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REDUCED EQUATIONS FOR THE HYDROELASTIC WAVES IN THE COCHLEA: THE MEMBRANE MODEL

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ABSTRACT. We consider the hydroelastic waves in the cochlea for the case where the elastic partition between the *scala media* and the *scala timpani* is modeled as an elastic membrane. Specifically the cochlea is modeled as an elongated box in the x direction, with the membrane dividing the cochlea into two fluid-filled chambers. Since the basilar membrane vibrates in the present model in the orthogonal y direction, it is not possible to reduce the problem to a one-dimension equation in the x variable. Instead, we provide a rigorous reduction of the fluid-membrane coupled system of three-dimensional partial differential equations to an ordinary differential equation in the lateral y direction for the membrane and a second ordinary differential equation in x for the fluid pressure, where the first equation does not depend on the second.

1. INTRODUCTION

The inner ear is a fascinating organ whose main role is to convert incoming acoustic waves into neural signals. This is performed via a complex chain of physiological processes that include converting the acoustic wave into elastic waves in the outer ear and the middle ear and then converting these waves into hydrodynamical waves in the cochlea. In the next step the cochlea converts the hydrodynamical waves into elastic waves in the basilar membrane (BM). The BM movements are converted (back!) to fluid motion in the organ of Corti, and this fluid in turn vibrates the inner hair cells that finally send a neural signal to the auditory nerve [4].

In this paper we focus on the fluid-elastic conversion that involves the BM. While the actual geometry of the cochlea is quite complicated, we use a simplified model where the cochlea is modeled as a fluid-filled elongated box, partitioned into two chambers by the BM. The BM itself is a nonuniform elastic body. At one end of the upper chamber of the box, called the *oval window*, incoming waves from the middle ear vibrate the fluid. As the wave propagates into the cochlea, it vibrates the BM. Since the BM is a nonuniform body and even has nonuniform width, the elastic wave along it is dispersive. It is well known that this wave exhibits resonance, with space-dependent frequency. This *place principle* gives rise to a spectral decomposition of the incoming sound on its way to the auditory nerve.

Although the model above is a simplified version of the actual hydro-elastic waves in the cochlea, many researchers seek a further simplification. Exploiting the large

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aspect ratio of the two chambers, they replace the three-dimensional (3D) model by one-dimensional (1D) models. See e.g. [1]. Such dimensional reductions are ubiquitous in many disciplines [3]. Are these one-dimensional cochlear models justified? In [9] we examined this question for the case where the BM is modeled as a continuous chain of springs. We proved a theorem that justifies the reduction to a 1D model, and then pointed out that just near resonance the dimensional reduction is not rigorous. Nevertheless, numerical comparisons between the full 3D model and the reduced 1D model show excellent agreement.

Our goal here is to examine the dimensional reduction for a more elaborate (and more realistic) elastic case, where the BM is modeled as a membrane. We shall denote by x, y coordinates along the cochlea and across it, respectively. Assuming that the membrane is fixed at its boundaries, so its deflection there must vanish, it is clear that it is not possible anymore to use just a single 1D model in the x direction, since there must be nonconstant deflection in the y direction. However, as we show below, it is still possible to simplify considerably the 3D model. Essentially, our main result is a rigorous reduction of the 3D model to a 1.5D model, where by 1.5D model we mean one canonical equation in the y direction, whose solution is then coupled to a 1D equation in the x direction. Therefore, although the final model involves two equations, one of them is solved independently of the other.

The process outlined above describes a passive BM. It is well known, though, that the cochlea includes an amplifier acting via a positive feedback, so that the actual wave problem is nonlinear [2]. However, in this paper we limit ourselves to the simpler passive case in order to capture the unusual dimensional reduction. The nonlinear active cochlea model for spring-like BM will be considered by us elsewhere.

Remark: It might seem somehow strange to model the BM as anything but a membrane, in spite of its name. However, there is experimental evidence that the BM exhibits to some extent properties of a plate. In principle the analysis performed here can be extended to the case of a plate [5], but we do not pursue the details.

2. The cochlea model

We will represent the cochlea as an elongated prism. Though the cross-sectional area of the cochlea tapers gradually towards the apex, in this analysis we will ignore this effect and consider it to occupy the region

$$0 < x < L, \ -c < y < c, \ -c < z < c. \tag{2.1}$$

for constants L and c to be discussed later.

The rest position of the basilar membrane (BM) is at the plane $z = 0$. Using this geometric simplification we neglect the effect of the cochlea coiling. It is believed in general that the coiling simply serves to store the elongated cochlea in the skull; some authors, though, argue that the coiling has a dynamical effect, essentially to enhance the low frequencies as the wave reaches the cochlea apex [8].

We assume that the cochlea is filled with a linear ideal fluid:

$$\rho \tilde{U}_t + \nabla \tilde{P} = 0, \ \ \nabla \cdot \tilde{U} = 0. \tag{2.2}$$

Here $\tilde{U}$ is the fluid velocity, $\tilde{P}$ is the pressure, and ρ is the density. The fluid equations hold in the upper and lower chambers of the cochlea. We use $\tilde{P}$ for the pressure in any of them. Later on we shall distinguish between them by using the notation $\tilde{P}^+$ and $\tilde{P}^-$. In this model we neglect the fluid viscosity. This can be justified by estimating the parameters in the problem, with further justification given in the work of Keller and Neu [7].

The boundary conditions that we use for the top ($z = c$), bottom ($z = -c$) and lateral sides ($y = \pm c$) of the cochlea are

$$\tilde{U}_\nu(x, \pm c, z, t) = \tilde{U}_\nu(x, y, \pm c, t) = 0 \tag{2.3}$$

where $\tilde{U}_\nu$ denotes the outer normal (to the boundary) component of the velocity.

