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ON THE RESIDUALITY OF CERTAIN CLASSES OF CONVEX FUNCTIONS

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Abstract. This note examines the residuality of various classes of convex functions, including the classes of functions that are respectively uniformly convex, strongly coercive or bounded on bounded sets.

1. Introduction and preliminaries

Minimizing functions is of fundamental importance in mathematics. So it is often first asked whether a function f attains a global minimum and then if it does, it is of interest to know whether (and how strongly or fast) (x_n) converges to x when f attains its minimum at x and $f(x_n) \to f(x)$. As in the recent work [10], our focus is on the entire class of proper lower semicontinuous convex functions on a Banach space X, which we denote by $\Gamma(X)$. In particular, we will examine whether certain classes of functions in $\Gamma(X)$ are *residual*, that is contain a dense G_δ-set, with respect to the Attouch-Wets topology. We work primarily with a metric based on Moreau envelopes, see [10], that is compatible with the Attouch-Wets topology. The primary motivation for this note comes from [10].

An important earlier work in this general topic is by Beer and Lucchetti who showed a generic well-posed minimization result for convex functions in [4]. The monograph [11] provides an overview of the convex function case, while [8] looks at more general settings. A forerunner to these works is the paper of Fabian *et. al.* [9]. That paper used category arguments for norms to obtain "Asplund Averaging" results, and in particular shows that various classes of rotund norms, including uniformly convex norms, are residual among all equivalent norms when one such norm exists on the Banach space.

Our aim is to examine the degree to which the residuality results for uniformly convex norms carry over to uniformly convex functions and related classes. To do this, we will build upon work initiated by Zălinescu on the study of uniformly convex functions [16, 17]. The study of residuality of classes of functions related to various forms of strict convexity is not new. Indeed, an important work of Butnariu *et al.* [7] establishes residuality results for strictly and totally convex functions in quite general settings, and moreover demonstrates the importance of the latter result for several types of optimization problems (see also the references in [7]). There is a

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broad spectrum of important work related to generic optimization, and we refer the reader to two such works by Reich and Zaslavski [13, 14] for additional context in this topic.

Some notation and results that we will need are as follows. Throughout, X will denote a real Banach space unless specified otherwise. We denote the closed unit ball of X by B_X, that is $B_X := \{x : \|x\| \leq 1\}$. Following [10], we define the *Moreau envelope* of $f \in \Gamma(X)$ by

$$e_\lambda f(x) := \inf_y \left\{ f(y) + \frac{1}{2\lambda}\|y-x\|^2 \right\}. \tag{1.1}$$

In particular, our attention will focus upon the case when $\lambda = 1$, so we will exclusively use $e_1 f$. One of the nice properties of $e_1 f$ is that it is a convex function that is bounded on bounded sets, so one can use natural metrics associated with uniform convergence on bounded sets for Moreau envelopes of functions. Given $f, g \in \Gamma(X)$, we let d_n be defined by $d_n(f,x) = \sup\{|e_1 f(x) - e_1 g(x)| : \|x\| \leq n\}$, and then we define the metric d on $\Gamma(X)$ by

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{d_n(e_1 f, e_1 g), 1\}. \tag{1.2}$$

This is a natural metric for uniform convergence on bounded sets of Moreau envelopes, another such, equivalent, metric is found in [10].

We next describe another compatible Attouch-Wets metric on $\Gamma(X)$; see [3, Chapter 3]. Given any Banach space X, we consider the norm $\|\cdot\|_\infty$ defined on $X \times \mathbb{R}$ by $\|(x,t)\|_\infty = \max\{\|x\|, |t|\}$. For any $\tilde{x} = (x,t) \in X \times \mathbb{R}$ and nonempty set $A \subset X \times \mathbb{R}$, we define the distance from $\tilde{x}$ to A by

$$\rho(\tilde{x}, A) = \inf\{\|(x,t) - (a,s)\|_\infty : (a,s) \in A\}. \tag{1.3}$$

Given functions $f, g \in \Gamma(X)$, a compatible *Attouch-Wets metric*, d_{AW} can be defined using distances to the epigraphs of f and g in $X \times \mathbb{R}$ as follows

$$d_{AW}(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\left\{ 1, \sup_{\|\tilde{x}\|_\infty \leq n} |\rho(\tilde{x}, \operatorname{epi} f) - \rho(\tilde{x}, \operatorname{epi} g)| \right\}. \tag{1.4}$$

See [3, p. 79] for the compatibility of the metric d_{AW} with the topology τ_{AW} on $\Gamma(X)$, and see [3, Exercise 6, p. 263] for the compatibility of the Moreau envelope metric d with τ_{AW} on $\Gamma(X)$. Moreover, it is shown in [3, Exercise 6, p. 241], that $(\Gamma(X), \tau_{AW})$ is a completely metrizable topological space.

The paper [10] uses a Moreau envelope metric similar to that in (1.2) as a natural and useful approach to study questions of category in $\Gamma(\mathbb{R}^n)$. In this direction, [10, Proposition 3.5] shows that $\Gamma(\mathbb{R}^n)$ is complete in that metric. In contrast to this, while $(\Gamma(\mathbb{R}), \tau_{AW})$ is completely metrizable as just mentioned, $(\Gamma(\mathbb{R}), d_{AW})$ is not a complete metric space. Indeed, consider the constant functions f_n defined by $f_n(t) = -n$ for all $t \in \mathbb{R}$. Then (f_n) is a Cauchy sequence, but it does not converge in $(\Gamma(\mathbb{R}), d_{AW})$.

Given $f \in \Gamma(X)$, we define its Fenchel conjugate f^* on X^* by

$$f^*(\phi) = \sup\{\phi(x) - f(x) : x \in X\}, \quad \text{for } \phi \in X^*.$$

We will say f is *coercive* if $\liminf_{\|x\|\to\infty} f(x) = \infty$, and more stringently we will say f is *strongly coercive* if $\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = \infty$. The book [17] is a beautiful modern work on convex functions in generality, and for those focusing on Hilbert spaces, [2] does a wonderful job presenting convex functions. We next include some basic results that we will use. A very concise and informative overview of properties of convex functions can be found in the paper [1], in particular, a simple proof of the following fact is contained therein.

