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CLASSIFICATION OF ε-ISOMETRIES BY STABILITY

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Abstract. Assume that X, Y are two Banach spaces, and $f : X \to Y$ is a standard ε-isometry. In this paper, we first study properties of the two specific subspaces $\mathbb{L}$ and $N^\perp$ of $L(f)^{**} \equiv \overline{\text{span}}[f(X)]^{**}$, which are deduced from the weak stability formula and play an essential rule in study of stability of f. Then we show that f is w^*-γ-stable if and only if there is a w^*-to-w^* projection $P : L(f)^{**} \to N^\perp$ so that $Pf : X \to N^\perp$ is a γ-approximate linear isometry, i.e. there exists a surjective linear isometry $U : X^{**} \to N^\perp$ so that

$$\|Pf(x) - Ux\| \le \gamma\varepsilon, \text{ for all } x \in X.$$

In particular, if $\varepsilon = 0$, then Pf is simply a linear isometry. Finally, we show a new characterization for f to be (resp. w^*) stable in terms of linear bounded selections respect to the weak stability formula of f.

1. Introduction

The study of stability properties of perturbed (or, ε-) isometries is moving to central stage of the research area about metric-preserving and perturbed metric-preserving mappings on Banach spaces. In this paper, we consider classification of ε-isometries from a Banach space X to another Banach space Y by their stability. Therefore, we study stability, w^*-stability and projective stability of ε-isometries. To begin with, we recall definitions of isometry and ε-isometry.

Definition 1.1. Let X, Y be two Banach spaces, $\varepsilon \ge 0$ be a constant, and $f : X \to Y$ be a mapping.

(1) f is said to be an ε-isometry if

$$\Big| \|f(x) - f(y)\| - \|x - y\| \Big| \le \varepsilon \text{ for all } x, y \in X;$$

(2) In particular, if $\varepsilon = 0$, then the 0-isometry f is simply called an isometry;
(3) We say that an (ε-) isometry f is standard if $f(0) = 0$.

Isometry and linear isometry. The study of properties of isometry between Banach spaces and of its generalizations has continued for 80 years. The first celebrated result was due to Mazur and Ulam [24, 1932]: Every surjective standard isometry between two Banach spaces is a linear isometry. This reveals a profound fact: Geometric preserving mapping is linear structure preserving! But the simple

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example: $f : \mathbb{R} \to \ell_\infty^2$ defined for t by $f(t) = (t, \sin t)$ shows that it is not true if an isometry is not surjective. For a general standard isometry $f : X \to Y$, Figiel [16, 1968] showed the following remarkable theorem: there is a linear operator $T : L(f) \to X$ with $\|T\| = 1$ such that $Tf = I_X$, the identity on X; i.e. every isometry admits a linear left inverse of norm one! We also call the operator T Figiel's operator associated with the isometry f. Godefroy and Kalton [18, 2003] studied the relationship between isometry and linear isometry, and showed the deep result for a standard isometry $f : X \to Y$, if X is separable then Y contains an isometric linear copy of X; and for every nonseparable weakly compactly generated space X, there exist a Banach space Y and an isometry $f : X \to Y$, so that X is not linearly isomorphic any subspace of Y. A further discussion about the relationship of (ε-) isometries and linear isometries including localized settings can be found in [13, 15], [32]-[36].

ε-isometry and stability. Hyers and Ulam [21, 1945] first studied ε-isometry and proposed the following question (see, also [25]): whether for every surjective ε-isometry $f : X \to Y$ with $f(0) = 0$ there exist a surjective linear isometry $U : X \to Y$ and $\gamma > 0$ such that

$$\|f(x) - Ux\| \leq \gamma\varepsilon, \text{ for all } x \in X. \tag{1.1}$$

An ε-isometry $f : X \to Y$ satisfying the inequality above is called a γ-approximate linear ε-isometry. Since then, some partial affirmative answers had been obtained by Hyers and Ulam [21, 22], D.G. Bourgin [3, 4, 5], R.D. Bourgin [6], Gruber [19] and Gevietz [17]. After 50 year efforts of a number of mathematicians, the following sharp estimate was finally obtained by Omladič and Šemrl [25, 1995]. (See, also, [2, pp.360].)

Theorem 1.2 (Omladič-Šemrl)**.** *If $f : X \to Y$ is a standard surjective ε-isometry, then there is a surjective linear isometry $U : X \to Y$ such that*

$$\|f(x) - Ux\| \leq 2\varepsilon, \text{ for all } x \in X.$$

We refer the reader to [10, 14, 29, 31, 32] for the recent development in this direction.

Since 90s of the last century, the study of stability property of non-surjective ε-isometry has become an active area. A standard ε-isometry $f : X \to Y$ is said to be (γ-) stable provided there exist a positive number γ and a bounded linear operator $T : L(f) \to X$ such that

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \text{ for all } x \in X. \tag{1.2}$$

Qian [26, 1995] first studied such a problem, showed that the answer to the problem is positive if both X and Y are L_p spaces for all $1 < p < \infty$. Šemrl and Väisälä [28, 2003] further presented a sharp estimate of (1.2) with $\gamma = 2$ if both X and Y are L^p spaces for $1 < p < \infty$. However, Qian [26] presented a simple counterexample showing that if a separable Banach space Y contains a uncomplemented closed subspace X then for every $\varepsilon > 0$ there is a standard ε-isometry $f : X \to Y$ which is unstable. This disappointment made us to search for some appropriate (but weaker) stability version. Cheng, Dong and Zhang [9, 2013] found a weak stability version for general standard ε-isometries. Cheng et al. [7, 2015] further improved it into the following sharp one:

Theorem 1.3 (Cheng-Cheng-Tu-Zhang)**.** *Let $f : X \to Y$ be a standard ε-isometry. Then for every $x^* \in X^*$, there exists $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\| \equiv r$ so that*

$$\big|\langle \varphi, f(x)\rangle - \langle x^*, x\rangle\big| \le 2\varepsilon r, \text{ for all } x \in X. \tag{1.3}$$

Making use of the weak stability versions, various properties concerning general ε-isometries have been extensively studied. It was shown in Cheng and Zhou [12] that if there is an ε-isometry from X to Y, then there is a closed subspace $N \subset Y^*$ so that X^* is linearly isometric to Y^*/N. In [1, 8], Banach spaces satisfying that every standard ε-isometry is stable, i.e. "universal stability spaces" were studied. Weak stability of ε-isometries defined on wedges of Banach spaces was also discussed in [11]. Theorem 1.3 will play an essential rule in this paper.

In this paper, we first introduce a concept about stability which is called w^*-stability of standard ε-isometries (Definition 2.1 ii)). w^*-stability is just the stability whenever the Banach spaces in question are reflexive. But it is weaker than stability in general (Example 2.3). Some evidence shows that this is perhaps the "right" notion to study. Making use of the weak stability formula established in [7], we show the two specific subspaces Ł and $N^\perp$ of Y^{**} associated with a standard ε-isometry $f : X \to Y$ have many nice properties (Section 3). They play an important rule in classification of ε-isometries. We prove that every such ε-isometry f deduces a w^*-to-w^* continuous linear isometry $S\big|_{Ł} : X^* \to Y^*$ whenever the domain of $S\big|_{Ł}(X^*)$ is restricted to Ł (Theorem 3.1). For a standard ε-isometry $f : X \to Y$, we show that if there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to N^\perp$ so that $Pf : X \to N^\perp$ is an approximate linear δ-isometry for some $\delta \ge 0$, then f is w^*-$2\delta/\varepsilon$-stable (Theorem 4.1); Conversely, if f is w^*-γ-stable, then there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to N^\perp$ so that $Pf : X \to N^\perp$ is a γ-approximate linear $2\gamma\varepsilon$-isometry (Theorem 4.4). As its application, we show that for every standard isometry $f : X \to Y$ there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to N^\perp$ so that Pf is a linear isometry (Corollary 4.5), and this can be regarded as a refinement of Figiel's theorem [16]. Stability (resp. w^*-stability) of the ε-isometry f is also equivalent to that there is a linear w^*-to-w^* continuous (resp. continuous) selection $x^* \to \varphi$ in the correspondence (1.3).

2. Preliminaries

All symbols and notations in this paper are standard. We use X to denote a real Banach space and X^* its dual. B_X and S_X denote the closed unit ball and the unit sphere of X, respectively. For a subspace $E \subset X$, $E^\perp$ denotes the annihilator of E, i.e. $E^\perp = \{x^* \in X^* : \langle x^*, e\rangle = 0 \text{ for all } e \in E\}$. If $E \subset X^*$, then we use ${}^\perp E$ to denote the pre-annihilator of E: $\{x \in X : \langle e, x\rangle = 0, \ \forall e \in E\}$. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$ (X^*), $\overline{A}$, (w^*-$\overline{A}$) and $\mathrm{co}(A)$ stand for the closure (the w^*-closure), and the convex hull of A, respectively. For a mapping $g : X \to Y$, we denote by $L(g)$, the subspace of Y generated by $g(X)$, i.e. $L(g) = \overline{\mathrm{span}}\, g(X)$.

