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WEAK AND STRONG CONVERGENCE THEOREMS FOR TWO GENERIC GENERALIZED NONSPREADING MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we first obtain a weak convergence theorem of Mann's type iteration for finding a common attractive point of noncommutative two generic generalized nonspreading mappings in a Banach space. Next, we obtain a strong convergence theorem of Halpern's type iteration for finding a common attractive point of the mappings in a Banach space. Using these results, we get well-known and new weak and strong convergence theorems in a Hilbert space and a Banach space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. We denote by $A(T)$ the set of *attractive points* [37] of T, i.e., $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$. We know from [37] that $A(T)$ is closed and convex. This concept of attractive points in a Hilbert space was extended to that in a Banach space by Lin and Takahashi [19]. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty, where $F(T)$ is the set of fixed points of T. Furthermore, from Baillon [3] we know the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space. Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

In 2010, Kocourek, Takahashi and Yao [11] defined a broad class of nonlinear mappings in a Hilbert space: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [11] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

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for all $x, y \in C$; see also [26]. Such a mapping T is called (α, β)-*generalized hybrid*. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [16, 17] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [34] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Kocourek, Takahashi and Yao [12] extended the concept of generalized hybrid mappings in a Hilbert space to that in a Banach space. A mapping $T : C \to E$ is called *generalized nonspreading* if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}(1.2)\quad \alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\}\end{aligned}$$

for all $x, y \in C$; see also [23] . Takahashi [35] proved the following nonlinear ergodic theorem for commutative two generalized nonspreading mappings in a Banach space.

Theorem 1.1 ([35]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty, closed and convex sunny generalized nonexpansive retract of E. Let S and T be commutative generalized nonspreading mappings of C into itself with $F(S) \cap F(T) \neq \emptyset$ such that $\phi(Sx, u) \leq \phi(x, u)$ and $\phi(Tx, v) \leq \phi(x, v)$ for all $x \in C$ and $u \in F(S)$ and $v \in F(T)$, respectively. Let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $F(S) \cap F(T)$, where $q = \lim_{(k,l)\in D} RS^kT^l x$.

This theorem extended Kohsaka's ergodic theorem [13] in a Hilbert space to that in a Banach space. On the other hand, in 1953, Mann [21] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Banach space E and let $T : C \to C$ be a nonexpansive mapping. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of Mann iterative process for finding fixed points of nonexpansive mappings. In 1967, Halpern [4] gave an iteration process as follows: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. There are many investigations of Halpern iterative process for finding fixed points of nonexpansive mappings.

In this paper, we first obtain a weak convergence theorem of Mann's type iteration for finding a common attractive point of noncommutative two generic generalized nonspreading mappings in a Banach space. Next, we obtain a strong convergence theorem of Halpern's type iteration for finding a common attractive point of the mappings in a Banach space. Using these results, we get well-known and new weak and strong convergence theorems in a Hilbert space and a Banach space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\}$$

for every ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [27, 28, 32, 33]. The following result is also well known; see [32].

Lemma 2.1 ([32]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi \colon E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [10]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

The following results are in Xu [41] and Kamimura and Takahashi [10].

Lemma 2.2 ([41]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 ([10]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* [19] of T, i.e.,

$$A(T) = \{z \in E : \phi(z, Tx) \leq \phi(z, x),\ \forall x \in C\}.$$

The following result by Lin and Takahashi [19] is crucial in our paper.

Lemma 2.4 ([19]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called *generalized nonexpansive* [8] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* [24] if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [7, 8, 18] for more details. The following results are in Ibaraki and Takahashi [8].

Lemma 2.5 ([8]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.6 ([8])**.** *Let C be a nonempty and closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz\rangle \leq 0$ *for all* $y \in C$;

(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [15] proved the following results:

Lemma 2.7 ([15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*

(b) *C is a generalized nonexpansive retract of E;*

(c) *JC is closed and convex.*

Lemma 2.8 ([15])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*

(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Ibaraki and Takahashi [9] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.9 ([9])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.7 and 2.9.

Lemma 2.10 ([9])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Using Lemma 2.7, we also have the following result.

Lemma 2.11 ([35])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let $\{C_i : i \in I\}$ be a family of sunny generalized nonexpansive retracts of E such that $\cap_{i \in I} C_i$ is nonempty. Then $\cap_{i \in I} C_i$ is a sunny generalized nonexpansive retract of E.*

Let E be a smooth, strictly convex and reflexive Banach space. We make use of the following mapping V studied in Alber [1] and Kohsaka and Takahashi [14]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$. Ibaraki and Takahashi [6] proved the following lemma by using this mapping V.

Lemma 2.12 ([6])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let V be as in (2.5). Then*

$$V(x, x^*) + 2\langle y, Jx - x^*\rangle \leq V(x + y, x^*)$$

for all $x, y \in E$ and $x^* \in E^*$.

To prove one of our main results, we also need the following lemmas by Aoyama, Kimura, Takahashi and Toyoda [2] and Maingé [20].

Lemma 2.13 ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}\beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty}\gamma_n \le 0$. Suppose that*

$$s_{n+1} \le (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.14 ([20]). *Let $\{X_n\}$ be a sequence of real numbers. Assume that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, in other words, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : X_k < X_{k+1}\} \neq \emptyset$. Define a sequence $\{\tau(n)\}_{n\ge n_0}$ of natural numbers as follows:*

$$\tau(n) = \max\{k \le n : X_k < X_{k+1}\}, \quad \forall n \ge n_0.$$

Then, the followings hold:

(i) $\tau(n) \to \infty$ *as* $n \to \infty$;

(ii) $X_n \le X_{\tau(n)+1}$ *and* $X_{\tau(n)} < X_{\tau(n)+1}, \quad \forall n \ge n_0$.

Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. A mapping $T : C \to E$ is called *generic generalized nonspreading* [38] if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha+\beta+\gamma+\delta \ge 0$, $\alpha+\beta > 0$ and

$$\begin{aligned}\alpha\phi(Tx,Ty)+&\beta\phi(x,Ty) + \gamma\phi(Tx,y) + \delta\phi(x,y) \\ &\le \varepsilon\{\phi(Ty,Tx) - \phi(Ty,x)\} + \zeta\{\phi(y,Tx) - \phi(y,x)\}\end{aligned} \tag{2.6}$$

for all $x, y \in C$. We call such a mapping a generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading mapping. A generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading mapping $T : C \to E$ is generalized nonspreading in the sense of Kocourek, Takahashi and Yao [12] if $\alpha+\beta = -\gamma-\delta = 1$. In particular, putting $\alpha = 1$, $\beta = \delta = 0$, $\gamma = \varepsilon = -1$ and $\zeta = 0$ in (2.6), we obtain that

$$\phi(Tx,Ty) + \phi(Ty,Tx) \le \phi(Tx,y) + \phi(Ty,x)$$

for all $x, y \in C$. Such a mapping is nonspreading in the sense of Kohsaka and Takahashi [17]. A nonspreading mapping is obtained from a resolvent of a maximal monotone operator in a Banach space; see [17].

Takahashi, Wong and Yao [38] proved the following attractive point theorem for generic generalized nonspreading mappings in a Banach space by using the technique developed by [31].

Theorem 2.15 ([38]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a generic generalized nonspreading mapping of C into itself. Then the following are equivalent:*

(a) $A(T) \neq \emptyset$;

(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Additionally, if C is closed and convex, then the following are equivalent:

(a) *$F(T) \neq \emptyset$;*
(b) *$\{T^n x\}$ is bounded for some $x \in C$.*

3. Weak convergence theorems of Mann's type iteration

In this section, we first obtain a weak convergence theorem of Mann's type iteration for finding a common attractive point of noncommutative two generic generalized nonspreading mappings in a Banach space; see also [25]. The following proposition was proved by Takahashi, Wong and Yao [38].

Proposition 3.1 ([38])**.** *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty subset of E and let T be a generic generalized nonspreading mapping of C into E. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in A(T)$. Additionally, if C is closed and convex, then $z \in F(T)$.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $T : C \to E$ be a generic generalized nonspreading mapping. This mapping with $F(T) \neq \emptyset$ has the property that $\phi(u, Ty) \leq \phi(u, y)$ for all $u \in F(T)$ and $y \in C$. This property can be revealed by putting $x = u \in F(T)$ in (2.6). In fact, putting $x = u \in F(T)$ in (2.6), we have that

$$\begin{aligned} \alpha\phi(u, Ty) &+ \beta\phi(u, Ty) + \gamma\phi(u, y) + \delta\phi(u, y) \\ &\leq \varepsilon\{\phi(Ty, u) - \phi(Ty, u)\} + \zeta\{\phi(y, u) - \phi(y, u)\} \end{aligned}$$

and hence

$$(\alpha + \beta)\phi(u, Ty) \leq -(\gamma + \delta)\phi(u, y).$$

Since $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta > 0$, we have

$$\phi(u, Ty) \leq \frac{-(\gamma + \delta)}{\alpha + \beta}\phi(u, y) \leq \phi(u, y).$$

Similarly, putting $y = u \in F(T)$ in (2.6), we obtain that for any $x \in C$,

$$\begin{aligned} \alpha\phi(Tx, u) &+ \beta\phi(x, u) + \gamma\phi(Tx, u) + \delta\phi(x, u) \\ &\leq \varepsilon\{\phi(u, Tx) - \phi(u, x)\} + \zeta\{\phi(u, Tx) - \phi(u, x)\} \end{aligned}$$

and hence

$$\begin{aligned} \alpha\{\phi(Tx, u) &- \phi(x, u) + \phi(x, u)\} + \beta\phi(x, u) \\ &+ \gamma\{\phi(Tx, u) - \phi(x, u) + \phi(x, u)\} + \delta\phi(x, u) \\ &\leq (\varepsilon + \zeta)\{\phi(u, Tx) - \phi(u, x)\}. \end{aligned}$$

From $\alpha + \beta + \gamma + \delta \geq 0$, we have that

$$\begin{aligned} \alpha\{\phi(Tx, u) - \phi(x, u)\} &+ \gamma\{\phi(Tx, u) - \phi(x, u)\} \\ &\leq (\varepsilon + \zeta)\{\phi(u, Tx) - \phi(u, x)\} \end{aligned}$$

and hence

$$(\alpha + \gamma)\{\phi(Tx, u) - \phi(x, u)\} \leq (\varepsilon + \zeta)\{\phi(u, Tx) - \phi(u, x)\}. \tag{3.1}$$

Since $\phi(u, Tx) \leq \phi(u, x)$, we have that $\alpha + \gamma > 0$ together with $\varepsilon + \zeta \geq 0$ implies

$$\phi(Tx, u) \leq \phi(x, u). \tag{3.2}$$

Motivated by this property of T and $F(T)$, we can give the following definition. Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $B(T)$ the set of *skew-attractive points* [19] of T, i.e.,

$$B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z),\ \forall x \in C\}.$$

The difference between the set $A(T)$ and the set $B(T)$ is strongly related to the differences between left and right Bregman operators. See, for example, [22, 30]. Lin and Takahashi [19] proved the following result.

