pi_{111} - 2\Pi_{221})],
\end{aligned} \tag{2.17}$$

which yields

$$\begin{aligned}
\phi_\theta^{(3)}(0) &= w_{22}\Pi_{11} - 2w_{12}\Pi_{12} - 2w_1\Pi_{122} - w\Pi_{1212} + [w_1(\Pi_{11} - 2\Pi_{22}) \\
&\quad + w(\Pi_{111} - 2\Pi_{221})]'(0) - \kappa_\theta'\phi(0) - \kappa'\phi_\theta(0) - \kappa\phi_\theta'(0) \\
&= w_{11}(\Pi_{11} - 2\Pi_{22}) + w_{22}\Pi_{11} - 2w_{12}\Pi_{12} + 2w_1(\Pi_{111} - 3\Pi_{221})
\end{aligned}$$

$$+w[\Pi_{1111} - 3\Pi_{2211} + 2\kappa(\Pi_{22} - \Pi_{11})]$$
$$- \langle \nabla\kappa, \dot\sigma(\theta)\rangle \langle W, \dot\sigma(\theta)\rangle + \langle \nabla\kappa, \sigma(\theta)\rangle \langle W, \sigma(\theta)\rangle, \tag{2.18}$$

where the superscripts $'$ and (3) denote the first derivative and the third derivative with respect to variable t, respectively, and the following formulas have been used

$$\Pi_{1212} = \Pi_{2211} + \mathbf{R}_{TE} D^2\Pi(T,E) = \Pi_{2211} + \kappa(\Pi_{11} - \Pi_{22}) \text{ (by (2.14))},$$
$$\phi'_\theta(0) = w\kappa(\Pi_{11} - \Pi_{22}).$$

On the other hand, using equation (2.5) and the first equation in (2.8), we obtain

$$\begin{aligned}(f_t\varphi)^{(3)}(0) &= [f^{(4)}\varphi + 3f^{(3)}\varphi' + 3f''\varphi'' + f'\varphi^{(3)}](0)\\ &= -2\kappa'\varphi(0) - 3\kappa(0)\varphi'(0) + \varphi^{(3)}(0)\\ &= -2\langle \nabla\kappa, \sigma(\theta)\rangle \langle W, \sigma(\theta)\rangle - w_{11}\Pi_{11} - 2w_1\Pi_{111} + w(3\kappa\Pi_{11} - \Pi_{1111})\end{aligned} \tag{2.19}$$

at o. Moreover, we have

$$\begin{aligned}(fw\Pi_{22})^{(3)}(0) &= f^{(3)}(0)w(o)\Pi_{22}(o) + 3(w\Pi_{22})''(0)\\ &= 3w_{11}\Pi_{22} + 6w_1\Pi_{221} + w(3\Pi_{2211} - \kappa\Pi_{22}) \quad \text{at} \quad o.\end{aligned} \tag{2.20}$$

Finally, using the third equation in (2.8), we obtain from (2.18)-(2.20)

$$\begin{aligned}0 &= (\phi_\theta + f_t\varphi + fw\Pi_{22})^{(3)}(0)\\ &= w_{11}\Pi_{22} - 2w_{12}\Pi_{12} + w_{22}\Pi_{11} + w\kappa(\Pi_{11} + \Pi_{22})\\ &\quad - \langle \nabla\kappa, \dot\sigma(\theta)\rangle \langle W, \dot\sigma(\theta)\rangle - \langle \nabla\kappa, \sigma(\theta)\rangle \langle W, \sigma(\theta)\rangle\\ &= \left\langle D^2 w, Q^*\Pi\right\rangle + w\kappa \operatorname{tr}\Pi - \langle \nabla\kappa, W\rangle \quad \text{at} \quad o.\end{aligned}$$

□

Let $s \geq 0$ be given. Let $\Phi_0(t)$ and $\Phi(t,s)$ solve the problem

$$\begin{cases} \Phi_{0tt}(t) + \kappa(t)\Phi_0(t) = 0 \quad \text{for} \quad t \geq 0,\\ \Phi_0(0) = 1, \quad \Phi_{0t}(0) = 0,\end{cases} \tag{2.21}$$

and

$$\begin{cases} \Phi_{tt}(t,s) + \kappa(t)\Phi(t,s) = 0 \quad \text{for} \quad t \geq s,\\ \Phi(s,s) = 0, \quad \Phi_t(s,s) = 1,\end{cases} \tag{2.22}$$

respectively. Note that

$$\Phi(t,0) = f.$$

Let w be a function on Ω and $W_o \in M_o$. Let

$$\phi = \Phi_0(t)\langle W_o, \dot\sigma(\theta)\rangle - w(o)\Pi(o)(\dot\sigma(\theta), \sigma(\theta))f + \int_0^t \Phi(t,s)P(w)(s)ds, \tag{2.23}$$

where $P(w)$ is given by (2.11). Then ϕ solves the problem (2.10) and (2.13).

We denote by $H^m_{\text{ob}}(\Omega)$ the set of all functions of $H^m(\Omega)$ which are in the form of

$$w = u(x) + \langle W_o, N\rangle(x) \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o), \tag{2.24}$$

where $W_o \in M_o$ are constant vectors and $u \in H^m(\Omega)$ satisfies the problem

$$\mathfrak{A}_o u + u(o)\Pi(o)(\sigma(\theta), \dot\sigma(\theta))\kappa_2 f = 0 \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o), \tag{2.25}$$

where

$$\mathfrak{A}_o u = \left\langle D^2 u, Q^*\Pi \right\rangle + u\kappa \operatorname{tr}\Pi + \kappa_1 \int_0^t u\Pi_{11} ds - \kappa_2 \int_0^t \Phi(t,s) P(u)(s) ds, \tag{2.26}$$

$\kappa_1 = \langle \nabla\kappa, T\rangle$, and $k_2 = \langle \nabla\kappa, E\rangle$.

Remark 2.2. Since Ω is star-shaped with respect to o, the imbedding theorem implies that for $m \geq 2$

$$|u(o)| \leq C(\|u\|_{H^m(\Omega)} + \|u\|_{L^2(\Gamma)}) \quad \text{for} \quad u \in H^m(\Omega), \tag{2.27}$$

where $\Gamma = \partial\Omega$ is the boundary of Ω. Thus the second term in the left hand side of (2.25) makes sense for $u \in H^m(\Omega)$ with $m \geq 2$. In general the above estimate is not true for $m = 1$.

Remark 2.3. As a constant vector W_o on Ω, or a translation displacement of Ω, $(\tilde{W}_o, \langle W_o, N\rangle)$ is a trivial infinitesimal isometry where $W_o = \tilde{W}_o + \langle W_o, N\rangle N$.

Remark 2.4. The formula (2.25) depends on the choice of the point $o \in \Omega$. If the point o can be chosen to be an umbilical point of M, then $\kappa(o) \geq 0$ and

$$\Pi(o) = \sqrt{\kappa(o)}g,$$

which yields

$$\Pi(o)(\sigma(\theta), \dot{\sigma}(\theta)) = 0 \quad \text{for} \quad \theta \in (0, 2\pi].$$

In this case the equation (2.25) becomes

$$\mathfrak{A}_o u = 0 \quad \text{for} \quad x \in \Omega. \tag{2.28}$$

Another case for which (2.28) holds true is that $o \in \Omega$ can be chosen such that $\kappa_2 = 0$ for $x \in \Omega$.

We have the following.

Theorem 2.5. *Let M be orientable and let Ω be a star-shaped domain with respect to a point $o \in \Omega$. Let $m \geq 2$. Then*

$$H^m_{\text{is}}(\Omega) \subset H^m_{\text{ob}}(\Omega). \tag{2.29}$$

Moreover, if

$$\bar{\Omega} \subset \exp_o \Sigma(o), \tag{2.30}$$

then

$$H^m_{\text{is}}(\Omega) = H^m_{\text{ob}}(\Omega). \tag{2.31}$$

Remark 2.6. From (2.7) vector fields T and E have singularities on the cut locus $\exp_o C(o)$. Without assumption (2.30) formula (2.31) may not be true. Later we will show that if Ω is a closed spherical shell, formula (2.31) holds (Theorem 3.11) but it is not true if Ω is a cylinder (Theorem 4.2) or conical shell (Theorem 4.4).

Proof of Theorem 2.5. Let $w \in H^m_{\text{is}}(\Omega)$ be given. Let a vector field $W \perp \mathrm{KF}(\Omega, I\!R^3)$ be such that $V = W + wN$ is an infinitesimal isometry. Let

$$W(o) = \hat{W}(o) + \langle W(o), N\rangle N \quad \text{for} \quad x \in \Omega,$$

where $\hat{W}(o) = W(o) - \langle W(o), N\rangle N$. Let

$$U = W - \hat{W}(o), \quad u = w - \langle W(o), N\rangle \quad \text{for} \quad x \in \Omega.$$

Then (U, u) is an infinitesimal isometry field with $U(o) = 0$. Using (2.9) and (2.23) in (2.16) where (W, w) is replaced by (U, u), we have formula (2.25) for u. Thus $w \in H^m_{\text{ob}}(\Omega)$.