Definition 2.1. Let $f : X \to Y$ be a standard ε-isometry, and $\gamma > 0$ be a constant. Then

i) f is said to be γ-stable provided there exist a bounded linear operator $T : L(f) \to X$ and a positive number γ so that

$$\|Tf(x) - x\| \le \gamma\varepsilon, \text{ for all } x \in X. \tag{2.1}$$

ii) We say that f is w^*-γ-stable provided there exist a continuous linear operator $T : L(f)^{**} \to X^{**}$ and a positive number γ so that

$$\|Tf(x) - x\| \le \gamma\varepsilon, \text{ for all } x \in X. \tag{2.2}$$

iii) f is called a γ-approximate linear isometry if there is a surjective linear isometry $U : X \to L(f)$ so that

$$\|f(x) - Ux\| \le \gamma\varepsilon, \text{ for all } x \in X. \tag{2.3}$$

Remark 2.2. With the stability notions above we should mention here that a) an approximate linear isometry is stable, but the simple example $f : \mathbb{R} \to \ell^2_\infty$ defined for $t \in \mathbb{R}$ by $f(t) = (t, |t|)$ says that a stable ε-isometry is not necessarily an approximate linear isometry; b) a stable ε-isometry must be w^*-stable. However, the following example shows that w^*-stability does not imply stability of an ε-isometry.

Example 2.3. Assume $g : c_0 \to B_{\ell_\infty}$ (the closed unit ball of ℓ_∞) is a bijective (not necessarily continuous) mapping with $g(0) = 0$. Given $\varepsilon > 0$, let $f : c_0 \to \ell_\infty$ be defined by

$$f(x) = x + (\varepsilon/2)g(x), \text{ for all } x \in c_0. \tag{2.4}$$

Clearly, f is an ε-isometry. Since $L(f) = \ell_\infty$ and since c_0 is not complemented in ℓ_∞, f is not stable. On the other hand, note $L(f)^{**} = \ell_\infty^{**} = \ell_\infty \oplus \ell_1^\perp$. Let $T : L(f)^{**} \to \ell_\infty = c_0^{**}$ be the projection along $\ell_1^\perp$. Then it satisfies

$$\|Tf(x) - x\| = \|f(x) - x\| = (\varepsilon/2)\|g(x)\| \le \varepsilon/2, \text{ for all } x \in c_0.$$

For an ε-isometry $f : X \to Y$, we define the following (set-valued) mapping $\imath : X^* \to L(f)^*$ by

$$\imath(x^*) = \{\varphi \in L(f)^* : x^* - \varphi \circ f \text{ is bounded on } X\}. \tag{2.5}$$

Definition 2.4. i) A filter $\mathcal{F}$ on a set Ω is a collection of subsets of Ω satisfying a. $\emptyset \notin \mathcal{F}$; b. $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and c. $A \in \mathcal{F}$ and $A \subset B \subset \Omega$ entail $B \in \mathcal{F}$.

ii) A filter $\mathcal{F}$ is said to be free if $\cap\{F \in \mathcal{F}\} = \emptyset$;

iii) A filter $\mathcal{U}$ is called an ultrafilter if for any $A \subset \Omega$, either $A \in \mathcal{U}$, or, $\Omega \setminus A \in \mathcal{U}$.

iv) Let K be a topological space, and $f : \Omega \to K$ be a function. We say f is convergent to some $k \in K$ with respect to a filter $\mathcal{F}$ if for every neighborhood U of k, we have $f^{-1}(U) \in \mathcal{F}$; in this case, we denote $\lim_{\mathcal{F}} f = k$.

The following property is classical and is easy to prove.

Proposition 2.5. *Suppose that Ω is a nonempty set , K is a Hausdorff compact space, and that $f : \Omega \to K$ is a mapping. Then for every free ultrafilter $\mathcal{U}$ on Ω the limit $\lim_{\mathcal{U}} f$ exists and is unique.*

Lemma 2.6. [7, Lemma 2.1] *Suppose that $f : X \to Y$ is a standard ε-isometry. Then for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the following w^*-free ultrafilter limits exist and define an isometry $\Phi : X \to L(f)^{**}$.*

$$\Phi(x) = w^*\text{-}\lim_{\mathcal{U}} \frac{f(nx)}{n}, \quad \text{for all } x \in X. \tag{2.6}$$

Invariant mean procedure is applied in this paper. Now, we recall definition of (left) mean of a semigroup and some related result, which can be found in Benyamini and Lindenstrauss' book [2] (pp.417-418).

Definition 2.7. Let G be a semigroup. A left-invariant mean on G is a linear functional μ on $\ell_\infty(G)$ such that:
(i) $\mu(1) = 1$,
(ii) $\mu(f) \geq 0$ for every $f \geq 0$,
(iii) $\forall f \in l_\infty(G), \forall g \in G$, $\mu(f_g) = \mu(f)$, where f_g is the left-translation of f by g; i.e., $f_g(h) = f(gh)$, $\forall h \in G$.

Analogously, we can define right-invariant mean of G. An invariant mean is a linear functional on $\ell_\infty(G)$ which is both left-invariant and right-invariant.

Clearly, an invariant mean of a semigroup G is just an index-translation invariant positive functional of norm one on $\ell_\infty(G)$.

Note that (i) and (ii) are equivalent to $\mu(1) = \|\mu\| = 1$.

Lemma 2.8. *Every Abelian semigroup G (in particular, every linear space) has an invariant mean.*

3. Several important subspaces associated with an ε-isometry

In this section, we shall discuss properties of the following three important subspaces associated with an ε-isometry.

3.1. **The subspace Ł.** Let $f : X \to Y$ be a standard ε isometry. Then, by Lemma 2.6, for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the following w^*-free ultrafilter limits exist and define an isometry $\Phi : X \to L(f)^{**}$.

$$\Phi(x) = w^*\text{-}\lim_{\mathcal{U}} \frac{f(nx)}{n}, \quad \text{for all } x \in X. \tag{3.1}$$

On the other hand, applying Theorem 1.3, for every $x^* \in X^*$, there exists $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\| \equiv r$ so that

$$\Big| \langle \varphi, f(x) \rangle - \langle x^*, x \rangle \Big| \leq 2\varepsilon r, \text{ for all } x \in X. \tag{3.2}$$

Note φ is w^*-continuous on Y^{**}. We substitute nx for x, divide the both sides by $n \in \mathbb{N}$ and take the w^*-$\mathcal{U}$ limit. Then we obtain

$$\langle \varphi, \Phi(x) \rangle = \langle x^*, x \rangle, \quad \text{for all } x \in X. \tag{3.3}$$

We denote by Ł, the w^*-closure of the subspace span$[\Phi(X)]$ in Y^{**}, i.e. Ł $= \overline{\text{span}}^{w^*}\Phi(X)$.

Theorem 3.1. *For a standard ε isometry $f: X \to Y$,*

i) the correspondence (3.3) defines a w^*-to-w^* continuous linear isometry $S_Ł$: $X^* \to L^*$, i.e. $S_Ł x^* = \psi \equiv \varphi|_Ł \in L^*$ with respect to the w^*-topologies of X^* and Y^*;

ii) the isometry $S_Ł$ is just the conjugate operator of Figiel's operator $F: L \to X$ associated with the isometry Φ defined as (3.1)

Proof. i) It is not difficult to check that for each $x^* \in X^*$, $\psi \equiv \varphi|_Ł$ (the corresponding functional φ in (3.2) restricted to Ł) is a unique. If we define $S_Ł x^* = \psi$ for every $x^* \in X^*$, then it follows again from (3.2) and (3.3) that $S_Ł : X^* \to Ł^*$ is a w^*-to-w^* continuous linear isometry with respect to the w^*-topologies of X^* and Y^*. Indeed, Assume that $\{x^*_\alpha\} \subset X^*$ is a bounded net w^*-converging to x^*; and assume that $\{\varphi_\alpha\} \subset Y^*$ satisfies

$$\big|\langle x^*_\alpha, x\rangle - \langle \varphi_\alpha, f(x)\rangle\big| \le 2\varepsilon r_\alpha, \text{ for all } x \in X.$$

It is easy to observe that for any w^*-cluster point ψ of $\{\varphi_\alpha\}$, we have

$$\big|\langle x^*, x\rangle - \langle \psi, f(x)\rangle\big| \le 2\varepsilon r, \text{ for all } x \in X,$$

where $r = \limsup_\alpha r_\alpha$. We substitute nx for x, divide the both sides by n, and take the w^*-ultrafilter limit. Then we get $\langle x^*, x\rangle = \langle \psi, \Phi(x)\rangle$ for all $x \in X$. Consequently, $S_Ł x^* = \psi|_Ł$, i.e. $S_Ł$ is a w^*-to-w^* continuous linear isometry with respect to the w^*-topologies of X^* and Y^*.

ii) It follows from i) that

$$\langle x^*, x\rangle = \langle \psi, \Phi(x)\rangle = \langle S_Ł x^*, \Phi(x)\rangle, \; x \in X, \; x^* \in X^*.$$

Let

$$\langle S_Ł x^*, \Phi(x)\rangle = \langle x^*, F\Phi(x)\rangle, \; x \in X, \; x^* \in X^*. \tag{3.4}$$

Then

$$\langle x^*, F\Phi(x)\rangle = \langle x^*, x\rangle, \; x \in X, \; x^* \in X^*. \tag{3.5}$$

Thus, (3.5) defines an operator $F : Ł \to X$ with $\|F\| = 1$ satisfying $F\Phi = I_X$, i.e. the operator $S_Ł$ is the conjugate operator of Figiel's operator F associated with the isometry Φ. □

3.2. **The subspace M.** For standard ε-isometry $f : X \to Y$, the subspace $M \subset Y^*$ is defined in [9]. Let $C(f)$ be the absolutely closed convex hull of the image $f(X)$ of f, i.e. $C(f) = \overline{\text{co}}(f(X) \cup -f(X))$. Let

$$M_\varepsilon = \{\varphi \in Y^* : \; \exists\, \beta > 0 \text{ so that } \varphi \text{ is bounded on } C(f) \text{ by } \beta\varepsilon\}. \tag{3.6}$$

Since $C(f) \subset Y$ is convex and symmetric, M_ε is a subspace of Y. The subspace M is the closure of M_ε, i.e.

$$M = \begin{cases} \overline{\{\varphi \in Y^* \text{ is bounded on } C(f)\}}\,, & \text{if } \varepsilon > 0; \\ \{\varphi \in Y^* : \; \langle \varphi, y\rangle = 0, \; \forall y \in C(f)\}\,, & \text{if } \varepsilon = 0. \end{cases} \tag{3.7}$$

Let

$$E = {}^{\perp}M = \{y \in Y : \langle m, y\rangle = 0, \text{ for all } m \in M\}. \tag{3.8}$$

Next, recall the mapping $\imath$ defined by (2.5), i.e.

$$\imath(x^*) = \{\varphi \in Y^* : |x^* - \varphi \circ f| \text{ is bounded by } \beta\varepsilon \text{ on } X \text{ for some } \beta > 0\}. \tag{3.9}$$

Finally, we define $Q : X^* \to Y^*/M$ for $x^* \in X^*$ by

$$Qx^* = \imath(x^*) + M. \tag{3.10}$$

Then it is easy to verify

$$Qx^* = \imath(x^*) + M = \varphi + M, \text{ for each } x^* \in X^* \text{ and for all } \varphi \in \imath(x^*). \tag{3.11}$$

The following theorem is [9, Theorem 4.4].