Lemma 3.2 ([19])**.** *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $B(T)$ is closed.*

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the *duality mapping* of T; see [5] and [39]. It is easy to show that if T is a mapping of C into itself, then T^* is a mapping of JC into itself. In fact, for any $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$ from the property of T. So we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then T^* is a mapping of JC into itself. Lin and Takahashi [19] also proved the following result by using the duality mapping T^* of T in a Banach space.

Lemma 3.3 ([19])**.** *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E and let T^* be the duality mapping of T. Then the following hold:*

(1) $JB(T) = A(T^*)$;
(2) $JA(T) = B(T^*)$.

In particular, $JB(T)$ is closed and convex.

Lemma 3.4. *Let E be a smooth and uniformly convex Banach space and let C be a nonempty and convex subset of E. Let S and T be generic generalized nonspreading mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$, $A(S) = B(S)$ and $A(T) = B(T)$. Let $R := R_{B(S)\cap B(T)}$ be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n Sx_n + (1 - \gamma_n)Tx_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1 \ \ \text{and} \ \ 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{Rx_n\}$ converges strongly to a point z of $B(S) \cap B(T)$.

Proof. From Lemmas 2.7 and 2.11, there exists the sunny generalized nonexpansive retraction R of E onto $B(S) \cap B(T)$. Put $T_n = \gamma_n S + (1 - \gamma_n)T$ for all $n \in \mathbb{N}$. We have that, for any $w \in B(S) \cap B(T)$,

$$\phi(T_n x_n, w) = \phi(\gamma_n Sx_n + (1 - \gamma_n)Tx_n, w)$$

$$\begin{aligned} &\le \gamma_n \phi(Sx_n, w) + (1-\gamma_n)\phi(Tx_n, w) \\ &\le \gamma_n \phi(x_n, w) + (1-\gamma_n)\phi(x_n, w) \\ &= \phi(x_n, w). \end{aligned} \tag{3.3}$$

Using (3.3), we have that

$$\begin{aligned} \phi(x_{n+1}, w) &= \phi(\alpha_n x_n + (1-\alpha_n)T_n x_n, w) \\ &\le \alpha_n \phi(x_n, w) + (1-\alpha_n)\phi(T_n x_n, w) \\ &\le \alpha_n \phi(x_n, w) + (1-\alpha_n)\phi(x_n, w) \\ &= \phi(x_n, w). \end{aligned}$$

Then, $\lim_{n\to\infty} \phi(x_n, w)$ exists. Since $\{\phi(x_n, w)\}$ is bounded, $\{x_n\}$ and $\{T_n x_n\}$ are bounded. Define $y_n = Rx_n$ for all $n \in \mathbb{N}$. Since $\phi(x_{n+1}, w) \le \phi(x_n, w)$ for all $w \in B(S) \cap B(T)$, from $y_n \in B(S) \cap B(T)$ we have

$$\phi(x_{n+1}, y_n) \le \phi(x_n, y_n). \tag{3.4}$$

From Lemma 2.6 and (3.4) we have

$$\begin{aligned} \phi(x_{n+1}, y_{n+1}) &= \phi(x_{n+1}, Rx_{n+1}) \\ &\le \phi(x_{n+1}, y_n) - \phi(Rx_{n+1}, y_n) \\ &\le \phi(x_{n+1}, y_n) \\ &\le \phi(x_n, y_n). \end{aligned}$$

So, $\phi(x_n, y_n)$ is a convergent sequence. We also have from (3.4) that, for all $m \in \mathbb{N}$,

$$\phi(x_{n+m}, y_n) \le \phi(x_n, y_n).$$

From $y_{n+m} = Rx_{n+m}$ and Lemma 2.6, we have

$$\phi(y_{n+m}, y_n) + \phi(x_{n+m}, y_{n+m}) \le \phi(x_{n+m}, y_n) \le \phi(x_n, y_n)$$

and hence

$$\phi(y_{n+m}, y_n) \le \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}).$$

Using Lemma 2.3, we have that

$$g(\|y_{n+m} - y_n\|) \le \phi(y_{n+m}, y_n) \le \phi(x_n, y_n) - \phi(x_{n+m}, y_{n+m}),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function such that $g(0) = 0$. Then, the properties of g yield that Rx_n converges strongly to an element z of $B(S) \cap B(T)$. □

Theorem 3.5. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and convex subset of E. Let S and T be generic generalized nonspreading mappings of C into itself such that $A(S) \cap A(T) \ne \emptyset$, $A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\big(\gamma_n Sx_n + (1-\gamma_n)Tx_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \le \gamma_n \le b < 1 \quad \text{and} \quad 0 < c \le \alpha_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $z \in A(S) \cap A(T)$, where $z = \lim_{n\to\infty} R_{B(S)\cap B(T)} x_n$. Additionally, if C is closed, then $\{x_n\}$ converges weakly to $z \in F(S) \cap F(T)$.

Proof. As in the proof of Lemma 3.4. for any $w \in B(S) \cap B(T)$, $\lim_{n\to\infty} \phi(x_n, w)$ exists. So, we have that the sequence $\{x_n\}$ is bounded. This implies that $\{T_n x_n\}$ is bounded. Put $r = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|T_n x_n\|\}$. Using Lemma 2.2, we have that there exists a strictly increasing, continuous and convex function $g : [0,\infty) \to [0,\infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. We have that, for any $w \in B(S) \cap B(T)$,

