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ON THE SOLVABILITY OF SOME SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH ANOMALOUS DIFFUSION IN TWO DIMENSIONS

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Abstract. The article deals with the existence of solutions of a system of integro-differential equations in the case of anomalous diffusion with the negative Laplacian in a fractional power in two dimensions. The proof of existence of solutions relies on a fixed point technique. Solvability conditions for elliptic operators without Fredholm property in unbounded domains are used.

1. Introduction

The present article is devoted to the studies of the existence of stationary solutions of the following system of integro-differential equations

$$\frac{\partial u_m}{\partial t} = -D_m(-\Delta)^{s_m} u_m + \int_{\mathbb{R}^2} K_m(x-y) g_m(u(y,t))dy + f_m(x), \tag{1.1}$$

$1 \le m \le N$, which appears in the cell population dynamics. The space variable x here corresponds to the cell genotype, functions $u_m(x,t)$ describe the cell density distributions for various groups of cells as functions of their genotype and time,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The right side of the system of equations (1.1) describes the evolution of cell densities by means of the cell proliferation, mutations and cell influx or efflux. The anomalous diffusion terms with positive coefficients D_m correspond to the change of genotype due to small random mutations, and the nonlocal production terms describe large mutations. Functions $g_m(u)$ stand for the rates of cell birth which depend on u (density dependent proliferation), and the kernels $K_m(x-y)$ express the proportions of newly born cells changing their genotype from y to x. Let us assume that they depend on the distance between the genotypes. The functions $f_m(x)$ describe the influx or efflux of cells for different genotypes.

The operators $(-\Delta)^{s_m}$, $1 \le m \le N$ in system (1.1) describe a particular case of anomalous diffusion actively treated in the context of various applications in plasma physics and turbulence [7], [20], surface diffusion [14], [18], semiconductors [19] and so on. Anomalous diffusion can be understood as a random process of particle motion characterized by the probability density distribution of jump length. The

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moments of this density distribution are finite in the case of normal diffusion, but this is not the case for the anomalous diffusion. The asymptotic behavior at infinity of the probability density function determines the value $s_m,\ 1 \leq m \leq N$ of the power of the negative Laplacian (see [17]). The operators $(-\Delta)^{s_m},\ 1 \leq m \leq N$ are defined by virtue of the spectral calculus. Let us consider the case of $0 < s_m < 1/2,\ 1 \leq m \leq N$ in the present article. A similar system in the case of the standard Laplacian in the diffusion term was treated recently in [32]. Let us note that the restriction on the powers $s_m,\ 1 \leq m \leq N$ here comes from the solvability conditions of our problem.

We set here all $D_m = 1$ and establish the existence of solutions of the system of equations

$$-(-\Delta)^{s_m} u_m + \int_{\mathbb{R}^2} K_m(x-y) g_m(u(y)) dy + f_m(x) = 0, \quad 0 < s_m < \frac{1}{2}, \tag{1.2}$$

where $1 \leq m \leq N$. We treat the case when the linear part of this operator fails to satisfy the Fredholm property. Consequently, the conventional methods of nonlinear analysis may not be applicable. Let us use the solvability conditions for the operators without Fredholm property along with the method of contraction mappings.

Let us consider the problem

$$-\Delta u + V(x)u - au = f, \tag{1.3}$$

where $u \in E = H^2(\mathbb{R}^d)$ and $f \in F = L^2(\mathbb{R}^d),\ d \in \mathbb{N}$, a is a constant and the scalar potential function $V(x)$ is either zero identically or converges to 0 at infinity. For $a \geq 0$, the essential spectrum of the operator $A : E \to F$ corresponding to the left side of equation (1.3) contains the origin. Consequently, this operator fails to satisfy the Fredholm property. Its image is not closed, for $d > 1$ the dimension of its kernel and the codimension of its image are not finite. The present article is devoted to the studies of certain properties of the operators of this kind. Note that elliptic equations with non Fredholm operators were studied actively in recent years. Approaches in weighted Sobolev and Hölder spaces were developed in [2], [3], [4], [5], [6]. The Schrödinger type operators without Fredholm property were treated with the methods of the spectral and the scattering theory in [21], [27], [26]. The Laplace operator with drift from the point of view of non Fredholm operators was studied in [29] and linearized Cahn-Hilliard equations in [24] and [30]. Nonlinear non Fredholm elliptic equations were treated in [28] and [31]. The significant applications to the theory of reaction-diffusion type problems were developed in [9], [10]. The operators without Fredholm property arise also when studying wave systems with an infinite number of localized traveling waves (see [1]). In particular, when $a = 0$ the operator A is Fredholm in some properly chosen weighted spaces (see [2], [3], [4], [5], [6]). However, the case of $a \neq 0$ is significantly different and the method developed in these works cannot be used. Front propagation problems with anomalous diffusion were treated largely in recent years (see e.g. [22], [23]). The form boundedness criterion for the relativistic Schrödinger operator was established in [16]. In article [15] the authors prove the imbedding theorems and study the spectrum of certain pseudodifferential operators.

Let us set $K_m(x) = \varepsilon_m H_m(x)$, where $\varepsilon_m \geq 0$, such that

$$\varepsilon := max_{1 \leq m \leq N} \varepsilon_m, \quad s := max_{1 \leq m \leq N} s_m$$

with $0 < s < \frac{1}{2}$ and suppose that the assumption below is fulfilled.