We assume that the oval window vibrates in a specified way, namely the system is driven by

$$\tilde{P}^+(0, y, z, t) = f(y, z, t) \tag{2.4}$$

for some given function f. Alternatively, one could provide a boundary condition on the velocity at the oval window. For the *round window* (the left face of the lower chamber) we assume

$$\tilde{P}^-(0, y, z, t) = 0, \tag{2.5}$$

and at the apex of the cochlea we take

$$\tilde{P}^+(L, y, z, t) = \tilde{P}^-(L, y, z, t) = 0. \tag{2.6}$$

We proceed to describe our elastic model for the BM. We take $\tilde{W} = \tilde{W}(x, y, t)$ to be the vertical deflection of the BM, and we use the fact that the deflection is small. Thus coupling between $\tilde{W}$ and the pressure/velocity variables is assumed to hold at $z = 0$. While in reality the BM is closer to a plate, we shall use here a simpler model where we take the BM to behave like a damped, vibrating membrane. Thus, we write the following model for the BM vibrations:

$$\begin{aligned} &m(x)\tilde{W}_{tt} + r(x)\tilde{W}_t - \kappa(x)\left(\tilde{W}_{xx} + \tilde{W}_{yy}\right) = l \text{ along } \{z = 0\}, \\ &\tilde{W} = 0 \text{ on the boundary of the BM.} \end{aligned} \tag{2.7}$$

Here $m(x)$ is the BM mass density, $r(x)$ and $\kappa(x)$ are the BM damping coefficient and elastic coefficient, respectively, and l is the load on the membrane, taken here to be simply minus the pressure jump $[\tilde{P}] := \tilde{P}^+(x, y, 0, t) - \tilde{P}^-(x, y, 0, t)$. Notice that we assume that m, r and κ might depend upon the longitudinal direction x. Later we shall write down two specific models for this dependency.

Throughout this article, we will in general employ the notation $[f]$ to denote the difference between two quantities f^+ and f^- defined in the upper and lower chambers, respectively.

At this point we introduce typical values for different parameters in the problem. The fluid density is taken to be $\rho = 1\ g/cm^3$. The mass density of the BM is about $m \sim 10^{-2} g/cm^2$ [10]. For the physical dimensions of the cochlea we take

$$L \sim 35 \text{ mm}, \quad c = 2 \text{ mm}.$$

Since we are interested mostly in the nature of the BM response to different frequencies, we assume the vibrations are driven by input f in (2.4) of the form

$$f = f(t) = e^{i\omega t} \quad \text{for some frequency } \omega. \tag{2.8}$$

At a frequency of about 1kHz for instance, we have $\omega \sim 10^3 - 10^4\ s^{-1}$. We then anticipate that the time dependence of the pressure, velocity and displacement of the BM is similarly maintained, so we seek solutions to our problem in the form

$$\tilde{P} = P(x,y,z)e^{i\omega t},\ \tilde{U} = U(x,y,z)e^{i\omega t} \text{ and } \tilde{W} = \hat{W}(x,y)e^{i\omega t}$$

for spatially dependent functions P, U and $\hat{W}$ to be determined. With these assumptions, the equations (2.2) and (2.7) transform to

$$i\omega\rho U^{\pm} = -\nabla P^{\pm},\ \ \nabla\cdot U^{\pm} = 0 \quad \text{in the upper and lower chambers} \tag{2.9}$$

and

$$-\omega^2 m\hat{W} + i\omega r\hat{W} - \kappa\left(\hat{W}_{xx} + \hat{W}_{yy}\right) = -[P] \quad \text{along } z = 0. \tag{2.10}$$

Since the fluid is incompressible, we can take the divergence of the first equation of (2.9), eliminate U altogether, and conclude that P is a harmonic function in each of the two chambers:

$$\Delta P^{\pm} = 0. \tag{2.11}$$

In light of the first relation in (2.9), the boundary conditions (2.3) for $\tilde{U}_\nu$ translate into homogeneous Neumann boundary conditions for P on the top, bottom and lateral sides of the cochlea.

Across the BM we must have continuity of the fluid velocity and acceleration which in turn must match the membrane velocity, so we see that $U_3^{\pm} = \hat{W}_t$. This, together with the momentum equation (2.9) implies

$$P_z^{\pm}(x,y,0) = \omega^2\rho\hat{W}(x,y) \quad \text{for } |x| < L,\ |y| < c. \tag{2.12}$$

Our goal is to use the large aspect ratio of the cochlea to approximate the three-dimensional model described in the previous section by a simpler one-dimensional model. It is useful for this purpose to scale x by L, and y, z by c. However, for simplicity of notation we retain the original notation x, y, z. Define also the small parameter $\delta = c/L$. Then (2.11) becomes

$$\Delta_\delta P^{\pm} = 0 \quad \text{in } D^{\pm}, \tag{2.13}$$

where we have introduced the notation $\Delta_\delta P^{\pm} := P^{\pm}_{xx} + \frac{1}{\delta^2}P^{\pm}_{yy} + \frac{1}{\delta^2}P^{\pm}_{zz}$ as well as

$$D^+ := \{(x,y,z) :\ 0 < x < 1,\ |y| < 1,\ 0 < z < 1\}$$

and

$$D^- := \{(x,y,z) :\ 0 < x < 1,\ |y| < 1,\ -1 < z < 0\}$$

to denote the scaled upper and lower chambers respectively.

