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CHARACTERIZING THE CONTINGENT CONE'S CONVEX KERNEL

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Abstract. In this paper we present a direct characterization of the convex kernel of the contingent cone and use this result to sharpen the assumptions under which the calculus of the contingent and adjacent cones is known to be valid. We apply this calculus to develop necessary optimality conditions for a nonsmooth mathematical program.

1. Introduction

Let X be a real normed space. We say that a set $C \subset X$ is a cone if whenever $x \in C$ and $\lambda > 0$, we have $\lambda x \in C$.

Among the fundamental objects of variational analysis are tangent cones, which provide local conical approximations to sets. Two of the most useful tangent cones are the contingent cone and adjacent cone.

Definition 1.1. Let $S \subset X$ and $x \in S$.
(a) The contingent cone to S at x is defined by

$$T(S,x) := \left\{ z \in X \;\middle|\; \exists \{(t_j, z^j)\} \to (0^+, z) \text{ such that } x + t_j z^j \in S \right\}.$$

(b) The adjacent cone to S at x is defined by

$$A(S,x) := \left\{ z \in X \;\middle|\; \forall \{t_j\} \to 0^+, \exists \{z^j\} \to z \text{ such that } x + t_j z^j \in S \right\}.$$

For all $S \subset X$ and $x \in S$, the contingent cone and adjacent cone are closed cones containing 0. (We consider the entire space X to be both closed and open.) From the definitions, we can see that the inclusion $A(S,x) \subset T(S,x)$ is always true. For more on the properties of these cones, see [1, 2, 4, 7, 10, 11] and their references.

In applying the contingent and adjacent cones in optimization theory, one would like to identify conditions under which the inclusions

$$T(S_1,x) \cap T(S_2,x) \subset T(S_1 \cap S_2, x), \tag{1.1}$$

$$T(S_1,x) \cap A(S_2,x) \subset T(S_1 \cap S_2, x), \tag{1.2}$$

$$A(S_1,x) \cap A(S_2,x) \subset A(S_1 \cap S_2, x) \tag{1.3}$$

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are satisfied for $x \in S_1 \cap S_2$. One method of deriving such conditions employs the *convex kernels* of these cones, defined for $S \subset X$ and $x \in S$ by

$$T^\infty(S,x) := \{z \in X \mid y + z \in T(S,x) \quad \forall y \in T(S,x)\}$$

and

$$A^\infty(S,x) := \{z \in X \mid y + z \in A(S,x) \quad \forall y \in A(S,x)\}.$$

Since $T(S,x)$ and $A(S,x)$ are closed cones, $T^\infty(S,x)$ and $A^\infty(S,x)$ are always closed convex cones [1, §4.5.1]. Note also that since $0 \in T(S,x)$ and $0 \in A(S,x)$, we have $T^\infty(S,x) \subset T(S,x)$ and $A^\infty(S,x) \subset A(S,x)$.

In [6] Penot gave the following direct characterization of the convex kernel of the adjacent cone:

Proposition 1.2 (6, Proposition 4.6)**.** *Let $S \subset X$, $x \in S$. Then*

$$A^\infty(S,x) = \{y \mid \forall x^j \to_S x, \forall t_j \to 0^+ \textit{ with } \lim_{j\to\infty}(x^j - x)/t_j \in A(S,x),$$
$$\exists\, y^j \to y \textit{ with } x^j + t_j y^j \in S\},$$

where $x^j \to_S x$ means that $x^j \to x$ with each $x^j \in S$.

Proposition 1.2 turns out to be quite useful in the development of the calculus of the contingent and adjacent cones, as Penot went on to show in [6]. Along with the characterization of A^∞ in Proposition 1.2, one can make use of a corresponding open tangent cone to establish this calculus.

Definition 1.3. Let S be a subset of X and $x \in S$. Define

$$IA^\infty(S,x) := \{y \mid \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } \lim_{j\to\infty}(x^j - x)/t_j \in A(S,x),$$
$$\forall\, y^j \to y,\ x^j + t_j y^j \in S \text{ for large enough j }\}.$$

For all $S \subset X$ and $x \in S$, $IA^\infty(S,x)$ is an open convex (possibly empty) cone. By definition, $IA^\infty(S,x) \subset A^\infty(S,x)$.

Proposition 1.2 raises the question of finding an analogous direct characterization for the convex kernel of the contingent cone. One natural candidate would be

$$P(S,x) := \{y \mid \forall x^j \to_S x, \forall t_j \to 0^+ \text{ such that } \lim_{j\to\infty}(x^j - x)/t_j \text{ exists},$$
$$\exists\, y^j \to y \text{ with } x^j + t_j y^j \in S\},$$

which is referred to in [6] as the prototangent cone or pseudo-strict tangent cone, and in [7] as the moderate tangent cone. Note that the existence of $\lim_{j\to\infty}(x^j - x)/t_j$ is equivalent to the requirement that $\lim_{j\to\infty}(x^j - x)/t_j \in T(S,x)$.

It is easy to show that $P(S,x)$ is a closed convex cone with $P(S,x) \subset T^\infty(S,x)$ for all $S \subset X$ and $x \in S$. As Penot notes in [7, p. 395], $P(S,x) = A^\infty(S,x) = T^\infty(S,x)$ whenever $A(S,x) = T(S,x)$. However, $P(S,x)$ is not equal to $T^\infty(S,x)$ in general. The following example illustrates this fact.

Example 1.4. Let $X = \mathbb{R}$, and let $\mathbb{Z}$ be the set of integers. Define $S := \{0\} \cup \{\pm 2^n \mid n \in \mathbb{Z}\}$. It is easy to see that $T(S,0) = \mathbb{R}$, so that $T^\infty(S,0) = \mathbb{R}$ as well. On the other hand, let $x^j = 2^{-j}$ and $t_j = 2^{-(j+1)}$ for $j \geq 1$. Then $\lim_{j\to\infty}(x^j - x)/t_j$ exists, but for the interval $(0, 2)$, we have

$$[x^j + t_j(0,2)] \cap S = \emptyset.$$

It follows that $1 \notin P(S, x)$. It can be shown similarly that $-1 \notin P(S, x)$, and we conclude that $P(S, x) = \{0\}$.

Example 1.4 shows that $P(S, x)$ is unsuccessful as a candidate for a sequential characterization of $T^\infty(S, x)$. In this paper, we present a successful candidate and use it to augment the calculus of the contingent and adjacent cones and their associated epiderivatives.

The paper is organized as follows. In §2, we begin with our sequential characterization of T^∞ in Theoerem 2.1 and use this result as the basis for the definition of a corresponding "interior cone" IT^∞. We then establish conditions under which inclusions (1.2) and (1.3) are satisfied, and we show that these conditions can be satisfied in some cases where the metric subregularity hypotheses of [2] do not hold. In §3 we develop the calculus of the epiderivatives associated with the contingent and adjacent cones, again making use of the convex kernels of these cones. Then in §4 we apply the calculus of epiderivatives to prove necessary optimality condtions of Fritz John and Karush-Kuhn-Tucker type for a mathematical program with inequality and set constraints.

We conclude this introduction with a listing of some notation used throughout the paper. Let X be a real normed space with dual space X^*. For a set $S \subset X$, we will denote the interior of S by $\operatorname{int} S$ and the closure of S by $\operatorname{cl} S$. The indicator function of S is defined by

$$i_S(x) := \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{otherwise,} \end{cases}$$

and the distance function is defined for $x \in X$ and $S \subset X$ by

$$d(x, S) := \inf\{\|x - s\| : s \in S\}.$$

The polar cone of S is the set

$$S^\circ := \{x^* \in X^* \mid \langle x^*, x\rangle \leq 0 \ \forall x \in S\}.$$

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the set of extended real numbers. For a function $f : X \to \bar{\mathbb{R}}$, define

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}.$$

We will say that f is *proper* if $\operatorname{dom} f$ is nonempty and f never takes on the value $-\infty$. The epigraph of f is the set

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}.$$

For a convex function $f : X \to \bar{\mathbb{R}}$ that is finite at $\bar{x}$, the subdifferential of f at $\bar{x}$ is the set

$$\partial f(\bar{x}) := \{x^* \in X^* \mid \langle x^*, x\rangle \leq f(x) - f(\bar{x}) \quad \forall x \in X\}.$$

2. The characterization and tangent cone calculus

We begin this section with our main result, a direct sequential characterization of the convex kernel of the contingent cone.