Next, suppose (2.30) is true. Then the vector fields T and E are smooth on $\bar{\Omega}$. Let $u \in H^m(\Omega)$ solve problem (2.25). It will suffice to prove that there is a vector field $U \in \mathfrak{X}(\bar{\Omega})$ such that $V = U + uN$ is an infinitesimal isometry. We define

$$U = \varphi T + \phi E \quad \text{for} \quad x \in \bar{\Omega},$$

where

$$\varphi = -\int_0^t u\Pi_{11} ds, \quad \phi = -u(o)\Pi(\dot{\sigma}(\theta), \sigma(\theta)) f + \int_0^t \Phi(t, s) P(u) ds. \tag{2.32}$$

Then equation (2.25) means that

$$\left\langle D^2 u, Q^*\Pi\right\rangle + u\kappa \operatorname{tr}\Pi = \langle \nabla\kappa, U\rangle \quad \text{for} \quad x \in \bar{\Omega}. \tag{2.33}$$

Clearly, φ and ϕ satisfy the first equation and the second equation in (2.8). To complete the proof, it remains to show that φ and ϕ, given by (2.32), solve the third equation in (2.8). For this end, we let

$$\eta = \phi_\theta + f_t\varphi + fu\Pi_{22} \quad \text{for} \quad x \in \bar{\Omega}.$$

Using (2.5), (2.17), and (2.33), we compute

$$\begin{aligned}
\eta'' &= \phi''_\theta + f^{(3)}\varphi + 2f''\varphi' + f'\varphi'' + f(u\Pi_{22})'' + 2f'(u\Pi_{22})' + f''u\Pi_{22} \\
&= \phi''_\theta - (f\kappa' + f'\kappa)\varphi - 2f\kappa\varphi' + f'\varphi'' + f(u\Pi_{22})'' + 2f'(u\Pi_{22})' - f\kappa u\Pi_{22} \\
&= \phi''_\theta + f[(u\Pi_{22})'' - \kappa u\Pi_{22} - 2\kappa\varphi' - \kappa'\varphi] + f'[2(u\Pi_{22})' - \kappa\varphi + \varphi''] \\
&= -(\kappa_\theta\phi + \kappa\phi_\theta + f\kappa'\varphi + f'\kappa\varphi) + f[(u\Pi_{22})'' - \kappa u\Pi_{22} - 2\kappa\varphi'] \\
&\quad + f'[2(u\Pi_{22})' + \varphi''] + f[u_{22}\Pi_{11} - 2u_{12}\Pi_{12} - 2u_1\Pi_{122} - u\Pi_{1122}] \\
&\quad + f'[u_1(\Pi_{11} - 2\Pi_{22}) + u(\Pi_{111} - 2\Pi_{122})] \\
&= -[f\langle \nabla\kappa, U\rangle + \kappa(\phi_\theta + f'\varphi + fu\Pi_{22})] + f(\left\langle D^2u, Q^*\Pi\right\rangle + u\kappa \operatorname{tr}\Pi) \\
&\quad + f'(u_1\Pi_{11} + u\Pi_{111} + \varphi'') \\
&= f(\left\langle D^2u, Q^*\Pi\right\rangle + u\kappa \operatorname{tr}\Pi - \langle \nabla\kappa, U\rangle) - \kappa\eta + f'(u\Pi_{11} + \varphi')' \\
&= -\kappa\eta, \qquad (2.34)
\end{aligned}$$

where the following formula has been used

$$\mathbf{R}_{E_1E_2}\Pi(E_1, E_2) = \kappa(\Pi_{11} - \Pi_{22})(\text{by } (2.14)).$$

Moreover, we have the initial data

$$\eta(0) = \phi_\theta(0) + \varphi(0) = 0, \quad \eta'(0) = \phi'_\theta(0) + \varphi'(0) + u(o)\Pi_{22}(o) = 0,$$

which imply by the equation (2.34) that $V = U + uN$ is an infinitesimal isometry. Then equations (2.8) hold true where w is replaced with u. In particular,

$$\varphi_t + u\Pi(T, T) = 0, \quad \phi_\theta + f_t\varphi + fu\Pi(E, E) = 0 \quad \text{for all} \quad x \in \bar{\Omega}. \tag{2.35}$$

Thus $u \in H^m(\Omega)$ implies that $\varphi, \phi \in H^m(\Omega)$, that is, $V \in \text{IS}^m(\Omega, I\!R^3)$. $\square$

Remark 2.7. If equations (2.35) just hold almost everywhere on Ω, the regularity of $u \in H^m(\Omega)$ may not imply $U \in H^m(\Omega, I\!R^3)$, see the proof of Theorem 4.2 later, where equation (4.7) has a solution (φ, ϕ) not in $C^0(\Omega) \times C^0(\Omega)$ when $w \in H^m(\Omega)$ with $\int_0^{2\pi} w d\theta \neq 0$.

If surface M is given as a graph, an infinitesimal isometry function $w \in H^m_{\text{is}}(\Omega)$ can be written as an explicit formula in the Cartesian orthogonal coordinate system. Let

$$M = \{ (x, h(x)) \,|\, x = (x_1, x_2) \in I\!R^2 \}, \tag{2.36}$$

where h is a smooth function on $I\!R^2$. Let

$$V(p) = (u_1, u_2, u) \quad \text{for} \quad p \in M. \tag{2.37}$$

We then have

Theorem 2.8 ([31, 34])**.** *Let $\tilde\Omega \subset I\!R^2$ be a star-shaped with respect to a point $\tilde o \in \tilde\Omega$. Then there are functions u_1, u_2 such that $V = (u_1, u_2, u)$ is an infinitesimal isometry on*

$$\Omega = \{ (x, h(x)) \,|\, x \in \tilde\Omega \}$$

if and only if u solves the problem

$$\tilde{\operatorname{div}}\, A(x) \tilde\nabla u = 0 \quad \text{for} \quad x \in \tilde\Omega, \tag{2.38}$$

where $\tilde{\operatorname{div}}$ and $\tilde\nabla$ are the divergence and gradient of $I\!R^2$ in the Euclidean metric, respectively, and

$$A(x) = \begin{pmatrix} h_{x_2x_2} & -h_{x_1x_2} \\ -h_{x_1x_2} & h_{x_1x_1} \end{pmatrix} \quad \text{for} \quad x \in \tilde\Omega. \tag{2.39}$$

3. Elliptic Surfaces

Let M be a surface in $I\!R^3$. M is said to be elliptic if the fundamental form Π is positive for all $x \in M$. Assume that M is elliptic throughout this section. Then problem (2.25) will become an elliptic one (Theorem 3.3). We introduce another metric on M by

$$\hat g = \Pi \quad \text{for} \quad x \in M.$$

Proposition 3.1. *Let M be elliptic. Then for $w \in C^2(M)$,*

$$\kappa \Delta_\Pi w + \frac{1}{2\kappa} Q^*\Pi(\nabla\kappa, \nabla w) = \langle D^2 w, Q^*\Pi \rangle \quad \text{for} \quad x \in M, \tag{3.1}$$

where Δ_Π is the Laplacian of the metric $\hat g = \Pi$ and $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the operator, given by (2.15).

Proof. Let $o \in M$ be fixed. Consider the polar coordinates in the induced metric g

$$\partial t = T, \quad \partial\theta = fE.$$

Note that the above $(\partial t, \partial\theta)$ is no longer the polar coordinates in the metric $\hat g = \Pi$.