For later use we also introduce here the notation

$$\Omega^+ := \{(y,z) :\ |y| < 1, 0 < z < 1\} \quad \text{and} \quad \Omega^- := \{(y,z) :\ |y| < 1, -1 < z < 0\}$$

and

$$\mathcal{B} := \{(x,y) :\ 0 < x < 1,\ |y| < 1\}.$$

We anticipate small vibrations of the BM so we also scale $\hat{W}$ by introducing W via $\hat{W} = \delta W$. Then in the rescaled z variable, the boundary condition (2.12) becomes

$$P_z^{\pm}(x,y,0) = \delta^2 L\rho\omega^2 W(x,y) \quad \text{for } (x,y) \in \mathcal{B}. \tag{2.14}$$

2.1. **First elastic model: Neglecting the inertial term and taking special forms for κ and r.** We turn next to the rescaled version of (2.10). We will assume that r and κ take the form

$$\kappa(x) = \frac{c^2\kappa_0}{\delta} e^{-\lambda x}, \;\; r(x) = \frac{r_0}{\delta} e^{-\lambda x}. \tag{2.15}$$

for a positive parameter λ. We have selected similar functional forms for κ and for r mostly for convenience. In Section 2.2 we indicate how to carry out the argument in more generality. Before proceeding we will also make the simplification to neglect the first term $-\omega^2 mW$. Neglecting the inertia is in no way an essential step; its only purpose is to simplify the calculations to follow. In a second model presented below we consider the full wave problem for the BM. Thus, we replace equation (2.10) by

$$i\omega r_0 W - \delta^2\kappa_0 W_{xx} - \kappa_0 W_{yy} = -e^{\lambda x}[P].$$

Our three-dimensional problem then consists of the system

$$\Delta_\delta P^{\pm} = 0 \quad \text{in } D^{\pm}, \tag{2.16}$$

$$i\omega r_0 W - \delta^2\kappa_0 W_{xx} - \kappa_0 W_{yy} = -e^{\lambda x}[P], \tag{2.17}$$

$$P_z^{\pm}(x,y,0) = \delta^2 L\rho\omega^2 W(x,y) \quad \text{for } (x,y) \in \mathcal{B}, \tag{2.18}$$

$$W(x,y) = 0 \quad \text{for } (x,y) \in \partial\mathcal{B}, \tag{2.19}$$

$$P^+(0,y,z) = 1, \; P^+(1,y,z) = 0 \text{ for } (y,z) \in \Omega^+, \tag{2.20}$$

$$P^-(0,y,z) = 0 = P^-(1,y,z) \text{ for } (y,z) \in \Omega^-, \tag{2.21}$$

$$P_y^+(x,\pm 1,z) = 0 \quad \text{for } 0 < x < 1, \; 0 < z < 1, \tag{2.22}$$

$$P_y^-(x,\pm 1,z) = 0 \quad \text{for } 0 < x < 1, \; -1 < z < 0 \quad \text{and} \tag{2.23}$$

$$P_z^+(x,y,1) = 0 = P_y^-(x,y,-1) \quad \text{for } (x,y) \in \mathcal{B}. \tag{2.24}$$

We point out that the boundary conditions on $P^{\pm}$, namely (2.20) and (2.21), follow from the assumptions given in (2.4), (2.5), (2.6) and (2.8). In particular, this linear problem is driven by the assumed pressure applied at the oval window, cf. (2.4).

Before stating our result on the $\delta \to 0$ limit of $P^{\pm}$ and W, we need to introduce two auxiliary functions. We denote by $T : [-1,1] \to \mathbb{C}$ the solution to the boundary value problem

$$ir_0\omega T - \kappa_0 T'' = -1 \text{ for } -1 < y < 1, \; T(\pm 1) = 0. \tag{2.25}$$

and we let $\bar{T} := \frac{1}{2}\int_{-1}^{1} T\,dy$ denote the integral average of T. Notice the crucial point that the equation for $T(y)$ is stand-alone; in particular it does not depend on the pressure in the two chambers. Though we will not need it, we note that of course one could solve for T explicitly. Aside from its existence, however, we will only need the fact that

$$\operatorname{Im} \bar{T} \neq 0, \tag{2.26}$$

a property established in the Appendix.

Then we let $\beta_\delta : [0,1] \to \mathbb{R}$ be any C^2 function such that $\beta_\delta(x) = 1$ for $\delta \leqslant x \leqslant 1$, $\beta_\delta(0) = 0$, $0 \leqslant \beta_\delta(x) \leqslant 1$, $|\beta'_\delta| \leqslant \frac{2}{\delta}$ and $|\beta''_\delta| \leqslant \frac{2}{\delta^2}$. This latter function will be used to handle the boundary layer that resides near $x = 0$ for the function W.

We will now establish convergence to a one-dimensional model in the small δ regime:

Theorem 2.1. *As $\delta \to 0$ the functions $P^\pm$ converge in $L^2(D^\pm)$ to the functions $p_0^\pm : [0,1] \to \mathbb{C}$ solving the following system of ODE's*

$$(p_0^\pm)'' \mp L\rho\omega^2 \bar{T} e^{\lambda x}[p_0] = 0 \quad \textit{for} \quad 0 < x < 1, \tag{2.27}$$

$$p_0^+(0) = 1,\ p_0^+(1) = 0 = p_0^-(0) = p_0^-(1). \tag{2.28}$$

Furthermore, the derivatives $P_y^\pm$ and $P_z^\pm$ converge to zero in $L^2(D^\pm)$. Finally, as $\delta \to 0$ one has

$$W - T\, e^{\lambda x}\,[p_0] \to 0 \quad \textit{and} \quad \Big(W - T\, e^{\lambda x}\,[p_0]\Big)_y \to 0 \quad \textit{in } L^2(\mathcal{B}). \tag{2.29}$$

Proof. We begin by taking the cross-sectional average of the PDE's in (2.16) over $D^\pm$. To this end, we introduce

$$\tilde{p}^\pm = \tilde{p}^\pm(x) := \frac{1}{2}\int_{\Omega^\pm} P^\pm(x,y,z)\,dy\,dz$$

and then define $p^\pm$ via

$$P^\pm = \tilde{p}^\pm + p^\pm. \tag{2.30}$$

For later use, we note that

$$\int_{\Omega^\pm} p^\pm(x,y,z)\,dy\,dz = 0 \quad \text{for each } x \in (0,1). \tag{2.31}$$

After an integration of (2.16) over $\Omega^\pm$ and the use of the boundary conditions we find

$$\tilde{p}_{xx}^\pm \mp \frac{1}{2} L\rho\omega^2 \int_{-1}^1 W(x,y)\,dy = 0. \tag{2.32}$$

Now define

$$\tilde{w} = \tilde{w}(x,y) := \beta_\delta(x) T(y) e^{\lambda x}[\tilde{p}](x) \tag{2.33}$$

and w via

$$W = \tilde{w} + w. \tag{2.34}$$

Note that both $\tilde{w}$ and w satisfy homogeneous Dirichlet boundary conditions on $\partial\mathcal{B}$.