$$\begin{aligned}
\phi(x_{n+1}, w) &= \phi(\alpha_n x_n + (1-\alpha_n)T_n x_n, w)\\
&= \|\alpha_n x_n + (1-\alpha_n)T_n x_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)T_n x_n, Jw\rangle + \|w\|^2\\
&\le \alpha_n\|x_n\|^2 + (1-\alpha_n)\|T_n x_n\|^2 - \alpha_n(1-\alpha_n)g(\|T_n x_n - x_n\|)\\
&\quad - 2\alpha_n\langle x_n, Jw\rangle - 2(1-\alpha_n)\langle T_n x_n, Jw\rangle + \|w\|^2\\
&= \alpha_n(\|x_n\|^2 - 2\langle x_n, Jw\rangle + \|w\|^2)\\
&\quad + (1-\alpha_n)(\|T_n x_n\|^2 - 2\langle T_n x_n, Jw\rangle + \|w\|^2) - \alpha_n(1-\alpha_n)g(\|T_n x_n - x_n\|)\\
&= \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(T_n x_n, w) - \alpha_n(1-\alpha_n)g(\|T_n x_n - x_n\|)\\
&\le \alpha_n\phi(x_n, w) + (1-\alpha_n)\phi(x_n, w) - \alpha_n(1-\alpha_n)g(\|T_n x_n - x_n\|)\\
&= \phi(x_n, w) - \alpha_n(1-\alpha_n)g(\|T_n x_n - x_n\|).
\end{aligned}$$

Then, we obtain that

$$\alpha_n(1-\alpha_n)g(\|T_n x_n - x_n\|) \le \phi(x_n, w) - \phi(x_{n+1}, w).$$

From the assumption of $\{\alpha_n\}$, we have

$$\lim_{n\to\infty} g(\|T_n x_n - x_n\|) = 0. \tag{3.5}$$

From the properties of g, we have $\lim_{n\to\infty} \|T_n x_n - x_n\| = 0$. Using this, we have from Lemma 2.12 that, for any $w \in B(S) \cap B(T)$,

$$\begin{aligned}
\phi(x_n, w) &= V(x_n, Jw)\\
&= V(x_n - T_n x_n + T_n x_n, Jw)\\
&\le V(T_n x_n, Jw) + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle\\
&= \phi(T_n x_n, w) + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle\\
&= \phi(\gamma_n S x_n + (1-\gamma_n)T x_n, w) + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle\\
&\le \gamma_n\|Sx_n\|^2 + (1-\gamma_n)\|Tx_n\|^2 - \gamma_n(1-\gamma_n)g(\|Sx_n - Tx_n\|)\\
&\quad - 2\langle \gamma_n S x_n + (1-\gamma_n)T x_n, Jw\rangle + \|w\|^2 + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle\\
&= \gamma_n\phi(Sx_n, w) + (1-\gamma_n)\phi(Tx_n, w) - \gamma_n(1-\gamma_n)g(\|Sx_n - Tx_n\|)\\
&\quad + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle\\
&\le \gamma_n\phi(x_n, w) + (1-\gamma_n)\phi(x_n, w) - \gamma_n(1-\gamma_n)g(\|Sx_n - Tx_n\|)\\
&\quad + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle
\end{aligned}$$

$$= \phi(x_n, w) - \gamma_n(1-\gamma_n)g(\|Sx_n - Tx_n\|) + 2\langle x_n - T_n x_n, Jx_n - Jw\rangle$$

and hence

$$\gamma_n(1-\gamma_n)g(\|Sx_n - Tx_n\|) \le 2\langle x_n - T_n x_n, Jx_n - Jw\rangle.$$

Since $x_n - T_n x_n \to 0$ and $\{x_n\}$ is bounded, we have that $g(\|Sx_n - Tx_n|) \to 0$ and then $\|Sx_n - Tx_n\| \to 0$. Using this, we have that

$$\begin{aligned}\|x_n - Sx_n\| &= \|x_n - T_n x_n + T_n x_n - Sx_n\| \\ &\le \|x_n - T_n x_n\| + \|T_n x_n - Sx_n\| \\ &= \|x_n - T_n x_n\| + (1-\gamma_n)\|Tx_n - Sx_n\| \\ &\to 0.\end{aligned}$$

Similarly, we have that $\|x_n - Tx_n\| \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in E$. From Lemma 3.1, we have that v is a point of $A(S)\cap A(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We have that $u, v \in A(S)\cap A(T) = B(S)\cap B(T)$. Put $a = \lim_{n\to\infty}(\phi(x_n, u) - \phi(x_n, v))$. Since

$$\phi(x_n, u) - \phi(x_n, v) = 2\langle x_n, Jv - Ju\rangle + \|u\|^2 - \|v\|^2,$$

we have $a = 2\langle u, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$ and $a = 2\langle v, Jv - Ju\rangle + \|u\|^2 - \|v\|^2$. From these equalities, we obtain $\langle u - v, Ju - Jv\rangle = 0$. From Lemma 2.1, it follows that $u = v$. Therefore, $\{x_n\}$ converges weakly to an element u of $A(S)\cap A(T)$. On the other hand, we know from Lemma 3.4 that $\{R_{B(S)\cap B(T)}x_n\}$ converges strongly to an element z of $B(S)\cap B(T)$. From the property of $R_{A(S)\cap A(T)}$, we also have

$$\langle x_n - R_{B(S)\cap B(T)}x_n, JR_{B(S)\cap B(T)}x_n - Ju\rangle \ge 0.$$

Taking $n\to\infty$, we have $\langle u - z, Jz - Ju\rangle \ge 0$. So, we have $-\|u - z\|^2 \ge 0$ and hence $z = u$. This implies that $\{x_n\}$ converges weakly to $z \in A(S)\cap A(T) = B(S)\cap B(T)$, where $z = \lim_{n\to\infty} R_{B(S)\cap B(T)}x_n$.

Additionally, if C is closed, then C is closed and convex. Then, $\{x_n\}$ converges weakly to an element u of $F(S)\cap F(T)$. This completes the proof. □

Using Theorem 3.5 we can prove the following weak convergence theorems.