Assumption 1.1. *Let $1 \leq m \leq N$ and consider $0 < s_m < \frac{1}{2}$. Let $f_m(x) : \mathbb{R}^2 \to \mathbb{R}$ be nontrivial for some m. Let $f_m(x) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and*

$$(-\Delta)^{1-s_m} f_m(x) \in L^2(\mathbb{R}^2).$$

We assume also that $H_m(x) : \mathbb{R}^2 \to \mathbb{R}$, such that $H_m(x) \in L^1(\mathbb{R}^2)$ and

$$(-\Delta)^{1-s_m} H_m(x) \in L^2(\mathbb{R}^2).$$

Moreover,

$$H^2 := \sum_{m=1}^{N} \|H_m(x)\|^2_{L^1(\mathbb{R}^2)} > 0$$

and

$$Q^2 := \sum_{m=1}^{N} \|(-\Delta)^{1-s_m} H_m(x)\|^2_{L^2(\mathbb{R}^2)} > 0.$$

We choose here the space dimension $d = 2$, which is related to the solvability conditions for the linear Poisson type equation (4.1) given in Lemma 4.1 below. For the applications, the space dimension is not restricted to $d = 2$, because the space variable here corresponds to the cell genotype but not to the usual physical space. In $d = 1$ our system was studied in [35] with all $0 < s_m = s < \frac{1}{4}$ based on the solvability conditions for the analog of (4.1) in one dimension. In $d = 3$ our system was treated in [33] with all $\frac{1}{4} < s_m = s < \frac{3}{4}$. As distinct from the situation in lower dimensions $d = 1, 2$, in $\mathbb{R}^3$ we were able to apply the Sobolev inequality for the fractional negative Laplacian (see Lemma 2.2 of [12], also [13]). Let us use the Sobolev spaces for the technical purposes with $0 < s \leq 1$, namely

$$H^{2s}(\mathbb{R}^2) := \{\phi(x) : \mathbb{R}^2 \to \mathbb{R} \mid \phi(x) \in L^2(\mathbb{R}^2), \ (-\Delta)^s \phi \in L^2(\mathbb{R}^2)\}$$

equipped with the norm

$$\|\phi\|^2_{H^{2s}(\mathbb{R}^2)} := \|\phi\|^2_{L^2(\mathbb{R}^2)} + \|(-\Delta)^s \phi\|^2_{L^2(\mathbb{R}^2)}. \tag{1.4}$$

For a vector vector function

$$u(x) = (u_1(x), u_2(x), ..., u_N(x))^T,$$

throughout the article we will use the norm

$$\|u\|^2_{H^2(\mathbb{R}^2,\mathbb{R}^N)} := \|u\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)} + \sum_{m=1}^{N} \|\Delta u_m\|^2_{L^2(\mathbb{R}^2)} \tag{1.5}$$

with

$$\|u\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)} := \sum_{m=1}^{N} \|u_m\|^2_{L^2(\mathbb{R}^2)}.$$

By virtue of the standard Sobolev embedding in two dimensions, we have

$$\|\phi\|_{L^\infty(\mathbb{R}^2)} \le c_e \|\phi\|_{H^2(\mathbb{R}^2)}, \tag{1.6}$$

where $c_e > 0$ is the constant of the embedding. When all the nonnegative parameters $\varepsilon_m = 0$, we arrive at the linear Poisson type equations

$$(-\Delta)^{s_m} u_m(x) = f_m(x), \quad 1 \le m \le N. \tag{1.7}$$

By virtue of Lemma 4.1 below along with Assumption 1.1 each equation (1.7) admits a unique solution

$$u_{0,m}(x) \in H^{2s_m}(\mathbb{R}^2), \quad 0 < s_m < \frac{1}{2}, \quad 1 \le m \le N,$$

such that no orthogonality conditions are required. According to Lemma 4.1 below, when $\frac{1}{2} \le s_m < 1$, a certain orthogonality condition (see formula (4.3)) is needed to be able to solve equation (1.7) in $H^{2s_m}(\mathbb{R}^2)$. Because

$$-\Delta u_{0,m}(x) = (-\Delta)^{1-s_m} f_m(x) \in L^2(\mathbb{R}^2), \quad 1 \le m \le N$$

due to Assumption 1.1, we obtain for the unique solution of linear problem (1.7) that each $u_{0,m}(x) \in H^2(\mathbb{R}^2)$, such that

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^2(\mathbb{R}^2, \mathbb{R}^N).$$

Let us look for the resulting solution of nonlinear system of equations (1.2) as

$$u(x) = u_0(x) + u_p(x), \tag{1.8}$$

with

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Evidently, we easily derive the perturbative system of equations

$$(-\Delta)^{s_m} u_{p,m}(x) = \varepsilon_m \int_{\mathbb{R}^2} H_m(x-y) g_m(u_0(y) + u_p(y)) dy, \quad 0 < s_m < \frac{1}{2}, \tag{1.9}$$

with $1 \le m \le N$. We introduce a closed ball in the Sobolev space

$$B_\rho := \{u(x) \in H^2(\mathbb{R}^2, \mathbb{R}^N) \mid \|u\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} \le \rho\}, \quad 0 < \rho \le 1. \tag{1.10}$$