Fact 1.1. *Suppose f is a proper lower semicontinuous convex function on a Banach space.*

(a) *f is strongly coercive if and only if f^* is bounded on bounded sets.*
(b) *f is bounded on bounded sets if and only if f^* is strongly coercive.*

We will say a function $f \in \Gamma(X)$ is *uniformly convex* if for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) \geq \delta \text{ whenever } x, y \in \operatorname{dom} f, \text{ and } \|x-y\| \geq \epsilon.$$

We note that a continuous convex function is *uniformly smooth* if it has a uniformly continuous derivative; see [5, Proposition 4.2.14]. Some results on uniformly convex functions that we will use frequently are the following.

Theorem 1.2. *Suppose f is a proper lower semicontinuous convex function on a Banach space.*

(a) *f is uniformly convex if and only if f^* is uniformly smooth.*
(b) *f is uniformly smooth if and only if f^* is uniformly convex.*

The previous theorem can be found in [17, Theorems 3.5.10 and 3.5.12] while the next theorem follows from [17, Propositions 3.6.3 and 3.6.4]. For this, will say a function $f \in \Gamma(X)$ is *uniformly convex on bounded sets* if for each bounded set $B \subset X$ and for each $\epsilon > 0$, there exists $\delta > 0$ depending on ϵ and B such that

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) \geq \delta \text{ whenever } x, y \in \operatorname{dom} f \cap B, \|x-y\| \geq \epsilon.$$

Theorem 1.3. *Suppose f is a proper lower semicontinuous convex function on a Banach space.*

(a) *f is strongly coercive, bounded and uniformly convex on bounded sets if and only if f^* is strongly coercive and uniformly smooth on bounded sets.*
(b) *f is strongly coercive and uniformly smooth on bounded sets if and only if f^* is strongly coercive, bounded and uniformly convex on bounded sets.*

The key reason why the previous theorem needs extra assumptions on f when compared to Theorem 1.2 is that a uniformly convex function is automatically strongly coercive [17, Proposition 3.5.8].

The following observations are also well-known (see [3, 5]).

Fact 1.4. *Let X be a Banach space.*

(a) *The strongly coercive functions are dense in $(\Gamma(X), \tau_{AW})$.*

(b) *Suppose X has a uniformly convex norm, then the functions that are uniformly convex on bounded sets are dense in* $(\Gamma(X), \tau_{AW})$.

Proof. Let $f \in \Gamma(X)$. For each $n \geq 1$, define f_n by $f_n(x) = f(x) + \frac{1}{n}\|x\|^2$. Then f_n is strongly coercive, and the sequence (f_n) converges to f uniformly on bounded subsets of the domain of f. If additionally, $\|\cdot\|$ is uniformly convex, then f_n is uniformly convex on bounded sets because $\|\cdot\|^2$ is a function that is uniformly convex on bounded sets (see [5, Example 5.3.11]).

□

Recall that a function f on X is said to attain its *strong minimum* at $\bar{x}$ if for each $\epsilon > 0$, there exists $\delta > 0$ so that $\|x - \bar{x}\| < \epsilon$ whenever $f(x) < f(\bar{x}) + \delta$. We will say a collection of functions $\{f_\alpha\}_{\alpha \in A}$ attain their *strong minimums uniformly* at $\{x_\alpha\}_{\alpha \in A}$ if for each $\epsilon > 0$, there exists $\delta > 0$ (independent of α) so that $\|x - x_\alpha\| < \epsilon$ whenever $f_\alpha(x) < f_\alpha(x_\alpha) + \delta$. The observations listed in the following result are elementary, and their proofs are included for completeness. In particular, we note that (c) will be used in Theorem 2.10 to show some classes of convex functions attaining certain types of strong minimums are *meagre*, where by meagre we mean a set contained in a countable union of closed nowhere dense sets.

Fact 1.5. (a) *There exists a sequence $(f_n) \subset \Gamma(\mathbb{R})$ that attain their strong minimums at (x_n) and (f_n) converges uniformly on bounded sets to some function $f \in \Gamma(\mathbb{R})$, but (x_n) is not bounded.*

(b) *Let X be a Banach space, and suppose $(f_n) \subset \Gamma(X)$ is a sequence of functions that attain their minimums at (x_n) and that the sequence (f_n) converges uniformly on bounded sets to a coercive function f. Then (x_n) is bounded. However, even when $X = \mathbb{R}$, (x_n) may not necessarily converge.*

(c) *Let X be a Banach space, and suppose $(f_n) \subset \Gamma(X)$ is a sequence functions that attain their strong minimums uniformly at (x_n). Suppose further that there is an $\alpha \in \mathbb{R}$ such that $f_n(x_n) \geq \alpha$ for all n, and an $f \in \Gamma(X)$ such that (f_n) converges uniformly on bounded sets to f. Then f attains its strong minimum at some $\bar{x} \in X$ and (x_n) converges to $\bar{x}$.*

Proof. (a) Let $f_n(x) = \frac{1}{n^2}|x - n|$, then f_n attains its strong minimum at $x = n$ and (f_n) converges uniformly on bounded sets to $f(x) = 0$.

(b) By translation we may assume $0 \in \operatorname{dom} f$. Because f is coercive we find a number $M > 0$ so that $f(x) \geq f(0) + 3$ whenever $\|x\| \geq M$. Because (f_n) converges uniformly on bounded sets to f, we choose $N > 0$ so that $|f_n(x) - f(x)| < 1$ for all $\|x\| \leq M$ and $n > N$. Then $f_n(0) < f(0) + 1$ and $f_n(x) > f(0) + 2$ for $\|x\| = M$ and $n > N$. The convexity of f_n then implies $f_n(x) \geq f(0) + 2$ for $\|x\| \geq M$ and $n > N$. Consequently $\|x_n\| \leq M$ for $n > N$.