In the coordinate system $(\partial t, \partial\theta)$, we have

$$\hat g = \hat g_{11} dt^2 + \hat g_{12}(dt d\theta + d\theta dt) + \hat g_{22} d\theta^2,$$

$$\hat{G} = \left(\hat{g}_{ij}\right) = \begin{pmatrix} \Pi_{11} & f\Pi_{12} \\ f\Pi_{12} & f^2\Pi_{22} \end{pmatrix}, \quad \det \hat{G} = \kappa f^2, \quad \hat{G}^{-1} = \frac{1}{\kappa f^2}\begin{pmatrix} f^2\Pi_{22} & -f\Pi_{12} \\ -f\Pi_{12} & \Pi_{11} \end{pmatrix}.$$

Moreover,

$$w_{t\theta} = f w_{12} + f' w_2, \quad w_{\theta\theta} = f^2 w_{22} - f f' w_1 + f_\theta w_2.$$

Using those formulas, we obtain

$$\begin{aligned} \kappa\Delta_\Pi w &= \frac{\kappa}{\sqrt{\kappa}f}[(\sqrt{\kappa}f\frac{\Pi_{22}}{\kappa}w_t)_t - (\sqrt{\kappa}f\frac{\Pi_{12}}{\kappa f}w_t)_\theta - (\sqrt{\kappa}f\frac{\Pi_{12}}{\kappa f}w_\theta)_t + (\sqrt{\kappa}f\frac{\Pi_{11}}{\kappa f^2}w_\theta)_\theta] \\ &= \left\langle D^2 w, Q^*\Pi\right\rangle + \{\frac{\sqrt{\kappa}}{f}[(\frac{f\Pi_{22}}{\sqrt{\kappa}})_t - (\frac{\Pi_{12}}{\sqrt{\kappa}})_\theta] - \Pi_{11}\}w_1 - 2\frac{f'}{f}\Pi_{12}w_2 \\ &\quad +\{\sqrt{\kappa}[(\frac{\Pi_{11}}{\sqrt{\kappa}f})_\theta - (\frac{\Pi_{12}}{\sqrt{\kappa}})_t] + \frac{f_\theta}{f^2}\Pi_{11}\}w_2 \\ &= \left\langle D^2 w, Q^*\Pi\right\rangle + \frac{1}{2\kappa}(\kappa_2\Pi_{12} - \kappa_1\Pi_{22})w_1 - 2\frac{f'}{f}\Pi_{12}w_2 \\ &\quad +[2\frac{f'}{f}\Pi_{12} + \frac{1}{2\kappa}(\kappa_1\Pi_{12} - \kappa_2\Pi_{11})]w_2 \\ &= \left\langle D^2 w, Q^*\Pi\right\rangle - \frac{1}{2\kappa}\Pi(Q\nabla\kappa, Q\nabla w). \end{aligned}$$

□

Let $o \in \Omega$ be fixed. For $w \in H^m(\Omega)$, we let

$$\mathfrak{B}w = \mathfrak{B}_o w + w(o)\frac{\kappa_2}{\kappa}\Pi(\sigma(\theta), \dot{\sigma}(\theta))f \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o), \tag{3.2}$$

where

$$\begin{aligned} \mathfrak{B}_o w &= \frac{1}{2\kappa^2}Q^*\Pi(\nabla\kappa, \nabla w) + w \operatorname{tr}\Pi + \frac{\kappa_1}{\kappa}\int_0^t w\Pi_{11}ds \\ &\quad -\frac{\kappa_2}{\kappa}\int_0^t \Phi(t,s)P(w)(s)ds \quad \text{for} \quad x \in \Omega \cap \exp_o \Sigma(o). \end{aligned} \tag{3.3}$$

Remark 3.2. Let $o \in \Omega$. Since $\Omega \cap \exp_o C(o)$ has measure zero and

$$\Omega = [\Omega \cap \exp_o \Sigma(o)] \cup [\Omega \cap \exp_o C(o)],$$

$\mathfrak{B}w$ is defined by (3.2) on Ω almost everywhere.

Consider operator $\mathfrak{A}_o$, defined by (2.26). It follows from (2.26) and (3.1) that

$$\mathfrak{A}_o w + w(o)\kappa_2\Pi(\sigma(\theta), \dot{\sigma}(\theta))f = \kappa(\Delta_\Pi w + \mathfrak{B}w). \tag{3.4}$$

Since for $W_o \in M_o$, $(\tilde{W}_o, \langle W_o, N\rangle)$ is a trivial, smooth infinitesimal isometry where $W_0 = \tilde{W}_o + \langle W_o, N\rangle\, N$, it follows from (3.4) that

Theorem 3.3. *Let $m \geq 2$. Let $\Omega \subset M$ be elliptic and star-shaped with respect to $o \in \Omega$. Then*

$$H^m_{\text{ob}}(\Omega) = \{\ w \mid w \in H^m(\Omega),\ \Delta_\Pi w + \mathfrak{B}w = 0\ \}. \tag{3.5}$$

We consider the structure of solutions to the equation $\Delta_\Pi w + \mathfrak{B} w = 0$ in $H^m(\Omega)$. For this purpose, we suppose that assumption (2.30) holds true. This assumption guarantees that $\Gamma \neq \emptyset$ and operator (3.2) satisfies

$$\mathfrak{B} w \in H^{m-1}(\Omega) \quad \text{for} \quad w \in H^m(\Omega).$$

Instead of the usual inner product of $L^2(\Omega)$, we use the following inner product on $L^2(\Omega)$

$$(w, v)_{L^2_\Pi(\Omega)} = \int_\Omega wv dg_\Pi \quad \text{for} \quad w,\ v \in L^2(\Omega).$$

We denote by $L^2_\Pi(\Omega)$ the above space. It is well known that the negative Laplacian operator $-\Delta_\Pi$ on Ω with the Dirichlet boundary condition is a positive selfadjoint operator on $L^2_\Pi(\Omega)$ with $D(\Delta_\Pi) = H^2(\Omega) \cap H^1_0(\Omega)$. Moreover,

$$\Delta_\Pi : \quad H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega) \tag{3.6}$$

is an isomorphism since $\Gamma \neq \emptyset$.

Since $\Delta_\Pi^{-1} w \in H^2(\Omega) \cap H^1_0(\Omega)$ for $w \in L^2(\Omega)$, using (3.3) and assumption (2.30) we have the following estimates

$$\|\mathfrak{B}_o \Delta_\Pi^{-1} w\|_{L^2_\Pi(\Omega)} \leq C\|\Delta_\Pi^{-1} w\|_{H^1(\Omega)} \leq C\|\Delta_\Pi^{-1} w\|_{H^2(\Omega)} \leq C\|w\|_{L^2_\Pi(\Omega)} \text{ for } w \in L^2_\Pi(\Omega),$$

which yield

Lemma 3.4. *Let assumption* (2.30) *hold true. The operator* $\mathfrak{B}_o \Delta_\Pi^{-1} :\ L^2_\Pi(\Omega) \to L^2_\Pi(\Omega)$ *is a compact operator.*

Let $\delta(o)$ be the Dirac function at o. Then for $w \in L^2_\Pi(\Omega)$, by the imbedding theorem we have

$$|(\Delta_\Pi^{-1}\delta(0),\, w)_{L^2_\Pi(\Omega)}| \leq c|(\Delta_\Pi^{-1} w)(o)| \leq c\|\Delta_\Pi^{-1} w\|_{H^2(\Omega)} \leq c\|w\|_{L^2_\Pi(\Omega)},$$

which imply that $\Delta_\Pi^{-1}\delta(o) \in L^2_\Pi(\Omega)$. Since the second term in (3.2) is an operator of rank one, $\mathfrak{B}\Delta_\Pi^{-1} :\ L^2_\Pi(\Omega) \to L^2_\Pi(\Omega)$ is also a compact operator.

Consider the operator $\Delta_\Pi + \mathfrak{B}$ with the domain $D(\Delta_\Pi + \mathfrak{B}) = H^2(\Omega) \cap H^1_0(\Omega)$. Denote by $\mathfrak{B}^*$ the adjoint operator of $\mathfrak{B}$ with respect to the inner product of $L^2_\Pi(\Omega)$. Then

$$\mathfrak{B}^* = \mathfrak{B}^*_o + \Big(\frac{\kappa_2}{\kappa}[\Pi(\sigma(\theta), \dot\sigma(\theta))f], \quad \cdot\Big)_{L^2_\Pi(\Omega)} \delta(o), \tag{3.7}$$

and $D(\Delta_\Pi + \mathfrak{B}^*) = H^2(\Omega) \cap H^1_0(\Omega)$. Let

$$\mathfrak{V}_0(\Omega) = \{\, \varphi \,|\, \varphi \in H^2(\Omega) \cap H^1_0(\Omega),\ \Delta_\Pi\varphi + \mathfrak{B}\varphi = 0 \,\}, \tag{3.8}$$

$$\mathfrak{V}_{0*}(\Omega) = \{\, \varphi \,|\, \varphi \in H^2(\Omega) \cap H^1_0(\Omega),\ \Delta_\Pi\varphi + \mathfrak{B}^*\varphi = 0 \,\}. \tag{3.9}$$

It follows from Lemma 3.4 and the formula (3.7) that

$$\Delta_\Pi^{-1}\mathfrak{B}^* = (\mathfrak{B}_o\Delta_\Pi^{-1})^* + \Big(\frac{\kappa_2}{\kappa}[\Pi(\sigma(\theta), \dot\sigma(\theta))f], \quad \cdot\Big)_{L^2_\Pi(\Omega)} \Delta_\Pi^{-1}\delta(o) :$$