Let us look for the solution of system (1.9) as the fixed point of the auxiliary nonlinear problem

$$(-\Delta)^{s_m} u_m(x) = \varepsilon_m \int_{\mathbb{R}^2} H_m(x-y) g_m(u_0(y) + v(y)) dy, \quad 0 < s_m < \frac{1}{2}, \tag{1.11}$$

where $1 \le m \le N$ in ball (1.10). For a given vector function $v(y)$ this is a system of equations with respect to $u(x)$. The left side of (1.11) contains the operators without the Fredholm property

$$(-\Delta)^{s_m} : H^{2s_m}(\mathbb{R}^2) \to L^2(\mathbb{R}^2).$$

Its essential spectrum fills the nonnegative semi-axis $[0, +\infty)$. Therefore, such operator has no bounded inverse. The similar situation appeared in works [28] and [31] but as distinct from the present case, the problems studied there required orthogonality conditions. The fixed point technique was used in [25] to estimate the perturbation to the standing solitary wave of the Nonlinear Schrödinger (NLS) equation when either the external potential or the nonlinear term in the NLS were perturbed but the Schrödinger operator involved in the nonlinear equation there possessed the Fredholm property (see Assumption 1 of [25], also [8]). Let us define the closed ball in the space of N dimensions as

$$I := \{z \in \mathbb{R}^N \mid |z| \le c_e \|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} + c_e\} \tag{1.12}$$

and the closed ball D_M in the space of $C^2(I, \mathbb{R}^N)$ vector functions given by

$$\{g(z) := (g_1(z), g_2(z), ..., g_N(z)) \in C^2(I, \mathbb{R}^N) \mid \|g\|_{C^2(I,\mathbb{R}^N)} \le M\}, \tag{1.13}$$

with $M > 0$. Here the norms

$$\|g\|_{C^2(I,\mathbb{R}^N)} := \sum_{m=1}^{N} \|g_m\|_{C^2(I)}, \tag{1.14}$$

$$\|g_m\|_{C^2(I)} := \|g_m\|_{C(I)} + \sum_{n=1}^{N} \Big\|\frac{\partial g_m}{\partial z_n}\Big\|_{C(I)} + \sum_{n,l=1}^{N} \Big\|\frac{\partial^2 g_m}{\partial z_n \partial z_l}\Big\|_{C(I)}, \tag{1.15}$$

where $\|g_m\|_{C(I)} := max_{z\in I}|g_m(z)|$. We make the following technical assumption on the nonlinear part of problem (1.2).

Assumption 1.2. *Let $1 \le m \le N$. Assume that $g_m(z) : \mathbb{R}^N \to \mathbb{R}$, such that $g_m(0) = 0$ and $\nabla g_m(0) = 0$. It is also assumed that $g(z) \in D_M$ and it is not equal to zero identically in the ball I.*

Let us introduce the operator T_g, such that $u = T_g v$, where u is a solution of problem (1.11). Our first main result is as follows.

Theorem 1.3. *Let Assumptions 1.1 and 1.2 hold. Then for every $\rho \in (0, 1]$ there exists $\varepsilon^* > 0$, such that system (1.11) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon^*$. The unique fixed point $u_p(x)$ of this map T_g is the only solution of problem (1.9) in B_ρ.*

Obviously, the resulting solution $u(x)$ of problem (1.2) will not vanish identically since the source terms $f_m(x)$ are nontrivial for some $1 \le m \le N$ and all $g_m(0) = 0$ due to the one of our assumptions. Let us make use of the following elementary lemma.

Lemma 1.4. *For $R \in (0, +\infty)$ consider the function*

$$\varphi(R) := \alpha R^{2-4s} + \frac{1}{R^{4s}}, \quad 0 < s < \frac{1}{2}, \quad \alpha > 0.$$

It attains the minimal value at $R^* := \sqrt{\dfrac{2s}{\alpha(1-2s)}}$, *which is given by*

$$\varphi(R^*) = \frac{(1-2s)^{2s-1}}{(2s)^{2s}} \alpha^{2s}.$$

Our second main proposition deals with the continuity of the fixed point of the map T_g which existence was established in Theorem 1.3 above with respect to the nonlinear vector function g.

Theorem 1.5. *Let $j = 1, 2$, the assumptions of Theorem 1.3 hold, such that $u_{p,j}(x)$ is the unique fixed point of the map $T_{g_j} : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon < \varepsilon_j^*$ and $\delta := min(\varepsilon_1^*, \varepsilon_2^*)$. Then for all $0 < \varepsilon < \delta$ the estimate*

$$\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} \le C\|g_1 - g_2\|_{C^2(I,\mathbb{R}^N)} \tag{1.16}$$

holds, where $C > 0$ is a constant.

Let us proceed to the proof of our first main statement.

2. The existence of the perturbed solution

Proof of Theorem 1.3. Let us choose an arbitrary vector function $v(x) \in B_\rho$ and denote the terms involved in the integral expressions in the right side of problem (1.11) as

$$G_m(x) := g_m(u_0(x) + v(x)), \quad 1 \le m \le N.$$

Throughout the article we will use the standard Fourier transform

$$\widehat{\phi}(p) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \phi(x)e^{-ipx}dx. \tag{2.1}$$

Clearly, we have the upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^2)} \le \frac{1}{2\pi}\|\phi(x)\|_{L^1(\mathbb{R}^2)}. \tag{2.2}$$

We apply (2.1) to both sides of problem (1.11). This yields

$$\widehat{u}_m(p) = \varepsilon_m 2\pi \frac{\widehat{H}_m(p)\widehat{G}_m(p)}{|p|^{2s_m}}, \quad 1 \le m \le N.$$

Then we express the norm as

$$\|u_m\|^2_{L^2(\mathbb{R}^2)} = 4\pi^2\varepsilon_m^2 \int_{\mathbb{R}^2} \frac{|\widehat{H}_m(p)|^2|\widehat{G}_m(p)|^2}{|p|^{4s_m}}dp, \quad 1 \le m \le N. \tag{2.3}$$