Theorem 3.2. *With the subspaces M, E, the mappings f and Q as the same as above, then*

i) $Q : X^* \to Y^*/M$ is a linear isometry;

ii) if, in addition, M is w^*-closed in Y^* (in particular, Y is reflexive), then Q is the conjugate operator for some surjective operator $F : E \to X$ of norm one.

iii) if $\varepsilon = 0$, then the corresponding operator F above is just Figiel's operator associated with the isometry f.

3.3. **The subspace N.** For a standard ε-isometry $f : X \to Y$, the subspace N associated with f is constructed in [12]. Because we shall need detailed discussion on it, a sketch constructive procedure is presented as follows.

Note X is an abelian group with respect to the vector addition of X. By Lemma 2.8, there exists a translation invariant mean μ on $\ell_\infty(X)$. Fix any $x \in X$. Since f is an ε-isometry,

$$g_x(z) = f(x + z) - f(z), \text{ for all } z \in X \tag{3.12}$$

defines a bounded mapping $g_x : X \to Y$. Therefore, $\langle \varphi, g_x\rangle \in \ell_\infty(X)$ for every $\varphi \in Y^*$. We also denote the invariant mean by μ_z or $\mu_z(\cdot)$, emphasizing that the mean is taken with respect to the variable z.

Next, we define a linear mapping $R : Y^* \to \mathbb{R}^X$ by

$$\langle R\varphi, x\rangle = \mu(\langle \varphi, g_x\rangle), \ \varphi \in Y^*, x \in X. \tag{3.13}$$

We claim that

(1) $R\varphi \in X^*$ for every $\varphi \in Y^*$;

(2) $\|R\varphi\| \le \|\phi\|$ for every $\varphi \in Y^*$;

Given $u, v \in X$,

$$\begin{aligned}\langle R\varphi, u + v\rangle &= \mu(\langle \varphi, g_{u+v}\rangle) = \mu_z(\langle \varphi, f(u + v + z) - f(z)\rangle) \\ &= \mu_z(\langle \varphi, f(u + v + z) - f(v + z)\rangle) + \mu_z(\langle \varphi, f(v + z) - f(z)\rangle) \\ &= \mu_z(\langle \varphi, f(u + z) - f(z)\rangle) + \mu_z(\langle \varphi, f(v + z) - f(z)\rangle) \\ &= \mu(\langle \varphi, g_u\rangle) + \mu(\langle \varphi, g_v\rangle) = \langle R\varphi, u\rangle + \langle R\varphi, v\rangle.\end{aligned} \tag{3.14}$$

That is, additivity of $R\varphi$ has been shown. It follows from additivity of $R\varphi$,

$$\langle R\varphi, \lambda u\rangle = \lambda\langle R\varphi, u\rangle, \text{ for all rational number } \lambda. \tag{3.15}$$

Therefore, (3.12)-(3.14) imply for all $u, v \in X$ and $k \in \mathbb{N}$,

$$|\langle R\varphi, u\rangle - \langle R\varphi, v\rangle| = \frac{1}{k}|\langle R\varphi, ku\rangle - \langle R\varphi, kv\rangle|$$

$$= |\mu_z(\langle\varphi, \frac{(f(ku+z)-f(z))}{k}\rangle) - \mu_z(\langle\varphi, \frac{(f(kv+z)-f(z))}{k}\rangle)|$$

$$= |\mu_z(\langle\varphi, \frac{(f(ku+z)-f(kv+z))}{k}\rangle)| \leq \|\mu\|\|\varphi\|\|\frac{(f(ku+z)-f(kv+z))}{k}\|$$

$$\leq \|\mu\|\|\varphi\|\frac{\|(ku+z)-(kv+z)\|+\varepsilon}{k} = \|\varphi\|\|u-v\| + \frac{\varepsilon}{k}$$

$$\to \|\varphi\|\|u-v\|, \text{ as } k\to\infty.$$

Hence

$$|\langle R\phi, u\rangle - \langle R\phi, v\rangle| \leq \|\phi\|\|u-v\|, \text{ for all } u, v \in X. \tag{3.16}$$

We have proven that $R\varphi$ is continuous on X. (3.12)–(3.15) together imply that $R\phi$ is linear and with $\|R\varphi\| \leq \|\phi\|$, that is, (1) and (2) hold. And this further entails that $R : Y^* \to X^*$ is a linear operator with $\|R\| \leq 1$.

The closed subspace N of Y^* is defined as

$$N = \ker(R) \equiv \{\varphi \in Y^* : R\varphi = 0\}. \tag{3.17}$$

The subspace N plays an essential rule in the study of stability of ε-isometries, and also, in this paper.

Theorem 3.3. [12, Theorem 2.3] *Suppose that $f : X \to Y$ is a standard ε-isometry, $N \subset Y^*$ is the subspace associated with f. Then*

i) the mapping $U : X^* \to Y^*/N$ defined by

$$Ux^* = \imath(x^*) + N, \quad \text{for all } x^* \in X^* \tag{3.18}$$

is a linear surjective isometry. Hence

ii) $V^* \equiv (U^*)^{-1} : X^{**} \to N^\perp \subset Y^{**}$ is a w^*-to-w^* continuous linear surjective isometry.

Let

$$\Theta(x^*) = \varphi\big|_{N^\perp},$$

where $x^* \in X^*$, $\varphi \in Y^*$ satisfy (3.2), and $\varphi\big|_{N^\perp}$ denote the functional φ acting as a functional on Y^{**} restricted to $N^\perp$. The following result is a consequence of the theorem above.

Corollary 3.4. *For any standard ε-isometry $f : X \to Y$, $\Theta : X^* \to (N^\perp)^*$ is a w^*-to-w^* continuous linear isometry with respect to the w^*-topologies of X^* and Y^*.*

Proof. Since $U : X^* \to Y^*/N$ is defined for $x^* \in X^*$ by

$$Ux^* = \varphi + N, \; \forall\, \varphi \in \imath(x^*),$$

for any $\varphi_1, \varphi_2 \in \imath(x^*)$, $\varphi_1 - \varphi_2 = 0$ on $N^\perp$. Thus, Θ is a linear isometry. Its w^*-to-w^* continuity follows from the same procedure of the proof of Theorem 3.1. □

Lemma 3.5. *For any standard ε-isometry $f : X \to Y$, the subspace M associated with f is a subspace of N.*

Proof. Since R is continuous, it suffices to show $M_\varepsilon \subset \ker R$, where M_ε is defined as (3.5). Given $\varphi \in M_\varepsilon$, we have $\langle \varphi, f\rangle \in \ell_\infty(X)$. For every $x \in X$, it follows from the translation invariance of μ,

$$\begin{aligned}\langle R\varphi, x\rangle &= \mu_z(\langle \varphi, f(x+z) - f(z)\rangle)\\ &= \mu_z(\langle \varphi, f(x+z)\rangle) - \mu_z(\langle \varphi, f(z)\rangle) = 0.\end{aligned}$$

□

Remark 3.6. From the proof we see that R can also be regarded as a linear operator from Y^*/M to X^* with $\|R\| \le 1$.

Theorem 3.7. *Suppose that $f : X \to Y$ is a standard-ε isometry, and suppose that the subspaces M, N, the linear operators $Q : X^* \to Y^*/M, R : Y^* \to X^*$ and $U : X^* \to Y^*/N$ associated with f are defined previously. Then i) Q and U are isometries and satisfying*

$$RQ = RU = I_{X^*}, \tag{3.19}$$

i.e R is the left inverses of the two linear isometries Q and U.

ii) Both $R^* : X^{**} \to M^\perp$ and $U^{*-1} : X^{**} \to N^\perp$ are w^*-to-w^* linear isometries.