An example where (x_n) does not converge is given by

$$f_n(t) = \max\left\{\frac{(-1)^n}{n}t, |t| - 1 - \frac{1}{n}\right\}$$

which attains its strong min at $x_n = (-1)^{n+1}$ and the sequence (f_n) converges uniformly to f defined by $f = \max\{|t| - 1, 0\}$

(c) Suppose (f_n) attain their strong minimums at (x_n) uniformly. This implies there exists $\epsilon > 0$ so that

$$\|x - x_n\| \geq 1 \text{ implies } f_n(x) \geq f_n(x_n) + \epsilon \text{ for each } n \in \mathbb{N}.$$

The convexity of f_n then implies that

$$f_n(x) \geq f_n(x_n) + \|x - x_n\|\epsilon \quad \text{whenever} \quad \|x - x_n\| \geq 1.$$

In particular, $f_n(0) \geq f_n(x_n) + \epsilon\|x_n\| \geq \alpha + \epsilon\|x_n\|$ for each n such that $\|x_n\| \geq 1$. Because $(f_n(0))$ is a convergent sequence this implies (x_n) is bounded. Thus we fix $K > 0$ so that $\|x_n\| \leq K$ for all $n \in \mathbb{N}$.

We will now show that (x_n) is a Cauchy sequence. For this, let $\epsilon > 0$. The uniform strong minimum property means we can choose $\delta > 0$ independent of $n \in \mathbb{N}$ so that

$$\|x - x_n\| < \epsilon \text{ whenever } n \text{ is such that } f_n(x) \leq f_n(x_n) + \delta. \tag{1.5}$$

Now choose $N > 0$ so that

$$|f_n(x) - f(x)| < \frac{\delta}{4} \text{ for all } n > N, \text{ and } \|x\| \leq K.$$

Then for $n, m > N$ we have

$$f_n(x_m) < f(x_m) + \frac{\delta}{4} < f_m(x_m) + \frac{\delta}{2} \leq f_m(x_n) + \frac{\delta}{2} \leq f(x_n) + \frac{3\delta}{4} \leq f_n(x_n) + \delta.$$

By (1.5), this implies $\|x_n - x_m\| < \epsilon$ for all $n, m > N$ and so (x_n) is a Cauchy sequence as desired.

Now let $\bar{x} = \lim_{n\to\infty} x_n$. Then $f(\bar{x}) = \lim_{n\to\infty} f_n(\bar{x}) \geq \liminf_{n\to\infty} f_n(x_n)$. On the other hand, the lower semicontinuity of f and the uniform convergence of (f_n) to f on KB_X ensures that

$$f(\bar{x}) \leq \liminf f(x_n) = \liminf_{n\to\infty} f_n(x_n).$$

Finally, suppose that $x \in X$ is such that $f(x) < f(\bar{x}) + \frac{\delta}{2}$. Then $f_n(x) < f_n(x_n) + \delta$ for sufficiently large n, and then (1.5) implies $\|x - x_n\| < \epsilon$ for all large n, and so $\|x - \bar{x}\| \leq \epsilon$. Therefore f attains its strong minimum at $\bar{x}$. $\square$

Note that part (c) in the previous fact can fail if we remove the bounded below condition. To see this, consider f_n defined on $\mathbb{R}$ by $f_n(x) = |x + n| - n$. Then the sequence of functions (f_n) attain their strong minimums uniformly at $x_n = -n$, and (f_n) converges uniformly on bounded sets to f defined by $f(x) = x$.

2. Uniformly convex functions

This section contains the main results. Specifically, it examines category results for classes of convex functions that are closely related to uniformly convex functions. For an interesting recent application of uniformly convex functions to the theory of Bregman distances, see the work of Reem and Reich [12].

Fact 2.1. *Let X be a Banach space, and suppose its norm, $\|\cdot\|$, has modulus of convexity of power type 2 (in particular when $\|\cdot\|$ is the standard norm on a Hilbert space). Then f is uniformly convex if and only if $e_1 f$ is uniformly convex.*

Proof. Suppose f is a uniformly convex function. Because $\|\cdot\|$ has modulus of convexity of power type 2, $\|\cdot\|^2$ is a uniformly convex function (see [5, Theorem 5.4.6]). Now $(e_1 f)^* = f^* + \frac{1}{2}\|\cdot\|_*^2$, where $\|\cdot\|_*$ denotes the dual norm of $\|\cdot\|$. Then by Theorem 1.2(a), $(e_1 f)^*$ is a sum of uniformly smooth functions, and is therefore uniformly smooth. Because $(e_1 f)^*$ is uniformly smooth, Theorem 1.2(a) ensures that $e_1 f$ is uniformly convex.

Conversely, if $e_1 f$ is uniformly convex, then $(e_1 f)^*$ is uniformly smooth by Theorem 1.2(a). Consequently $f^* = (e_1 f)^* - \frac{1}{2}\|\cdot\|_*^2$ is a difference of uniformly smooth functions, and is therefore uniformly smooth. According to Theorem 1.2(a) the function f is uniformly convex. □

The next example shows that the previous result does not hold for functions that are uniformly convex on bounded sets.

Example 2.2. Consider $\mathbb{R}^2$ endowed with the Euclidean norm. There exists a function $f \in \Gamma(\mathbb{R}^2)$ such that f is strongly coercive and $e_1 f$ is uniformly convex on bounded sets, but f is not uniformly convex on bounded sets.

Proof. In [5, p. 249, Exercise 5.3.10] a lsc convex function f is constructed whose domain is bounded in $\mathbb{R}^2$ such that f is not strictly convex (and hence it is not uniformly convex on bounded sets), but f^* is a Lipschitz function that is uniformly smooth on bounded sets. Therefore $(e_1 f)^* = f^* + \frac{1}{2}\|\cdot\|_*^2$ is strongly coercive, bounded on bounded sets and uniformly smooth on bounded sets. Consequently, Theorem 1.3(a) implies $e_1 f$ is uniformly convex on bounded sets. □

The following fact shows that the reverse situation to the preceding example cannot occur.