As distinct from works [28] and [31] with the standard Laplacian in the diffusion term, here we do not try to control the norms

$$\left\|\frac{\widehat{H}_m(p)}{|p|^{2s_m}}\right\|_{L^\infty(\mathbb{R}^2)}, \quad 1 \le m \le N.$$

Instead, let us estimate the right side of (2.3) via the analog of bound (2.2) applied to functions H_m and G_m with $R \in (0, +\infty)$ as

$$4\pi^2\varepsilon_m^2\Big[\int_{|p|\le R} \frac{|\widehat{H}_m(p)|^2|\widehat{G}_m(p)|^2}{|p|^{4s_m}}dp + \int_{|p|>R} \frac{|\widehat{H}_m(p)|^2|\widehat{G}_m(p)|^2}{|p|^{4s_m}}dp\Big] \le$$

$$\leq \varepsilon_m^2 \|H_m\|_{L^1(\mathbb{R}^2)}^2 \left\{ \frac{1}{4\pi} \|G_m(x)\|_{L^1(\mathbb{R}^2)}^2 \frac{R^{2-4s_m}}{1-2s_m} + \frac{1}{R^{4s_m}} \|G_m(x)\|_{L^2(\mathbb{R}^2)}^2 \right\}. \tag{2.4}$$

Be means of norm definition (1.5) along with the triangle inequality and since $v(x) \in B_\rho$, we easily arrive at

$$\|u_0 + v\|_{L^2(\mathbb{R}^2, \mathbb{R}^N)} \leq \|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1.$$

Sobolev embedding (1.6) gives us

$$|u_0 + v| \leq c_e(\|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1).$$

Let the dot stand for the scalar product of two vectors in $\mathbb{R}^N$. The representation

$$G_m(x) = \int_0^1 \nabla g_m(t(u_0(x) + v(x))).(u_0(x) + v(x))dt, \quad 1 \leq m \leq N,$$

where the ball I is defined in (1.12) implies

$$|G_m(x)| \leq sup_{z \in I} |\nabla g_m(z)||u_0(x) + v(x)| \leq M|u_0(x) + v(x)|.$$

Therefore,

$$\|G_m(x)\|_{L^2(\mathbb{R}^2)} \leq M\|u_0 + v\|_{L^2(\mathbb{R}^2, \mathbb{R}^N)} \leq M(\|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1).$$

Evidently, for $t \in [0, 1]$ and $1 \leq m, j \leq N$, we have the representation

$$\frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x))) = \int_0^t \nabla \frac{\partial g_m}{\partial z_j}(\tau(u_0(x) + v(x))).(u_0(x) + v(x))d\tau.$$

This gives us

$$\left| \frac{\partial g_m}{\partial z_j}(t(u_0(x) + v(x))) \right| \leq sup_{z \in I} \left| \nabla \frac{\partial g_m}{\partial z_j} \right| |u_0(x) + v(x)| \leq$$

$$\leq \sum_{n=1}^N \left\| \frac{\partial^2 g_m}{\partial z_n \partial z_j} \right\|_{C(I)} |u_0(x) + v(x)|.$$

Thus, $|G_m(x)| \leq$

$$\leq |u_0(x) + v(x)| \sum_{n,j=1}^N \left\| \frac{\partial^2 g_m}{\partial z_n \partial z_j} \right\|_{C(I)} |u_{0,j}(x) + v_j(x)| \leq M|u_0(x) + v(x)|^2.$$

Therefore,

$$\|G_m(x)\|_{L^1(\mathbb{R}^2)} \leq M\|u_0 + v\|_{L^2(\mathbb{R}^2, \mathbb{R}^N)}^2 \leq M(\|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1)^2. \tag{2.5}$$

This allows us to derive the estimate from above for the right side of (2.4) as $\varepsilon_m^2 M^2 \|H_m\|_{L^1(\mathbb{R}^2)}^2 \times$

$$\times(\|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1)^2 \left\{ \frac{(\|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1)^2 R^{2-4s_m}}{4\pi(1-2s_m)} + \frac{1}{R^{4s_m}} \right\},$$

where $R \in (0, +\infty)$. Lemma 1.4 yields the minimal value of the expression above. Hence, $\|u_m\|_{L^2(\mathbb{R}^2)}^2 \leq$

$$\leq {\varepsilon_m}^2 \|H_m\|_{L^1(\mathbb{R}^2)}^2 (\|u_0\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} + 1)^{2+4s_m} \frac{M^2}{(1-2s_m)(8\pi s_m)^{2s_m}}.$$

We introduce

$$\frac{1}{(8\pi S)^{2S}} := max_{1\le m\le N}\frac{1}{(8\pi s_m)^{2s_m}},$$

with $0<S<\frac{1}{2}$. Hence

$$\|u\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)} \le \varepsilon^2 H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{2+4s}\frac{M^2}{(1-2s)(8\pi S)^{2S}}. \tag{2.6}$$

Evidently, (1.11) gives us

$$-\Delta u_m(x) = \varepsilon_m(-\Delta)^{1-s_m}\int_{\mathbb{R}^2} H_m(x-y)G_m(y)dy, \quad 1\le m\le N,$$

with $0<s_m<\frac{1}{2}$. By virtue of the analog of estimate (2.2) applied to function G_m along with (2.5) we arrive at

$$\|\Delta u_m\|^2_{L^2(\mathbb{R}^2)} \le \varepsilon_m^2\|G_m\|^2_{L^1(\mathbb{R}^2)}\|(-\Delta)^{1-s_m}H_m\|^2_{L^2(\mathbb{R}^2)} \le$$

$$\le \varepsilon^2 M^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^4\|(-\Delta)^{1-s_m}H_m\|^2_{L^2(\mathbb{R}^2)}.$$