$L^2_\Pi(\Omega) \to L^2_\Pi(\Omega)$ is a compact operator. By the first Fredholm theorem [18], $\mathfrak{V}_0(\Omega)$ and $\mathfrak{V}_{0*}(\Omega)$ are subspaces of finite dimension and $\dim \mathfrak{V}_0(\Omega) = \dim \mathfrak{V}_{0*}(\Omega)$. Let

$$\mathfrak{V}^{m-1/2}(\Gamma) = \{\, \psi \in H^{m-1/2}(\Gamma) \,|\, (\psi, \varphi_\nu)_{L^2_\Pi(\Gamma)} = 0,\ \varphi \in \mathfrak{V}_{0*}(\Omega) \,\}.$$

Theorem 3.5. *Let $m \geq 2$. Suppose assumption* (2.30) *is true. Then $w \in H^m_{\rm is}(\Omega)$ if and only if w has a form of*

$$w = \varphi + \hat{w}, \tag{3.10}$$

where $\varphi \in \mathfrak{V}_0(\Omega)$ and $\hat{w}$ is given by

$$\hat{w} = w_0 - \Delta_\Pi^{-1}(I + \mathfrak{B}\Delta_\Pi^{-1})^{-1}\mathfrak{B}w_0, \tag{3.11}$$

where $w_0 \in H^m(\Omega)$ is a solution to problem

$$\begin{cases} \Delta_\Pi w_0 = 0 \quad \textit{for} \quad x \in \Omega, \\ w_0 = \psi \quad \textit{for} \quad x \in \Gamma, \end{cases} \tag{3.12}$$

for some $\psi \in \mathfrak{V}^{m-1/2}(\Gamma)$.

Proof. We use induction in $m \geq 2$. Let $m = 2$. By Theorems 2.5 and 3.3, $w \in H^2_{\rm is}(\Omega)$ is given by (3.10) if and only if $\hat{w} \in H^2(\Omega)$ solves problem

$$\begin{cases} \Delta_\Pi \hat{w} + \mathfrak{B}\hat{w} = 0 \quad \text{for} \quad x \in \Omega, \\ \hat{w} = \psi \quad \text{for} \quad x \in \Gamma, \end{cases} \tag{3.13}$$

where $\psi = w|_\Gamma \in H^{3/2}(\Gamma)$. Let $w_0 \in H^2(\Omega)$ be the solution to problem (3.12) and let $v = \hat{w} - w_0$. Then problem (3.13) is equivalent to solve

$$\Delta_\Pi v + \mathfrak{B}v = -\mathfrak{B}w_0 \quad \text{for some} \quad v \in H^2(\Omega) \cap H^1_0(\Omega). \tag{3.14}$$

Let $u = \Delta_\Pi v$. Then problem (3.14) is the same to problem

$$u + \mathfrak{B}\Delta_\Pi^{-1}u = -\mathfrak{B}w_0 \quad \text{for some} \quad u \in L^2_\Pi(\Omega). \tag{3.15}$$

By the second Fredholm theorem [18], problem (3.15) is solvable if and only if

$$(\mathfrak{B}w_0,\ \varphi)_{L^2_\Pi(\Omega)} = 0 \tag{3.16}$$

for all $\varphi \in \mathfrak{V}$ where

$$\mathfrak{V} = \{\, \varphi \in L^2_\Pi(\Omega) \,|\, \varphi + (\mathfrak{B}\Delta_\Pi^{-1})^*\varphi = 0 \,\}. \tag{3.17}$$

It is easy to check that

$$\mathfrak{V} = \mathfrak{V}_{0*}(\Omega) = \{\, \varphi \in H^2(\Omega) \cap H^1_0(\Omega) \,|\, \Delta_\Pi\varphi + \mathfrak{B}^*\varphi = 0 \,\}.$$

Thus,

$$(\mathfrak{B}w_0,\ \varphi)_{L^2_\Pi(\Omega)} = (w_0, \mathfrak{B}^*\varphi)_{L^2_\Pi(\Omega)} = -(w_0, \Delta_\Pi\varphi)_{L^2_\Pi(\Omega)} = -(\psi, \varphi_\nu)_{L^2_\Pi(\Gamma)},$$

for all $\varphi \in \mathfrak{V}_{0*}(\Omega)$. It follows from (3.16) that problem (3.14) is solvable if and only if $\psi \in \mathfrak{V}^{3/2}(\Gamma)$.

Suppose the equivalent relationship holds true for some $m \geq 2$. We prove it is true for $m+1$. Let $\hat{w}$ be given by (3.11) and (3.12) for some $\psi \in \mathfrak{V}^{m+1/2}(\Gamma)$. Since $H^{m+1}_{\rm is}(\Omega) \subset H^m_{\rm is}(\Omega)$, it suffices to show $\hat{w} \in H^{m+1}(\Omega)$. By the induction assumption, $\hat{w} \in H^m(\Omega)$. In addition from (3.12), $w_0 \in H^{m+1}(\Omega)$. Thus, $v = \hat{w} - w_0 \in H^m(\Omega)$. By (3.14), we obtain

$$\|v\|_{H^{m+1}(\Omega)} \leq C[\|\mathfrak{B}v + \mathfrak{B}w_0\|_{H^{m-1}(\Omega)} + \|v\|_{H^m(\Omega)}],$$

which implies $\hat{w} \in H^{m+1}(\Omega)$. The proof is complete. $\square$

From the proof Theorem 3.5 we have

$$\|\hat{w}\|_{H^m(\Omega)} \le c_m \|\psi\|_{H^{m-1/2}(\Gamma)} \quad \text{for} \quad \psi \in \mathfrak{V}^{m-1/2}(\Gamma). \tag{3.18}$$

Thus, if $\psi \in \mathfrak{V}^{m-/2}(\Gamma) \cap C^\infty(\Gamma)$, then $w \in H^m_{\text{is}}(\Omega) \cap C^\infty(\Omega)$. Since $\mathfrak{V}^{m-/2}(\Gamma) \cap C^\infty(\Gamma)$ is dense in $\mathfrak{V}^{m-1/2}(\Gamma)$, estimate (3.18) yields the following density result.

Theorem 3.6. *Let $m \ge 2$. Suppose assumption* (2.30) *is true. Let $\Omega \subset M$ be elliptic which is star-shaped with respect to $o \in \Omega$. Then the strong $H^m(\Omega)$ closure of*

$$H^m_{\text{is}}(\Omega) \cap C^\infty(\Omega)$$

agrees with $H^m_{\text{is}}(\Omega)$.

Remark 3.7. If $\bar{\Omega} \cap \exp_o C(o) \neq \emptyset$, the term $\mathfrak{B}w$ may have singularities on $\bar{\Omega} \cap \exp_o C(o)$. Thus estimate (3.18) may not be true. An interesting example is given in [9] (see [22] or [34]), where Ω is a closed smooth surface of non-negative curvature for which C^∞ infinitesimal isometries consist only of trivial fields, whereas there exist non-trivial C^2 infinitesimal isometries. Therefore $H^2_{\text{is}}(\Omega) \cap C^\infty(\Omega)$ is not dense in $H^2_{\text{is}}(\Omega)$ for this surface.

By Theorem 3.5, if $\mathfrak{V}_0(\Omega) = \{ 0 \}$, an infinitesimal isometry function $w \in H^m_{\text{is}}(\Omega)$ is completely given by its boundary trace $w \in H^{m-1/2}(\Gamma)$. However, in general $\mathfrak{V}_0(\Omega) \neq \{ 0 \}$ even for a spherical cap, see Theorem 3.8 later.

A Spherical Cap Let M be a sphere of constant curvature $\kappa > 0$ with the induced metric g from $I\!R^3$. Then the second fundamental form of M is given by

$$\Pi = \sqrt{\kappa} g. \tag{3.19}$$

Then

$$\sqrt{\kappa}\Delta_\Pi w = \Delta w, \quad \mathfrak{B}w = 2\sqrt{\kappa} w,$$

where Δ is the Laplacian of M in the induced metric g from $I\!R^3$.

Let $o \in M$ be given. Let $\rho(x) = \rho(x, o)$ be the distance from $x \in M$ to o in the induced metric g of M. Set

$$\Omega(a) = \{ x \,|\, x \in M,\ \rho(x) < a \} \quad \text{for} \quad 0 < a \le \frac{\pi}{\sqrt{\kappa}}.$$

Then for $0 < a < \dfrac{\pi}{\sqrt{\kappa}}$, $\Omega(a)$ is a spherical cap with a nonempty smooth boundary

$$\Gamma(a) = \{ x \,|\, x \in M,\ \rho(x) = a \},$$

where assumption (2.30) holds true. It follows Theorems 2.5 and 3.3 that $w \in H^m_{\text{is}}(\Omega(a))$ if and only if $w \in H^m(\Omega)$ satisfies problem

$$\Delta w + 2\kappa w = 0 \quad \text{for} \quad x \in \Omega(a), \quad 0 < a < \frac{\pi}{\sqrt{\kappa}}. \tag{3.20}$$

Moreover,

$$\mathfrak{V}_0(\Omega(a)) = \mathfrak{V}_{0*}(\Omega(a)) = \{ \varphi \in H^2(\Omega(a)) \,|\, \Delta\varphi + 2\kappa\varphi = 0,\ \varphi|_{\Gamma(a)} = 0 \}.$$

We have the following. Its proof is omitted.