Fact 2.3. *Consider $\mathbb{R}^n$ endowed with its Euclidean norm. Suppose $f \in \Gamma(\mathbb{R}^n)$ is strongly coercive and uniformly convex on bounded sets, then $e_1 f$ is uniformly convex on bounded sets.*

Proof. Because f is strongly coercive, Fact 1.1(a) ensures that f^* is bounded on bounded sets and has full domain. Now $f^{**} = f$ is strictly convex, and thus f^* is differentiable on the interior of its domain [5, Proposition 5.3.6], and because $\mathbb{R}^n$ is finite-dimensional, that derivative is automatically continuous (see e.g. [5, Theorem 2.2.2]). Therefore, f^* is uniformly smooth on bounded subsets of $\mathbb{R}^n$. Now $(e_1 f)^* = f^* + \frac{1}{2}\|\cdot\|^2$ is a convex function that is strongly coercive, bounded on bounded sets and uniformly smooth on bounded sets. According to Theorem 1.3(a), the function $e_1 f$ is uniformly convex on bounded sets. □

Fact 2.4. *Let X be a Banach space, and let $f \in \Gamma(X)$. Then f is strongly coercive if and only if $e_1 f$ is strongly coercive.*

Proof. Suppose $e_1 f$ is strongly coercive. Then $f \geq e_1 f$, and so f is strongly coercive.

Conversely, suppose f is strongly coercive. Then Fact 1.1(a) implies f^* is bounded on bounded sets, and it follows that $(e_1 f)^*$ is bounded on bounded sets because $(e_1 f)^* = f^* + \frac{1}{2}\|\cdot\|_*^2$. Consequently, $e_1 f$ is strongly coercive by Fact 1.1(a). □

Fact 2.5. *Let X be a Banach space and suppose its norm $\|\cdot\|$ is uniformly convex. Let $f \in \Gamma(X)$ be strongly coercive and bounded on bounded sets. Then f is uniformly convex on bounded sets if and only if $e_1 f$ is uniformly convex on bounded sets.*

Proof. Because $\|\cdot\|$ is uniformly convex, $\|\cdot\|^2$ is a function that is uniformly convex on bounded sets (see [5, Example 5.3.11]). It follows from Theorem 1.3(a) that $\|\cdot\|_*^2$ is uniformly smooth on bounded sets.

Suppose f is uniformly convex on bounded sets. Because f is also strongly coercive and bounded on bounded sets, Theorem 1.3(a) ensures that f^* is uniformly smooth on bounded sets. Therefore $(e_1 f)^* = f^* + \frac{1}{2}\|\cdot\|_*^2$ is uniformly smooth on bounded sets because it is a sum of functions that are uniformly smooth on bounded sets. Applying Theorem 1.3(a), it follows that $e_1 f$ is uniformly convex on bounded sets.

Conversely, suppose $e_1 f$ is uniformly convex on bounded sets. Because f is strongly coercive, Fact 2.4 ensures that $e_1 f$ is strongly coercive, and it is bounded on bounded sets. Using Theorem 1.3(a), $(e_1 f)^*$ is uniformly smooth on bounded sets. Therefore, $f^* = (e_1 f)^* - \frac{1}{2}\|\cdot\|_*^2$ is uniformly smooth on bounded sets, and then f is uniformly convex on bounded sets by Theorem 1.3(a). □

We are now ready for the central results.

Theorem 2.6. *Let X be any Banach space, and let*

$$S = \{f \in \Gamma(X) \,:\, f \text{ is strongly coercive}\}.$$

Then S is a dense G_δ-set in $(\Gamma(X), \tau_{AW})$.

Proof. We will work with the compatible Moreau envelope metric d on $\Gamma(X)$. Let

$$O_n = \left\{ f \in \Gamma(X) \,:\, \text{ There exists } K > 1, \text{ such that } \inf_{\|x\|=K} e_1 f(x) > Kn \right\}.$$

First we will show that O_n is open. For this, let $f \in O_n$, and choose $K > 1$ and $\epsilon > 0$ so that $e_1 f(x) > Kn + \epsilon$ whenever $\|x\| = K$. Now choose $\eta > 0$ so that $d(f,g) < \eta$ implies $|e_1 f(x) - e_1 g(x)| < \epsilon/2$ when $\|x\| = K$. Therefore, $\inf_{\|x\|=K} e_1 g \geq Kn + \epsilon/2$ when $d(f,g) < \eta$. Thus f is an interior point of O_n, and it follows that O_n is open.

Let $G = \bigcap_{n=1}^{\infty} O_n$. We will show that $f \in G$ if and only if $e_1 f$ is strongly coercive. Clearly, if $e_1 f$ is strongly coercive, then $f \in O_n$ for each n.

Conversely, suppose $f \in O_n$ for each n, and fix any $M > 0$. Then we choose n so that $n > M + |e_1 f(0)|$. Now choose $K > 1$ so that $e_1 f(x) \geq nK$ whenever $\|x\| = K$. For $\|x\| \geq K$, the convexity of $e_1 f$ implies

$$e_1 f\left(K \frac{x}{\|x\|}\right) \leq \frac{\|x\| - K}{\|x\|} e_1 f(0) + \frac{K}{\|x\|} e_1 f(x).$$

Therefore, whenever $\|x\| \geq K$, we have

$$e_1 f(x) \geq \frac{\|x\|}{K}\left(nK - |e_1 f(0)|\right) \geq \frac{\|x\|}{K}(nK - |e_1 f(0)|K) \geq M\|x\|.$$

Therefore, $e_1 f$ is strongly coercive.