Theorem 2.1. *Let $S \subset X$ and $x \in S$. Then*

$$T^\infty(S,x) = \{y \,|\, \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } (x^j - x)/t_j \text{ convergent},$$
$$\exists\, t_j{}' \to 0^+, \exists\, y^j \to y \text{ with } x + t_j{}'(y^j + (x^j - x)/t_j) \in S\}.$$

Proof. Denote the set on the right-hand side by Ω. To show that $T^\infty(S,x) \subset \Omega$, let $y \in T^\infty(S,x)$, and let $x^j \to_S x$ and $t_j \to 0^+$ with $(x^j - x)/t_j \to z$ for some $z \in X$. Then $z \in T(S,x)$, and so $y + z \in T(S,x)$. Hence there exist sequences $t_j{}' \to 0^+$ and $w^j \to y + z$ such that $x + t_j{}' w^j \in S$. Let $z^j = (x^j - x)/t_j$ and $y^j = w^j - z^j$. Then $y^j \to y$ and

$$x + t_j{}'(y^j + (x^j - x)/t_j) = x + t_j{}'(y^j + z^j) = x + t_j{}' w^j \in S,$$

establishing that $y \in \Omega$. Therefore $T^\infty(S,x) \subset \Omega$.

To demonstrate the opposite inclusion, let $y \in \Omega$ and $w \in T(S,x)$. We want to show that $y + w \in T(S,x)$. Since $w \in T(S,x)$, there exist $t_j \to 0^+$ and $w^j \to w$ such that $x + t_j w^j \in S$. Now set $x^j = x + t_j w^j$. We have $x^j \to_S x$ and $(x^j - x)/t_j$ convergent. Since $y \in \Omega$, there exist sequences $t_j{}' \to 0^+$ and $y^j \to y$ satisfying $x + t_j{}'(y^j + (x^j - x)/t_j) \in S$. Noting that $y^j + (x^j - x)/t_j \to y + w$, we see that $y + w \in T(S,x)$, implying that $y \in T^\infty(S,x)$. Therefore $\Omega \subset T^\infty(S,x)$. ☐

Theorem 2.1 can help us to identify conditions under which inclusion (1.2) is satisfied. Some of those conditions involve an "interior cone" corresponding to T^∞.

Definition 2.2. For $S \subset X$ and $x \in S$, define

$$IT^\infty(S,x) := \{y \,|\, \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } (x^j - x)/t_j \text{ convergent}, \exists t_j{}' \to 0^+$$
$$\text{such that } \forall y^j \to y,\ x + t_j{}'(y^j + (x^j - x)/t_j) \in S \text{ for large enough j}\}.$$

We observe that $IT^\infty(S,x)$ is a (possibly empty) cone. To see this, suppose that $y \in IT^\infty(S,x)$ and $\alpha > 0$, and let $x^j \to_S x$, $t_j \to 0^+$ with $(x^j - x)/t_j$ convergent. Then there exists $t_j{}' \to 0^+$ such that for all $y^j \to y$, we have $x + t_j{}'(y^j + (x^j - x)/t_j) \in S$ for j large enough. Now let $z^j \to \alpha y$. Since $z^j/\alpha \to y$, it follows that

$$x + \frac{t_j{}'}{\alpha}(z^j + (x^j - x)/t_j) = x + t_j{}'(z^j/\alpha + (x^j - x)/t_j) \in S$$

for large enough j. Noting that $t_j{}'/\alpha \to 0^+$, we conclude that $\alpha y \in IT^\infty(S,x)$.

We can give an alternate characterization of $IT^\infty(S,x)$ in terms of neighborhoods.

Proposition 2.3. *Let $S \subset X$ and $x \in S$. For $\varepsilon > 0$, let*

$$B_\varepsilon(x) := \{y \,|\, \|y - x\| < \varepsilon\}.$$

Then $IT^\infty(S,x) = \Omega(S,x)$, where

$$\Omega(S,x) := \{y \,|\, \forall z \in T(S,x),\ \exists \varepsilon > 0 \text{ such that } \forall \lambda > 0,\ t \in (0,\varepsilon), v \in B_\varepsilon(0)$$
$$\text{with } x + t(z + v) \in S,\ \exists t' \in (0,\lambda) \text{ with } x + t'(z + v + y + B_\varepsilon(0)) \subset S\,\}.$$

Proof. To show $\Omega(S,x) \subset IT^\infty(S,x)$, suppose $y \in \Omega(S,x)$. Let $x^j \to_S x$, $t_j \to 0^+$ with $z^j := (x^j - x)/t_j \to z \in T(S,x)$, and let $y^j \to y$. There exists $\varepsilon > 0$ such that for all $\lambda > 0$ and for all $t \in (0,\varepsilon)$, $v \in B_\varepsilon(0)$ with $x + t(z+v) \in S$, there exists $t' \in (0,\lambda)$ with $x + t'(z + v + y + B_\varepsilon(0)) \subset S$.

Let $v^j := z^j - z$. For j large enough, we have $v^j \in B_\varepsilon(0)$, $t_j \in (0,\varepsilon)$, and $y^j - y \in B_\varepsilon(0)$. Moreover, for each j, $x + t_j(z + v^j) = x^j \in S$. So for all j large enough, there exists $t_j{}' \in (0,1/j)$ such that

$$x+t_j{}'(y^j+(x^j-x)/t_j) = x+t_j{}'(z+v^j+y+y^j-y) \in x+t_j{}'(z+v^j+y+B_\varepsilon(0)) \subset S.$$

Therefore $y \in IT^\infty(S,x)$.

To prove the opposite inclusion, suppose $y \notin \Omega(S,x)$. Then there exists $z \in T(S,x)$ such that for all $\varepsilon > 0$, there exist $\lambda > 0$, $t \in (0,\varepsilon)$, and $v \in B_\varepsilon(0)$ with $x + t(z+v) \in S$ such that for all $t' \in (0,\lambda)$, there exists $v' \in B_\varepsilon(0)$ with

$$x + t'(z + v + y + v') \notin S.$$

To show that $y \notin IT^\infty(S,x)$, let $t_j{}' \to 0^+$ and N a natural number. For $\varepsilon = 1/N$, there exist $\lambda_N > 0$, $t_N \in (0,1/N)$, and $v_N \in B_{1/N}(0)$ with $x + t_N(z+v_N) \in S$ such that for all $t' \in (0,\lambda_N)$, there exists $v_N{}' \in B_{1/N}(0)$ with $x+t'(z+v_N+y+v_N{}') \notin S$.

Now let $j(0) = 1$, and define $j(N)$ for each natural number N to be a natural number such that

- $j(N) \geq N$;
- $j(N) > j(N-1)$;
- $t_{j(N)}{}' \in (0,\lambda_N)$.