Therefore,

$$\sum_{m=1}^N \|\Delta u_m\|^2_{L^2(\mathbb{R}^2)} \le \varepsilon^2 M^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^4 Q^2. \tag{2.7}$$

Thus, by means of the definition of the norm (1.5) along with estimates (2.6) and (2.7) we obtain the upper bound for $\|u\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}$ as

$$\varepsilon M(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2\left[\frac{H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{4s-2}}{(1-2s)(8\pi S)^{2S}}+Q^2\right]^{\frac{1}{2}} \le \rho \tag{2.8}$$

for all $\varepsilon>0$ small enough. Therefore, $u(x)\in B_\rho$ as well. If for some $v(x)\in B_\rho$ there exist two solutions $u_{1,2}(x)\in B_\rho$ of problem (1.11), their difference $w(x) := u_1(x)-u_2(x)\in L^2(\mathbb{R}^2,\mathbb{R}^N)$ satisfies

$$(-\Delta)^{s_m}w_m(x) = 0, \quad 0<s_m<\frac{1}{2}, \quad 1\le m\le N.$$

Since the operator $(-\Delta)^{s_m}$ considered on the whole $\mathbb{R}^2$ does not have any nontrivial square integrable zero modes, $w(x)=0$ a.e. on $\mathbb{R}^2$. Hence, system (1.11) defines a map $T_g: B_\rho\to B_\rho$ for all $\varepsilon>0$ sufficiently small.

Our aim is to prove that this map is a strict contraction. We choose arbitrarily $v_{1,2}(x)\in B_\rho$. The argument above yields $u_{1,2} := T_g v_{1,2}\in B_\rho$ as well. By virtue of (1.11) we have for $1\le m\le N$

$$(-\Delta)^{s_m}u_{1,m}(x) = \varepsilon_m\int_{\mathbb{R}^2} H_m(x-y)g_m(u_0(y)+v_1(y))dy, \tag{2.9}$$

$$(-\Delta)^{s_m}u_{2,m}(x) = \varepsilon_m\int_{\mathbb{R}^2} H_m(x-y)g_m(u_0(y)+v_2(y))dy, \tag{2.10}$$

where all $0 < s_m < \frac{1}{2}$. Let us define

$$G_{1,m}(x) := g_m(u_0(x) + v_1(x)), \quad G_{2,m}(x) := g_m(u_0(x) + v_2(x)), \quad 1 \le m \le N$$

and apply the standard Fourier transform (2.1) to both sides of problems (2.9) and (2.10). This gives us

$$\widehat{u}_{1,m}(p) = \varepsilon_m 2\pi \frac{\widehat{H}_m(p)\widehat{G}_{1,m}(p)}{|p|^{2s_m}}, \quad \widehat{u}_{2,m}(p) = \varepsilon_m 2\pi \frac{\widehat{H}_m(p)\widehat{G}_{2,m}(p)}{|p|^{2s_m}}.$$

Evidently,

$$\|u_{1,m} - u_{2,m}\|^2_{L^2(\mathbb{R}^2)} = \varepsilon_m^2 4\pi^2 \int_{\mathbb{R}^2} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,m}(p) - \widehat{G}_{2,m}(p)|^2}{|p|^{4s_m}} dp.$$

Clearly, it can be bounded from above via estimate (2.2) by $\varepsilon^2 \|H_m\|^2_{L^1(\mathbb{R}^2)} \times$

$$\times \left\{ \frac{\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^1(\mathbb{R}^2)}}{4\pi} \frac{R^{2-4s_m}}{1 - 2s_m} + \frac{\|G_{1,m}(x) - G_{2,m}(x)\|^2_{L^2(\mathbb{R}^2)}}{R^{4s_m}} \right\},$$

where $R \in (0, +\infty)$. Let us make use of the representation for $1 \le m \le N$

$$G_{1,m}(x) - G_{2,m}(x) = \int_0^1 \nabla g_m(u_0(x) + tv_1(x) + (1-t)v_2(x)).(v_1(x) - v_2(x))dt.$$

Evidently, for $t \in [0, 1]$

$$\|v_2(x) + t(v_1(x) - v_2(x))\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} \le t\|v_1(x)\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} +$$
$$+ (1-t)\|v_2(x)\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)} \le \rho,$$

which yields that $v_2(x) + t(v_1(x) - v_2(x)) \in B_\rho$. Thus,

$$|G_{1,m}(x) - G_{2,m}(x)| \le sup_{z \in I} |\nabla g_m(z)| |v_1(x) - v_2(x)| \le M|v_1(x) - v_2(x)|.$$

This gives us

$$\|G_{1,m}(x) - G_{2,m}(x)\|_{L^2(\mathbb{R}^2)} \le M\|v_1 - v_2\|_{L^2(\mathbb{R}^2, \mathbb{R}^N)} \le M\|v_1 - v_2\|_{H^2(\mathbb{R}^2, \mathbb{R}^N)}.$$

Let us express $\frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x))$ for $1 \le m, j \le N$ as

$$\int_0^1 \nabla \frac{\partial g_m}{\partial z_j}(\tau[u_0(x) + tv_1(x) + (1-t)v_2(x)]).[u_0(x) + tv_1(x) + (1-t)v_2(x)]d\tau.$$