Thus we have shown that $f \in G$ if and only if $e_1 f$ is strongly coercive. Then using Fact 2.4, we conclude $f \in G$ if and only if f is strongly coercive. Because the strongly coercive functions are dense in $\Gamma(X)$ (see Fact 1.4), the proof is complete. □

For most of the results in this section, the metric d based upon Moreau envelopes is very convenient to use. However, the following result seems to be an exception to this, where instead we work directly with the compatible metric d_{AW}. However, it should be noted that the following result is dual to the previous theorem, and so one could derive it from the previous theorem along with the bicontinuity of the mapping $f \mapsto f^*$; see [3, Theorem 7.2.11, p. 247].

Theorem 2.7. *Let $S = \{f \in \Gamma(X) : f$ is bounded on bounded sets$\}$ where X is a Banach space. Then is S residual in $(\Gamma(X), \tau_{AW})$.*

Proof. Given any f that is bounded on bounded sets, for $n \in \mathbb{N}$ choose integers $M_{n,f} > 0$ so that $|f(x)| \leq M_{n,f}$ for $\|x\| \leq n$. Then define $K_{n,f} = \max\{M_{n+2,f} + 2, n+2\}$. Now define the sets O_n by

$$O_n = \bigcup_{f \in S} N(f,n) \quad \text{where } N(f,n) = \left\{ g \in \Gamma(X) : d_{AW}(f,g) < \frac{1}{2^{K_{n,f}+1}} \right\}$$

Then O_n is open as a union of open sets, and clearly $S \subset \bigcap_{n=1}^{\infty} O_n$.

Now suppose $g \notin S$. So we fix $n \in \mathbb{N}$ such that g is unbounded on nB_X. To complete the proof, it will suffice to show that $g \notin O_n$. For this, we will show that $g \notin N(f,n)$ for any $f \in S$. So we fix an arbitrary $f \in S$. Let

$$F = \{x : g(x) \leq M_{n+2,f}\}$$

Because g is unbounded on nB_X we choose $x_0 \in nB_X$ such that $x_0 \notin F$. Then by the separation theorem, we choose $\phi \in X^*$ with $\|\phi\|_* = 1$ so that

$$\sup\{\phi(x) : x \in F\} = \alpha < \phi(x_0) \leq n.$$

We now fix $y \in X$ with $\|y\| \leq n+1$ such that $\phi(y) \geq \alpha + 1$. Let $\tilde{x} = (y, M_{n+2,f}) \in X \times \mathbb{R}$. Then $\tilde{x} \in \operatorname{epi} f$, and so with ρ as in (1.3), we have $\rho(\tilde{x}, \operatorname{epi} f) = 0$. On the other hand $\rho(\tilde{x}, \operatorname{epi} g) \geq 1$, since $\|y - v\| \geq 1$ for all $v \in F$, and $g(u) \geq M_{n+2,f} + 2$ if $u \notin F$. Because $\|\tilde{x}\| \leq K_{n,f}$, it follows from the definition of d_{AW} given in (1.4) that $d_{AW}(f,g) \geq \frac{1}{2^{K_{n,f}}}$, and so $g \notin N(f,n)$. It follows that $g \notin O_n$, and so $S = \bigcap_{n=1}^{\infty} O_n$. $\square$

The following is the natural convex function version of the residuality of uniformly convex norms as given in [9], and its proof was motivated by [9] along with the approach of [10] in the use of the Moreau envelope metric. At first one might hope this result would involve functions that are uniformly convex functions on the entire space, but we will see below in Corollary 2.13 that class of functions is not residual in $(\Gamma(X), \tau_{AW})$. Moreover, the uniform convexity of a norm is completely determined on its sphere, so it really is a bounded set phenomenon.

Theorem 2.8. *Suppose X is endowed with a uniformly convex norm. The set of functions that are strongly coercive, bounded and uniformly convex on bounded sets is residual in $(\Gamma(X), \tau_{AW})$.*

Proof. Define the sets

$$\begin{aligned} G_{n,m} \quad = \quad & \left\{ f : \inf \left\{ \frac{1}{2} e_1 f(x) + \frac{1}{2} e_1 f(y) - e_1 f\left(\frac{x+y}{2}\right) : \right.\right. \\ & \left.\left. \|x - y\| > \frac{1}{n}, \|x\|, \|y\| \leq m \right\} > 0 \right\}. \end{aligned}$$

It is easy to see that $G_{n,m}$ is open, and $f \in \bigcap G_{n,m}$ if and only if $e_1 f$ is uniformly convex on bounded sets. Therefore, the set $\{f \in \Gamma(X) : e_1 f$ is uniformly convex on bounded sets$\}$ is a G_δ-set, and it is a dense set by Fact 1.4(b).

However, $e_1 f$ may be uniformly convex on bounded sets without f being so, thus we intersect this dense G_δ-set with the residual sets in $\Gamma(X)$ of strongly coercive functions, and bounded on bounded set functions and apply Fact 2.5 to get the conclusion of the theorem. □

It is not difficult to show that a coercive function on a Banach space that is uniformly convex on bounded sets attains a strong minimum on the space, so the previous result provides a generic well-posed optimization result, however, it is only applicable in Banach spaces with equivalent uniformly convex norms, that is in *superreflexive* Banach spaces. In this direction it is important to note that Beer and Lucchetti (see [4, Theorem 4.5]) proved a generic well-posed minimization result for $(\Gamma(X), \tau_{AW})$ where X is an *arbitrary* Banach space.

In the paper [10] it is shown that the strongly convex functions form a meagre set in $\Gamma(\mathbb{R}^n)$. That leaves the natural question as to whether the collection of convex functions that attain a stronger type of minimum is a residual set in $(\Gamma(X), \tau_{AW})$. We will now turn our attention to this question.