Let $t_{j(N)} = t_N \in (0,1/N)$, $v_{j(N)} = v_N \in B_{1/N}(0)$ with $x + t_{j(N)}(z + v_{j(N)}) \in S$ and $v_{j(N)}{}' \in B_{1/N}(0)$ such that

$$x + t_{j(N)}{}'(z + v_{j(N)} + y + v_{j(N)}{}') \notin S.$$

For $j \in (j(N-1), j(N))$, let $t_j = t_{j(N)}$, $v_j = v_{j(N)}$, $v_j{}' = v_{j(N)}{}'$. Define $x^j = x + t_j(z + v_j)$, $y^j = y + v_j{}'$. Then $t_j \to 0^+$, $x^j \in S$, $(x^j - x)/t_j \to z$, and for each $j = j(N)$,

$$x + t_j{}'(y^j + (x^j - x)/t_j) = x + t_j{}'(z + v_j + y + v_j{}') \notin S.$$

Therefore $y \notin IT^\infty(S,x)$. □

The following fact follows quickly from Proposition 2.3.

Corollary 2.4. *Let $S \subset X$ and $x \in S$. Then $IT^\infty(S,x)$ is an open set.*

Proof. Let $y \in IT^\infty(S,x) = \Omega(S,x)$. There exists $\varepsilon > 0$ such that for all $\lambda > 0$ and for all $t \in (0,\varepsilon)$, $v \in B_\varepsilon(0)$ with $x + t(z+v) \in S$, there exists $t' \in (0,\lambda)$ with $x + t'(z + v + y + B_\varepsilon(0)) \subset S$. Let $y' \in y + B_{\varepsilon/2}(0)$, $z \in T(S,x)$, $\lambda > 0$. Let $t \in (0,\varepsilon/2)$, $v \in B_{\varepsilon/2}(0)$ with $x + t(z+v) \in S$. Then there exists $t' \in (0,\lambda)$ with

$$x + t'(z + v + y' + B_{\varepsilon/2}(0)) \subset x + t'(z + v + y + B_\varepsilon(0)) \subset S.$$

Therefore $y' \in IT^\infty(S,x)$. We conclude that $y \in \operatorname{int} IT^\infty(S,x)$, which means that $IT^\infty(S,x)$ is open. □

It is instructive to compare IT^{∞} with the interior Clarke tangent cone

$$IC(S,x) = \{y \,|\, \forall x^j \to_S x, \forall t_j \to 0^+, \forall y^j \to y,\ x^j + t_j y^j \in S \text{ for large enough j}\}.$$

Taking $t_j{}' = t_j$, we can see that $IC(S,x) \subset IT^{\infty}(S,x)$, which means in particular that $IT^{\infty}(S,x)$ is nonempty whenever $IC(S,x)$ is nonempty. Sets for which $IC(S,x)$ is nonempty are said to be "epi-Lipschitzian at x" [8, Theorem 3; 9, pp. 20-21].

We next present some simple calculus rules involving the contingent and adjacent cones and their convex kernels. Proposition 2.5 gives information about the relationships between closed tangent cones and their corresponding interior cones. For $S \subset X$ and $x \in S$, we define interior cones corresponding to the contingent and adjacent cones by

$$IT(S,x) := \{z \in X \,|\, \exists\, t_j \to 0^+ \text{ such that } \forall z^j \to z, x + t_j z^j \in S \text{ for large enough j }\}$$

and

$$IA(S,x) := \{z \in X \,|\, \forall\, t_j \to 0^+, \forall\, z^j \to z, x + t_j z^j \in S \text{ for large enough j }\},$$

respectively.

Proposition 2.5. *Let $S \subset X$ and $x \in S$. Then*

(a) $T(S,x) + IT^{\infty}(S,x) \subset IT(S,x)$.
(b) $T^{\infty}(S,x) + IT^{\infty}(S,x) \subset IT^{\infty}(S,x)$.
(c) $A(S,x) + IA^{\infty}(S,x) \subset IA(S,x)$.
(d) $A^{\infty}(S,x) + IA^{\infty}(S,x) \subset IA^{\infty}(S,x)$.

Proof. To establish (a), suppose $y \in T(S,x)$, $w \in IT^{\infty}(S,x)$. There exist $t_j \to 0^+$, $y^j \to y$ with $x + t_j y^j \in S$. Let $x^j := x + t_j y^j$. Then $x^j \to_S x$ and $(x^j - x)/t_j = y^j \to y$. Since $w \in IT^{\infty}(S,x), \exists t_j{}' \to 0^+$ such that $\forall w^j \to w$, $x + t_j{}'(w^j + (x^j - x)/t_j) \in S$ for large enough j. Let $b^j \to y + w$. Then $b^j - y^j \to w$. Thus

$$x + t_j{}' b^j = x + t_j{}'(b^j - y^j + y^j) = x + t_j{}'(b^j - y^j + (x^j - x)/t_j) \in S \text{ for } j \text{ large enough.}$$

Therefore $y + w \in IT(S,x)$.

To prove (b), suppose that $y \in T^{\infty}(S,x)$, $w \in IT^{\infty}(S,x)$. Let $x^j \to_S x$, $t_j \to 0^+$ with $(x^j - x)/t_j$ convergent. There exist $t_j{}' \to 0^+$ and $y^j \to y$ such that $z^j := x + t_j{}'(y^j + (x^j - x)/t_j) \in S$. Since $z^j \to_S x$ and $(z^j - x)/t_j{}'$ converges, there exists $t_j{}'' \to 0^+$ such that $\forall w^j \to w, x + t_j{}''(w^j + (z^j - x)/t_j{}') \in S$ for j large enough. Let $b^j \to y + w$. Since $b^j - y^j$ converges to w,

$$x + t_j{}''(b^j + (x^j - x)/t_j) = x + t_j{}''(b^j - y^j + (z^j - x)/t_j{}') \in S \text{ for } j \text{ large enough.}$$

So $y + w \in IT^{\infty}(S,x)$.

Parts (c) and (d), which are stated in Propositions 4.3 and 4.6 of [6], have proofs analogous to those of (a) and (b). $\square$

Note that since $IT^{\infty}(S,x) \subset T^{\infty}(S,x)$ and $IA^{\infty}(S,x) \subset A^{\infty}(S,x)$, parts (b) and (d) of Proposition 2.5 imply that

$$IT^{\infty}(S,x) + IT^{\infty}(S,x) \subset IT^{\infty}(S,x)$$

and

$$IA^{\infty}(S,x) + IA^{\infty}(S,x) \subset IA^{\infty}(S,x),$$

from which it follows that $IT^\infty(S,x)$ and $IA^\infty(S,x)$ are always convex.

Corollary 2.6. *Let $S \subset X$ and $x \in S$. If $IT^\infty(S,x) \neq \emptyset$, then $T(S,x) = \operatorname{cl} IT(S,x)$ and $T^\infty(S,x) = \operatorname{cl} IT^\infty(S,x)$. Analogously, if $IA^\infty(S,x) \neq \emptyset$, then $A(S,x) = \operatorname{cl} IA(S,x)$ and $A^\infty(S,x) = \operatorname{cl} IA^\infty(S,x)$.*

Proof. Since $IT(S,x) \subset T(S,x)$ and $T(S,x)$ is closed, it follows that $\operatorname{cl} IT(S,x) \subset T(S,x)$. To prove the opposite inclusion, suppose that $IT^\infty(S,x) \neq \emptyset$, and let $y \in T(S,x)$, $v \in IT^\infty(S,x)$. Then for all $t > 0$, $y + tv \in IT(S,x)$, implying that $y \in \operatorname{cl} IT(S,x)$. The proofs of the other assertions are analogous to this one. $\square$

Corollary 2.7. *Let $S \subset X$ and $x \in S$. If $IT^\infty(S,x) \neq \emptyset$, then $IT^\infty(S,x) = \operatorname{int} T^\infty(S,x)$. Similarly, if $IA^\infty(S,x) \neq \emptyset$, then $IA^\infty(S,x) = \operatorname{int} A^\infty(S,x)$.*

Proof. Since $IT^\infty(S,x) \subset T^\infty(S,x)$ and $IT^\infty(S,x)$ is open, we have $IT^\infty(S,x) \subset \operatorname{int} T^\infty(S,x)$. To prove the opposite inclusion, let $y \in \operatorname{int} T^\infty(S,x)$. Since $IT^\infty(S,x) \neq \emptyset$, there exists $z \in IT^\infty(S,x)$. Choose $\lambda > 0$ small enough so that for all $t \in (0,\lambda)$, $y - tz \in T^\infty(S,x)$. Then by Proposition 2.5(b), for all $t \in (0,\lambda)$ it follows that

$$y = y - tz + tz \in T^\infty(S,x) + IT^\infty(S,x) \subset IT^\infty(S,x).$$

Hence $\operatorname{int} T^\infty(S,x) \subset IT^\infty(S,x)$, completing the proof of the first assertion. The proof of the second assertion is analogous (see Corollary 4.4 in [6]). $\square$

We next utilize Proposition 2.5 to present conditions under which inclusions (1.2) and (1.3) are satisfied.