Theorem 3.6. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and convex subset of E. Let S be a generic generalized nonspreading mapping of C into itself such that $A(S) \neq \emptyset$ and $A(S) = B(S)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n, \quad \forall n \in \mathbb{N},$$

where $c, d \in \mathbb{R}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < c \le \alpha_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $z \in A(S)$, where $z = \lim_{n\to\infty} R_{B(S)}x_n$. Additionally, if C is closed, then $\{x_n\}$ converges weakly to $z \in F(S)$.

Proof. Putting $S = T$ and $\gamma_n = \frac{1}{2}$ in Theorem 3.5, we obtain the desired result from Theorem 3.5. □

Theorem 3.7. *Let E be a uniformly convex and uniformly smooth Banach space. Let $S, T : E \to E$ be generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ and $(\alpha', \beta', \gamma', \delta', \varepsilon', \zeta')$-generalized nonspreading mappings such that $\alpha + \gamma > 0$ and $\varepsilon + \zeta \geq 0$, and $\alpha' + \gamma' > 0$ and $\varepsilon' + \zeta' \geq 0$, respectively. Assume that $F(S) \cap F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $z \in F(S) \cap F(T)$.

Proof. We know that $\alpha + \gamma > 0$ together with $\varepsilon + \zeta \geq 0$ implies that $\phi(Sx, u) \leq \phi(x, u)$ for all $x \in E$ and $u \in F(S)$. Similarly, we also have

$$\phi(Tx, u) \leq \phi(x, u)$$

for all $x \in E$ and $u \in F(T)$. We also have that $A(T) = A(T) \cap E = F(T)$ and $B(T) = B(T) \cap E = F(T)$. Then $A(S) = B(S)$ and $A(T) = B(T)$. Therefore we have the desired result from Theorem 3.5. ☐

Using Theorem 3.5, we can prove the following theorem in a Hilbert space which was obtained by Takahashi [36].

Theorem 3.8 ([36])**.** *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let $S, T : C \to C$ be generic generalized hybrid mappings with $A(S) \cap A(T) \neq \emptyset$ and let P be the mertic projection of H onto $A(S) \cap A(T)$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

converges weakly to $z \in A(S) \cap A(T)$, where $z = \lim_{n\to\infty} P x_n$.

Proof. Since generalized hybrid mappings in a Hilbert space are in the class of generic generalized nonspreading mappings in a Banach space, we obtain the desired result from Theorem 3.5. ☐

4. Strong convergence theorems of Halpern's type iteration

In this section, we prove a strong convergence theorem of Halpern's type iteration for finding a common attractive point of noncommutative two generic generalized nonspreading mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space such that the duality mapping J is weakly sequentially continuous. Let C be a nonempty and convex subset of E. Let S and T be generic generalized nonspreading* *mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$, $A(S) = B(S)$ and $A(T) = B(T)$. Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)(\gamma_n S x_n + (1-\gamma_n)T x_n)\Big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\gamma_n\} \subset [0,1]$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < a \le \gamma_n \le b < 1 \quad \text{and} \quad 0 < c \le \beta_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to Ru, where R is a sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a point of $F(S) \cap F(T)$.

Proof. Put $T_n x_n = \gamma_n S x_n + (1-\gamma_n)T x_n$ and

$$z_n = \beta_n x_n + (1-\beta_n)\big(\gamma_n S x_n + (1-\gamma_n)T x_n\big)$$

for all $n \in \mathbb{N}$. Since C is convex, $\{z_n\}$ is a sequence in C. We have that, for any $w \in B(S) \cap B(T)$,

$$\begin{aligned}\phi(T_n x_n, w) &= \phi(\gamma_n S x_n + (1-\gamma_n)T x_n, w)\\ &\le \gamma_n \phi(S x_n, w) + (1-\gamma_n)\phi(T x_n, w)\\ &\le \beta_n \phi(x_n, w) + (1-\beta_n)\phi(x_n, w)\\ &= \phi(x_n, w)\end{aligned}$$

and hence

$$\begin{aligned}\phi(z_n, w) &= \phi(\beta_n x_n + (1-\beta_n)T_n x_n, w)\\ &\le \beta_n \phi(x_n, w) + (1-\beta_n)\phi(T_n x_n, w)\\ &\le \beta_n \phi(x_n, w) + (1-\beta_n)\phi(x_n, w)\\ &= \phi(x_n, w).\end{aligned}$$

Then we have that

$$\begin{aligned}\phi(x_{n+1}, w) &= \phi(\alpha_n u + (1-\alpha_n)z_n, w)\\ &\le \alpha_n \phi(u, w) + (1-\alpha_n)\phi(z_n, w)\\ &\le \alpha_n \phi(u, w) + (1-\alpha_n)\phi(x_n, w).\end{aligned}$$

Putting $K = \max\{\phi(x_1, w), \phi(u, w)\}$, we have that $\phi(x_n, w) \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\phi(x_1, w) \le K$. Suppose that $\phi(x_k, w) \le K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}\phi(x_{k+1}, w) &\le \alpha_k \phi(u, w) + (1-\alpha_k)\phi(x_k, w)\\ &\le \alpha_k K + (1-\alpha_k)K = K.\end{aligned}$$

Hence, by induction, we obtain that $\phi(x_n, w) \le K$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{z_n\}$ are bounded.