We will say $f \in \Gamma(X)$ attains a *strong p-minimum* at x_0 if $f(x) \geq f(x_0)$ for all $x \in X$, and there exist $C > 0$ and $\delta > 0$ such that

$$f(x) \geq f(x_0) + C\|x - x_0\|^p \quad \text{whenever } \|x - x_0\| \leq \delta.$$

Proposition 2.9. *Let $f \in \Gamma(X)$ and $p \geq 2$. Then f attains a strong p-minimum at $\bar{x}$ if and only if $e_1 f$ attains a strong p-minimum at $\bar{x}$.*

Proof. Assume $e_1 f$ attains a strong p-minimum at $\bar{x}$, so we fix $\delta > 0$ and $C > 0$ so that

$$e_1 f(x) \geq e_1 f(\bar{x}) + C\|x - \bar{x}\|^p, \quad \text{whenever } \|x - \bar{x}\| < \delta. \tag{2.1}$$

Because $e_1 f$ attains its minimum at $\bar{x}$, it follows from the definition of $e_1 f$ and the lower semicontinuity of f that $e_1 f(\bar{x}) = f(\bar{x})$. Now for $\|x - \bar{x}\| < \delta$, using (2.1), we have

$$f(x) \geq e_1 f(x) \geq e_1 f(\bar{x}) + C\|x - \bar{x}\|^p = f(\bar{x}) + C\|x - \bar{x}\|^p.$$

Thus, f attains its strong p-minimum at $\bar{x}$.

Conversely suppose f attains a strong p-minimum at $\bar{x}$. Then $f(\bar{x}) \leq \inf_{x \in X} f \leq e_1 f(\bar{x}) \leq f(\bar{x})$, and so $e_1 f(\bar{x}) = f(\bar{x})$. Now we fix δ with $0 < \delta < 1$ and $C > 0$ so that

$$f(x) \geq f(\bar{x}) + C\|x - \bar{x}\|^p, \quad \text{whenever } \|x - \bar{x}\| < \delta. \tag{2.2}$$

Let $\epsilon > 0$ and for each $x \in X$, choose v_x such that $e_1 f(x) \geq f(v_x) + \frac{1}{2}\|x - v_x\|^2 - \epsilon$. Suppose $x \neq \bar{x}$, and $\|x - \bar{x}\| < \delta/2$. In the case $\|x - v_x\| \geq \frac{1}{2}\|x - \bar{x}\|$ we have

$$e_1 f(x) \geq f(v_x) + \frac{1}{2}\|x - v_x\|^2 - \epsilon \geq e_1 f(\bar{x}) + \frac{1}{8}\|x - \bar{x}\|^2 - \epsilon \geq e_1 f(\bar{x}) + \frac{1}{8}\|x - \bar{x}\|^p - \epsilon. \tag{2.3}$$

In the other case, $\|x - v_x\| < \frac{1}{2}\|x - \bar{x}\|$, and so $\frac{1}{2}\|x - \bar{x}\| < \|v_x - \bar{x}\| < \delta$. Therefore,

$$e_1 f(x) \quad \geq \quad f(v_x) + \frac{1}{2}\|x - v_x\|^2 - \epsilon$$

$$\geq \quad f(\bar{x}) + C\|v_x - \bar{x}\|^p + \frac{1}{2}\|x - v_x\|^2 - \epsilon \quad \text{[by (2.2)]}$$

$$\geq \quad e_1 f(\bar{x}) + \frac{C}{2^p}\|x - \bar{x}\|^p - \epsilon. \tag{2.4}$$

Let $K = \min\{1/8, C/2^p\}$, because $\epsilon > 0$ was arbitrary, (2.3) and (2.4) imply

$$e_1 f(x) \geq e_1 f(\bar{x}) + K\|x - \bar{x}\|^p \quad \text{for} \quad 0 \leq \|x - \bar{x}\| < \delta$$

as desired. □

Theorem 2.10. *The collection of functions in* $\Gamma(X)$ *that attain their strong* p*-minimums is a meager subset of* $(\Gamma(X), \tau_{AW})$.

Proof. For this, let

$$F_{n,m} \quad = \quad \Big\{ f \;:\; e_1 f \text{ attains a strong } p\text{-minimum at some } \bar{x},\; e_1 f(\bar{x}) \geq -n,$$
$$\text{and} \;\; e_1 f(x) \geq e_1 f(\bar{x}) + \frac{1}{n}\|x - \bar{x}\|^p, \text{ for } \|x - \bar{x}\| < \frac{1}{m} \Big\}.$$

We first show that $F_{n,m}$ is closed. For this suppose $(f_k) \subset F_{n,m}$ and $d(f_k, f) \to 0$. Let $\bar{x}_k$ be the point where $e_1 f_k$ attains its strong p minimum. Because $(e_1 f_k)$ converges uniformly on bounded sets to $e_1 f$ and because they attain their strong minimums uniformly, it follows from Fact 1.5(c) that $(\bar{x}_k)$ is a Cauchy sequence. Thus we let $\bar{x}$ be the limit of $(\bar{x}_k)$. Then $e_1 f(\bar{x}) = \lim_{k\to\infty} e_1 f_k(\bar{x}_k)$ and for $\|x - \bar{x}\| < \frac{1}{m}$ we have

$$\begin{aligned} e_1 f(x) - e_1 f(\bar{x}) &= \lim_{k\to\infty} (e_1 f_k(x) - e_1 f_k(\bar{x})) \\ &= \lim_{k\to\infty} (e_1 f_k(x) - e_1 f_k(\bar{x}_k)) \\ &\geq \lim_{k\to\infty} \frac{1}{n}\|x - \bar{x}_k\|^p \quad \text{[eventually } \|x - \bar{x}_k\| < 1/m] \\ &= \frac{1}{n}\|x - \bar{x}\|^p \end{aligned}$$

This shows $F_{n,m}$ is closed.

If f attains a strong p-minimum, then $e_1 f$ attains a strong p-minimum by Proposition 2.9 and it is not hard to check that $f \in F_{n,m}$ for some n, m.

Finally we check that $F_{n,m}$ has no interior points. Suppose $g \in F_{n,m}$, and so g attains a strong p-minimum at $\bar{x}$. Now let $I_{\frac{1}{n}B_X}$ be the indicator function of $\frac{1}{n}B_X$ and let $g_n(x) = g \,\square\, I_{\frac{1}{n}B_X}$. Then g_n does not attain a strong minimum, but $d(g_n, g) \to 0$. □

We now turn from examining classes of functions that attain their minima "fast" to those that functions that globally "grow" fast. In particular, a contrast to the fact that the set of strongly coercive functions is residual is that those with a power type p-growth for some $p > 1$ are meager.