Theorem 2.8. *Let $S_1, S_2 \subset X$ and $x \in S_1 \cap S_2$.*

(a) *If $A^\infty(S_1,x) \cap IT^\infty(S_2,x) \neq \emptyset$ or $IA^\infty(S_1,x) \cap T^\infty(S_2,x) \neq \emptyset$, then*

$$A(S_1,x) \cap T(S_2,x) \subset T(S_1 \cap S_2,x) \subset T(S_1,x) \cap T(S_2,x). \tag{2.1}$$

(b) *If $A^\infty(S_1,x) \cap IA^\infty(S_2,x) \neq \emptyset$, then*

$$A(S_1,x) \cap A(S_2,x) = A(S_1 \cap S_2,x). \tag{2.2}$$

Proof. In (a) the right-hand inclusion always holds. To prove the left-hand inclusion, suppose that $A^\infty(S_1,x) \cap IT^\infty(S_2,x) \neq \emptyset$, and let $y \in A(S_1,x) \cap T(S_2,x)$, $v \in A^\infty(S_1,x) \cap IT^\infty(S_2,x)$. Then for all $t > 0$,

$$y + tv \in A(S_1,x) \cap IT(S_2,x)$$

by Proposition 2.5 (a). Since the inclusion $A(S_1,x) \cap IT(S_2,x) \subset T(S_1 \cap S_2,x)$ is always satisfied, we then have $y + tv \in T(S_1 \cap S_2,x)$. It follows that $y \in \operatorname{cl} T(S_1 \cap S_2,x)$, and since $T(S_1 \cap S_2,x)$ is closed, the left-hand inclusion in (a) is true.

Similarly, suppose that $IA^\infty(S_1,x) \cap T^\infty(S_2,x) \neq \emptyset$, and let $y \in A(S_1,x) \cap T(S_2,x)$, $v \in IA^\infty(S_1,x) \cap T^\infty(S_2,x)$. Then for all $t > 0$,

$$y + tv \in IA(S_1,x) \cap T(S_2,x)$$

by Proposition 2.5 (c). It is easy to see that $IA(S_1,x) \cap T(S_2,x) \subset T(S_1 \cap S_2,x)$, and so $y + tv \in T(S_1 \cap S_2,x)$. Then $y \in \operatorname{cl} T(S_1 \cap S_2,x) = T(S_1 \cap S_2,x)$, and again the left-hand inclusion in (a) is true. The proof of (b) is similar. $\square$

The inclusions in Theorem 2.8 also have been derived under metric subregularity conditions [2]. The following result is a special case of [2, Theorem 3.1]:

Theorem 2.9. *Let X be a real Banach space, and let $S_1, S_2 \subset X$ be closed sets with $x \in S_1 \cap S_2$. Suppose that there exist $\delta > 0$, $M > 0$, such that for all $(y,z) \in B((x,x),\delta) \cap (S_1 \times S_2)$,*

$$d((y,z), \{(w,w) \mid w \in S_1 \cap S_2\}) \le M\|y - z\|. \tag{2.3}$$

Then (2.1) and (2.2) hold.

We will contrast (2.3) with the hypotheses of Theorem 2.8 in the next two examples. In these examples, for $(y,z) \in X \times X$ in (2.3), we will use the norm $\|(y,z)\| := \|y\| + \|z\|$, following [2]. Our first example illustrates the fact that there are instances in which the hypotheses of Theorem 2.9 are satisfied but those of Theorem 2.8 are not.

Example 2.10. Let $X := \mathbb{R}^2$, $S_1 := \{(y_1, 0) \mid y_1 \in \mathbb{R}\}$, $S_2 := \{(0, z_2) \mid z_2 \in \mathbb{R}\}$, $x := (0,0)$. In this example, $T(S_i, x) = A(S_i, x) = S_i$ for $i = 1, 2$, and $T(S_1 \cap S_2, x) = A(S_1 \cap S_2, x) = \{(0,0)\}$, so (2.1) and (2.2) hold. For $(y_1, 0) \in S_1$, $(0, z_2) \in S_2$, inequality (2.3) reduces to

$$|y_1| + |z_2| \le M\sqrt{{y_1}^2 + {z_2}^2},$$

which is satisfied with $M = \sqrt{2}$. On the other hand, $IT^\infty(S_i, x) = IA^\infty(S_i, x) = \emptyset$, so the hypotheses of Theorem 2.8 are not satisfied.

There are other examples in which the hypotheses of Theorem 2.8 are satisfied but the metric subregularity condition (2.3) does not hold.

Example 2.11. Let $X := \mathbb{R}^2$, $S_1 := \{(y_1, y_2) \mid y_2 \le -y_1\} \cup \{(y, -y^2) \mid y = 2^{-n}, n = 1, 2, 3, \dots\}$, $S_2 := \{(z_1, z_2) \mid z_1 \le 0, z_2 \ge z_1\} \cup \{(z_1, z_2) \mid z_1 \ge 0, z_2 \ge {z_1}^2\}$, and $x := (0,0)$. In this example $A(S_1, x) = \{(y_1, y_2) \mid y_2 \le -y_1\}$ and

$$T(S_2, x) = A(S_2, x) = \{(z_1, z_2) \mid z_1 \le 0, z_2 \ge z_1\} \cup \{(z_1, z_2) \mid z_1 \ge 0, z_2 \ge 0\},$$

while $T(S_1 \cap S_2, x) = A(S_1 \cap S_2, x) = S_1 \cap S_2$, so (2.1) and (2.2) hold. Moreover, $A^\infty(S_1, x) = A(S_1, x)$ and

$$IT^\infty(S_2, x) = IA^\infty(S_2, x) = \{(z_1, z_2) \mid z_1 \le 0, z_2 > 0\} \cup \{(z_1, z_2) \mid z_1 \ge 0, z_2 > z_1\},$$

so that conditions

$$A^\infty(S_1, x) \cap IT^\infty(S_2, x) \ne \emptyset \tag{2.4}$$

and

$$A^\infty(S_1, x) \cap IA^\infty(S_2, x) \ne \emptyset \tag{2.5}$$

are satisfied. On the other hand, if we take $y = (2^{-n}, -2^{-2n})$ and $z = (2^{-n}, 2^{-2n})$ in Theorem 2.9, inequality (2.3) reduces to

$$2^n\sqrt{1 + \frac{1}{2^{2n}}} \le M,$$

which is not satisfied for any real number M.

We can extend Theorem 2.8 inductively to a situation involving any finite number of sets by making use of the following lemma.