Proof. i) We first show $RQ = I_{X^*}$. Let $x^* \in X^*$. Note $Qx^* = \imath x^* + M$. Since $M \subset N \equiv \ker(R)$,

$$RQx^* = R(\imath(x^*) + M) = R(\imath(x^*)) = R(\varphi) \text{ for all } \varphi \in \imath(x^*). \tag{3.20}$$

By Theorem 1.3, there exists $\varphi \in \imath(x^*)$ so that

$$\big|\langle \varphi, f(z)\rangle - \langle x^*, z\rangle\big| \le 2\varepsilon\|x^*\|, \text{ for all } z \in X. \tag{3.21}$$

Therefore, for all $x \in X$,

$$\begin{aligned}\langle R\varphi, x\rangle &= \mu_z(\langle \varphi, f(x+z) - f(z)\rangle)\\ &= \mu_z\{(\langle \varphi, f(x+z)\rangle - \langle x^*, x+z\rangle) - (\langle \varphi, f(z)\rangle - \langle x^*, z\rangle) + \langle x^*, x\rangle\}\\ &\le \mu(2\varepsilon\|x^*\|) + \le \mu(2\varepsilon\|x^*\|) + \mu_z(\langle x^*, x\rangle)\\ &= 4\varepsilon\|x^*\| + \langle x^*, x\rangle\end{aligned}$$

or, equivalently,

$$\langle R\varphi - x^*, x\rangle \le 4\varepsilon\|x^*\| \text{ for all } x \in X.$$

Consequently, $R\varphi - x^* = 0$. Therefore,

$$RQ(x^*) = (R \circ \imath)(x^*) = x^*, \ i.e.\ RQ = I_{X^*}.$$

Analogously, we can show $RU = Id_{X^*}$ by substituting N for M in the procedure above.

ii) According to Theorem 3.3, it suffices to show that R^* is a w^*-to-w^* continuous linear isometry.

By definition of Q and by Theorem 1.3, we know that $\|Q\| \le 1$. By the facts $\|R\| \le 1$, and $RQ = I_{X^*}$ that we have just proven, we obtain $Q^* : (Y^*/M)^* = M^\perp \to X^{**}$, $R^* : X^{**} \to M^\perp$ with $\|R^*\| = \|R\| \le 1$, $\|Q^*\| = \|Q\| \le 1$ and $Q^*R^* = (RQ)^* = I_{X^{**}}$. Therefore, for all $x^{**} \in X^{**}$, we have

$$\|x^{**}\| = \|(Q^*R^*)x^{**}\| \le \|Q^*\|\|R^*(x^{**})\| \le \|R^*(x^{**})\|.$$

On the other hand,

$$\|R^*(x^{**})\| \le \|R^*\|\|x^{**}\| \le \|x^{**}\|.$$

Hence, $R^* : X^{**} \to M^\perp$ is a w^*-to-w^* continuous linear isometry. □

For a standard ε-isometry $f : X \to Y$ and for every $n \in \mathbb{N}$, let $f_n(x) = \frac{f(nx)}{n}$, $x \in X$. Then f_n is an $\frac{\varepsilon}{n}$-isometry. We denote by $C(f_n)^{**}$ (resp. $C(f)^{**}$) the w^*-closure of $C(f_n) \equiv \overline{\text{co}}[f_n(X)]$ (resp. $C(f)$)) in Y^{**}, and let $C^{**} = \bigcap_{n=1}^{\infty} C^{**}(f_n)$.

Theorem 3.8. *With the standard ε-isometry f and associated subsets $C(f)^{**}$, $C(f_n)^{**}$ ($n \in \mathbb{N}$) and the subspaces $M, N \subset Y^*$ and $Ł \subset Y^{**}$ as previously defined, we have*
i) C^{**} is the maximum w^*-closed subspace contained in $C(f)^{**}$;
ii) the three subspaces $M^\perp, N^\perp$ and $Ł$ are contained in C^{**}.

Proof. i) Clearly, C^{**} is w^*-closed, convex and symmetric. Note $C(f_n)^{**} = \frac{1}{n}C(f)^{**}$. We observe that $y^{**} \in C^{**}$ implies $ny^{**} \in C^{**}$ for all $n \in \mathbb{N}$. This and symmetric convexity of C^{**} entail that every $\mathbb{R}y^{**} \equiv \{ry^{**} : r \in \mathbb{R}\} \subset C^{**}$, i.e. C^{**} is a subspace. C^{**} is maximum because every subspace of $C(f)^{**}$ is again contained in C^{**}.

ii) To show that $Ł \subset C^{**}$, it suffices to prove $\Phi(X) \subset C^{**}$. By the definition of Φ (3.1), each $y^{**} \in \Phi(X)$ is a w^*-cluster point of $\{f(nx)/n\}$ for some $x \in X$. Therefore, $y^{**} \in C(f_n)^{**}$ for all $n \in \mathbb{N}$. Consequently, $y^{**} \in C^{**}$.

Since $M \subset N$ (Lemma 3.5), to show $M^\perp, N^\perp \subset C^{**}$, we need only to show $M^\perp \subset C^{**}$. Since C^{**} is the maximum subspace of $C(f)^{**}$, it suffices to prove that $M^\perp \subset C(f)^{**}$.

Suppose, to the contrary, that there is $y^{**} \in M^\perp \setminus C^{**}(f)$. Since the two sets are w^*-closed convex in Y^{**}, there is $\varphi \in Y^*$ so that

$$\langle \varphi, y^{**} \rangle > \sup\{\langle \varphi, z^{**} \rangle : z^{**} \in C^{**}(f)\} \ge 0. \tag{3.22}$$

Therefore, φ is bounded on $C^{**}(f)$. Consequently, $\varphi \in M_\varepsilon \subset M$, and $\langle \varphi, y^{**} \rangle = 0$. This contradicts to (3.22).

□

4. A projective characterization of stable ε-isometries

Suppose again that $f : X \to Y$ is a standard ε-isometry. In this section, we shall show that a sufficient and necessary condition for f being w^*-stable (Definition 2.1 ii)) is that there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to N^\perp$ so that $Pf : X \to N^\perp$ is an approximate linear isometry.

Theorem 4.1. *Suppose that X, Y are Banach spaces, and $f : X \to Y$ is a standard ε-isometry for some $\varepsilon > 0$, and that $N \subset L(f)^*$ is the subspace associated with the ε-isometry f. If there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to N^\perp$ so that $g \equiv Pf$ is a δ-isometry for some $\delta \ge 0$, then there is a bounded linear operator $T : L(f)^{**} \to X^{**}$ with $\|T\| \le \|P\|$ so that*

$$\|Tf(x) - x\| \le 2\delta, \quad \text{for all } x \in X, \tag{4.1}$$

i.e. f is w^*-$\frac{2\delta}{\varepsilon}$-stable.

Proof. Without loss of generality we assume $L(f) = Y$; otherwise, we substitute Y for $L(f)$. Let $N \subset Y^*$ be the subspace associated with the ε-isometry f.

Since $P : Y^{**} \to N^\perp$ is a w^*-to-w^* continuous projection, there is a closed subspace N_c of Y^* with $N \cap N_c = \{0\}$ so that $Y^* = N + N_c = N \oplus N_c$, and the projection P is just from $Y^{**} = N^\perp \oplus N_c^\perp$ to $N^\perp$ along $N_c^\perp$.

Note that, the ε-isometry f can regard as an ε-isometry from X to Y^{**}. Suppose that the subspace corresponding to $f : X \to Y^{**}$ is $N_1 \subset Y^{***}$. Then

$$N_1 = N + Y^\perp = N \oplus Y^\perp \subset Y^{***} = Y^* \oplus Y^\perp. \tag{4.2}$$

Note $Y^{\perp\perp} = Y^{**}$, and note if N is acting as a subspace of Y^{***}, the annihilator of N is $N^{\perp\perp\perp} \subset Y^{****}$. Therefore,

$$N_1^\perp = N^\perp \cap Y^{\perp\perp} = N^{\perp\perp\perp} \cap Y^{**} = N^\perp \subset Y^{**}, \tag{4.3}$$

that is,

$$N_1^\perp = N^\perp. \tag{4.4}$$

We define

$$\jmath(x^*) = \{\varphi \in Y^{***} : |x^* - \varphi \circ f| \text{ is bounded on } X\}. \tag{4.5}$$

Then $U_1 : X^* \to Y^{***}/N_1$ defined for $x^* \in X^*$ by

$$U_1(x^*) = \jmath(x^*) + N_1 \tag{4.6}$$

is a surjective linear isometry, and it satisfies

$$U_1(x^*) = \varphi + N_1 \tag{4.7}$$

for every $\varphi \in \jmath(x^*)$. Therefore, $U^* : N_1^\perp = N^\perp \to X^{**}$ is again a linear surjective isometry.

Since $Pf : X \to N^\perp \subset Y^{**}$ is a δ-isometry, by [7, Theorem 2.3], for every $x^* \in X^*$ there is $\psi \in Y^{***}$ with $\|\psi\| = \|x^*\|$ so that

$$\big|\langle x^*, x\rangle - \langle \psi, Pf(x)\rangle\big| \le 2\delta, \quad \text{for all } x \in X; \tag{4.8}$$

or, equivalently,

$$\big|\langle x^*, x\rangle - \langle \psi \circ P, f(x)\rangle\big| \le 2\delta, \quad \text{for all } x \in X. \tag{4.9}$$

Hence, $\psi \circ P \in \jmath(x^*)$, and

$$U_1(x^*) = \psi \circ P + N_1. \tag{4.10}$$

Let $T = U_1^* P$. Then $\|T\| \le \|P\|$. It follows from (4.4) that T is from Y^{**} to X^{**}, and is a surjective isometry restricted to $N^\perp$. For any fixed $x \in X$ and $\alpha > 0$, let $x^* \in S_{X^*}$ so that

$$\begin{aligned}\|Tf(x) - x\| - \alpha &< \langle x^*, Tf(x) - x\rangle \\ &= \langle T^* x^*, f(x)\rangle - \langle x^*, x\rangle \\ &= \langle U_1^{**} x^*, Pf(x)\rangle - \langle x^*, x\rangle.\end{aligned}$$

Since U_1^{**} is just an extension of U_1 from X^* to X^{***}, we obtain

$$\|Tf(x) - x\| - \alpha < \langle U_1^{**} x^*, Pf(x)\rangle - \langle x^*, x\rangle = \langle U_1 x^*, Pf(x)\rangle - \langle x^*, x\rangle. \tag{4.11}$$

Let $\psi \in Y^{***}$ with $\|\psi\| = \|x^*\|$ satisfy (4.8). Then it follows from $U_1(x^*) = \psi \circ P + N_1$ that

$$\begin{aligned}\langle U_1 x^*, Pf(x)\rangle - \langle x^*, x\rangle &= \langle \psi \circ P + N_1, Pf(x)\rangle - \langle x^*, x\rangle \\ &= \langle \psi, Pf(x)\rangle - \langle x^*, x\rangle \le 2\delta.\end{aligned}$$

Since $x \in X$ and $\alpha > 0$ are arbitrary, we have shown

$$\|Tf(x) - x\| \le 2\delta, \quad \text{for all } \ x \in X,$$

and this says that f is $2\delta/\varepsilon$ stable. □

Remark 4.2. 1. If $\varepsilon = 0$ in the theorem above, i.e. f is a standard isometry, by Figiel's theorem we know that f is 0-stable, since the Figiel operator $T : L(f) \to X$ satisfy $Tf = I_X$.