Theorem 3.8.

$$\begin{cases} \mathfrak{V}_0(\Omega(a)) = \{\, 0 \,\} & \textit{for} \quad 0 < a < \dfrac{\pi}{2\sqrt{\kappa}}, \\ \mathfrak{V}_0(\Omega(a)) \neq \{\, 0 \,\} & \textit{for} \quad \dfrac{\pi}{2\sqrt{\kappa}} \leq a \leq \dfrac{\pi}{\sqrt{\kappa}}. \end{cases} \tag{3.21}$$

Remark 3.9. The relations (3.21) mean that, for the first eigenvalue $\lambda_1(a)$ of $-\Delta$ on $\Omega(a)$ with the Dirichlet boundary condition on $\Gamma(a)$,

$$\begin{cases} \lambda_1(a) > 2\kappa & \text{for} \quad 0 < a < \dfrac{\pi}{2\sqrt{\kappa}}, \\ \lambda_1(a) = 2\kappa & \text{for} \quad \dfrac{\pi}{2\sqrt{\kappa}} \leq a \leq \dfrac{\pi}{\sqrt{\kappa}}. \end{cases}$$

A Closed Spherical Shell For simplicity, we assume that $\Omega = M$ is the unit closed spherical shell. By Theorem 3.3, for $m \geq 2$,

$$H^m_{\text{ob}}(\Omega) = \{\, w \,|\, w \in H^2(\Omega),\ \Delta w + 2w = 0 \,\}, \tag{3.22}$$

which is the subspace of eigenfunctions of $-\Delta$ corresponding to the eigenvalue 2. It is well-known that $H^m_{\text{ob}}(\Omega)$ is finitely dimensional and

$$H^m_{\text{ob}}(\Omega) = H^2_{\text{ob}}(\Omega) = C^\infty_{\text{ob}}(\Omega) \quad \text{for all} \quad m \geq 2.$$

Let $o \in \Omega$ be fixed and $m \geq 2$. We have

Theorem 3.10.

$$H^m_{\text{ob}}(\Omega) = \ \text{span}\,\{\, \cos t,\ \sin t \cos\theta,\ \sin t \sin\theta \,\}. \tag{3.23}$$

Proof. In the polar coordinates (t, θ), we have

$$w_{tt} + \frac{f_t}{f} w_t + \frac{1}{f^2} w_{\theta\theta} + 2w = 0 \quad \text{for} \quad w \in H^m_{\text{ob}}(\Omega), \tag{3.24}$$

which imply that $w_\theta \in H^m_{\text{ob}}(\Omega)$ for $w \in H^m_{\text{ob}}(\Omega)$. Moreover, the operator A : $H^m_{\text{ob}}(\Omega) \to H^m_{\text{ob}}(\Omega)$, given by

$$Aw = -w_{\theta\theta} \quad \text{for} \quad w \in H^m_{\text{ob}}(\Omega),$$

is a self-adjoint, nonnegative operator in the norm of $L^2(\Omega)$. Thus, all the eigenfunctions of A span $H^m_{\text{ob}}(\Omega)$.

Next, we compute the eigenfunctions of A. Let $w \in H^m_{\text{ob}}(\Omega)$ be such that $Aw = 0$. From (3.24), w satisfies

$$f w_{tt} + f_t w_t + 2 f w = 0 \quad \text{for} \quad 0 \leq t < \pi. \tag{3.25}$$

It follows from (3.25) that $w_t(0) = 0$ since $f(0) = 0$ and $f_t(0) = 1$. Thus, equation (3.25) has a dimensional subspace of solutions in $L^2(0, \pi)$ with $Aw = 0$. In particular, $w = \cos t \in H^m_{\text{ob}}(\Omega)$ is an eigenfunction of A corresponding to the zero eigenvalue.

Let $\lambda > 0$ be an eigenvalue of A. Then its eigenfunctions have the form of

$$w = \alpha(t) \cos k\theta, \quad \text{or} \quad w = \alpha(t) \sin k\theta,$$

where $k = \sqrt{\lambda}$ is an integer. Using the relation $-w_{\theta\theta} = \lambda w$ in (3.24), we obtain

$$f^2 \alpha'' + f f' \alpha' + (2f^2 - \lambda)\alpha = 0 \quad \text{for} \quad 0 \leq t < \pi. \tag{3.26}$$

Setting $t=0$ in the above equation yields $\alpha(0)=0$. Moreover, after differentiating the above equation, we let $t=0$ to have $(1-\lambda)\alpha'(0)=0$. Thus equation (3.26) has a nonzero solution if and only if $\lambda=1$ and $\alpha'(0)\neq 0$. Then a linearly independent solution of (3.26) is $\alpha(t)=\sin t$, where $\lambda=1$. Thus, we obtain (3.23). $\square$

For a closed spherical shell, assumption (2.30) fails. However, we still have the following.

Theorem 3.11. *Let $m\geq 2$ and $o\in\Omega$. Then*

$$H^m_{\text{is}}(\Omega)=H^m_{\text{ob}}(\Omega).$$

Proof. The proof consists of a simple computation by (2.8) and (3.23) as follows. Let

$$\Omega=\{\,(x_1,x_2,x_3)\in I\!R^3\,|\,x_1^2+x_2^2+(x_3-1)^2=1\,\},$$

and $o=(0,0,0)$. Then

$$T=\Big(\sigma(\theta)\cos t,\ \sin t\Big),\quad E=\Big(\dot\sigma(\theta),\ 0\Big),\quad N=\Big(\sigma(\theta)\sin t,\ -\cos t\Big),$$

where $\sigma(\theta)=(\cos\theta,\sin\theta)$.

(1) Let $w=\cos t$. The corresponding infinitesimal isometry is

$$V=\Big(\dot\sigma(\theta)\sin t,-1\Big).$$

Then

$$\hat D_X V=AX\quad\text{for}\quad X\in\Omega_x,\quad x\in\Omega,$$

where

$$A=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},$$

that is, V is trivial.

(2) For $w=\sin t\cos\theta$, or $\sin t\sin\theta$, the corresponding infinitesimal isometry is

$$V=(1,0,0),\quad\text{or}\quad(0,1,0),$$

respectively. $\square$

It follows from the above proof that

Corollary 3.12 ([31]). *Let $m\geq 2$. For the unit closed spherical shell Ω, all $H^m(\Omega,I\!R^3)$ infinitesimal isometries are trivial.*

4. Parabolic Surfaces

A surface M is said to be parabolic if

$$\kappa=0,\quad\Pi\neq 0\quad\text{for all}\quad x\in M.$$

Let M be parabolic and orientable. Let $m\geq 2$. It follows from equation (2.25) that $w\in H^m_{\text{ob}}(M)$ if and only if $w\in H^m(M)$ solves the problem

$$\big\langle D^2w,Q^*\Pi\big\rangle=0\quad\text{for}\quad x\in M.$$

We assume that there is a vector field $E\in\mathfrak{X}(M)$ such that

$$\hat D_E N=0,\quad |E|=1\quad\text{for}\quad x\in M. \tag{4.1}$$

Let $p_0 \in M$ be given. We consider the parabolic coordinates (t,s) on M as follows. Let curves r and $\zeta : I\!R \to M$ be given by

$$\begin{cases} \dot r(t) = E(r(t)) \quad \text{for} \quad t \in I\!R, \\ r(0) = p_0, \end{cases}$$

and

$$\begin{cases} \dot\zeta(s) = QE(\zeta(s)) \quad \text{for} \quad s \in I\!R, \\ \zeta(0) = p_0, \end{cases}$$

respectively, where operator $Q:\ M_p \to M_p$ for $p \in M$ is given by (2.15). Let two-parameters families $\alpha(t,s)$ and $\beta(t,s)$ be given by

$$\begin{cases} \dfrac{\partial\alpha}{\partial t}(t,s) = E(\alpha(t,s)) \quad \text{for} \quad t \in I\!R, \\ \alpha(0,s) = \zeta(s), \end{cases}$$

and

$$\begin{cases} \dfrac{\partial\beta}{\partial s}(t,s) = QE(\beta(t,s)) \quad \text{for} \quad s \in I\!R, \\ \zeta(t,0) = r(t), \end{cases}$$

respectively. Then

$$\alpha(t,s) = \beta(t,s) \quad \text{for} \quad (t,s) \in I\!R^2,$$

$$\partial t = \frac{\partial\alpha}{\partial t}(t,s) = E, \quad \partial s = \frac{\partial\beta}{\partial s}(t,s) = Q\frac{\partial\alpha}{\partial t}(t,s).$$

We have the following.