Lemma 2.12. *Let $S_i, 1 \leq i \leq n$ be subsets of X and $x \in \cap_{i=1}^n S_i$. Then*

$$A^\infty(S_1, x) \cap \cap_{i=2}^n IA^\infty(S_i, x) \subset A^\infty(\cap_{i=1}^n S_i, x) \tag{2.6}$$

and

$$\cap_{i=1}^n IA^\infty(S_i, x) \subset IA^\infty(\cap_{i=1}^n S_i, x). \tag{2.7}$$

Proof. To prove (2.6), let $y \in A^\infty(S_1, x) \cap \cap_{i=2}^n IA^\infty(S_i, x)$, and let $x^j \to_{\cap_{i=1}^n S_i} x$, $t_j \to 0^+$ with $(x^j - x)/t_j \to w \in A(\cap_{i=1}^n S_i, x)$. Then $w \in A(S_1, x)$, so there exists $y^j \to y$ with $x^j + t_j y^j \in S_1$. For each $i = 2, \ldots, n$ we have $w \in A(S_i, x)$, so $x^j + t_j y^j \in S_i$ for large enough j. Therefore we have for large enough j that $x^j + t_j y^j \in \cap_{i=1}^n S_i$, which means that $y \in A^\infty(\cap_{i=1}^n S_i, x)$. The proof of (2.7) is analogous to this one. □

Theorem 2.13. *Let $E_i, 1 \leq i \leq n$ be subsets of X and $x \in \cap_{i=1}^n E_i$.*
(a) *If*

$$A^\infty(E_1, x) \cap \cap_{i=2}^{n-1} IA^\infty(E_i, x) \cap IT^\infty(E_n, x) \neq \emptyset \tag{2.8}$$

or

$$\cap_{i=1}^{n-1} IA^\infty(E_i, x) \cap T^\infty(E_n, x) \neq \emptyset, \tag{2.9}$$

then

$$\cap_{i=1}^{n-1} A(E_i, x) \cap T(E_n, x) \subset T(\cap_{i=1}^n E_i, x) \subset \cap_{i=1}^n T(E_i, x). \tag{2.10}$$

(b) *If*

$$A^\infty(E_1, x) \cap_{i=2}^n IA^\infty(E_i, x) \neq \emptyset, \tag{2.11}$$

then

$$\cap_{i=1}^n A(E_i, x) = A(\cap_{i=1}^n E_i, x). \tag{2.12}$$

Proof. The right-hand inclusion in (2.10) is easily seen to be true. In deriving the remaining inclusions, we begin with the case where $n = 3$. To prove (a), suppose that (2.8) holds, and let $S_1 = E_1 \cap E_2$, $S_2 = E_3$.. By Lemma 2.12, $A^\infty(E_1, x) \cap IA^\infty(E_2, x) \subset A^\infty(S_1, x)$, so (2.8) implies that $A^\infty(S_1, x) \cap IT^\infty(S_2, x) \neq \emptyset$. By Theorem 2.8(a), $A(E_1 \cap E_2, x) \cap T(E_3, x) \subset T(E_1 \cap E_2 \cap E_3, x)$. It also follows from (2.8) that

$$A^\infty(E_1, x) \cap IA^\infty(E_2, x) \neq \emptyset,$$

so by Theorem 2.8(b), $A(E_1, x) \cap A(E_2, x) = A(E_1 \cap E_2, x)$. Therefore (2.10) holds for $n = 3$. The proofs of the remaining assertions when $n = 3$ are similar.

Now proceed inductively, assuming that (a) and (b) are true for the case of $n-1$ sets. To prove (a), suppose that (2.8) holds, and let $S_1 = \cap_{i=1}^{n-1} E_i$, $S_2 = E_n$. By Lemma 2.12,

$$A^\infty(E_1, x) \cap \cap_{i=2}^{n-1} IA^\infty(E_i, x) \subset A^\infty(\cap_{i=1}^{n-1} E_i, x),$$

and so (2.8) implies that

$$A^\infty(S_1, x) \cap IT^\infty(S_2, x) \neq \emptyset.$$

By Theorem 2.8(a),

$$A(\cap_{i=1}^{n-1} E_i, x) \cap T(E_n, x) \subset \cap_{i=1}^{n} T(E_i, x).$$

Moreover, since (2.8) also gives

$$A^\infty(E_1, x) \cap \cap_{i=2}^{n-1} IA^\infty(E_i, x) \neq \emptyset,$$

we have by the induction hypothesis in part (b) that

$$\cap_{i=1}^{n-1} A(E_i, x) = A(\cap_{i=1}^{n-1} E_i, x).$$

Therefore (2.10) holds. Again, the proofs of the remaining assertions are similar. □

3. Epiderivative calculus

Convex kernels and their corresponding interior cones can also be used in developing the calculus of directional derivative functions associated with the contingent and adjacent cones. To explain how this is done, we begin by reviewing the concept of the epiderivative associated with a tangent cone [1].

Let R denote some concept of tangent cone. (Examples defined in this paper include $R = T, A, P, IT, IA, T^\infty, A^\infty, IT^\infty, IA^\infty$.) Suppose that $f : X \to \bar{\mathbb{R}}$ is finite at x. We define the R-epiderivative of f at x in the direction $y \in X$ by

$$f^R(x; y) = \inf\{r \mid (y, r) \in R(\operatorname{epi} f, (x, f(x)))\}. \tag{3.1}$$

Notice that in cases where R is always a closed cone (e.g., $R = T, A, P, T^\infty, A^\infty$), then (3.1) implies that

$$\operatorname{epi} f^R(x; \cdot) = R(\operatorname{epi} f, (x, f(x))). \tag{3.2}$$

The case of the indicator function is also worth noting. When $C \subset X$ and $x \in C$, we have

$$i_C{}^R(x; y) = \inf\{r \mid (y, r) \in R(C \times [0, +\infty), (x, 0))\} = i_{R(C,x)}(y) \quad \forall y \in X$$

for $R = T, A, P, IT, IA, T^\infty, A^\infty, IT^\infty, IA^\infty$.

More explicit formulae for a number of epiderivatives can readily be obtained [1, Chapter 6; 11, Theorem 5.4]. For example,

$$\begin{aligned}
f^T(x; y) &= \sup_{\epsilon>0} \sup_{\lambda>0} \inf_{0<t<\lambda} \inf_{\|v-y\|<\epsilon} (f(x+tv) - f(x))/t;\\
f^A(x; y) &= \sup_{\epsilon>0} \inf_{\lambda>0} \sup_{0<t<\lambda} \inf_{\|v-y\|<\epsilon} (f(x+tv) - f(x))/t;\\
f^{IA}(x; y) &= \inf_{\epsilon>0} \inf_{\lambda>0} \sup_{0<t<\lambda} \sup_{\|v-y\|<\epsilon} (f(x+tv) - f(x))/t;\\
f^{IT}(x; y) &= \inf_{\epsilon>0} \sup_{\lambda>0} \inf_{0<t<\lambda} \sup_{\|v-y\|<\epsilon} (f(x+tv) - f(x))/t.
\end{aligned}$$

The results of §2 can be applied to derive calculus rules for epiderivatives of pointwise maxima of functions. Let $f_i : X \to \bar{\mathbb{R}}$, $i = 1, \cdots m$, and $I = \{1, \cdots, m\}$. Define

$$f(x) = \max_{i \in I} f_i(x). \tag{3.3}$$

It follows quickly from (3.3) that

$$\operatorname{epi} f = \cap_{i\in I} \operatorname{epi} f_i. \tag{3.4}$$

Now define

$$I(x) = \{i \in I \mid f(x) = f_i(x)\}.$$

Then if $i \in I\backslash I(x)$ and f_i is finite and continuous at x, we have $(x, f(x)) \in \operatorname{int}\operatorname{epi} f_i$, which implies that

$$R(\operatorname{epi} f_i, (x, f(x))) = X \times \mathbb{R} \tag{3.5}$$

for $R = T, A, P, IT, IA, T^\infty, A^\infty, IT^\infty, IA^\infty$.

Keeping (3.4) and (3.5) in mind, we can deduce the following result from Theorem 2.13.