Fact 2.11. *Let* $f \in \Gamma(X)$ *and* $1 < p \leq 2$. *Then*

$$\liminf_{\|x\|\to\infty} \frac{e_1 f(x)}{\|x\|^p} > 0 \quad \textit{if and only if} \quad \liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^p} > 0.$$

Proof. One direction is clear because $f \geq e_1 f$. To prove the converse suppose $\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^p} > 0$. Choose $C > 0$ and $M > 0$ so that $f(x) \geq C\|x\|^p$ whenever $\|x\| \geq M$. Because $f \in \Gamma(X)$ it follows that there are exists $\alpha \in \mathbb{R}$ and $C > 0$ so that $f \geq C\|\cdot\|^p + \alpha$. Now it follows that $f^* \leq D\|\cdot\|^q + \beta$ for some $D > 0$ and $\beta \in \mathbb{R}$, where $q^{-1} + p^{-1} = 2$. Because $q \geq 2$, it also follows that $(e_1 f)^* \leq E\|\cdot\|^q + \gamma$ for some $E > 0$ and $\gamma \in \mathbb{R}$. Therefore, by duality (again)

$$\liminf_{\|x\|\to\infty} \frac{e_1 f(x)}{\|x\|^p} > 0,$$

as desired. □

Theorem 2.12. *Let X be a Banach space. Let*

$$S = \left\{ f \in \Gamma(X) \,:\, \liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^p} > 0 \quad \textit{for some } p > 1 \right\}.$$

Then S is a meager subset of $(\Gamma(X), \tau_{AW})$.

Proof. By Fact 2.11, we can write

$$S = \left\{ f \in \Gamma(X) \,:\, \liminf_{\|x\|\to\infty} \frac{e_1 f(x)}{\|x\|^p} > 0, \text{ for some } p > 1 \right\}.$$

Let

$$A_{k,n,m} = \left\{ f \,:\, e_1 f(x) \geq \frac{1}{n}\|x\|^{1+1/k} - m, \text{ for all } x \in X \right\}.$$

Now suppose $(f_j) \subset A_{k,n,m}$ and (f_j) converges to f. Then for $x \in X$,

$$e_1 f(x) = \lim_{j\to\infty} e_1 f_j(x) \geq \frac{1}{n}\|x\|^{1+1/k} - m$$

and so $f \in A_{k,n,m}$. Thus $A_{k,n,m}$ is closed.

Since $A_{k,n,m}$ is closed, to show it is nowhere dense it will suffice to show it has no interior points. So we let $f \in A_{k,n,m}$ be arbitrary, and define $g_j = f \,\square\, j\|\cdot\|$. Then g_j is Lipschitz with Lipschitz constant j, so clearly $g_j \notin A_{k,n,m}$. By [3, Theorem 7.3.8] the sequence (g_j) converges Attouch-Wets to f. Therefore, by [3, Exercise 6, p. 263], $(e_1 g_j)$ converges to $e_1 f$ uniformly on bounded sets, and so f is not an interior point of $A_{k,n,m}$. □

Using the previous result along with an important result of Zălinescu's established in [16], we can now show that Theorem 2.8 cannot be extended to the class of uniformly convex functions.

Corollary 2.13. *Let X be any Banach space. Then the collection of uniformly convex functions in $\Gamma(X)$ is a meager subset of* $(\Gamma(X), \tau_{AW})$.

Proof. This follows from Theorem 2.12 and Zălinescu's result (see [17, Proposition 3.5.8]) that shows $\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^2} > 0$ when f is uniformly convex on X.

□

3. Residuality of strictly and locally uniformly convex functions

This section follows the work of [9] in the norm case, but some care is needed with convex functions to obtain appropriate uniform convergence on a bounded set from Attouch-Wets convergence. Before proceeding we recall that a function $f \in \Gamma(X)$ is said to be *locally uniformly convex* if $\|x_n - x\| \to 0$ whenever $x_n, x \in \operatorname{dom} f$ and

$$\frac{1}{2}f(x_n) + \frac{1}{2}f(x) - f\left(\frac{x + x_n}{2}\right) \to 0.$$

We will say $f \in \Gamma(X)$ is *strictly convex* if

$$f\left(\frac{x+y}{2}\right) < \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

whenever $x, y \in \operatorname{dom} f$ and $x \neq y$ (because $f \in \Gamma(X)$ this is equivalent to the requirement $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$ for $0 < \lambda < 1$, $x \neq y$ with $x, y \in \operatorname{dom} f$).

It is important to note that the results herein are closely related to those of [7, Theorems 1.2 and 2.1] which establish residuality of strictly and totally convex functions respectively in various spaces of convex functions defined on a nonempty closed convex subset of a Banach space. Moreover, the class of totally convex functions is intermediate to the classes of strictly convex and locally uniformly convex functions and lends itself nicely to a variety of applications ([6, 7]). In particular, Theorem 3.3 as given below in the locally uniformly convex case does provide a residuality result for the important class of totally convex functions. However, the assumption of a locally uniformly convex norm in Theorem 3.3, while applying to wide classes of Banach spaces, is much more restrictive than the assumptions in [7, Theorem 2.1]. On the other hand, a nice benefit of using the Attouch-Wets topology on $\Gamma(X)$ is the availability of continuity properties of the mapping $f \mapsto f^*$ that are useful for averaging results: see Corollary 3.5.

The following basic results concerning uniform convergence on bounded sets and Attouch-Wets convergence will be useful and sufficient for our purposes.

Lemma 3.1. *Suppose $K \geq 1$ and $N, M \in \mathbb{N}$ are such that $M \geq N+1$, $|f|$, $|g|$ are bounded by M on $(N+1)B_X$, and f, g satisfy a Lipschitz condition with constant K on $(N+1)B_X$. If*

$$d_{AW}(f,g) < \epsilon \quad \textit{where} \quad 0 < \epsilon < 2^{-M},$$

then $\sup_{x \in NB_X} |f(x) - g(x)| \leq 3K2^M\epsilon$.