Put $r = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|Sx_n\|, \|Tx_n\|\}$. Using Lemma 2.2, we have that there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. We have that, for any $w \in B(S) \cap B(T)$,

$$\begin{aligned}
\phi(x_{n+1}, w) &= \phi(\alpha_n u + (1-\alpha_n)z_n, w) \\
&\le \alpha_n\phi(u, w) + (1-\alpha_n)\phi(z_n, w) \\
&\le \alpha_n\phi(u, w) + \phi(z_n, w) \\
&= \alpha_n\phi(u, w) + \|\beta_n x_n + (1-\beta_n)T_n x_n\|^2 \\
&\quad - 2\langle\beta_n x_n + (1-\beta_n)T_n x_n, Jw\rangle + \|w\|^2 \\
&\le \alpha_n\phi(u, w) + \beta_n\|x_n\|^2 + (1-\beta_n)\|T_n x_n\|^2 \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - T_n x_n\|) \\
&\quad - 2\langle\beta_n x_n + (1-\beta_n)T_n x_n, Jw\rangle + \|w\|^2 \\
&= \alpha_n\phi(u, w) + \beta_n(\|x_n\|^2 - 2\langle x_n, Jw\rangle + \|w\|^2) \\
&\quad + (1-\beta_n)(\|T_n x_n\|^2 - 2\langle T_n x_n, Jw\rangle + \|w\|^2) \\
&\quad - \beta_n(1-\beta_n)g(\|x_n - T_n x_n\|) \\
&= \alpha_n\phi(u, w) + \beta_n\phi(x_n, w) \\
&\quad + (1-\beta_n)\phi(T_n x_n, w) - \beta_n(1-\beta_n)g(\|x_n - T_n x_n\|) \\
&\le \alpha_n\phi(u, w) + \beta_n\phi(x_n, w) \\
&\quad + (1-\beta_n)\phi(x_n, w) - \beta_n(1-\beta_n)g(\|x_n - T_n x_n\|) \\
&= \alpha_n\phi(u, w) + \phi(x_n, w) - \beta_n(1-\beta_n)g(\|x_n - T_n x_n\|)
\end{aligned}$$

and hence

$$\beta_n(1-\beta_n)g(\|x_n - T_n x_n\|) \le \alpha_n\phi(u, w) + \phi(x_n, w) - \phi(x_{n+1}, w). \tag{4.1}$$

We also have that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n u + (1-\alpha_n)z_n - x_n\| \\
&= \|\alpha_n(u - x_n) + (1-\alpha_n)(z_n - x_n)\| \\
&\le \alpha_n\|u - x_n\| + (1-\alpha_n)\|z_n - x_n\| \\
&= \alpha_n\|u - x_n\| + (1-\alpha_n)\|\beta_n x_n + (1-\beta_n)T_n x_n - x_n\| \\
&= \alpha_n\|u - x_n\| + (1-\alpha_n)\|(1-\beta_n)(T_n x_n - x_n)\| \\
&= \alpha_n\|u - x_n\| + (1-\alpha_n)(1-\beta_n)\|T_n x_n - x_n\| \\
&\le \alpha_n\|u - x_n\| + (1-\beta_n)\|T_n x_n - x_n\|.
\end{aligned} \tag{4.2}$$

Define $X_n = \phi(x_n, z_0)$, where $z_0 = Ru = R_{B(S)\cap B(T)}u$. Our aim is to show that $X_n \to 0$. The rest of the proof is divided into two cases.

Case (A). Suppose that there exists a natural number N such that $X_{n+1} \le X_n$ for all $n \ge N$. In this case, the sequence $\{X_n\}$ is convergent. It holds from $\alpha_n \to 0$,

$0 < c \leq \beta_n \leq d < 1$ and (4.1) that $g(\|x_n - T_n x_n\|) \to 0$ and then

$$\|x_n - T_n x_n\| \to 0 \tag{4.3}$$

as $n \to \infty$. As in the proof of Theorem 3.5, we have that $Sx_n - Tx_n \to 0$. Then we get that

$$\begin{aligned}\|x_n - Sx_n\| &= \|x_n - T_n x_n + T_n x_n - Sx_n\| \\ &\leq \|x_n - T_n x_n\| + \|T_n x_n - Sx_n\| \\ &= \|x_n - T_n x_n\| + (1-\gamma_n)\|Tx_n - Sx_n\| \\ &\to 0.\end{aligned}$$

Similarly, we have that $\|x_n - Tx_n\| \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in E$. From Lemma 3.1, we have that v is a point of $A(S) \cap A(T)$. Using this, we prove that

$$\limsup_{n\to\infty} \langle u - z_0, Jx_n - Jz_0\rangle \leq 0. \tag{4.4}$$

Since $\{x_n\}$ is bounded, without loss of generality, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, Jx_n - Jz_0\rangle = \lim_{i\to\infty}\langle u - z_0, Jx_{n_i} - Jz_0\rangle$$

and $x_{n_i} \rightharpoonup v$ for some $v \in E$. Since $x_{n_i} \rightharpoonup v \in A(S) \cap A(T)$ and J is weakly sequentially continuous, we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z_0, Jx_n - Jz_0\rangle &= \lim_{i\to\infty}\langle u - z_0, Jx_{n_i} - Jz_0\rangle \\ &= \langle u - z_0, Jv - Jz_0\rangle \leq 0.\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}X_{n+1} &= \phi(x_{n+1}, z_0) \\ &= \phi(\alpha_n u + (1-\alpha_n)z_n, z_0) \\ &= V(\alpha_n u + (1-\alpha_n)z_n, Jz_0) \\ &\leq V(\alpha_n u + (1-\alpha_n)z_n - \alpha_n(u - z_0), Jz_0) \\ &\quad - 2\langle -\alpha_n(u - z_0), Jx_{n+1} - Jz_0\rangle \\ &= V(\alpha_n z_0 + (1-\alpha_n)z_n, Jz_0) \\ &\quad + 2\alpha_n\langle u - z_0, Jx_{n+1} - Jz_0\rangle \\ &= \phi(\alpha_n z_0 + (1-\alpha_n)z_n, z_0) \\ &\quad + 2\alpha_n\langle u - z_0, Jx_{n+1} - Jz_0\rangle \\ &\leq \alpha_n\phi(z_0, z_0) + (1-\alpha_n)\phi(z_n, z_0) \\ &\quad + 2\alpha_n\langle u - z_0, Jx_{n+1} - Jz_0\rangle \\ &= (1-\alpha_n)\phi(z_n, z_0) + 2\alpha_n\langle u - z_0, Jx_{n+1} - Jz_0\rangle \\ &\leq (1-\alpha_n)\phi(x_n, z_0) + 2\alpha_n\langle u - z_0, Jx_{n+1} - Jz_0\rangle.\end{aligned}$$