2. Making use of the weak stability theorem, i.e. Theorem 1.3, we can show that the operator T in (4.1) restricted to Ł is just Figiel's operator associated with the isometry $\Phi : X \to Y^{**}$, i.e. $T\Phi = I_X$.

Before presenting the converse version of Theorem 4.1, we need the following lemma. This is an analogous but slightly different result appeared in [12, Theorems 3.1 and 4.1].

Lemma 4.3. *Suppose that $f : X \to Y$ is a standard w^*-stable ε-isometry, i.e. there exist a bounded linear operator $T : L(f)^{**} \to X^{**}$ and a constant $\gamma > 0$ so that*

$$\|Tf(x) - x\| \le \gamma\varepsilon, \quad \text{for all } x \in X. \tag{4.12}$$

Then $N^\perp$ is complemented in $L(f)^{**}$.

Proof. We assume $L(f) = Y$ and $T : Y^{**} \to X^{**}$ is w^*-to-w^* continuous; otherwise, we denote by $T|_Y : Y \to X^{**}$, the restriction of T from Y^{**} to Y. Then $T|_Y{}^{**} : Y^{**} \to X^{****}$ satisfies $\|T|_Y{}^{**}\| = \|T|_Y\|$ and with $(T|_Y{}^{**})|_Y = T|_Y$. Note $X^{****} = X^{**} \oplus X^{*\perp}$. Let $P_1 : X^{****} \to X^{**}$ be the natural projection (along $X^{*\perp}$). Then it is w^*-to-w^* continuous with respect to the w^*-topologies of X^{****} and X^{**}. We are done by letting $\tilde{T} = P_1(T|_Y)^{**}$ and substituting $\tilde{T}$ for T.

Let the subspace N and the linear isometry U (defined by (3.17)) be associated with f, and let $V = U^{-1}$. Then $V^* = U^{-1*} = U^{*-1}$, and $V^* : X^{**} \to N^\perp$ is a w^*-to-w^* continuous linear isometry. We first show

$$V^* T|_{N^\perp} = I_{N^\perp}, \tag{4.13}$$

or, equivalently,

$$TV^* = I_{X^{**}}. \tag{4.14}$$

Note $V^*(X^{**}) = N^\perp \subset C(f)^{**}$ (the w^*-closure of $C(f)$ in Y^{**} [12, Corllary 2.5]). For every $x_0 \in X^{**}$, there exists a net $(y_\alpha) \subset \mathrm{co}(f(X) \cup -f(X))$ of the form: for each α, there exist three finite sets J_α, $(\lambda_j^\alpha)_{j \in J_\alpha} \subset \mathbb{R}$ with $\sum_{j \in J_\alpha} |\lambda_j^\alpha| = 1$, and $(x_j^\alpha)_{j \in J_\alpha} \subset X$ such that

$$y_\alpha = \sum_{j \in J_\alpha} \lambda_j^\alpha f(x_j^\alpha) \to V^* x_0, \text{ in the } w^*\text{-topology of } Y^{**}. \tag{4.15}$$

On the other hand, by Theorem 1.3, for each $x^* \in X^*$, there exists $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\| \equiv r$ so that

$$|\langle \varphi, f(x)\rangle - \langle x^*, x\rangle| \le 2\varepsilon r, \text{ for all } x \in X. \tag{4.16}$$

Let $x_\alpha = \sum_{j\in J_\alpha}^n \lambda_j^\alpha x_j^\alpha$. Then (4.15), (4.16) and $\sum_{j\in J_\alpha} |\lambda_j^\alpha| = 1$ together entail

$$\begin{aligned}
&|\langle \phi, V^*x_0 - y_\alpha\rangle| = |\langle \varphi, V^*x_0\rangle - \langle \varphi, y_\alpha\rangle| \\
&= |\langle V\phi, x_0\rangle - \langle \varphi, y_\alpha\rangle| = |\langle x^*, x_0 - x_\alpha\rangle - (\langle \varphi, y_\alpha\rangle - \langle x^*, x_\alpha\rangle)| \\
&\ge |\langle x^*, x_0 - x_\alpha\rangle| - |\langle \varphi, y_\alpha\rangle - \langle x^*, x_\alpha\rangle)| \\
&= |\langle x^*, x_0 - x_\alpha\rangle| - |\sum_{j=1}^n \lambda_j(\langle \varphi, f(x_j^\alpha)\rangle - \langle x^*, x_j^\alpha\rangle)| \\
&\ge |\langle x^*, x_0 - x_\alpha\rangle| - 2\varepsilon r. \qquad .
\end{aligned}$$

Consequently,

$$|\langle x^*, x_0 - x_\alpha\rangle| \le 4\varepsilon\|x^*\| + |\langle \varphi, V^*x_0 - y_\alpha\rangle|, \text{ for all } x^* \in X^*. \tag{4.17}$$

Since $T : Y^{**} \to X^{**}$ is bounded and w^*-to-w^* continuous, there is $S : X^* \to Y^*$ so that $T = S^*$ and with $\|S\| = \|T\|$. This and (4.12) imply

$$|\langle Sx^*, f(x)\rangle - \langle x^*, x\rangle| = |\langle x^*, Tf(x) - x\rangle| \le \gamma\varepsilon\|x^*\|. \tag{4.18}$$

Given $\delta > 0$, let $x^* \in X^*$ with $\|x^*\| = 1$ so that

$$\langle x^*, TV^*x_0 - x_0\rangle \ge \|TV^*x_0 - x_0\| - \delta. \tag{4.19}$$

Then (4.15)–(4.19) together imply

$$\begin{aligned}
&\|TV^*x_0 - x_0\| - \delta \le |\langle x^*, TV^*x_0 - x_0\rangle| \\
&\le |\langle x^*, TV^*x_0 - Ty_\alpha\rangle| + |\langle x^*, Ty_\alpha - x_\alpha\rangle| + |\langle x^*, x_\alpha - x_0\rangle| \\
&= |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + |\langle Sx^*, y_\alpha\rangle - \langle x^*, x_\alpha\rangle| + |\langle x^*, x_\alpha - x_0\rangle| \\
&\le |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + \sum_{j\in J_\alpha} |\lambda_j^\alpha(\langle Sx^*, f(x_j^\alpha)\rangle - \langle x^*, x_j^\alpha\rangle)| + \\
&\qquad + |\langle x^*, x_\alpha - x_0\rangle| \\
&\le |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + \gamma\varepsilon + |\langle x^*, x_\alpha - x_0\rangle| \\
&\le |\langle Sx^*, V^*x_0 - y_\alpha\rangle| + |\langle \varphi, V^*x_0 - y_\alpha\rangle| + (2+\gamma)\varepsilon \\
&\longrightarrow (2+\gamma)\varepsilon.
\end{aligned}$$

Therefore, $\|TV^*x_0 - x_0\| \le (\gamma + 2)\varepsilon$. Arbitrariness of $x_0 \in X^{**}$ entails

$$\|(TV^* - I_{X^{**}})x\| \le (\gamma + 2)\varepsilon, \text{ for all } x \in X^{**},$$

which, in turn, implies $TV^* = I_{X^{**}}$. Hence, (4.14) has been proven.

Let $P = V^*T$. Then

$$P(Y^{**}) = (V^*T)(Y^{**}) = V^*(X^{**}) = N^\perp.$$

This and (4.13) together entail that $P : Y^{**} \to N^\perp$ is a w^*-to-w^* continuous projection and with $\|P\| \le \|T\|$. □

Theorem 4.4. *Suppose that X, Y are Banach spaces, and $f : X \to Y$ is a standard ε-isometry for some $\varepsilon \geq 0$, and that $N \subset L(f)^*$ is the subspace associated with f. If f is w^*-stable, i.e. there exist a constant $\gamma > 0$ and a bounded linear operator $T : L(f)^{**} \to X^{**}$ so that*

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \quad \text{for all } x \in X, \tag{4.20}$$

then there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to N^\perp$ so that $g \equiv Pf : X \to N^\perp$ is a γ-approximate linear $2\gamma\varepsilon$-isometry.