Hence for $t \in [0, 1]$

$$\left| \frac{\partial g_m}{\partial z_j}(u_0(x) + tv_1(x) + (1-t)v_2(x)) \right| \le$$
$$\le \sum_{n=1}^N \left\| \frac{\partial^2 g_m}{\partial z_n \partial z_j} \right\|_{C(I)} (|u_0(x)| + t|v_1(x)| + (1-t)|v_2(x)|).$$

Thus we derive the estimate from above for $G_{1,m}(x) - G_{2,m}(x)$ in the absolute value as

$$M|v_1(x) - v_2(x)| \Big(|u_0(x)| + \frac{1}{2}|v_1(x)| + \frac{1}{2}|v_2(x)| \Big).$$

By virtue of the Schwarz inequality we obtain at the upper bound for the norm $\|G_{1,m}(x)-G_{2,m}(x)\|_{L^1(\mathbb{R}^2)}$ as

$$M\|v_1-v_2\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\Big(\|u_0\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}+\frac{1}{2}\|v_1\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}+\frac{1}{2}\|v_2\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\Big)\leq$$

$$\leq M\|v_1-v_2\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1). \tag{2.11}$$

Hence we obtain the estimate from above for the norm $\|u_{1,m}-u_{2,m}\|^2_{L^2(\mathbb{R}^2)}$ given by $\varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^2)}M^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^2,\mathbb{R}^N)}\times$

$$\times\Big\{\frac{1}{4\pi}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2\frac{R^{2-4s_m}}{1-2s_m}+\frac{1}{R^{4s_m}}\Big\}.$$

Let us minimize the expression above over $R\in(0,+\infty)$ by virtue of Lemma 1.4. Thus, we arrive at $\|u_{1,m}(x)-u_{2,m}(x)\|^2_{L^2(\mathbb{R}^2)}\leq$

$$\leq \varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^2)}M^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^2,\mathbb{R}^N)}\frac{(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{4s_m}}{(1-2s_m)(8\pi s_m)^{2s_m}},$$

such that $\|u_1(x)-u_2(x)\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\leq$

$$\leq \varepsilon^2H^2M^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^2,\mathbb{R}^N)}\frac{(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{4s}}{1-2s}\frac{1}{(8\pi S)^{2S}}. \tag{2.12}$$

Formulas (2.9) and (2.10) with $1\leq m\leq N$ give us

$$(-\Delta)(u_{1,m}(x)-u_{2,m}(x))=$$

$$=\varepsilon_m(-\Delta)^{1-s_m}\int_{\mathbb{R}^2}H_m(x-y)[G_{1,m}(y)-G_{2,m}(y)]dy.$$

By means of inequalities (2.2) and (2.11) we derive

$$\|\Delta(u_{1,m}(x)-u_{2,m}(x))\|^2_{L^2(\mathbb{R}^2)}\leq$$

$$\leq\varepsilon^2\|G_{1,m}-G_{2,m}\|^2_{L^1(\mathbb{R}^2)}\|(-\Delta)^{1-s_m}H_m\|^2_{L^2(\mathbb{R}^2)}\leq$$

$$\leq\varepsilon^2M^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^2,\mathbb{R}^N)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2\|(-\Delta)^{1-s_m}H_m\|^2_{L^2(\mathbb{R}^2)}.$$

Therefore, $\sum_{m=1}^N\|\Delta(u_{1,m}(x)-u_{2,m}(x))\|^2_{L^2(\mathbb{R}^2)}\leq$

$$\leq\varepsilon^2M^2\|v_1-v_2\|^2_{H^2(\mathbb{R}^2,\mathbb{R}^N)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2Q^2. \tag{2.13}$$

Estimates (2.12) and (2.13) imply that the norm $\|u_1-u_2\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}$ can be bounded from above by the expression $\varepsilon M(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)\times$

$$\times\left\{\frac{H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{4s-2}}{(1-2s)(8\pi S)^{2S}}+Q^2\right\}^{\frac{1}{2}}\|v_1-v_2\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}. \tag{2.14}$$

This implies that the map $T_g:B_\rho\to B_\rho$ defined by problem (1.11) is a strict contraction for all values of $\varepsilon>0$ sufficiently small. Its unique fixed point $u_p(x)$ is the only solution of system (1.9) in the ball B_ρ. The resulting $u(x)\in H^2(\mathbb{R}^2,\mathbb{R}^N)$ given by (1.8) is a solution of problem (1.2). Note that by virtue of (2.8) $u_p(x)$ converges to zero in the $H^2(\mathbb{R}^2,\mathbb{R}^N)$ norm as ε tends to zero.

Let us turn our attention to the proof of the second main proposition of our article.