Theorem 3.1. *Suppose that each f_i is finite at x, and that f_i is continuous at x for all $i \in I\backslash I(x)$.*

(a) *If there exist $j, k \in I(x)$ such that*

$$\operatorname{dom} {f_j}^{A^\infty}(x;\cdot) \cap \operatorname{dom} {f_k}^{IT^\infty}(x;\cdot) \cap \cap_{i\in I(x)\backslash\{j,k\}} \operatorname{dom} {f_i}^{IA^\infty}(x;\cdot) \neq \emptyset, \tag{3.6}$$

then

$$f^T(x;y) \leq \max\{{f_k}^T(x;y), {f_i}^A(x;y), i \in I(x)\backslash\{k\}\} \quad \forall y \in X. \tag{3.7}$$

(b) *If there exists $k \in I(x)$ such that*

$$\operatorname{dom} {f_k}^{T^\infty}(x;\cdot) \cap \cap_{i\in I(x)\backslash\{k\}} \operatorname{dom} {f_i}^{IA^\infty}(x;\cdot) \neq \emptyset, \tag{3.8}$$

then

$$f^T(x;y) \leq \max\{{f_k}^T(x;y), {f_i}^A(x;y), i \in I(x)\backslash\{k\}\} \quad \forall y \in X. \tag{3.9}$$

(c) *If there exists $j \in I(x)$ such that*

$$\operatorname{dom} {f_j}^{A^\infty}(x;\cdot) \cap \cap_{i\in I(x)\backslash\{j\}} \operatorname{dom} {f_i}^{IA^\infty}(x;\cdot) \neq \emptyset, \tag{3.10}$$

then

$$f^A(x;y) = \max_{i\in I(x)} {f_i}^A(x;y) \quad \forall y \in X. \tag{3.11}$$

Proof. To prove (a), suppose that z is an element of the intersection of sets in (3.6). Then there exists a real number r such that

$$(z, r) \in A^\infty(\operatorname{epi} f_j, (x, f(x))) \cap IT^\infty(\operatorname{epi} f_k, (x, f(x))) \cap \cap_{i\in I(x)\backslash\{j,k\}} IA^\infty(\operatorname{epi} f_i, (x, f(x))).$$

By Theorem 2.13, we have

$$T(\operatorname{epi} f_k, (x, f(x))) \cap \cap_{i\in I(x)\backslash\{k\}} A(\operatorname{epi} f_i, (x, f(x))) \subset T(\cap_{i\in I(x)} \operatorname{epi} f_i, (x, f(x))).$$

By (3.4) and (3.5),

$$T(\operatorname{epi} f, (x, f(x))) = T(\cap_{i\in I} \operatorname{epi} f_i, (x, f(x))) = T(\cap_{i\in I(x)} \operatorname{epi} f_i, (x, f(x))),$$

and so

$$T(\operatorname{epi} f_k, (x, f(x))) \cap_{i\in I(x)\backslash\{k\}} A(\operatorname{epi} f_i, (x, f(x))) \subset T(\operatorname{epi} f, (x, f(x))),$$

which implies (3.7). The proofs of (b) and (c) are analogous to the proof of (a). $\square$

To put Theorem 3.1 in context, it is helpful to keep in mind that if f is Lipschitzian near x, then $\operatorname{dom} f^R(x;\cdot) = X$ for $R = T, A, P, IT, IA, T^\infty, A^\infty, IT^\infty, IA^\infty$. As a result, Theorem 3.1 covers the case where the f_i are locally Lipschitzian, along with much more.

We can also develop epiderivative calculus rules for sums of functions. We start with some basic inequalities that follow quickly from the definitions of the relevant tangent cones.

Proposition 3.2. *Let $f_1 : X \to \bar{\mathbb{R}}$, $f_2 : X \to \bar{\mathbb{R}}$ be finite at $x \in X$. Then $\forall y \in X$,*

$$(f_1 + f_2)^T(x;y) \leq {f_1}^A(x;y) + {f_2}^{IT}(x;y). \tag{3.12}$$

$$(f_1 + f_2)^A(x;y) \leq {f_1}^A(x;y) + {f_2}^{IA}(x;y). \tag{3.13}$$

Next we make use of Propositions 3.2 and 2.5 to derive sum rules for contingent and adjacent epiderivatives.

Theorem 3.3. *Let $f_1 : X \to \bar{\mathbb{R}}$, $f_2 : X \to \bar{\mathbb{R}}$ be finite at $x \in X$.*

(a) *If $\operatorname{dom} {f_1}^{A^\infty}(x;\cdot) \cap \operatorname{dom} {f_2}^{IT^\infty}(x;\cdot) \neq \emptyset$ or $\operatorname{dom} {f_1}^{IA^\infty}(x;\cdot) \cap \operatorname{dom} {f_2}^{T^\infty}(x;\cdot) \neq \emptyset$, then $\forall y \in X$,*

$$(f_1 + f_2)^T(x;y) \leq {f_1}^A(x;y) + {f_2}^T(x;y). \tag{3.14}$$

(b) *If $\operatorname{dom} {f_1}^{A^\infty}(x;\cdot) \cap \operatorname{dom} {f_2}^{IA^\infty}(x;\cdot) \neq \emptyset$, then $\forall y \in X$,*

$$(f_1 + f_2)^A(x;y) \leq {f_1}^A(x;y) + {f_2}^A(x;y). \tag{3.15}$$

Proof. To prove (a), suppose that ${f_1}^A(x;y) + {f_2}^T(x;y) \leq r$. Then there exist $r_1, r_2 \in \mathbb{R}$ with $r_1 + r_2 \leq r$ and ${f_1}^A(x;y) \leq r_1$, ${f_1}^T(x;y) \leq r_2$. It follows that $(y, r_1) \in A(\operatorname{epi} f_1, (x, f_1(x)))$ and $(y, r_2) \in T(\operatorname{epi} f_2, (x, f_2(x)))$. If

$$\operatorname{dom} {f_1}^{A^\infty}(x;\cdot) \cap \operatorname{dom} {f_2}^{IT^\infty}(x;\cdot) \neq \emptyset,$$

there exist $d \in X$, $s \in \mathbb{R}$ such that

$$(d, s) \in A^\infty(\operatorname{epi} f_1, (x, f_1(x))) \cap IT^\infty(\operatorname{epi} f_2, (x, f_2(x))).$$

So for all $t > 0$,the definition of A^∞ and Proposition 2.5(a) imply that $(y, r_1) + t(d, s) \in A(\operatorname{epi} f_1, (x, f_1(x)))$ and $(y, r_2) + t(d, s) \in IT(\operatorname{epi} f_2, (x, f_2(x)))$. By (3.12),

$$(f_1 + f_2)^T(x; y + td) \leq {f_1}^A(x; y + td) + {f_2}^{IT}(x; y + td) \leq r_1 + r_2 + 2ts,$$

so $(y + td, r_1 + r_2 + 2ts) \in T(\operatorname{epi}(f_1 + f_2), (x, (f_1 + f_2)(x)))$. Since T is closed, $(y, r_1 + r_2) \in T(\operatorname{epi}(f_1 + f_2), (x, (f_1 + f_2)(x)))$, and therefore $(f_1 + f_2)^T(x;y) \leq r$. We conclude that (3.14) holds. The proofs of the remaining assertions are analogous to this one. $\square$

Chain rules for contingent and adjacent epiderivatives can also be developed via the approach of this paper, as in [12].

We close this section with a technical result that will come into play in §4. The analogous result for the epiderivatives associated with A^∞ and IA^∞ is given in [5, Theorem 2.8].

Proposition 3.4. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be finite at $x \in X$. If there exists $y \in \operatorname{dom} f^{IT^\infty}(x;\cdot)$ such that $-\infty < f^{T^\infty}(x;y)$, then*

(a) $f^{T^\infty}(x;\cdot)$ *is continuous on the interior of its domain, which is nonempty; and*
(b) $\operatorname{dom} f^{IT^\infty}(x;\cdot) = \operatorname{int}\operatorname{dom} f^{T^\infty}(x;\cdot)$.