Proof. Without loss of generality, we can assume again that $L(f) = Y$ and $T : Y^{**} \to X^{**}$ is w^*-to-w^* continuous. It is easy to check that $T : Y^{**} \to X^{**}$ is also surjective. By Lemma 4.3, the corresponding projection $P = V^*T : Y^{**} \to N^\perp$ is w^*-to-w^* continuous with $\|P\| \leq \|T\|$. Therefore,

$$\begin{aligned} \|Pf(x) - Pf(y)\| - \|x - y\| &= \|V^*(Tf(x) - Tf(y))\| - \|x - y\| \\ &\leq \|Tf(x) - Tf(y)\| - \|x - y\| \\ &\leq \|Tf(x) - x\| + \|Tf(y) - y\| \leq 2\gamma\varepsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|x - y\| - \|Pf(x) - Pf(y)\| &= \|x - y\| - \|V^*(Tf(x) - Tf(y))\| \\ &\leq \|x - y\| - \|Tf(x) - Tf(y)\| \\ &\geq -\|Tf(x) - x\| - \|Tf(y) - y\| \geq -2\gamma\varepsilon. \end{aligned}$$

Consequently, $Pf : X \to N^\perp$ is a $2\gamma\varepsilon$-isometry.

It remains to show that $g \equiv Pf$ is γ-approximate linear. Note $V^* = (U^*)^{-1} : X^{**} \to N^\perp$ is a w^*-to-w^* continuous linear surjective isometry (Theorem 3.3 ii)). Then by (4.20), for all $x \in X$, we have

$$\begin{aligned} \gamma\varepsilon \geq \|Tf(x) - x\| &= \|(V^*T)f(x) - V^*x\| \\ &= \|Pf(x) - V^*x\|. \end{aligned}$$

□

Corollary 4.5. *Suppose that $f : X \to Y$ is an isometry. Then there is a projection $P : L(f)^{**} \to N^\perp$ of norm one so that*

$$Pf : X \to N^\perp \subset Y^{**} \tag{4.21}$$

is a linear isometry, where $P = V^*F^{**}$ and F is the Figiel operator associated with f, i.e. $Ff = I_X$ with $\|F\| = 1$.

Proof. Note that every standard isometry is 0-stable. Let $F : L(f) \to X$ be Figiel's operator, i.e. $\|F\| = 1$ satisfies $Ff = I_X$, the linear isometry $V^{-1} = U : X^* \to L(f)^*/N$ be defined as in Theorem 3.3, and $N \subset L(f)^*$ be the the subspace associated with f. Then by Theorem 4.4, $P = V^*F^{**} : L^{**}(f) \to N^\perp$ is a projection so that $Pf : X \to N^\perp$ is an isometry. It is linear because $Pf = V^*(F^{**}f) = V^*(Ff) = V^*I_X$. □

Remark 4.6. For a standard isometry $f : X \to Y$, let $F : L(f) \to X$ be Figiel's operator associated with f. It follows from (3.1) and Corollary 4.5 that f produces another isometry $\Phi : X \to Ł \subset L(f)^{**} \subset Y^{**}$. Though f and Φ are different in general, they have the same linear left inverse $F^{**} : L(f)^{**} \to X^{**}$, i.e.

$$F^{**}\Phi(x) = F^{**}|_{L(f)}f(x) = (Ff)(x) = x, \text{ for all } x \in X, \tag{4.22}$$

Consequently, they further deduce the same projective linear isometries Pf and $P\Phi$, where $P = V^*F^{**} : L(f)^{**} \to N^\perp$ is the corresponding projection.

Theorem 4.7. *Suppose that the three subspaces $M^\perp$, $N^\perp$ and $Ł$ are defined in Section 3 associated with a standard ε-isometry $f : X \to Y$. Then, up to a w^*-to-w^* continuous linear isometry, we have*
i) $Ł \subset M^\perp$; and
ii) $N^\perp$ is isometry to a quotient space of $Ł$.

Proof. i) Suppose, to the contrary, that there is $y^{**} \in Ł \setminus M^\perp$. Then by separation theorem, there is $\varphi \in Y^*$ so that $\langle \varphi, y^{**}\rangle > \sup\langle\varphi, M^\perp\rangle(= 0)$. Thus, $\varphi \in M$. Density of M_ε in M ((3.5) and (3.6)) allows us to assume $\varphi \in M_\varepsilon$, i.e. φ is bounded on $C(f)$. Consequently, φ is bounded on $C(f)^{**} \supset Ł$. This is a contradiction.

ii) Since $M \subset N \subset Y^*$, $N^\perp \subset M^\perp$. To show $N^\perp \subset Ł$ up to a w^*-to-w^* continuous linear isometry, let $\Phi : X \to Ł$ be the isometry associated with f. Then, by Theorem 3.1 ii), f induces a w^*-to-w^* continuous linear isometry $S_Ł : X^* \to Ł^*$ with respect to the w^*-topologies of X^* and Y^*, respectively. And there is a linear operator $F : Ł \to X$ so that $(F\Phi)(x) = x$ for all $x \in X$. Therefore, $F^{**} = S_Ł^* : Ł^{**} \to X^{**}$ is (w^*-to-w^* continuous) linear surjective. Note that $Z^{***} = Z^* \oplus Z^\perp$ for every Banach space Z and the projection $P_{Z^*} : Z^{***}$ along $Z^\perp$ is of norm one. Since $Ł$ is w^*-closed in Y^{**}, $Ł^{**} = Ł \oplus (Y^*/E)^\perp$, where $E = {}^\perp Ł \equiv \{\psi \in Y^* : \psi \text{ is vanishing on } Ł\}$. Let $P_Ł : Ł^{**} \to Ł$ be the projection along $(Y^*/E)^\perp$. Then it is w^*-to-w^* continuous with respect to the w^*-topologies of $Ł^{**}$ and $Ł$ (the w^*-topology of Y^{**} but restricted to $Ł$). Therefore, $F^{**} \circ P_Ł : L^{**} \to X^{**}$ is again w^*-to-w^* continuous. Consequently,

$$(F^{**} \circ P_{|Ł})\big|_Ł = F^{**}\big|_Ł = F$$

is w^*-to-w^* continuous. Since the w^*-topology on $Ł^{**}$ is just the w^*-topology of $Ł$ whenever it is restricted to $Ł$. This says that $F : Ł \to X^{**}$ is w^*-to-w^* continuous. $F(B_Ł) \supset B_X$ and w^*-compactness together imply that $F(B_Ł) = B_{X^{**}}$.

On the other hand, let $P = V^*F$, where $V^* = U^{*-1} : X^{**} \to N^\perp$ is a w^*-to-w^* continuous linear isometry (Theorem 3.3). Then $P : Ł \to N^\perp$ is surjective, and $N^\perp$ is isometric to the quotient space $Ł/\ker P$.

□

5. STABILITY CHARACTERIZATIONS OF ε ISOMETRIES

In this section, we shall present some characterizations for w^*-stability of ε-isometries. To begin with, we recall that every standard ε-isometry $f : X \to Y$ induces an isometry $\Phi : X \to Ł$, where $Ł = \overline{\text{span}}^{w^*}\Phi(X) \subset L(f)^{**} \subset Y^{**}$. With respect to the set-valued mapping $\imath : X^* \to 2^{L(f)^*}$ defined as in (3.8), for any fixed $\gamma \ge 2$, we further define a mapping $\imath_\gamma$ as follows.

$$\imath_\gamma(x^*) = \{\varphi \in L(f)^* : \sup|\langle x^*, x\rangle - \langle\varphi, f(x)\rangle| \le \gamma\varepsilon\|x^*\|, \ \forall x \in X\}. \tag{5.1}$$

By Theorem 1.3, $\imath_\gamma$ is an everywhere nonempty set-valued mapping.

Theorem 5.1. *Suppose that $f : X \to Y$ is a standard ε-isometry. Then it is w^*-stable (resp. stable) if and only if there is a bounded linear (resp. w^*-to-w^*) continuous selection $S : X^* \to L(f)^*$ for the set-valued mapping $\imath_\gamma : X^* \to 2^{L(f)^*}$ for some $\gamma \geq 2$.*

Proof. We assume again $L(f) = Y$.

Sufficiency. Let $S : X^* \to Y^*$ be a linear bounded w^*-to-w^* continuous selection for the set-valued mapping $\imath_\gamma : X^* \to 2^{Y^*}$ defined as (5.1). Since S is injective, there is a surjective linear operator $T : Y \to X$ so that $T^* = S$ and $\|T\| = \|S\|$. Thus, for all $x^* \in S_{X^*}$,

$$\Big|\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle = \Big|\langle x^*, x\rangle - \langle Sx^*, f(x)\rangle\Big| \leq \gamma\varepsilon, \text{ for all } x \in X. \tag{5.2}$$

Clearly, (5.2) is equivalent to

$$\|Tf(x) - x\| \leq \gamma\varepsilon, \text{ for all } x \in X, \tag{5.3}$$

i.e. f is stable.

If $S : X^* \to Y^*$ is a linear bounded selection for the set-valued mapping $\imath_\gamma : X^* \to 2^{Y^*}$, then $T \equiv S^* : Y^{**} \to X^{**}$ is surjective, and for all $x^* \in S_{X^*}$,

$$\Big|\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle = \Big|\langle x^*, x\rangle - \langle Sx^*, f(x)\rangle\Big| \leq \gamma\varepsilon, \text{ for all } x \in X. \tag{5.4}$$

i.e. f is w^*-stable.

Necessity. If f is stable, i.e. there is a linear bounded operator $T : Y \to X$ so that (5.3) holds, then we take $S = T^*$. It is easy to observe that $S : X^* \to Y^*$ is a bounded w^*-to-w^* continuous selection for $\imath_\gamma$. If f is w^*- stable, i.e. there is a continuous linear operator $T : Y^{**} \to X^{**}$, without loss of generality, we assume that T is w^*-to-w^* continuous, so that (5.3) holds, then we take $S : X^* \to Y^*$ to be the pre-conjugate operator of T. □

As its application of the characterization above, we show the following stability theorem for standard ε- isometries.