Since $x_{n+1} - x_n \to 0$ and E is uniformly smooth, we have that $Jx_{n+1} - Jx_n \to 0$. Using (4.4) and Lemma 2.13, we have that $X_n \to 0$. This completes the proof for Case (A).

Case (B). Suppose that there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : X_k < X_{k+1}\} \ne \emptyset$. Define

$$\tau(n) = \max\{k \le n : X_k < X_{k+1}\}$$

for all $n \ge n_0$. From Lemma 2.14, the followings hold:

$$\tau(n) \to \infty \text{ as } n \to \infty; \tag{4.5}$$

$$X_n \le X_{\tau(n)+1} \quad \text{and} \quad X_{\tau(n)} < X_{\tau(n)+1}, \quad \forall n \ge n_0. \tag{4.6}$$

From (4.6), it is sufficient to prove that $X_{\tau(n)+1} = \phi(x_{\tau(n)+1}, z_0) \to 0$. From (4.1), we have that

$$\begin{aligned} &\beta_{\tau(n)}(1-\beta_{\tau(n)})g(\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\|) \\ &\qquad \le \alpha_{\tau(n)}\phi(u_{\tau(n)}, z_0) + \phi(x_{\tau(n)}, z_0) - \phi(x_{\tau(n)+1}, z_0) \end{aligned} \tag{4.7}$$

for all $n \ge n_0$. From (4.6), we have that $g(\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\|) \to 0$ and hence

$$\|x_{\tau(n)} - T_{\tau(n)}x_{\tau(n)}\| \to 0.$$

From (4.2), we obtain that

$$\|x_{\tau(n)+1} - x_{\tau(n)}\| \to 0. \tag{4.8}$$

For $z_0 = R_{B(S)\cap B(T)}u$, let us show that $\limsup_{n\to\infty}\langle z_0 - u, Jx_{\tau(n)} - Jz_0\rangle \ge 0$. Put

$$l = \limsup_{n\to\infty}\langle z_0 - u, Jx_{\tau(n)} - Jz_0\rangle.$$

Without loss of generality, there exists a subsequence $\{x_{\tau(n_i)}\}$ of $\{x_{\tau(n)}\}$ such that $l = \lim_{i\to\infty}\langle z_0 - u, Jx_{\tau(n_i)} - Jz_0\rangle$ and $\{x_{\tau(n_i)}\}$ converges weakly to some point $w \in E$. As in the proof of Case (A), we have that $w \in A(S) \cap A(T)$ and hence

$$l = \lim_{i\to\infty}\langle z_0 - u, Jx_{\tau(n_i)} - Jz_0\rangle = \langle z_0 - u, Jw - Jz_0\rangle \ge 0. \tag{4.9}$$

As in the proof of Case (A), we also have that

$$\begin{aligned} X_{\tau(n)+1} &= \phi(x_{\tau(n)+1}, z_0) \\ &= \phi(\alpha_{\tau(n)}u + (1-\alpha_{\tau(n)})z_{\tau(n)}, z_0) \\ &= V(\alpha_{\tau(n)}u + (1-\alpha_{\tau(n)})z_{\tau(n)}, Jz_0) \\ &\le V(\alpha_{\tau(n)}u + (1-\alpha_{\tau(n)})z_{\tau(n)} - \alpha_{\tau(n)}(u - z_0), Jz_0) \\ &\quad - 2\langle -\alpha_{\tau(n)}(u - z_0), Jx_{\tau(n)+1} - Jz_0\rangle \\ &= V(\alpha_{\tau(n)}z_0 + (1-\alpha_{\tau(n)})z_{\tau(n)}, Jz_0) \\ &\quad + 2\alpha_{\tau(n)}\langle u - z_0, Jx_{\tau(n)+1} - Jz_0\rangle \\ &= \phi(\alpha_{\tau(n)}z_0 + (1-\alpha_{\tau(n)})z_{\tau(n)}, z_0) \\ &\quad + 2\alpha_{\tau(n)}\langle u - z_0, Jx_{\tau(n)+1} - Jz_0\rangle \\ &\le \alpha_{\tau(n)}\phi(z_0, z_0) + (1-\alpha_{\tau(n)})\phi(z_{\tau(n)}, z_0) \\ &\quad + 2\alpha_{\tau(n)}\langle u - z_0, Jx_{\tau(n)+1} - Jz_0\rangle \\ &= (1-\alpha_{\tau(n)})\phi(x_{\tau(n)}, z_0) \\ &\qquad + 2\alpha_{\tau(n)}\langle u - z_0, Jx_{\tau(n)+1} - Jz_0\rangle. \end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$\alpha_{\tau(n)}\phi(x_{\tau(n)}, z_0) \leq 2\alpha_{\tau(n)}\langle u - z_0, Jx_{\tau(n)+1} - Jz_0\rangle.$$

Since $\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}\phi(x_{\tau(n)}, z_0) &\leq 2\langle u - z_0, Jx_{\tau(n)+1} - Jz_0\rangle \\ &= 2(\langle u - z_0, Jx_{\tau(n)+1} - Jx_{\tau(n)}\rangle + \langle u - z_0, Jx_{\tau(n)} - Jz_0\rangle).\end{aligned}$$

Using $x_{\tau(n)+1} - x_{\tau(n)} \to 0$ and (4.9), we have that

$$\limsup_{n\to\infty} \phi(x_{\tau(n)}, z_0) \leq 0$$

and hence $x_{\tau(n)} \to z_0$. We have from (4.8) that $x_{\tau(n)+1} \to z_0$ as $n \to \infty$. Using (4.6), we obtain that

$$\phi(x_n, z_0) \leq \phi(x_{\tau(n)+1}, z_0) \to 0$$

as $n \to \infty$. This implies that $x_n \to z_0$.