Then through substitution into (2.32) we can decompose $\tilde{p}^\pm$ as $\tilde{p}^\pm = p_0^\pm + g^\pm$ where the functions $p_0^\pm$ are given by (2.27)-(2.28) and $g^\pm : [0,1] \to \mathbb{C}$ solve the system

$$\frac{1}{L\rho\omega^2} g_{xx}^\pm \mp \beta_\delta \bar{T} e^{\lambda x}[g] = \pm\frac{1}{2}\int_{-1}^1 w(x,y)\,dy \pm (\beta_\delta - 1)\,\bar{T} e^{\lambda x}[p_0] \text{ for } 0 < x < 1, \tag{2.35}$$

subject to the boundary conditions

$$g^\pm(0) = 0 = g^\pm(1). \tag{2.36}$$

For later use, we note that subtraction of the ODE's for g^+ and g^- gives an ODE for $[g]$:

(2.37)

$$\frac{1}{L\rho\omega^2}[g]^{\pm}_{xx} - 2\beta_\delta \bar{T} e^{\lambda x}[g] = \int_{-1}^{1} w(x,y)\,dy + 2\,(\beta_\delta - 1)\,\bar{T} e^{\lambda x}[p_0] \quad \text{for} \quad 0 < x < 1.$$

Ultimately then, we have decomposed $P^{\pm}$ as

$$P^{\pm}(x,y,z) = p_0^{\pm}(x) + g^{\pm}(x) + p^{\pm}(x,y,z). \tag{2.38}$$

A primary goal of this argument is to show that $g^{\pm}$ and $p^{\pm}$ vanish in the $\delta \to 0$ limit.

We break the remainder of the proof into a series of energy identities and estimates.

1. Energy identities for $\nabla p^{\pm}$.
Let us now return to (2.16) and use (2.30) to conclude that

$$p^{\pm}_{xx} + \frac{1}{\delta^2}\left(p^{\pm}_{yy} + p^{\pm}_{zz}\right) = -\tilde{p}^{\pm}_{xx}. \tag{2.39}$$

Invoking the boundary conditions (2.18), (2.20)-(2.24) as well as (2.34) we then integrate by parts over the regions $D^{\pm}$ in the expressions

$$\int_{D^{\pm}} (2.39)\cdot(-p^{\pm})^* + (2.39)^*\cdot(-p^{\pm})$$

(where $*$ denotes complex conjugation) to arrive at the identities

$$\begin{aligned}
&\int_{D^{\pm}} \left\{ \left|p^{\pm}_x\right|^2 + \frac{1}{\delta^2}\left(\left|p^{\pm}_y\right|^2 + \left|p^{\pm}_z\right|^2\right) \right\} dx\,dy\,dz = \\
&\int_0^1 \left(\mathrm{Re}\left\{ \tilde{p}^{\pm}_{xx}(x) \int_{\Omega^{\pm}} (p^{\pm}(x,y,z))^*\,dy\,dz \right\} \right) dx \\
&\mp L\rho\omega^2 \int_{\mathcal{B}} \left(\mathrm{Re}\left\{p^{\pm}(x,y,0)^*\tilde{w}\right\} + \mathrm{Re}\,p^{\pm}(x,y,0)^* w\} \right) dx\,dy = \\
&\mp L\rho\omega^2 \int_{\mathcal{B}} \left(\mathrm{Re}\left\{p^{\pm}(x,y,0)^* T(y)([p_0]+[g])\right\}\beta_\delta\, e^{\lambda x} + \mathrm{Re}\,p^{\pm}(x,y,0)^* w\} \right) dx\,dy,
\end{aligned}$$

(2.40)

where the last line follows from (2.31) and (2.33).

2. Energy identities for w and ∇w.
Turning now to the equation for W we use (2.25), (2.34) and (2.38) to rewrite (2.17) as

$$\begin{aligned}
&i\omega r_0 w - \delta^2\kappa_0 w_{xx} - \kappa_0 w_{yy} = \\
&(\beta_\delta - 1)e^{\lambda x}[p_0] + (\beta_\delta - 1)e^{\lambda x}[g] - e^{\lambda x}[p] + \delta^2\kappa_0 T\left(\beta_\delta\, e^{\lambda x}\left([p_0]+[g]\right)\right)_{xx}.
\end{aligned}$$

(2.41)

Using the zero Dirichlet boundary conditions satisfied by w we then carry out the integration

$$\int_{\mathcal{B}} (2.41) \cdot w^* + (2.41)^* \cdot w \, dx \, dy.$$

This leads to the identity

$$\int_{\mathcal{B}} \delta^2 \kappa_0 \, |w_x|^2 + \kappa_0 \, |w_y|^2 \, dx \, dy = \int_{\mathcal{B}} (\beta_\delta - 1) e^{\lambda x} \mathrm{Re} \, \{w^* [p_0]\} \, dx \, dy +$$
$$\int_{\mathcal{B}} (\beta_\delta - 1) e^{\lambda x} \mathrm{Re} \, \{w^* [g]\} \, dx \, dy + \delta^2 \kappa_0 \int_{\mathcal{B}} \mathrm{Re} \left\{ w^* T \left(\beta_\delta \, e^{\lambda x} \, [p_0] \right)_{xx} \right\} dx \, dy +$$
$$\delta^2 \kappa_0 \int_{\mathcal{B}} \mathrm{Re} \left\{ w^* T \left(\beta_\delta \, e^{\lambda x} \, [g] \right)_{xx} \right\} dx \, dy - \int_{\mathcal{B}} e^{\lambda x} \mathrm{Re} \, \{w^* [p]\} \, dx \, dy.$$