Theorem 5.2. *Suppose that X, Y are Banach spaces, and $f : X \to Y$ is a standard ε-isometry. If Y^* is strictly convex, and L is w^* 1-complemented in $L(f)^{**}$, then f is 2-stable.*

Proof. By Theorem 1.3, for each $x^* \in X^*$, there is $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\| \equiv r$ so that

$$\Big|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle\Big| \leq 2\varepsilon r, \text{ for all } x \in X. \tag{5.5}$$

Since Y^* is strictly convex, the functional φ corresponding to x^* in the inequality above is unique. Indeed, Assume that $\varphi_1, \varphi_2 \in Y^*$ with $\|\varphi_1\| = \|\varphi_2\| = r$ satisfy (5.5). Then

$$\Big|\langle x^*, x\rangle - \langle \frac{\varphi_1 + \varphi_2}{2}, f(x)\rangle\Big| \leq 2\varepsilon r, \text{ for all } x \in X. \tag{5.6}$$

Choose any $\{x_n\} \subset X$ with $\|x_n\| = 1$ so that $\langle x^*, x_n\rangle \to \|x^*\| = r$. Then it follows from

$$(5.7) \qquad \Big|\langle x^*, x_n\rangle - \langle \frac{\varphi_1 + \varphi_2}{2}, \frac{f(nx_n)}{n}\rangle\Big| \le \frac{2\varepsilon r}{n}, \text{ for all } x \in X,\ n \in \mathbb{N},$$

and $\|f(nx_n)/n\| \to 1$ that $\|\frac{\varphi_1+\varphi_2}{2}\| \ge r$. Strict convexity of Y^* further entails $\varphi_1 = \varphi_2$. Let $Sx^* = \varphi$ for every $x^* \in X^*$, where φ satisfies (5.5). Then it is easy to observe that $S : X^* \to Y^*$ is a norm-to-w^* continuous norm-preserving mapping.

To show that S is linear, note S can be regarded as a mapping from X^* to $Ł^* \subset L(f)^{***}$. Let $S_Ł : X^* \to Ł^*$ be defined for $x^* \in X^*$ by

$$(5.8) \qquad S_Ł(x^*) = S(x^*)\Big|_Ł, \text{ the restriction of } S(x^*) \text{ from } L(f)^{**} \text{ to } Ł.$$

Then Theorem 3.1 says that $S_Ł$ is a w^*-to-w^* continuous linear isometry with respect to the w^*- topology of $L(f)^*$. Since Ł is w^* 1-complemented in $L(f)^{**}$, there is a w^*-to-w^* continuous projection $P : L(f)^{**} \to Ł$ of norm one. Therefore, for each $x^* \in X^*$, $S(x^*) \circ P$ satisfies

$$(5.9) \qquad \|S(x^*) \circ P\| = \|S(x^*)\| = \|x^*\|.$$

Without any difficulty to check that $S(x^*) \circ P$ restricted to $L(f)$ has the same norm as $\|S(x^*) \circ P\|$. Strict convexity of Y^* implies $S(x^*) \circ P = S(x^*)$. This entails that $S : X^* \to L(f)^*$ is a w^*-to-w^* continuous linear isometry. Let $T : L(f) \to X$ be the pre-conjugate operator of S. Then, $\|T\| = \|S\| = 1$. By definition of S, we obtain

$$\begin{aligned} 2\varepsilon\|x^*\| &\ge \Big|\langle x^*, x\rangle - \langle S(x^*), f(x)\rangle\Big| \\ &= \Big|\langle x^*, x\rangle - \langle x^*, Tf(x)\rangle\Big|. \end{aligned}$$

Clearly, the inequality above is equivalent to

$$\|Tf(x) - x\| \le 2\varepsilon, \text{ for all } x \in X.$$

□

Applying Corollary 3.4 and by the same procedure of the proof of Theorem 5.2 but substituting $N^\perp$ for Ł , we can show the following theorem.

Theorem 5.3. *Suppose that X, Y are Banach spaces, and $f : X \to Y$ is a standard ε-isometry. If Y^* is strictly convex, and $N^\perp$ is w^* 1-complemented in $L(f)^{**}$, then f is 2-stable.*

The following theorem is an improvement of [9, Theorem 4.3].

Theorem 5.4. *Suppose that X is a n-dimensional Banach space. Then every standard ε-isometry is $2n$-stable.*

Proof. Suppose $\dim X = n < \infty$, and $f : X \to Y$ is a standard ε-isometry. By Theorem 3.1 i), we obtain a w^*-to-w^* continuous linear isometry $S_Ł : X^* \to Ł^*$ with

$$(5.10) \qquad S_Ł x^* = \varphi|_Ł \equiv \psi, \text{ where } \varphi \text{ satisfies (5.5)}.$$

We can choose n linearly independent extreme points $x_1^*, x_2^*, \dots, x_n^*$ of the closed unit ball B_{X^*} of X^* with $\frac{1}{n}B_{X^*} \subset C_{X^*} \equiv \text{co}\{\pm x_j^*\}_{j=1}^n$. It is easy to see that $\psi_1, \psi_2, \dots, \psi_n$ are also n linearly independent extreme points of the closed unit ball B_Z of $Z \equiv \text{span}\{\psi_j : 1 \le j \le n\}$.

Let

$$C_{Y^*} = \text{co}\{\pm\varphi_j\}_{j=1}^n, \text{ and } C_Z = \text{co}\{\pm\psi_j : 1 \le j \le n\}).$$

Then they are convex, symmetric and have (relative) nonempty interiors, where

$$\{x_j^*, \varphi_j\} \text{ satisfy (5.5)}, \quad \psi_j = S_{\text{Ł}}(x_j^*) = \varphi_j|_{\text{Ł}}, \quad \text{for } 1 \le j \le n,$$

Note $\frac{1}{n}B_{X^*} \subset C_{X^*}$. Next, we define a (continuous) linear operator

$$\tilde{S} : X^* \to \tilde{Z} \equiv \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_n\}$$

by

$$\tilde{S}\Big(\sum_{j=1}^n \alpha_j x^*\Big) = \sum_{j=1}^n \alpha_j \varphi_j, \ \forall \alpha_j \in \mathbb{R}, 1 \le j \le n. \tag{5.11}$$

Let $T : Y \to X$ be the pre-conjugate operator of $\tilde{S}$. In the following, we show $\tilde{S}$ is a linear operator required. Given $x \in X$, let $x^* \in \frac{1}{n}S_{X^*} \subset C_{X^*}$ so that $\langle x^*, x - Tf(x)\rangle \ge \frac{1}{n}\|x - Tf(x)\|$. Then there exists $\{\alpha_j\}_{j=1}^n \subset \mathbb{R}$ with $\sum_j |\alpha_j| \le 1$ so that $x^* = \sum_j \alpha_j x_j^*$. Therefore,

$$\begin{aligned}\frac{1}{n}\|x - Tf(x)\| \le \langle x^*, x - Tf(x)\rangle &= \langle x^*, x\rangle - \langle \tilde{S}x^*, f(x)\rangle \\ &= \langle x^*, x\rangle - \langle \varphi, f(x)\rangle \le 2\varepsilon.\end{aligned}$$

This says that f is $2n$-stable. □

Theorem 5.5. *Suppose that $f : X \to Y$ is a standard ε-isometry. If the corresponding subspace $L \subset L(f)^{**}$ is complemented, i.e. there is a continuous projection $P : L(f)^{**} \to L$ along some closed subspace of $L(f)^{**}$, then for every $\alpha \ge 2$ there exist a selection $\hbar : X^* \to L(f)^*$ of the set-valued mapping*

$$\text{ı}_{\alpha\|x^*\|}(x^*) = \{\varphi \in Y^* : |\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \le \alpha\varepsilon\|x^*\|, \ \forall x \in X\}, \tag{5.12}$$

and a bounded w^*-to-w^* continuous linear operator T from the w^*-closure of $\text{span}[\hbar(X^*)] \subset L(f)^*$ *onto X^* so that $T\hbar = I_{X^*}$, i.e. $\hbar$ has a linear w^*-to-w^* continuous left inverse.*

Proof. We assume again that $L(f) = Y$. Since $f : X \to Y$ is a standard ε-isometry, by Theorem 1.3, for each $x^* \in X^*$ there is $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\| \equiv r$ so that

$$\Big|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle\Big| \le 2\varepsilon r, \quad \text{for all } x \in X. \tag{5.13}$$

According to Theorem 3.1,

$$Sx^* = \psi \equiv \varphi|_{\text{Ł}}, \quad x^* \in X^* \tag{5.14}$$

defines a w^*-to-w^* continuous linear isometry $S : X^* \to \text{Ł}^*$, and the isometry S is just the conjugate operator of the Figiel operator F associated with the isometry Φ defined as (3.1); where x^*, φ satisfy (5.5), and $\varphi|_{\text{Ł}}$ denotes the restriction of the functional φ to Ł.

Since Ł is $\|P\|$-complemented in Y^{**}, there is a closed subspace $Ł_c$ of Y^{**} so that $Ł + Ł_c = Ł \oplus Ł_c = Y^{**}$ and the projection $P : Y^{**} \to Ł$ is just along $Ł_c$.

Step 1. By (5.6), we claim that there is a selection $\hbar : S_{X^*} \to Y^*$ for the set-valued mapping

$$\imath_{2\|x^*\|}(x^*) = \{\varphi \in Y^* : |\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \le 2\varepsilon\|x^*\|,\ \forall x \in X\} \tag{5.15}$$

satisfying the following conditions:

a) $\|\hbar(x^*)\| = \|x^*\|$;

b) $\hbar(kx^*) = k\hbar(x^*), \forall k \in \mathbb{R},\ x^* \in X^*$.