Additionally, if C is closed, then C is a closed and convex subset of E. From $A(S) \cap A(T) \cap C \subset F(S) \cap F(T)$, $\{x_n\}$ converges strongly to a point of $F(S) \cap F(T)$. This completes the proof. □

Remark We know that the duality mappings J on the sequence spaces $l^p, 1 < p < \infty$ and smooth finite dimensional Banach spaces are weakly sequentially continuous. While the duality mapping J in l^p is demicontinuous and the duality mapping J_ϕ with the gauge function $\phi(t) = t^{p-1}$ is weakly sequentially continuous; see, for example, [29]. However, we do not know whether Theorem 4.1 holds or not without assuming that J is weakly sequentially continuous.

Using Theorem 4.1, we can obtain the following strong convergence theorems in a Banach space.

Theorem 4.2. *Let E be a uniformly smooth and uniformly convex Banach space such that the duality mapping J is weakly sequentially continuous. Let C be a nonempty and convex subset of E. Let S and T be generalized nonspreading mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$, $A(S) = B(S)$ and $A(T) = B(T)$. Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)(\gamma_n S x_n + (1 - \gamma_n)T x_n)), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{\gamma_n\} \subset [0, 1]$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \beta_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to Ru, where R is a sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a point of $F(S) \cap F(T)$.

Proof. Since generalized nonspreading mappings are generic generalized nonspreading mappings, we obtain the desired result from Theorem 4.1. □

Theorem 4.3. *Let E be a uniformly smooth and uniformly convex Banach space such that the duality mapping J is weakly sequentially continuous. Let C be a nonempty and convex subset of E. Let S be a generic generalized nonspreading mapping of C into itself such that $A(S) \neq \emptyset$ and $A(S) = B(S)$. Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)Sx_n\Big), \quad \forall n \in \mathbb{N},$$

where $c, d \in \mathbb{R}$, $\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset [0,1]$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \ \text{and} \ \ 0 < c \leq \beta_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to Ru, where R is a sunny generalized nonexpansive retraction of E onto $B(S)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a point of $F(S)$.

Proof. Putting $S = T$ and $\gamma_n = \frac{1}{2}$ in Theorem 4.1, we obtain the desired result from Theorem 4.1.. □

As in the proof of Theorem 3.7, from Theorem 4.1 we get the following theorem.

Theorem 4.4. *Let E be a uniformly smooth and uniformly convex Banach space such that the duality mapping J is weakly sequentially continuous. Let S and T be generic $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta)$ and $(\alpha',\beta',\gamma',\delta',\varepsilon',\zeta')$-generalized nonspreading mappings of E into itself such that $\alpha+\gamma > 0$ and $\varepsilon+\zeta \geq 0$, and $\alpha'+\gamma' > 0$ and $\varepsilon'+\zeta' \geq 0$, respectively. Assume that $F(S) \cap F(T) \neq \emptyset$ and let R be the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Let $u \in E$ and define a sequence $\{x_n\}$ in E as follows: $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)(\gamma_n Sx_n + (1-\gamma_n)Tx_n)\Big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\gamma_n\} \subset [0,1]$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < a \leq \gamma_n \leq b < 1 \ \ \text{and} \ \ 0 < c \leq \beta_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to Ru, where R is a sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$.

As in the proof of Theorem 3.8, from Theorem 4.1 we can prove the following theorem in a Hilbert space which was obtained by Takahashi [36].

Theorem 4.5 ([36])**.** *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let S and T be generalized hybrid mappings of C into itself with $A(S) \cap A(T) \neq \emptyset$. Given $x_1 \in C$ and $u \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)(\gamma_n Sx_n + (1-\gamma_n)Tx_n)\Big)$$

for all $n \in \mathbb{N}$, where $a, b, c, d \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\gamma_n\} \subset [0,1]$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \beta_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{A(S)\cap A(T)}u$, where $P_{A(S)\cap A(T)}$ is the metric projection from H onto $A(S) \cap A(T)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}u$, where $P_{F(S)\cap F(T)}$ is the metric projection from H onto $F(S) \cap F(T)$.

Using Theorem 4.5, we have the following result for nonexpansive and hybrid mappings in a Hilbert space.

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let S and T be nonexpansive and hybrid mappings of C into itself, respectively, with $A(S) \cap A(T) \neq \emptyset$. Given $x_1 \in C$ and $u \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)\big(\gamma_n S x_n + (1-\gamma_n) T x_n\big)\Big)$$

for all $n \in \mathbb{N}$, where $a, b, c, d \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{\gamma_n\} \subset [0,1]$ satisfy the following:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \beta_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $P_{A(S)\cap A(T)}u$, where $P_{A(S)\cap A(T)}$ is the metric projection from H onto $A(S) \cap A(T)$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a point of $F(S) \cap F(T)$.

Proof. Since nonexpansive mappings and hybrid mappings are contaiined in the class of generalized hybrid mappings, we obtain the desired result from Theorem 4.5. ☐

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