(2.42)

Similarly, from the integration

$$\int_{\mathcal{B}} (2.41) \cdot w^* - (2.41)^* \cdot w \, dx \, dy.$$

we derive

$$\omega \, r_0 \int_{\mathcal{B}} |w|^2 \, dx \, dy = \int_{\mathcal{B}} (\beta_\delta - 1) e^{\lambda x} \mathrm{Im} \, \{w^* [p_0]\} \, dx \, dy +$$
$$\int_{\mathcal{B}} (\beta_\delta - 1) e^{\lambda x} \mathrm{Im} \, \{w^* [g]\} \, dx \, dy + \delta^2 \kappa_0 \int_{\mathcal{B}} \mathrm{Im} \left\{ w^* T \left(\beta_\delta \, e^{\lambda x} \, [p_0] \right)_{xx} \right\} dx \, dy +$$
$$\delta^2 \kappa_0 \int_{\mathcal{B}} \mathrm{Im} \left\{ w^* T \left(\beta_\delta \, e^{\lambda x} \, [g] \right)_{xx} \right\} dx \, dy - \int_{\mathcal{B}} e^{\lambda x} \mathrm{Im} \, \{w^* [p]\} \, dx \, dy.$$

(2.43)

3. Energy identities for $[g]$ and $[g]'$.

Next we return to (2.37) and derive from the expression

$$\int_0^1 (2.37) \cdot [g]^* + (2.37)^* [g] \, dx$$

the identity

$$\frac{1}{\rho \omega^2} \int_0^1 \left| [g]' \right|^2 \, dx = 2 (\mathrm{Re} \, \bar{T}) \int_0^1 e^{\lambda x} \beta_\delta \, |[g]|^2 \, dx$$
$$+ \int_{\mathcal{B}} \mathrm{Re} \, \{w [g]^*\} + (\beta_\delta - 1) e^{\lambda x} \mathrm{Re} \, \{T [p_0][g]\} \, dx \, dy, \tag{2.44}$$

along with its imaginary counterpart $\int_0^1 (2.37) \cdot [g]^* - (2.37)^* [g] \, dx$ which takes the form

(2.45)

$$-2 (\mathrm{Im} \, \bar{T}) \int_0^1 \beta_\delta \, e^{\lambda x} \, |[g]|^2 \, dx = \int_{\mathcal{B}} \mathrm{Im} \, \{w [g]^*\} + (\beta_\delta - 1) e^{\lambda x} \mathrm{Im} \, \{T [p_0][g]\} \, dx \, dy.$$

4. Estimates for $[g]$ and $[g]'$ in terms of w.

In light of (2.36) we see that for $x \in [0,\delta]$ one has $|[g](x)| \leqslant \int_0^\delta |[g]'(s)|\,ds$ from which one readily checks that

$$\int_0^\delta |[g]|^2\,dx \leqslant \delta^2 \int_0^1 \left|[g]'\right|^2\,dx. \tag{2.46}$$

From (2.44) we can estimate that

$$\int_0^1 \left|[g]'\right|^2\,dx \leqslant C\left(\int_0^1 |[g]|^2\,dx + \int_{\mathcal{B}} |w|^2\,dx\,dy + \int_0^\delta |[g]|\,dx\right)$$

so that, writing say $|[g]| \leqslant 1 + |[g]|^2$ we have

$$\int_0^1 \left|[g]'\right|^2\,dx \leqslant C\left(\int_0^1 |[g]|^2\,dx + \int_{\mathcal{B}} |w|^2\,dx\,dy + \delta\right) \tag{2.47}$$

in light of (2.46), where here and in what follows we use C to denote any positive constant independent of δ.

Then invoking (2.26) we see from (2.45) that for a $C > 0$ depending on $\left|\operatorname{Im}\overline{\mathrm{T}}\right|$ we have

$$\begin{aligned}\int_\delta^1 |[g]|^2\,dx \quad &\leqslant C\int_{\mathcal{B}} |w|^2\,dx\,dy + \frac{1}{2}\int_0^1 |[g]|^2\,dx + C\int_0^\delta |[g]|\,dx\\ &\leqslant C\int_{\mathcal{B}} |w|^2\,dx\,dy + \frac{1}{2}\int_0^1 |[g]|^2\,dx + \int_0^\delta |[g]|^2\,dx + C\delta.\end{aligned}$$

Combining this last estimate with (2.46) and (2.47) we find that

$$\int_0^1 |[g]|^2\,dx \leqslant C\left(\int_{\mathcal{B}} |w|^2\,dx\,dy + \delta\right) \tag{2.48}$$

and

$$\int_0^1 \left|[g]'\right|^2\,dx \leqslant C\left(\int_{\mathcal{B}} |w|^2\,dx\,dy + \delta\right). \tag{2.49}$$

5. Control of the trace of $p^\pm$ on $\mathcal{B}$

In light of the standard trace inequality for Sobolev functions, we know that for some $C > 0$

$$\int_{-1}^1 \left|p^+(x,y,0)\right|^2 dy \leqslant C\int_0^1\int_{-1}^1 \left(\left|p^+(x,y,z)\right|^2 + \left|p_y^+(x,y,z)\right|^2 + \left|p_z^+(x,y,z)\right|^2\right) dy\,dz$$

for every $x \in (0,1)$. Hence, appealing to (2.31) in order to apply the Poincaré inequality in Ω^+, and then integrating with respect to x, we obtain (for a different C) that

$$\int_{\mathcal{B}} \left|p^+(x,y,0)\right|^2\,dx\,dy \leqslant C\int_{D^+} \left(\left|p_y^+\right|^2 + \left|p_z^+\right|^2\right) dx\,dy\,dz, \tag{2.50}$$

with a similar inequality holding with p^+ replaced by p^- and D^+ replaced by D^-.