Theorem 4.1. *Let M be a parabolic surface and orientable and let $m \geq 2$.. Let (t,s) be the parabolic coordinates on M. Then*

$$H^m_{\text{ob}}(M) = \{\, w_0(s) + w_1(s)t \mid w_1,\ w_0 \in H^m(I\!R),\ t \in I\!R \,\}. \tag{4.2}$$

Proof. Let $w \in H^m_{\text{ob}}(M)$ be given. Consider the frame field $E_1 = E$, $E_2 = QE$. By (4.1), we have

$$\begin{aligned} D_{\partial t}\partial t &= \hat D_{\partial t}\partial t + \Pi(\partial t,\partial t)N = \frac{\partial^2\alpha}{\partial t^2}(t,s) \\ &= \left\langle \frac{\partial^2\alpha}{\partial t^2}(t,s), \frac{\partial\alpha}{\partial t}(t,s)\right\rangle \frac{\partial\alpha}{\partial t}(t,s) + \left\langle \frac{\partial^2\alpha}{\partial t^2}(t,s), \frac{\partial\beta}{\partial s}(t,s)\right\rangle \frac{\partial\beta}{\partial s}(t,s) \\ &\quad + \left\langle \frac{\partial^2\alpha}{\partial t^2}(t,s), N\right\rangle N \\ &= \frac{1}{2}\frac{\partial}{\partial t}\left|\frac{\partial\alpha}{\partial t}(t,s)\right|^2 \frac{\partial\alpha}{\partial t}(t,s) + \frac{1}{2}\frac{\partial}{\partial t}\left\langle \frac{\partial\alpha}{\partial t}(t,s), Q\frac{\partial\alpha}{\partial s}(t,s)\right\rangle \frac{\partial\beta}{\partial s}(t,s) \\ &\quad + \frac{\partial}{\partial t}\left\langle \frac{\partial\alpha}{\partial t}(t,s), N\right\rangle N = 0. \end{aligned} \tag{4.3}$$

It follows from (4.1) and (4.3) that

$$0 = \left\langle D^2w, Q^*\Pi\right\rangle = D^2w(E,E)\Pi(QE,QE) = \frac{\partial^2 w}{\partial t^2}\Pi(QE,QE).$$

Since $\Pi(QE,QE) \neq 0$, we have the formula (4.2). $\square$

A Cylinder Consider a cylinder

$$M = \{\, (x, z) \,|\, x = (x_1, x_2) \in I\!R^2,\ |x| = 1,\ z \in I\!R \,\}.$$

Then

$$N = (x, 0).$$

Let $E = (0, 0, 1) = \partial z$. Then

$$\hat{D}_E N = 0, \quad |E| = 1.$$

Consider the parabolic coordinates (z, θ), given by

$$(x, z) = (\cos\theta, \sin\theta, z).$$

Let $b > 0$ be given and let

$$\Omega = \{\, (x, z) \,|\, |x| = 1,\ |z| < b \,\}, \quad \mathbf{T} = \{\ x \mid x \in I\!R^2,\ |x| = 1\ \}. \tag{4.4}$$

Then, by Theorem 4.1,

$$H^m_{\text{ob}}(\Omega) = \{\, w_0 + w_1 z \,|\, w_0,\ w_1 \in H^m(\mathbf{T}),\ |z| < b\}. \tag{4.5}$$

We have the following.

Theorem 4.2. *For* $m \geq 2$,

$$H^m_{\text{is}}(\Omega) = \left\{ w \,\middle|\, w \in H^m_{\text{ob}}(\Omega),\ \int_0^{2\pi} w d\theta = 0 \right\}. \tag{4.6}$$

Proof. By Theorem 2.5, $H^m_{\text{is}}(\Omega) \subset H^m_{\text{ob}}(\Omega)$.
We have

$$\partial z = (0, 0, 1), \quad \partial\theta = (-\sin\theta, \cos\theta, 0), \quad N = (\cos\theta, \sin\theta, 0).$$

For $w \in H^m_{\text{ob}}(\Omega)$, let

$$V = \varphi\partial z + \phi\partial\theta + wN = (-\phi\sin\theta + w\cos\theta, \phi\cos\theta + w\sin\theta, \varphi).$$

Then V is an infinitesimal isometry in $\text{IS}^{\,m}(\Omega, I\!R^3)$ if and only if φ and ϕ in $H^m(\Omega)$ solve the problem

$$\begin{cases} \varphi_z = 0, \\ \phi_\theta + w = 0, \\ \varphi_\theta + \phi_z = 0. \end{cases} \tag{4.7}$$

Clearly, (4.7) has a solution in $H^m(\Omega)$ if and only if $\int_0^{2\pi} w d\theta = 0$. □

Remark 4.3. Let $0 < \varepsilon < 2\pi$. If the middle surface is given by

$$\Omega = \{\, (\cos\theta, \sin\theta, z) \,|\, 0 \leq \theta \leq 2\pi - \varepsilon,\ |z| < b \,\},$$

then assumption (2.30) holds and, by Theorem 2.5,

$$H^m_{\text{is}}(\Omega) = H^m_{\text{ob}}(\Omega) = \{\, w_0 + z w_1 \,|\, w_0,\ w_1 \in H^m(0, 2\pi - \varepsilon),\ |z| < b \,\}.$$

A Conical Surface Let

$$M = \{\ (x,z)\mid |x| = |z|,\ x = (x_1,x_2)\in I\!R^2,\ z\in I\!R\ \}.$$

Then

$$N = \frac{1}{\sqrt{2}}(\frac{x}{|x|}, -1).$$

Consider the parabolic coordinates (z,θ), given by

$$(x,z) = z(\cos\theta, \sin\theta, 1).$$

Let $b_1,\ b_2 > 0$ be given and let

$$\Omega = \{\ (x,z)\mid |x| = z,\ \ b_1 < z < b_2\ \},\quad \mathbf{T} = \{\ x\mid x\in I\!R^2,\ |x| = 1\ \}. \tag{4.8}$$

Since $\hat{D}_{\partial z}N = 0$, we have from Theorem 4.1

$$H^m_{\text{ob}}(\Omega) = \{\ w_0 + w_1 z\mid w_0,\, w_1\in H^m(\mathbf{T}),\ b_1 < z < b_2\ \}. \tag{4.9}$$

By Theorem 2.5, $H^m_{\text{is}}(\Omega)\subset H^m_{\text{ob}}(\Omega)$.

For $w = w_0 + zw_1\in H^m_{\text{ob}}(\Omega)$, let

$$\begin{aligned} V &= \varphi\frac{\partial z}{\sqrt{2}} + \phi\frac{\partial\theta}{z} + wN \\ &= \Big(\frac{1}{\sqrt{2}}(\varphi + w)\cos\theta - \phi\sin\theta,\, \frac{1}{\sqrt{2}}(\varphi + w)\sin\theta + \phi\cos\theta,\, \frac{1}{\sqrt{2}}(\varphi - w)\Big), \end{aligned}$$

where

$$\partial z = (\cos\theta,\sin\theta,1),\quad \partial\theta = z(-\sin\theta,\cos\theta,0),\quad N = \frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,-1).$$

Thus V is an infinitesimal isometry in $\text{IS}^{\,m}(\Omega, I\!R^3)$ if and only if φ and ϕ in $H^m(\Omega)$ solve the problem

$$\begin{cases} \varphi_z = 0, \\ \phi_\theta + \frac{1}{\sqrt{2}}(\varphi + w) = 0, \\ z\phi_z - \phi + \sqrt{2}\varphi_\theta = 0. \end{cases} \tag{4.10}$$

Clearly, (4.7) has a solution if and only if $\int_0^{2\pi} wd\theta = 0$. It is easy to check that (4.10) has a solution (φ,ϕ) in $H^m(\Omega)\times H^m(\Omega)$ if and only if $\int_0^{2\pi} w_1 d\theta = 0$.

We have the following.

Theorem 4.4. *If* $m\geq 2$ *and* Ω *is given by* (4.8), *then*

$$H^m_{\text{is}}(\Omega) = \{\, w_0 + zw_1\,|\, w_0,\, w_1\in H^m(\mathbf{T}),\ \int_0^{2\pi} w_1 d\theta = 0\,\}.$$

5. Bending of Shells

We shall apply the results in Sections 2-4 to the limit energy functionals of the Γ-convergence to reduce bending of shells to a one-dimensional problem in the elliptic case and the parabolic case, respectively.

Let M be a connected, oriented surface in $I\!R^3$ with the normal field N. Suppose that g is the induced metric of the surface M from the standard metric of $I\!R^3$. A family $\{\mathrm{S}^h\}_{h>0}$ of shells of small thickness h around Ω is given through

$$\mathrm{S}^h = \{\,p\,|\,p = x + zN(x),\ x \in \Omega,\ -h/2 < z < h/2\,\}, \quad 0 < h < h_0.$$

The projection onto Ω along N will be denoted by π. We will assume that $0 < h < h_0$, with h_0 sufficiently small to have π well defined on each S^h.