Thus, the isometry $S : X^* \to Ł^*$ defined as (5.14) satisfies

$$Sx^* = \hbar(x^*)\big|_{Ł},\ \ \forall x^* \in X^*. \tag{5.16}$$

Step 2. Let $C = \mathrm{co}[\hbar(B_{X^*})]$, the convex hull of $K \equiv \hbar(B_{X^*})$. Then $C \subset B_{Y^*}$, and the set $\mathrm{ext}(C)$ of all extreme points of C is contained in K. We use $C|_{Ł}$ to denote the restriction of C, i.e. (each $\varphi \in C$ acting as a w^*-to-w^* continuous functional on Y^{**})

$$C_{Ł} = \{\varphi|_{Ł} : \varphi \in C\}. \tag{5.17}$$

This and (5.14) entail that $S_{Ł}(B_{X^*}) \subset C_{Ł}$, in particular,

$$\mathrm{ext}[S_{Ł}(B_{X^*})] \subset \mathrm{ext}(C_{Ł}). \tag{5.18}$$

Step 3. Now, we define a linear operator Θ from $H \equiv \mathrm{span}(C) = \cup_{\alpha>0}\alpha C$ to $Ł^*$ as follows.

Let p be the Minkowski functional on H generated by C, i.e. $p(\varphi) = \inf\{\alpha > 0 : \varphi \in \alpha C\}$. Then it is a (not necessarily equivalent) norm on H. Note $\varphi \in C$ if and only if $p(\varphi) \le 1$; and note $\mathrm{co}\big(\mathrm{ext}(C)\big) = C$.

$$\Theta(\varphi) = \begin{cases} \sum_j \lambda_j\varphi_j\Big|_{Ł}, & \text{if } \varphi = \sum_j \lambda_j\varphi_j,\ \lambda_j \ge 0, \sum_j \lambda_j = 1, \varphi_j \in \mathrm{ext}(C); \\ p(\varphi)\Theta\Big(\varphi/p(\varphi)\Big), & \text{if } p(\varphi) > 1. \end{cases} \tag{5.19}$$

Since $\Theta : H \to Ł^*$ is just a restriction, it is necessarily continuous and of norm one with respect to the new norm p. In the following, we show that Θ is also continuous with respect to the original norm of Y^* on H whenever Ł is complemented in Y^{**}.

Suppose that $\{\varphi_n\} \subset H$ is a null sequence. Then both $\{\varphi_n \circ P\}$ and $\{\varphi_n \circ (I-P)\}$ are null sequences. Since $\varphi_n \circ P$ restricted to Ł is just $\Theta(\varphi_n)$, Θ is continuous. By Theorem 3.1, $S_{Ł}^{-1} : S(X^*) \to X^*$ is a w^*-to-w^* continuous isometry with respect to the w^*-topologies of Y^* and X^*. It is easy to check that Θ is also w^*-to-w^* continuous with respect to the w^*-topology of Y^* . Since $S_{Ł} : X^* \to Ł^*$ is an isometry, it maps each extreme point of B_{X^*} into an extreme point of $B_{S_{Ł}(B_{X^*})}$. Consequently, for every extreme point x^* of B_{X^*}, $\hbar(x^*) \equiv \varphi$ is an extreme point of C. Therefore, $S^{-1}\Theta(\hbar(x^*)) = x^*$ for every $x^* \in \mathrm{ext}(B_{X^*})$. Let $T = S^{-1}\Theta$. By the Krein-Milman theorem (for instance, [27, Theorem 3.23, p.75]), $S_{Ł}^{-1}\Theta(\hbar(x^*)) = x^*$ for every x^* in the w^* dense subspace $\mathrm{span}[\mathrm{ext}(B_{X^*})]$. Let $T = S_{Ł}^{-1}\Theta$. Then, (up to the natural extension) T is a bounded linear w^*-to-w^* continuous operator from the w^*- closure of H in Y^* onto X^*. □

Again by Corollary 3.4, similarly, we can show the following result.

Theorem 5.6. *Suppose that $f : X \to Y$ is a standard ε-isometry. If the corresponding subspace $N^\perp \subset L(f)^{**}$ is w^* complemented, then for every $\alpha \geq 2$ there exist a selection $\hbar : X^* \to L(f)^*$ of the set-valued mapping*

$$1_{\alpha\|x^*\|}(x^*) = \{\varphi \in Y^* : |\langle x^*, x\rangle - \langle \varphi, f(x)\rangle| \leq \alpha\varepsilon\|x^*\|, \ \forall x \in X\}, \tag{5.20}$$

and a bounded w^*-to-w^* continuous linear operator T from the w^*-closure of $\text{span}[\hbar(X^*)] \subset L(f)^*$ *onto X^* so that $T\hbar = I_{X^*}$, i.e. $\hbar$ has a linear w^*-to-w^* continuous left inverse.*

6. Classification of ε isometries by their stability

Now, we can classify standard ε-isometries f from a Banach space X to another Banach space Y by their w^*-stability, stability and approximate property. Characterizations and properties are listed as follows.

6.1. **Properties of standard ε-isometries.** a) Every standard ε-isometry $f : X \to Y$ admits w-stability [7, Theorem 2.3]: For each $x^* \in X^*$, there is $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\| \equiv r$ so that

$$\big|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle\big| \leq 2\varepsilon r, \ \text{ for all } x \in X. \tag{6.1}$$

b) [7, Lemma 2.1] The stability formula above induces an isometry $\Phi : X \to L(f)^{**}$ satisfying

$$\langle x^*, x\rangle = \langle \varphi, \Phi(x)\rangle, \ \text{ for all } x \in X, \tag{6.2}$$

where the pair $x^* \in X^*, \varphi \in Y^*$ satisfy (6.1) and

c) (Theorem 3.1) the correspondence $x^* \to \varphi\big|_{Ł}$ (the restriction of φ from Y^{**} to $Ł \equiv \overline{\text{span}}^{w^*}[\Phi(X)]$) defines a w^*-to-w^* continuous linear isometry $S_{Ł} : X^* \to Ł^*$ with respect to the w^* topologies of X^* and Y^*.

d) [12] Making use of (6.1) and invariant mean procedure, we can further induce a closed subspace N of Y^* (also, $L(f)^*$) so that there is a surjective linear isometry $U : X^* \to Y^*/N$, hence, $U^* : N^\perp \to X^{**}$ is a w^*-to-w^* continuous linear isometry [12, Theorem 2.3].

e) $Ł \subset M^\perp$ and $N^\perp$ is linear isometry to a quotient space of $Ł$ (Theorem 4.7).

6.2. **w^*-stable ε-isometries.** a) If a standard ε-isometry $f : X \to Y$ is w^*-stable, then the subspace $N^\perp$ is w^*-complemented in $L(f)^{**}$ (Lemma 4.3).

b) Every w^*-γ-stable ε-isometry $f : X \to Y$ admits a projection $P : L^{**}(f) \to N^\perp$ so that $Pf : X \to N^\perp$ is a w^*-γ approximate $2\gamma\varepsilon$-isometry (Theorem 4.4). Thus, every standard isometry admits a projection $P : L(f)^{**} \to N^\perp$ of norm one so that $Pf : X \to N^\perp$ is a linear isometry (Corollary 4.5). Conversely, if a standard ε-isometry $f : X \to Y$ admits a projection $P : L^{**}(f) \to N^\perp$ so that $Pf : X \to N^\perp$ is a δ-isometry, then f is w^*-β-stable, where $\beta = 0$, if $\varepsilon = 0$;$=(2\delta)/\varepsilon$, if $\varepsilon > 0$ (Theorem 4.1).

c) (Theorem 5.1) A standard ε-isometry $f : X \to Y$ is w^*-stable if and only if there exist $\gamma > 0$ and a continuous linear selection $\hbar : X^* \to L(f)^*$ for the set-valued mapping

$$\imath_\gamma(x^*) = \{\varphi \in \|x^*\| S_{L^*(f)} : \big|\langle x^*, x\rangle - \langle \varphi, f(x)\rangle\big| \le \gamma\varepsilon \|x^*\|,\ \forall x \in X\}. \tag{6.3}$$

6.3. **Stable ε-isometries.** a) Every stable ε-isometry is w^*-stable, but the converse is not true (Example 2.3). If Y is reflexive, then the two notions coincide.

b) A standard ε-isometry $f : X \to Y$ is stable if and only if there exist $\gamma > 0$ and a continuous linear selection $\hbar : X^* \to L(f)^*$ for the set-valued mapping $\imath_\gamma$ defined as (6.3) [Theorem 5.1].

c) If Y^* is strictly convex, and the subspace Ł, or, $N^\perp$ associated with f is 1-complemented in $L(f)^{**}$, then f is 2-stable (Theorems 5.2 and 5.3).

d) If both X and Y are L_p-spaces ($1 < p < \infty$), then every standard ε-isometry $f : X \to Y$ is 2-stable [28].

6.4. **Approximate linear ε-isometries.** Every standard surjective ε-isometry is 2-approximate linear ([25]). More general, every standard ε-isometry $f : X \to Y$ admitting a sublinear net (i.e.

$$\tau(f) \equiv \sup_{y \in S_Y} \inf_{t\to\infty, x\in X} \|ty - f(x)\|/|t| = 0)$$

is 2-approximate linear ([7, 31, 34]), which is equivalent to $\tau(f) < \frac{1}{2}$ [10].

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