Proof. We begin by showing that (a) implies (b). Suppose that (a) holds, and let $y \in \operatorname{dom} f^{IT^\infty}(x;\cdot)$. Then there exists $r \in \mathbb{R}$ with $(y,r) \in IT^\infty(\operatorname{epi} f, (x, f(x)))$. By Corollary 2.7, $(y,r) \in \operatorname{int} T^\infty(\operatorname{epi} f, (x, f(x)))$, and so $y \in \operatorname{int}\operatorname{dom} f^{T^\infty}(x;\cdot)$. Therefore $\operatorname{dom} f^{IT^\infty}(x;\cdot) \subset \operatorname{int}\operatorname{dom} f^{T^\infty}(x;\cdot)$.

Now let $y \in \operatorname{int}\operatorname{dom} f^{T^\infty}(x;\cdot)$. By (a), there exists $r \in \mathbb{R}$ such that $f^{T^\infty}(x;\cdot)$ is bounded above by r on a neighborhood of y, so that $(y,r) \in \operatorname{int} T^\infty(\operatorname{epi} f, (x, f(x))) = IT^\infty(\operatorname{epi} f, (x, f(x)))$. Therefore $y \in \operatorname{dom} f^{IT^\infty}(x;\cdot)$ and $\operatorname{int}\operatorname{dom} f^{T^\infty}(x;\cdot) \subset \operatorname{dom} f^{IT^\infty}(x;\cdot)$.

To prove part (a), let $y \in \operatorname{dom} f^{IT^\infty}(x;\cdot)$ with $-\infty < f^{T^\infty}(x;y) \le f^{IT^\infty}(x;y) < r$. By [3, Proposition 2.5, p. 12], it suffices to show that $f^{IT^\infty}(x;\cdot)$ is bounded above on a neighborhood of y. Let $(z,s) \in T(\operatorname{epi} f, (x, f(x)))$. Since $(y,r) \in IT^\infty(\operatorname{epi} f, (x, f(x)))$, by Proposition 2.3 there exists $\varepsilon > 0$ such that for all $\lambda > 0$, $t \in (0,\varepsilon)$, and $(v,s') \in B_\varepsilon(0) \times (-\varepsilon,\varepsilon)$ with $(x,f(x)) + t(z+v, s+s') \in \operatorname{epi} f$, there exists $t' \in (0,\lambda)$ such that

$$(x, f(x)) + t'(z + v + y + B_\varepsilon(0), s + s' + r + (-\varepsilon,\varepsilon)) \subset \operatorname{epi} f.$$

Let $(\hat{y},\hat{r}) \in B_{\varepsilon/2}(y) \times (r - \varepsilon/2, r + \varepsilon/2)$. Let $\lambda > 0$. Suppose $(v,s') \in B_{\varepsilon/2}(0) \times (-\varepsilon/2, \varepsilon/2)$ with $(x, f(x)) + t(z+v, s+s') \in \operatorname{epi} f$. Then with $t' \in (0,\lambda)$ chosen as above, we have

$$(x, f(x)) + t'(z + v + \hat{y} + B_{\varepsilon/2}(0), s + s' + \hat{r} + (-\varepsilon/2, \varepsilon/2))$$

$$\subset (x, f(x)) + t'(z + v + y + B_\varepsilon(0), s + s' + r + (-\varepsilon,\varepsilon)) \subset \operatorname{epi} f.$$

Therefore $(\hat{y},\hat{r}) \in IT^\infty(\operatorname{epi} f, (x, f(x)))$, and it follows that $f^{IT^\infty}(x;y') < r$ for all $y' \in y + B_{\varepsilon/2}(0)$. $\square$

4. Necessary optimality conditions for a nonsmooth program

We next consider the mathematical program

$$\min\{f(x) \mid g_i(x) \le 0,\ i \in J, x \in C\}, \tag{4.1}$$

where $f, g_i : X \to \bar{\mathbb{R}}$, $J = \{1, \cdots, m\}$, and $C \subset X$. In this section, we establish necessary conditions for local optimality in problem (4.1). These conditions will be stated in terms of subdifferentials associated with closed convex tangent cones that are contained in the contingent or adjacent cone.

Definition 4.1. (a) Let R_1 and R_2 be tangent cones. We will say that $R_1 \subset R_2$ if $R_1(S,x) \subset R_2(S,x)$ for all $S \subset X$ and $x \in S$.
(b) We will say that a tangent cone R has a certain property (e.g., "R is closed", or "R is convex") if $R(S,x)$ has that property for all $S \subset X$ and $x \in S$.
(c) Let $f : X \to \bar{\mathbb{R}}$ be finite at $x \in X$, and let R be a convex tangent cone such that $f^R(x;\cdot)$ is proper. We define the R-subdifferential of f at x by

$$\partial^R f(x) := \{p \in X^* \mid \langle p, y\rangle \le f^R(x;y) \quad \forall y \in X\}.$$

Remark 4.2. (a) In Definition 4.1(c), since $f^R(x;\cdot)$ is proper, we have $f^R(x;0)=0$. This means that

$$\partial^R f(x) = \{p \in X^* \mid \langle p, y\rangle \le f^R(x;y) - f^R(x;0)\, \forall y \in X\},$$

and so $\partial^R f(x)$ is $\partial(f^R(x;\cdot))(0)$, the subdifferential of the convex function $f^R(x;\cdot)$ at 0.
(b) Closed convex tangent cones contained in the contingent cone include, in particular, T^∞, P, and the Clarke tangent cone

$$C(S,x) := \{z \in X \mid \forall x^j \to_S x, \forall t_j \to 0^+, \exists \{z^j\} \to z \text{ with } x^j + t_j z^j \in S\}.$$

Closed convex tangent cones contained in the adjacent cone include A^∞, P, and C.

In deriving our optimality condtions, we will need to work with expressions of the form $\lambda f^R(x;y)$, where $\lambda \ge 0$. When $\lambda = 0$, we will follow the convention that $0f^R(x;y) = i_{\operatorname{dom} f^R(x;\cdot)}(y)$, so that $\partial^R(0f)(x) = (\operatorname{dom} f^R(x;\cdot))^\circ$. The equation $\partial^R(\lambda f)(x) = \lambda\partial^R f(x)$ will then hold for all $\lambda \ge 0$ if we define $0\partial f(x)$ to be $(\operatorname{dom} f^R(x;\cdot))^\circ$. Keeping this convention in mind, we can state the following result.

Theorem 4.3. *Let $\bar{x}$ be a local minimizer for (4.1), and define $I(\bar{x}) = \{i \in J \mid g_i(\bar{x}) = 0\}$. Assume that g_i is continuous at $\bar{x}$ for each $i \in J\backslash I(\bar{x})$, that $f^T(\bar{x};\cdot)$ and ${g_i}^A(\bar{x};\cdot)$, $i \in I(\bar{x})$, are proper, and that*

$$\operatorname{dom} f^{IT^\infty}(\bar{x};\cdot) \cap A^\infty(C,\bar{x}) \cap \cap_{i\in I(\bar{x})} \operatorname{dom} {g_i}^{IA^\infty}(\bar{x};\cdot) \ne \emptyset. \tag{4.2}$$

Suppose that $R_0 \subset T$, $R, R_i \subset A$ for $i \in I(\bar{x})$, are convex tangent cones containing the origin. There there exist $\lambda_i \ge 0$, $i \in I(\bar{x}) \cup \{0\}$, not all equal to zero, such that

$$0 \in \lambda_0 \partial^{R_0} f(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \partial^{R_i} g_i(\bar{x}) + (R(C,\bar{x}))^\circ. \tag{4.3}$$

If in addition

$$\operatorname{dom} f^{IT^\infty}(\bar{x};\dot{)} \cap A^\infty(C,\bar{x}) \cap \cap_{i\in I(\bar{x})} \{y \mid {g_i}^{IA^\infty}(\bar{x};y) < 0\} \ne \emptyset, \tag{4.4}$$

then (4.3) holds with $\lambda_0 = 1$.