3. The continuity of the fixed point of the map T_g

Proof of Theorem 1.5. Evidently, for all $0 < \varepsilon < \delta$ we have

$$u_{p,1} = T_{g_1} u_{p,1}, \quad u_{p,2} = T_{g_2} u_{p,2}.$$

Thus,

$$u_{p,1} - u_{p,2} = T_{g_1} u_{p,1} - T_{g_1} u_{p,2} + T_{g_1} u_{p,2} - T_{g_2} u_{p,2}.$$

Obviously, $\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} \leq$

$$\leq \|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} + \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}.$$

Estimate (2.14) gives us

$$\|T_{g_1} u_{p,1} - T_{g_1} u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} \leq \varepsilon\sigma \|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)},$$

where $\varepsilon\sigma < 1$ because the map $T_{g_1} : B_\rho \to B_\rho$ under the given assumptions is a strict contraction. Here and further down we use the positive constant

$$\sigma := M(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} + 1)\left\{\frac{H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} + 1)^{4s-2}}{(1-2s)(8\pi S)^{2S}} + Q^2\right\}^{\frac{1}{2}}.$$

Hence, we arrive at

$$(1 - \varepsilon\sigma)\|u_{p,1} - u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)} \leq \|T_{g_1} u_{p,2} - T_{g_2} u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}. \tag{3.1}$$

Evidently, for our fixed point $T_{g_2} u_{p,2} = u_{p,2}$. We designate $\xi(x) := T_{g_1} u_{p,2}$. For $1 \leq m \leq N$, we obtain

$$(-\Delta)^{s_m} \xi_m(x) = \varepsilon_m \int_{\mathbb{R}^2} H_m(x-y) g_{1,m}(u_0(y) + u_{p,2}(y)) dy, \tag{3.2}$$

$$(-\Delta)^{s_m} u_{p,2,m}(x) = \varepsilon_m \int_{\mathbb{R}^2} H_m(x-y) g_{2,m}(u_0(y) + u_{p,2}(y)) dy, \tag{3.3}$$

with all $0 < s_m < \dfrac{1}{2}$. We denote here

$$G_{1,2,m}(x) := g_{1,m}(u_0(x) + u_{p,2}(x)), \quad G_{2,2,m}(x) := g_{2,m}(u_0(x) + u_{p,2}(x)).$$

Let us apply the standard Fourier transform (2.1) to both sides of formulas (3.2) and (3.3). This gives us

$$\widehat{\xi}_m(p) = \varepsilon_m 2\pi \frac{\widehat{H}_m(p)\widehat{G}_{1,2,m}(p)}{|p|^{2s_m}}, \quad \widehat{u}_{p,2,m}(p) = \varepsilon_m 2\pi \frac{\widehat{H}_m(p)\widehat{G}_{2,2,m}(p)}{|p|^{2s_m}}.$$

Clearly,

$$\|\xi_m(x) - u_{p,2,m}(x)\|^2_{L^2(\mathbb{R}^2)} = \varepsilon_m^2 4\pi^2 \int_{\mathbb{R}^2} \frac{|\widehat{H}_m(p)|^2 |\widehat{G}_{1,2,m}(p) - \widehat{G}_{2,2,m}(p)|^2}{|p|^{4s_m}} dp.$$

Obviously, it can be estimated from above via (2.2) by

$$\varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^2)}\Bigg\{\frac{\|G_{1,2,m}-G_{2,2,m}\|^2_{L^1(\mathbb{R}^2)}}{4\pi}\frac{R^{2-4s_m}}{1-2s_m}+\frac{\|G_{1,2,m}-G_{2,2,m}\|^2_{L^2(\mathbb{R}^2)}}{R^{4s_m}}\Bigg\},$$

where $R\in(0,+\infty)$. Let us use the identity

$$G_{1,2,m}(x)-G_{2,2,m}(x)=\int_0^1\nabla[g_{1,m}-g_{2,m}](t(u_0(x)+u_{p,2}(x))).$$
$$(u_0(x)+u_{p,2}(x))dt.$$

Hence

$$|G_{1,2,m}(x)-G_{2,2,m}(x)|\leq\|g_{1,m}-g_{2,m}\|_{C^2(I)}|u_0(x)+u_{p,2}(x)|.$$

This yields

$$\|G_{1,2,m}-G_{2,2,m}\|_{L^2(\mathbb{R}^2)}\leq\|g_{1,m}-g_{2,m}\|_{C^2(I)}\|u_0+u_{p,2}\|_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\leq$$
$$\leq\|g_{1,m}-g_{2,m}\|_{C^2(I)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1).$$

We apply another useful representation formula for $1\leq j\leq N$ and $t\in[0,1]$, namely

$$\frac{\partial}{\partial z_j}(g_{1,m}-g_{2,m})(t(u_0(x)+u_{p,2}(x)))=$$
$$=\int_0^t\nabla\Big[\frac{\partial}{\partial z_j}(g_{1,m}-g_{2,m})\Big](\tau(u_0(x)+u_{p,2}(x))).(u_0(x)+u_{p,2}(x))d\tau.$$

Hence, we arrive at

$$\Big|\frac{\partial}{\partial z_j}(g_{1,m}-g_{2,m})(t(u_0(x)+u_{p,2}(x)))\Big|\leq$$
$$\leq\sum_{n=1}^N\left\|\frac{\partial^2(g_{1,m}-g_{2,m})}{\partial z_n\partial z_j}\right\|_{C(I)}|u_0(x)+u_{p,2}(x)|.$$

Therefore,

$$|G_{1,2,m}(x)-G_{2,2,m}(x)|\leq\|g_{1,m}-g_{2,m}\|_{C^2(I)}|u_0(x)+u_{p,2}(x)|^2,$$

such that

$$\|G_{1,2,m}-G_{2,2,m}\|_{L^1(\mathbb{R}^2)}\leq\|g_{1,m}-g_{2,m}\|_{C^2(I)}\|u_0+u_{p,2}\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\leq$$
$$\leq\|g_{1,m}-g_{2,m}\|_{C^2(I)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2. \tag{3.4}$$

This allows us to obtain the estimate from above for the norm $\|\xi_m-u_{p,2,m}\|^2_{L^2(\mathbb{R}^2)}$ as $\varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2\times$

$$\times\|g_{1,m}-g_{2,m}\|^2_{C^2(I)}\left[\frac{(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2R^{2-4s_m}}{4\pi(1-2s_m)}+\frac{1}{R^{4s_m}}\right].$$