To a deformation $u \in H^2(\mathrm{S}^h, I\!R^3)$, we associate its elastic energy (scaled per unit thickness):

$$E^h(u) = \frac{1}{h}\int_{\mathrm{S}^h} W(\hat\nabla u)dp, \tag{5.1}$$

where $\hat\nabla$ denotes the gradient of the Euclidean space $I\!R^3$. Here, the stored energy density $W : I\!R^{3\times 3} \to [0,\infty]$ is assumed to be C^2 in a neighborhood of $\mathrm{SO}\,(3)$, and to satisfy the following normalization, frame indifference, and nondegeneracy conditions

$$\forall\, F \in I\!R^{3\times3},\ \forall\, R \in \mathrm{SO}\,(3), \quad W(R) = 0, \quad W(RF) = W(F),$$
$$W(F) \geq C\,\mathrm{dist}^2(F, \mathrm{SO}\,(3)),$$

with a uniform constant $C > 0$.

Suppose $E^h(u) \sim h^\beta$ and consider the Γ-limit of $E^h(u)$. In [23], the limiting model has been identified for the range of scalings $\beta \geq 4$, based on some estimates in [16]. In these cases, the admissible deformations u are only those which are close to a rigid motion R and whose first order term in the expansion of $u - R$ with respect to h is given by RV, where $V \in \mathrm{IS}^1(\Omega, I\!R^3)$ is an infinitesimal isometry on Ω.

Let $V \in \mathrm{IS}^2(\Omega, I\!R^3)$. Then there exists a matrix A such that

$$A^\tau(x) = -A(x), \quad \hat D_X V = A(x)X, \quad X \in M_x, \quad x \in \Omega. \tag{5.2}$$

For $\beta > 4$ the limiting energy is given only by a bending term, that is, the first order change in the second fundamental form of Ω, produced by V,

$$I(V) = \frac{1}{24}\int_\Omega \mathfrak{Q}_2\Big(x,\ \Xi(V)\Big)dg \quad \text{for} \quad V \in \mathrm{IS}^2(\Omega, I\!R^3), \tag{5.3}$$

where

$$\Xi(V) = (\hat D^*(AN) - A\Pi)_{\tan}. \tag{5.4}$$

In (5.4), $\hat D^*(AN)$ is the transpose of $\hat D(AN)$, given by

$$\hat D^*(AN)(\tau,\eta) = \Big\langle \hat D_\tau(AN), \eta\Big\rangle \quad \text{for} \quad \tau,\ \eta \in M_x,\ x \in \Omega.$$

In (5.3), the quadratic forms $\mathfrak{Q}_2(x,\cdot)$ are defined as follows:

$$\mathfrak{Q}_2(x, F_{\tan}) = \min_{a\in I\!R^3} \mathfrak{Q}_3(F + a\otimes N), \quad \mathfrak{Q}_3(F) = D^2W(I)(F,F).$$

The form $\mathfrak{Q}_3$ is defined for all $F \in I\!R^{3\times3}$, while $\mathfrak{Q}_2(x,\cdot)$ for a given $x \in \Omega$ is defined on tangential minors $F_{\tan} = (\langle F\tau, \eta\rangle)_{\tau,\ \eta\in M_x}$ of such matrices.

It was further shown in [23] that for a certain class of surfaces, referred to as approximately robust surfaces, the limiting energy for $\beta = 4$ reduces to the purely linear bending functional (5.3). Elliptic surfaces happen to belong to this class [23].

Moreover, [22] has proved that the limit energy of the range of scalings $2 < \beta < 4$ for elliptic surfaces is still given by (5.3).

Here we focus on the limit energy (5.3) and reduce it from over the space $\mathrm{IS}^2(\Omega, I\!R^3)$ to over the space $H^2_{\mathrm{is}}(\Omega)$ to give mathematical formulas, as in [37].

We now describe the limiting energy formula (5.3) in the common notation in Riemannian geometry. For simplicity, we restrict ourselves to the case when the stored-energy function is isotropic (that is to say, $W(F) = W(R_1FR_2)$ for all $F \in M^{3\times 3}$ and all $R_1,\ R_2 \in \mathrm{SO}(3)$). In this case, the second derivative of W at the identity is

$$D^2W(I)(A, A) = 2\mu|E|^2 + \lambda(\hat{\mathrm{tr}}\, E)^2, \quad E = \frac{A + A^\tau}{2},$$

for some constants $\mu,\ \lambda \in I\!R$. Let G be a second-order tensor field on Ω. We define

$$|G|^2_{T^2_x} = \sum_{i=1}^{2} G^2(e_i, e_j) \quad \text{for} \quad x \in \Omega,$$

where $e_1,\ e_2$ is an orthonormal basis of M_x in the induced metric g. We have

Lemma 5.1. *Let $\mu > 0$ and $2\mu + \lambda > 0$. For $G \in T^2(\Omega)$ symmetric,*

$$\mathfrak{Q}_2(x, G) = 2\mu|G|^2_{T^2_x} + \frac{\lambda\mu}{\mu + \lambda/2}\,\mathrm{tr}^2 G \quad \textit{for} \quad x \in \Omega. \tag{5.5}$$

Let k be a nonnegative integer and let $T \in T^k(\Omega)$ be a kth-order tensor field on Ω. The internal product of X with T is a $k-1$-th order tensor field $\mathrm{i}\,(X)T$, defined by

$$\mathrm{i}\,(X)T(X_1, \cdots, X_{k-1}) = T(X, X_1, \cdots, X_{k-1}) \quad \text{for} \quad X_1,\ \cdots,\ X_{k-1} \in \mathfrak{X}(M). \tag{5.6}$$

Let $T_0 \in T^2(M)$ be the third fundamental form of surface M, given by

$$T_0(\tau, \eta) = \left\langle \hat{D}_\tau N, \hat{D}_\eta N \right\rangle \quad \text{for} \quad \tau,\ \eta \in M_x, \quad x \in M.$$

A simple computation yields

Lemma 5.2. *Let $V \in \mathrm{IS}^2(\Omega, I\!R^3)$ with $V = W + wN$. Then*

$$\Xi(V) = \mathrm{i}\,(W)D\Pi + \Pi(D.W, \cdot) + \Pi(\cdot, D.W) + wT_0 - D^2w, \tag{5.7}$$

where $\Xi(V)$ is given by (5.4)*, D is the Levi-Civita connection of the induced metric g, and $\cdot$ denotes the position of variables.*

Let $\Omega \subset M$ be elliptic and star-shaped with respect to $o \in \Omega$ such that assumption (2.30) holds true. We further assume that for any $\psi \in H^{3/2}(\Gamma)$ problem (3.13) has a unique solution $w = \lambda(\psi) \in H^2(\Omega)$. By Theorem 3.5, there is a unique $W = \Lambda(\psi) \in H^2(\Omega, I\!R^3)$ which is perpendicular to $\mathrm{KF}(\Omega, I\!R^3)$ such that $V = \Lambda(\psi) + \lambda(\psi)N$ is an infinitesimal isometry. Then for any $V \in \mathrm{IS}^2(\Omega, I\!R^3)$, we have a formula in the form of

$$V = W + \Lambda(\psi) + \lambda(\psi)N \quad \text{for} \quad W \in H^2_{\mathrm{kf}}(\Omega, I\!R^3),\ \psi \in H^{3/2}(\Gamma).$$

Since $\dim \mathrm{KF}(\Omega, I\!R^3) \le 3$, the limit energy (5.3) of the Γ-convergence becomes a functional over a one-dimensional space

$$I(V) = \tilde{I}(\alpha, \psi) \quad \text{for} \quad (\alpha, \psi) \in I\!R^3 \times H^{3/2}(\Gamma). \tag{5.8}$$

Similar situations happen when the middle surface Ω is parabolic. It follows from Theorems 3.5 and 4.1 that

Theorem 5.3. *Let Ω be star-shaped with respect to $o \in \Omega$ such that*

$$\bar{\Omega} \subset \exp_o \Sigma(o).$$

If M is parabolic such that condition (4.1) *holds and $\Omega \subset M$, or the middle surface Ω is elliptic, then the limit energy formula* (5.3) *of the Γ-convergence reduces to be a one-dimensional problem.*

We shall write out explicit formulas of (5.8) for spherical shells and cylinder shells, respectively, before ending this section.

Bending of Spherical Shells Let M be the sphere of curvature $\kappa > 0$ and let g be the induced metric of M from $I\!R^3$. Then the third fundamental form of M is given by $T_0 = \kappa g$.