Proof. For $S \subset X$, $x \in S$, let $D_0(S,x) = \operatorname{cl}(R_0(S,x) + T^\infty(S,x))$, $D(S,x) = \operatorname{cl}(R(S,x) + A^\infty(S,x))$, and $D_i(S,x) = \operatorname{cl}(R_i(S,x) + A^\infty(S,x))$, $i \in I(\bar{x})$. Then D and each D_i are nonempty closed, convex tangent cones with $R_0 \subset D_0$, $T^\infty \subset D_0$, $R \subset D$, $A^\infty \subset D$, and $R_i \subset D_i$, $A^\infty \subset D_i$ for $i \in I(\bar{x})$. In addition, we have $D_0 \subset T$, since

$$D_0(S,x) \subset \operatorname{cl}(T(S,x) + T^\infty(S,x)) = \operatorname{cl} T(S,x) = T(S,x).$$

Similarly, $D \subset A$ and $D_i \subset A$, $i \in I(\bar{x})$.

Next define

$$F(x) = \max\{f(x) + i_C(x) - f(\bar{x}), g_1(x), \ldots, g_m(x)\}.$$

Since $\bar{x}$ is a local minimizer of (4.1), $\bar{x}$ is also a local minimizer of F. Then for all $y \in X$, we have by (4.2) and Theorem 3.1 that

$$0 \le F^T(\bar{x};y) \le \max\{(f+i_C)^T(\bar{x};y), {g_i}^A(\bar{x};y), i \in I(\bar{x})\}$$

$$\le \max\{f^T(\bar{x};y) + i_{A(C,\bar{x})}(y), {g_i}^A(\bar{x};y), i \in I(\bar{x})\} \text{ (by Theorem 3.3)}$$
$$\le \max\{f^{D_0}(\bar{x};y) + i_{D(C,\bar{x})}(y), {g_i}^{D_i}(\bar{x};y), i \in I(\bar{x})\}.$$

Now let $q = |I(\bar{x})| + 1$, where $|I(\bar{x})|$ is the cardinality of $I(\bar{x})$. Define sets $S_1 = \{z \in \mathbb{R}^q \,|\, z \le 0\}$ and

$$S_2 = \{z \in \mathbb{R}^q \,|\, (f^{D_0}(\bar{x};y) + i_{D(C,\bar{x})}(y), {g_i}^{D_i}(\bar{x};y), i \in I(\bar{x})) \le z\},$$

where $\le$ denotes the coordinate-wise ordering on $\mathbb{R}^q$. Note that S_1 and S_2 are convex cones with $\operatorname{int} S_1 \cap S_2 = \emptyset$, so we may separate them with a hyperplane. Then there exist $\lambda_i \in \mathbb{R}$, $i \in \{0\} \cup I(\bar{x})$, at least one of which is nonzero, such that for all $z = (z_i) \in S_1$ and $y \in X$,

$$\sum_{i \in \{0\} \cup I(\bar{x})} \lambda_i z_i \le 0 \le \lambda_0((f^{D_0}(\bar{x};y) + i_{D(C,\bar{x})}(y)) + \sum_{i \in I(\bar{x})} \lambda_i {g_i}^{D_i}(\bar{x};y). \tag{4.5}$$

The left-hand inequality in (4.5) implies that each $\lambda_i \ge 0$. The right-hand inequality gives

$$0 \in \partial(\lambda_0(f^{D_0}(\bar{x};\cdot) + i_{D(C,\bar{x})}(\cdot)) + \sum_{i \in I(\bar{x})} \lambda_i {g_i}^{D_i}(\bar{x};\cdot))(0). \tag{4.6}$$

Using Proposition 3.4 and [5, Theorem 2.8] along with (4.2), we note that $f^{D_0}(\bar{x};\cdot)$ and each ${g_i}^{D_i}(\bar{x};\cdot)$ are continuous on the interiors of their domains with

$$D(C,\bar{x}) \cap \operatorname{int} \operatorname{dom} f^{D_0}(\bar{x};\cdot) \cap \cap_{i \in I(\bar{x})} \operatorname{int} \operatorname{dom} {g_i}^{D_i}(\bar{x};\cdot) \neq \emptyset. \tag{4.7}$$

We may then apply a formula for subdifferentials of sums of convex functions [3, p. 26] to obtain

$$\begin{aligned} 0 &\in \partial(\lambda_0 f)^{D_0}(\bar{x};\cdot)(0) + \sum_{i \in I(\bar{x})} \partial(\lambda_i g_i)^{D_i}(\bar{x};\cdot)(0) + (D(C,\bar{x}))^\circ \\ &= \lambda_0 \partial^{D_0} f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^{D_i} g_i(\bar{x}) + (D(C,\bar{x}))^\circ \\ &\subset \lambda_0 \partial^{R_0} f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^{R_i} g_i(\bar{x}) + (R(C,\bar{x}))^\circ, \end{aligned}$$

so that (4.3) holds.

Finally, suppose that (4.4) holds, and let v be an element of the set in (4.4). If $\lambda_0 = 0$ in (4.3), then (4.5) implies that

$$0 \le \sum_{i \in I(\bar{x})} \lambda_i {g_i}^{D_i}(\bar{x};y) \quad \forall y \in D(C,\bar{x}). \tag{4.8}$$

Since $v \in A^\infty(C,\bar{x}) \subset D(C,\bar{x})$ and at least one $\lambda_i > 0$, (4.8) gives

$$0 \le \sum_{i \in I(\bar{x})} \lambda_i {g_i}^{R_i}(\bar{x};v) < 0,$$

a contradiction. Therefore $\lambda_0 > 0$, and we conclude that (4.3) is satisfied with $\lambda_0 = 1$. $\square$

One notable feature of Theorem 4.3 is the fact that hypotheses (4.2) and (4.4) are independent of the choice of R and R_i. It is also important to observe that (4.2) and (4.4) do not require f or $g_i, i \in I(\bar{x})$, to be locally Lipschitzian. Actually, Theorem 4.3 takes on a simple form in the locally Lipschitzian case.

Corollary 4.4. *Let $\bar{x}$ be a local minimizer for (4.1), where we assume that f and each g_i are Lipschitzian near $\bar{x}$. Suppose that $R_0 \subset T$, $R, R_i \subset A$ for $i \in I(\bar{x})$, are convex tangent cones containing the origin. Then there exist $\lambda_i \geq 0$, $i \in I(\bar{x}) \cup \{0\}$, not all equal to zero, such that*

$$0 \in \lambda_0 \partial^{R_0} f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial^{R_i} g_i(\bar{x}) + (R(C, \bar{x}))^\circ. \tag{4.9}$$

If in addition

$$A^\infty(C, \bar{x}) \cap \cap_{i \in I(\bar{x})} \{y \mid {g_i}^{IA^\infty}(\bar{x}; y) < 0\} \neq \emptyset, \tag{4.10}$$

then (4.9) holds with $\lambda_0 = 1$.

Proof. For a function $h : X \to \bar{\mathbb{R}}$ that is finite at x and Lipschitzian near x, $\operatorname{dom} h^R(x; \cdot) = X$ for $R = T, A, IT, IA, T^\infty, A^\infty, IT^\infty, IA^\infty$. Hence 0 is an element of the intersection of (4.2), while (4.4) reduces to (4.10). $\square$

Our direct characterization of the convex kernel of the contingent cone in Theorem 4.3 enables us to contribute to a calculus of tangent cones and epiderivatives begun by Penot [6] in the 1980s. We believe that this calculus is worth revisiting and investigating further. The theorems are valid for large classes of sets and functions and do not require metric subregularity or local Lipschitz hypotheses.

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