This expression can be easily minimized over $R\in(0,+\infty)$ due to Lemma 1.4. We derive the upper bound $\|\xi_m(x)-u_{p,2,m}(x)\|^2_{L^2(\mathbb{R}^2)}\leq$

$$\leq\varepsilon^2\|H_m\|^2_{L^1(\mathbb{R}^2)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{2+4s_m}\frac{\|g_{1,m}-g_{2,m}\|^2_{C^2(I)}}{(1-2s_m)(8\pi s_m)^{2s_m}},$$

such that

$$\|\xi(x)-u_{p,2}(x)\|^2_{L^2(\mathbb{R}^2,\mathbb{R}^N)}\le \varepsilon^2 H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{2+4s}\frac{\|g_1-g_2\|^2_{C^2(I,\mathbb{R}^N)}}{(1-2s)(8\pi S)^{2S}}.$$

Equalities (3.2) and (3.3) with $1\le m\le N$ give us

$$-\Delta\xi_m(x)=\varepsilon_m(-\Delta)^{1-s_m}\int_{\mathbb{R}^2}H_m(x-y)G_{1,2,m}(y)dy,$$

$$-\Delta u_{p,2,m}(x)=\varepsilon_m(-\Delta)^{1-s_m}\int_{\mathbb{R}^2}H_m(x-y)G_{2,2,m}(y)dy.$$

Therefore, by virtue of (2.2) and (3.4) the norm $\|\Delta(\xi_m(x)-u_{p,2,m}(x))\|^2_{L^2(\mathbb{R}^2)}$ can be bounded from above by

$$\varepsilon^2\|G_{1,2,m}-G_{2,2,m}\|^2_{L^1(\mathbb{R}^2)}\|(-\Delta)^{1-s_m}H_m\|^2_{L^2(\mathbb{R}^2)}\le$$

$$\le\varepsilon^2\|g_{1,m}-g_{2,m}\|^2_{C^2(I)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^4\|(-\Delta)^{1-s_m}H_m\|^2_{L^2(\mathbb{R}^2)}.$$

Therefore, $\sum_{m=1}^N\|\Delta(\xi_m(x)-u_{p,2,m}(x))\|^2_{L^2(\mathbb{R}^2)}\le$

$$\varepsilon^2\|g_1-g_2\|^2_{C^2(I,\mathbb{R}^N)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^4Q^2.$$

Hence, we obtain $\|\xi(x)-u_{p,2}(x)\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}\le$

$$\le\varepsilon\|g_1-g_2\|_{C^2(I,\mathbb{R}^N)}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2\left[\frac{H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{4s-2}}{(1-2s)(8\pi S)^{2S}}+Q^2\right]^{\frac12}.$$

By means of inequality (3.1), the norm $\|u_{p,1}-u_{p,2}\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}$ can be estimated from above by $\dfrac{\varepsilon}{1-\varepsilon\sigma}(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^2\times$

$$\times\left[\frac{H^2(\|u_0\|_{H^2(\mathbb{R}^2,\mathbb{R}^N)}+1)^{4s-2}}{(1-2s)(8\pi S)^{2S}}+Q^2\right]^{\frac12}\|g_1-g_2\|_{C^2(I,\mathbb{R}^N)},$$

which completes the proof of our theorem. □

4. Auxiliary results

Let us state here the solvability conditions for the linear Poisson type equation with a square integrable right side

$$(-\Delta)^s\phi=f(x),\quad x\in\mathbb{R}^2,\quad 0<s<1. \tag{4.1}$$

The inner product can be designated as

$$(f(x),g(x))_{L^2(\mathbb{R}^2)}:=\int_{\mathbb{R}^2}f(x)\bar{g}(x)dx, \tag{4.2}$$

with a slight abuse of notations when the functions involved in (4.2) are not square integrable, like for instance the one involved in orthogonality condition (4.3) of Lemma 4.1 below. Indeed, if $f(x)\in L^1(\mathbb{R}^2)$ and $g(x)\in L^\infty(\mathbb{R}^2)$, then the integral in the right side of (4.2) is well defined. We have the following technical proposition, which can be easily established by applying the standard Fourier transform (2.1) to both sides of problem (4.1) (see the part b) of the first theorem of [36] and for $s = \frac{1}{2}$ the part 2) of Lemma 3.1 of [34]).

Lemma 4.1. *Let $f(x) : \mathbb{R}^2 \to \mathbb{R}$ and $f(x) \in L^2(\mathbb{R}^2)$.*

1) When $0 < s < \frac{1}{2}$ and additionally $f(x) \in L^1(\mathbb{R}^2)$, equation (4.1) possesses a unique solution $\phi(x) \in H^{2s}(\mathbb{R}^2)$.

2) When $\frac{1}{2} \le s < 1$ and in addition $|x|f(x) \in L^1(\mathbb{R}^2)$, problem (4.1) has a unique solution $\phi(x) \in H^{2s}(\mathbb{R}^2)$ if and only if the orthogonality relation

$$(f(x), 1)_{L^2(\mathbb{R}^2)} = 0 \tag{4.3}$$

holds.

Let us note that for the lower values of the power of the negative Laplacian $0 < s < \frac{1}{2}$ under the conditions stated above no orthogonality relations are required to solve the linear Poisson type problem (4.1) in $H^{2s}(\mathbb{R}^2)$.

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