Let $o \in M$ be fixed. Let $\rho(x) = \rho(x, o)$ be the distance function from $x \in M$ to o in the induced metric g. For $0 < a < \dfrac{\pi}{\sqrt{\kappa}}$, let

$$\Omega(a) = \{ \ x \mid x \in M, \ \rho(x) < a \ \}, \quad \Gamma(a) = \{ \ x \mid x \in M, \ \rho(x) = a \ \}. \tag{5.9}$$

Let $V = W + wN$ be an infinitesimal isometry on $\Omega(a)$. By the formulas (5.7) and (3.19), we have

$$\Xi(V) = \sqrt{\kappa}(DW + D^*W) + \kappa wg - D^2 w = -\kappa wg - D^2 w. \tag{5.10}$$

In particular, for $V = W \in \mathrm{KF}\,(\Omega, I\!R^3)$ a Killing field, $\Xi(V) = 0$.

By Theorem 3.5, $w \in H^2_{\mathrm{is}}(\Omega)$ if and only if w solves the problem

$$\begin{cases} \Delta w + 2\kappa w = 0 \quad \text{for} \quad x \in \Omega(a), \\ w = \psi \in H^{3/2}(\Gamma(a)). \end{cases} \tag{5.11}$$

Then it follows from (5.10) and (5.11) that

$$\operatorname{tr} \Xi(V) = 0 \quad \text{for} \quad x \in \Omega. \tag{5.12}$$

Furthermore, we have

Lemma 5.4. *Let $V = W + wN$ be an infinitesimal isometry with $w \in H^2_{\mathrm{is}}(\Omega)$. Then*

$$|\Xi(V)|^2_{T^2_x} = \frac{1}{2}\Delta|Dw|^2 + \kappa \operatorname{div} w\nabla w \quad \textit{for} \quad x \in \Omega. \tag{5.13}$$

Proof. Recall that the Weitzenböck formula(Theorem 1.27 in [37]) reads

$$|D^2 w|^2_{T^2_x} = \frac{1}{2}\Delta|Dw|^2 + \langle \mathbf{\Delta} Dw, Dw\rangle - \operatorname{Ric}(Dw, Dw) \quad \text{for} \quad x \in \Omega, \tag{5.14}$$

where $\mathbf{\Delta}$ is the Hodge-Laplacian in the metric g applying to vector fields and $\operatorname{Ric}(\cdot,\cdot)$ is the Ricci curvature tensor. Since $\operatorname{Ric} = \kappa g$ and $\langle \mathbf{\Delta} Dw, Dw\rangle = -\langle D(\Delta w), Dw\rangle$, we have, by (5.11) and (5.14),

$$|D^2 w|^2_{T^2_x} = \frac{1}{2}\Delta|Dw|^2 + \kappa|Dw|^2 \quad \text{for} \quad x \in \Omega. \tag{5.15}$$

From (5.10) and (5.15), we obtain

$$|\Xi(V)|^2_{T^2_x} = |\kappa wg + D^2 w|^2_{T^2_x} = 2\kappa^2 w^2 + 2\kappa w \langle g, D^2 w\rangle_{T^2_x} + |D^2 w|^2_{T^2_x}$$

$$= \frac{1}{2}\Delta|Dw|^2 + \kappa(|Dw|^2 - 2\kappa w^2)$$
$$= \frac{1}{2}\Delta|Dw|^2 + \kappa \operatorname{div} w\nabla w \quad \text{for} \quad x \in \Omega.$$

□

We define a linear operator $\Theta : H^{3/2}(\Gamma(a)) \to H^{1/2}(\Gamma(a))$ by

$$\Theta\psi = w_\rho,$$

where $w \in H^2(\Omega(a))$ is the solution to problem (5.11). Then $D(\Theta) = H^{3/2}(\Gamma(a))$ for $0 < a \leq \frac{\pi}{\sqrt{\kappa}}$.

Theorem 5.5. *Let $\Omega(a)$ and $\Gamma(a)$ be given in (5.9). Then the bending energy (5.3) of the Γ-convergence becomes the following one-dimensional problem*

$$\tilde{I}(\psi) = \frac{\mu}{12}\int_{\Gamma(a)}[2\psi_\tau(\Theta\psi)_\tau - \kappa\psi\Theta\psi - \sqrt{\kappa}a \operatorname{ctg}(\sqrt{\kappa}a)(|\Theta\psi|^2 + |\psi_\tau|^2)]d\Gamma \tag{5.16}$$

for $\psi \in \mathfrak{V}^{3/2}(\Gamma(a))$, where τ is the unit tangential vector field along $\Gamma(a)$.

Proof. Let $\tau = \tau(\rho)$ be the unit tangential vector field along $\Gamma(\rho)$ for $0 < \rho \leq a$. Then $D\rho$, τ forms a frame field on $\Omega(a)$. We have

$$D_{D\rho}D\rho = 0, \quad D_{D\rho}\tau = 0, \tag{5.17}$$

$$D_\tau D\rho = \sqrt{\kappa}\rho \operatorname{ctg}(\sqrt{\kappa}\rho)\tau, \quad D_\tau\tau = -\sqrt{\kappa}\rho \operatorname{ctg}(\sqrt{\kappa}\rho)D\rho. \tag{5.18}$$

Moreover, the equation in (5.11) gives

$$\begin{aligned} D^2w(D\rho, D\rho) &= -2\kappa w - D^2w(\tau,\tau) \\ &= -w_{\tau\tau} - \sqrt{\kappa}a \operatorname{ctg}(\sqrt{\kappa}a)w_\rho - 2\kappa w \quad \text{for} \quad x \in \Gamma(a). \end{aligned} \tag{5.19}$$

It follows from the formulas (5.13) and (5.17)-(5.19) that

$$\begin{aligned} \int_{\Omega(a)}|\Xi(V)|^2_{T^2_x}dg &= \int_{\Gamma(a)}[D^2w(D\rho, Dw) + \kappa w w_\rho]d\Gamma \\ &= \int_{\Gamma(a)}[w_\rho D^2w(D\rho,D\rho) + w_\tau(w_{\rho\tau} - \langle Dw, D_\tau D\rho\rangle) + \kappa w w_\rho]d\Gamma \\ &= \int_{\Gamma(a)}[2w_\tau w_{\rho\tau} - \sqrt{\kappa}a \operatorname{ctg}(\sqrt{\kappa}a)(w_\rho^2 + w_\tau^2) - \kappa w w_\rho]d\Gamma. \end{aligned} \tag{5.20}$$

Finally, we use the formulas (5.20), (5.12) and (5.5) in the formula (5.3) to obtain (5.16). □

Bending of a Cylinder Shell Let $a > 0$ and let

$$\Omega = \{\ (\cos\theta, \sin\theta, z) \mid \theta \in [0, 2\pi),\ |z| < a\ \}. \tag{5.21}$$

Then

$$\partial z = (0,0,1), \quad \partial\theta = (-\sin\theta, \cos\theta, 0).$$

Let $w \in H^2_{\text{is}}(\Omega)$ be given. By Theorem 4.2,

$$w = w_0 + w_1 z, \quad w_0,\ w_1 \in H^2(\mathbf{T}), \quad \int_0^{2\pi} w_0 d\theta = \int_0^{2\pi} w_1 d\theta = 0.$$

Let $W \in \mathfrak{X}(\Omega)$ be such that $V = W + wN$ is an infinitesimal isometry. A simple computation shows that

$$W = [\int_0^\theta (\theta - \eta) w_1(\eta) d\eta + c_1]\partial z - [\int_0^\theta [w_0(\eta) + w_1(\eta) z] d\eta + c_2]\partial\theta,$$

where c_1, c_2 are constants and

$$\Xi(V)(\partial z, \partial z) = 0, \quad \Xi(V)(\partial z, \partial\theta) = -\int_0^\theta w_1(\eta) d\eta - w_{1\theta}, \quad \Xi(V)(\partial\theta, \partial\theta) = -w - w_{\theta\theta}.$$

Using the above formulas, we obtain

Theorem 5.6. *Let* Ω *be given by* (5.21). *Then the bending energy* (5.3) *of the* Γ*-convergence becomes the following one-dimensional formula*

$$\begin{aligned}\tilde{I}(w_0, w_1) \ &= \int_{-\pi}^{\pi} \Big\{ \frac{\mu a}{3} [\frac{\mu + \lambda}{2\mu + \lambda}(w_0 + w_{0\theta\theta})^2 + (w_{1\theta} + \int_0^\theta w_1(\eta) d\eta)^2] \\ &\quad + \frac{\mu(\mu + \lambda) a^3}{3(2\mu + \lambda)} (w_1 + w_{1\theta\theta})^2 \Big\} d\theta \quad \textit{for} \quad (w_0, w_1) \in H^2(\mathbf{T}) \times H^2(\mathbf{T}).\end{aligned}$$

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