6. Bounds on $\|p_x^\pm\|_{L^2(D^\pm)}$, $\frac{1}{\delta}\left\|p_y^\pm\right\|_{L^2(D^\pm)}$, $\frac{1}{\delta}\|p_z^\pm\|_{L^2(D^\pm)}$ in terms of $\|w\|_{L^2(\mathcal{B})}$

We return to the identity (2.40) to see that

$$\int_{D^\pm}\left\{|p_x^\pm|^2+\frac{1}{\delta^2}\left(|p_y^\pm|^2+|p_z^\pm|^2\right)\right\}dx\,dy\,dz\leqslant C\int_{\mathcal{B}}\left(|p^\pm|+|p^\pm|\,|[g]|+|p^\pm|\,|w|\right)\,dx\,dy.$$

Consequently, through the use of (2.48) and (2.50) there exists a $C>0$ such that

$$\int_{D^\pm}\left\{|p_x^\pm|^2+\frac{1}{\delta^2}\left(|p_y^\pm|^2+|p_z^\pm|^2\right)\right\}dx\,dy\,dz\leqslant C+\frac{1}{2}\int_{\mathcal{B}}|w|^2\,dx\,dy. \tag{2.51}$$

7. Estimate on $\|w\|_{L^2(\mathcal{B})}$ in terms of $\|[p]\|_{L^2(\mathcal{B})}$

To close this estimate, we consider the energy identity for w, (2.43). Note that there are five terms on the right-hand side of this identity. We comment briefly on how each one is handled.

For the first one we have

$$\begin{aligned}\left|\int_{\mathcal{B}}(\beta_\delta-1)e^{\lambda x}\mathrm{Im}\,\{w^*[p_0]\}\,dx\,dy\right|&\leqslant C\int_{\mathcal{B}}|\beta_\delta-1|\,|w|\,dx\,dy\\ &\leqslant C\delta^{1/2}\|w\|_{L^2(\mathcal{B})}\leqslant\frac{\omega r_0}{10}\int_{\mathcal{B}}|w|^2\,dx\,dy+C\delta\end{aligned} \tag{2.52}$$

for an appropriate C.

For the second one we apply Cauchy-Schwarz and then combine (2.46) and (2.49) to make the estimate

$$\begin{aligned}&\left|\int_{\mathcal{B}}(\beta_\delta-1)e^{\lambda x}\mathrm{Im}\,\{w^*[g]\}\,dx\,dy\right|\\ &\leqslant C\left(\int_{\mathcal{B}}|w|^2\,dx\,dy\right)^{1/2}\left(\int_{\mathcal{B}}|\beta_\delta-1|^2\,|[g]|^2\,dx\,dy\right)^{1/2}\\ &\leqslant C\left(\int_{\mathcal{B}}|w|^2\,dx\,dy\right)^{1/2}\left(\int_0^\delta|[g]|^2\,dx\right)^{1/2}\\ &\leqslant C\left(\int_{\mathcal{B}}|w|^2\,dx\,dy\right)^{1/2}\left(\delta^2\int_{\mathcal{B}}|w|^2\,dx\,dy+\delta^3\right)^{1/2}\\ &\leqslant C\delta\left(\int_{\mathcal{B}}|w|^2\,dx\,dy+\delta\right).\end{aligned} \tag{2.53}$$

For the third term on the right-hand side of (2.43)

$$\delta^2\kappa_0\int_{\mathcal{B}}\mathrm{Im}\,\left\{w^*T\left(\beta_\delta\,e^{\lambda x}\,[p_0]\right)_{xx}\right\}\,dx\,dy$$

we observe that all terms not involving derivatives of β_δ can be bounded in absolute value by $C\delta^2\int_{\mathcal{B}}|w|\,dx\,dy$. and so, via Cauchy-Schwarz, by $C\delta^2\|w\|_{L^2(\mathcal{B})}$. As for the terms involving differentiation of β_δ, in light of the bounds $|\beta_\delta'|\leqslant\frac{2}{\delta}$, $|\beta_\delta''|\leqslant\frac{2}{\delta^2}$ and the fact that these derivatives are supported on the interval $0<x<\delta$, all of these can be bounded by $C\int_0^\delta|w|$ which in turn can be bounded by say $C\delta+\frac{\omega r_0}{10}\|w\|^2_{L^2(\mathcal{B})}$

for an appropriate C. Consequently, we have

$$\left|\delta^2\kappa_0 \int_{\mathcal{B}} \operatorname{Im}\left\{w^* T\left(\beta_\delta\, e^{\lambda x}\,[p_0]\right)_{xx}\right\} dx\, dy\right| \leqslant C\delta + \frac{\omega r_0}{10}\int_{\mathcal{B}} |w|^2\; dx\, dy. \tag{2.54}$$

The fourth term, namely

$$\delta^2\kappa_0 \int_{\mathcal{B}} \operatorname{Re}\left\{w^* T\left(\beta_\delta\, e^{\lambda x}\,[g]\right)_{xx}\right\} dx\, dy,$$

is handled in a similar manner except that one must substitute for the term containing $[g]_{xx}$ using the ODE (2.37) and make use of (2.46)-(2.49). Carrying this out, we again can bound it by the right-hand side of (2.54).

For the fifth and final term on the right-hand side of (2.43) we have

$$\begin{aligned}\int_{\mathcal{B}} \left|e^{\lambda x}\operatorname{Im}\{w^*[p]\}\, dx\, dy\right| &\leqslant C\, \|w\|_{L^2(\mathcal{B})}\, \|[p]\|_{L^2(\mathcal{B})} \\ &\leqslant \frac{\omega r_0}{10}\int_{\mathcal{B}} |w|^2\; dx\, dy + C\int_{\mathcal{B}} |[p]|^2\; dx\, dy\end{aligned} \tag{2.55}$$

for an appropriately chosen C.

Combining the estimates (2.52)-(2.55) and absorbing the square L^2-norms of w into the left-hand side of (2.43) we conclude that

$$\int_{\mathcal{B}} |w|^2\; dx\, dy \leqslant C\left(\int_{\mathcal{B}} |[p]|^2\; dx\, dy + \delta\right). \tag{2.56}$$

8. Closing the estimates on $\|p_x^\pm\|_{L^2(D^\pm)}$, $\frac{1}{\delta}\left\|p_y^\pm\right\|_{L^2(D^\pm)}$, $\frac{1}{\delta}\|p_z^\pm\|_{L^2(D^\pm)}$ and $\|w\|_{L^2(\mathcal{B})}$

Now we add (2.51) and (2.56) and through an appeal to (2.50) and its counterpart for p^- we arrive at the uniform estimate

$$\int_{D^\pm}\left\{\left|p_x^\pm\right|^2 + \frac{1}{\delta^2}\left(\left|p_y^\pm\right|^2 + \left|p_z^\pm\right|^2\right)\right\} dx\, dy\, dz + \int_{\mathcal{B}} |w|^2\; dx\, dy \leqslant C. \tag{2.57}$$

In particular, then, we have shown that $\left\|p_y^\pm\right\|_{L^2(D^\pm)}$ and $\left\|p_y^\pm\right\|_{L^2(D^\pm)}$ are $O(\delta)$.

From this, (2.50) and the Poincaré inequality it then immediately follows that $\|p^\pm\|_{L^2(\mathcal{B})}$ and $\|p^\pm\|_{L^2(D^\pm)}$ tend to zero at the same order. Therefore, by (2.56), $\|w\|_{L^2(\mathcal{B})} = O(\delta^{1/2})$. Necessarily, $\|[g]\|_{L^2(0,1)}$ approaches zero as well, in view of (2.48).

Since the right-hand side of (2.35) approaches zero in $L^2(0,1)$, we observe that $g^\pm$ both satisfy ODE's of the form $g_{xx}^\pm = f$ where f approaches zero in L^2. Given the homogeneous boundary conditions (2.36), it easily follows that $g^\pm$ also approach zero. Hence, recalling (2.38), we have shown that $P^\pm \to p_0^\pm$ in $L^2(D^\pm)$.

Finally, these estimates along with (2.52)-(2.55) applied to (2.42) imply that $\|w_y\|_{L^2(\mathcal{B})} \to 0$ as $\delta \to 0$ as well, completing the proof of (2.29).

□

2.2. **Second elastic model: Full wave equation with resonance and friction.** Suppose now that instead of neglecting the inertial term in (2.10) and instead of assuming (2.15) we take m to be constant and we assume

$$\kappa(x) = \frac{c^2\kappa_0(x)}{\delta}, \quad r = \frac{r_0}{\delta}, \tag{2.58}$$

where κ_0 is a positive, decreasing function of x and r_0 is a positive constant. Now if we do not scale m with δ then equation (2.17) is replaced by

$$-\omega^2\delta m W + i\omega r_0 W - \delta^2\kappa_0 W_{xx} - \kappa_0 W_{yy} = -[P]. \tag{2.59}$$

For now, let us scale m with δ by defining m_0 via $m = \frac{m_0}{\delta}$, leading to the equation

$$-\omega^2 m_0 W + i\omega r_0 W - \delta^2\kappa_0 W_{xx} - \kappa_0 W_{yy} = -[P]. \tag{2.60}$$

Then we can proceed as before, but now we replace $T(y)$ given in (2.25) by $T(x,y)$ mapping $[0,1]\times[-1,1]$ into $\mathbb{C}$ as the solution to the boundary value problem

$$(ir_0\omega - \omega^2 m_0)T - \kappa_0(x)T_{yy} = -1 \text{ for } 0<x<1,\ -1<y<1,\ T(x,\pm 1) = 0. \tag{2.61}$$

If we let $\bar{T}(x) := \frac{1}{2}\int_{-1}^{1} T(x,y)\,dy$ denote the integral average in y of T, then the analogue of property (2.26), namely

$$\inf_{x\in[0,1]} \operatorname{Im}\bar{T}(x) > 0, \tag{2.62}$$

follows by a simple calculation given in the Appendix and the same approach based on energy estimates yields:

Theorem 2.2. *As $\delta \to 0$ the functions $P^{\pm}$ converge in $L^2(D^{\pm})$ to the functions $p_0^{\pm} : [0,1] \to \mathbb{C}$ solving the following system of ODE's*

$$(p_0^{\pm})'' \mp L\rho\omega^2\bar{T}(x)[p_0] = 0 \quad \textit{for} \quad 0<x<1, \tag{2.63}$$

$$p_0^+(0) = 1,\ p_0^+(1) = 0 = p_0^-(0) = p_0^-(1). \tag{2.64}$$

Furthermore, the derivatives $P_y^{\pm}$ and $P_z^{\pm}$ converge to zero in $L^2(D^{\pm})$. Finally, as $\delta \to 0$ one has

$$W - T(x,y)\,[p_0] \to 0 \quad \textit{and} \quad (W - T(x,y)\,[p_0])_y \to 0 \quad \textit{in } L^2(\mathcal{B}). \tag{2.65}$$

3. Discussion

We derived 1.5D reduced models for the hydro-elastic waves in the cochlea where the BM is modeled as an elastic membrane. We presented two models. The first one, where the inertia of the BM was neglected, while the friction and elastic coefficient took a special form, is an extension of the spring model that was considered by us in [9]; see also [6]. The reduced model in this case consists of equations (2.26) for the function $T(y)$ and equation (2.27) for the fluid pressure. We recall that the place principle determining the location along the cochlea excited by a given frequency is expressed via an experimentally derived function $x = G(\omega)$ where the Greenwood function G is specific to a given mammal. To see how the 1.5D model above is related to the function G we compare the reduced model in the present setting with the spring model of [9], and refer to the formula for $\bar{T}$ derived in the Appendix. If we maintain only the leading order term in the sums for the two expressions for $\operatorname{Re} T$ and $\operatorname{Im} T$, (cf. (4.6) and (4.5)) and set $m = 0$, we recover the 1D model of [9]. We can then perform a WKB expansion of equation (2.27) and obtain, for an appropriate choice of the parameters r_0 and κ_0, the Greenwood function G. Maintaining further terms in the sums appearing in the formula for $\operatorname{Re} T$ and $\operatorname{Im} T$ will provide corrections to the classical place principle formula.

In the second model the equation for the BM motion has a built-in expression for the resonance. To obtain the place principle formula $x = G(\omega)$, we can again maintain just the first term in the infinite sum for $\operatorname{Re} T$ and $\operatorname{Im} T$. Then the resonance formula becomes

$$m_0\omega^2 = \left(\frac{\pi}{2}\right)^2 \kappa_0(x).$$

Therefore, selecting an appropriate function $\kappa_0(x)$ would imply the experimentally observed G.

4. Appendix

In this appendix we provide a simple proof of (2.62). The condition (2.26) is established similarly. Writing $T = \operatorname{Re} T + i\operatorname{Im} T$, it follows from (2.61) that

$$r_0\omega \operatorname{Re} T - m_0\omega^2 \operatorname{Im} T - \kappa_0(x) \operatorname{Im} T_{yy} = 0 \tag{4.1}$$

and

$$r_0\omega \operatorname{Im} T + m_0\omega^2 \operatorname{Re} T + \kappa_0(x) \operatorname{Re} T_{yy} = 1, \tag{4.2}$$

with $\operatorname{Re} T(x, \pm 1) = \operatorname{Im} T(\pm 1) = 0$. It also readily follows from (2.61) and the boundary conditions that T, and therefore $\operatorname{Re} T$ and $\operatorname{Im} T$, are even functions of y so that we may instead work on the interval $0 \leqslant y \leqslant 1$ and replace the boundary conditions by

$$\operatorname{Re} T_y(x, 0) = 0 = \operatorname{Im} T_y(x, 0) \quad \text{and} \quad \operatorname{Re} T(x, 1) = 0 = \operatorname{Im} T(x, 1).$$

In light of these boundary conditions we seek Fourier series expansions for $\operatorname{Re} T$ and $\operatorname{Im} T$ of the form

$$\operatorname{Re} T(x, y) = \sum_{k=1}^{\infty} a_n(x) \cos\left[\frac{(2n-1)\pi}{2} y\right]$$

and

$$\operatorname{Im} T(x, y) = \sum_{k=1}^{\infty} b_n(x) \cos\left[\frac{(2n-1)\pi}{2} y\right].$$

Expanding

$$1 = \sum_{n=1}^{\infty} A_n \cos\left[\frac{(2n-1)\pi}{2} y\right]$$

so that

$$A_n = \frac{4}{(2n-1)\pi}(-1)^{n+1}$$

we then substitute these expansions into (4.1) and (4.2) to obtain the system

$$r_0\omega\, a_n + \left(\kappa_0 \left[\frac{(2n-1)\pi}{2}\right]^2 - m_0\omega^2\right) b_n = 0, \tag{4.3}$$

$$r_0\omega\, b_n - \left(\kappa_0 \left[\frac{(2n-1)\pi}{2}\right]^2 - m_0\omega^2\right) a_n = \frac{4}{(2n-1)\pi}(-1)^{n+1}. \tag{4.4}$$

Solving for b_n we find that $\operatorname{Im} T$ is given by

(4.5) $\operatorname{Im} T(x,y)$

$$= \frac{4r_0\omega}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)\left\{r_0^2\omega^2+\left(\kappa_0\left[\frac{(2n-1)\pi}{2}\right]^2-m_0\omega^2\right)^2\right\}}\cos\left[\frac{(2n-1)\pi}{2}y\right].$$

As this series is clearly uniformly convergent in y, we may integrate term-wise to obtain

$$\operatorname{Im}\bar{T}(x)=\frac{1}{2}\int_{-1}^{1}\operatorname{Im} T(x,y)\,dy=\int_{0}^{1}\operatorname{Im} T(x,y)\,dy=$$
$$\frac{8r_0\omega}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2\left\{r_0^2\omega^2+\left(\kappa_0\left[\frac{(2n-1)\pi}{2}\right]^2-m_0\omega^2\right)^2\right\}}>0.$$

Hence (2.62) is verified.

It is useful to write also the associated formula for $\operatorname{Re} T$ that follows similarly from equations (4.3)-(4.4):

(4.6)

$$\operatorname{Re} T(x,y)=\frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\left(\kappa_0\left[\frac{(2n-1)\pi}{2}\right]^2-m_0\omega^2\right)^2}{(2n-1)\left\{r_0^2\omega^2+\left(\kappa_0\left[\frac{(2n-1)\pi}{2}\right]^2-m_0\omega^2\right)^2\right\}}\cos\left[\frac{(2n-1)\pi}{2}y\right].$$

and after integration:

$$\int_{-1}^{1}\operatorname{Re} T(x,y)\,dy=\frac{-16}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(\kappa_0\left[\frac{(2n-1)\pi}{2}\right]^2-m_0\omega^2\right)}{(2n-1)^2\left\{r_0^2\omega^2+\left(\kappa_0\left[\frac{(2n-1)\pi}{2}\right]^2-m_0\omega^2\right)^2\right\}}.$$

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