Proof. Let $0 < \epsilon < 2^{-M}$ and suppose to the contrary that there exists $x_0 \in NB_X$ such that $|f(x_0) - g(x_0)| > 3K2^M\epsilon$. Without loss of generality, say $g(x_0) > f(x_0) + 3K2^M\epsilon$. Then for $\|x - x_0\| \leq 2^M\epsilon$, we have $x \in (N+1)B_X$ and it follows that

$$g(x) > f(x) + 3K2^M\epsilon - 2K(2^M\epsilon) \geq f(x) + 2^M\epsilon \quad \text{for } \|x - x_0\| \leq 2^M\epsilon.$$

Therefore $\|(x_0, f(x_0))\|_\infty \leq M$ and $\rho((x_0, f(x_0)), \operatorname{epi} g) \geq 2^M\epsilon$ where ρ is as defined in (1.3). Consequently, $d_{AW}(f,g) \geq \frac{1}{2^M}(2^M\epsilon) = \epsilon$. ☐

Lemma 3.2. *Suppose $f, g \in \Gamma(X)$ and g is bounded on bounded sets. Let $N \in \mathbb{N}$ and $M \in \mathbb{N}$ be such that $M = \max\{N, \sup_{\|x\| \leq N+2} |g(x)|\}$. If $d_{AW}(f,g) \leq 2^{-M-2}$, then f is bounded above by $M+2$ on NB_X.*

Proof. Suppose to the contrary that there exists x_0 with $\|x_0\| \leq N$ such that $f(x_0) > M+2$. As in the proof of Theorem 2.7, let $C = \{x : f(x) \leq M+2\}$. Because C is closed and convex, by the separation theorem, we choose $\phi \in X^*$ such that $\|\phi\| = 1$ and $\sup_C \phi < \phi(x_0)$. Now fix $\bar{x}$ with $\|\bar{x}\| \leq M+1$ and $\phi(\bar{x}) > \sup_C \phi + 1$. Let $U = \{x \in X : \|x - \bar{x}\| \leq 1\}$. Then $f(x) > M+2$ for all $x \in U$ while $g(x) \leq M$ for all $x \in U$. Then the distance $\rho\left((\bar{x}, g(\bar{x})), \operatorname{epi} f\right) \geq 1$ where ρ is as defined in (1.3). It then follows that $d_{AW}(f,g) > 2^{-M-2}$ which is a contradiction. $\square$

The following is the main result of this section. It now follows by adapting the proof from [9] from the norm setting to the convex function setting.

Theorem 3.3. *Suppose the Banach space X admits a locally uniformly convex (resp. strictly convex) norm $\|\cdot\|$. Then the collection of locally uniformly convex (resp. striclty convex) functions is a residual set in $(\Gamma(X), \tau_{AW})$.*

Proof. Suppose $\|\cdot\|$ is locally uniformly convex. Define the function r on X by $r = \|\cdot\|^2$. Then r is a locally uniformly convex function [5, Example 5.3.11]. Following [9], for $h \in \Gamma(X)$ and $j \in \mathbb{N}$, define

$$G(h,j) = \{f \in \Gamma(X) : d_{AW}(f, h + j^{-1}r) < j^{-2}\}$$

and then let $G_k = \bigcup\{G(h,j) : j \geq k, h \in \Gamma(X)\}$.

Clearly the sets G_k are dense open sets in the τ_{AW}-topology. Let S denote the functions in $\Gamma(X)$ that are bounded on bounded sets. Then S is residual according to Theorem 2.7, and so $A = \bigcap_k G_k \cap S$ is a residual set in $(\Gamma(X), \tau_{AW})$.

Now suppose $g \in A$. It remains to show g is locally uniformly convex; for this suppose (x_n) is bounded and

$$\frac{1}{2}g(x_n) + \frac{1}{2}g(x) - g\left(\frac{x_n + x}{2}\right) \to 0. \tag{3.1}$$

Fix $N \in \mathbb{N}$ so that $\|x\| \leq N$ and $\|x_n\| \leq N$ for all $n \in \mathbb{N}$. Since g is bounded on bounded sets, according to Lemma 3.2, there is a $k_0 \in \mathbb{N}$ and a number $L > 0$ such that $\sup\{f(x) : \|x\| \leq N+2\} \leq L$ whenever $d_{AW}(f,g) < k_0^{-2}$. There is a constant $K \geq 1$ such that f and g satisfy a Lipschitz condition with constant K on $(N+1)B_X$ whenever $d_{AW}(f,g) < k_0^{-2}$ (this follows because in that case f and g are both bounded on $(N+2)B_X$). Applying this to $f = h + k^{-1}r$, Lemma 3.1 then implies there exist $M > 0$ so that for sufficiently large k we have

$$d_{AW}(g, h + \frac{1}{k}r) < \frac{1}{k^2} \quad \Rightarrow \quad \sup_{x \in NB_X} |g - (h + k^{-1}r)| < \frac{3K2^M}{k^2} \tag{3.2}$$

Using (3.1) and (3.2), for sufficiently large k, one has

$$\limsup_{n \to \infty} \frac{1}{2}(h + k^{-1}r)(x_n) + \frac{1}{2}(h + k^{-1}r)(x) - (h + k^{-1}r)\left(\frac{x_n + x}{2}\right) \leq \frac{3K2^{M+1}}{k^2}.$$

The convexity of h implies that

$$\frac{1}{2}h(x_n) + \frac{1}{2}h(x) - h\left(\frac{x_n + x}{2}\right) \geq 0$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{k}\left[\frac{1}{2}r(x_n) + \frac{1}{2}r(x) - r\left(\frac{x + x_n}{2}\right)\right] \leq \frac{3K2^{M+1}}{k^2}$$

for all sufficiently large k. Consequently,

$$\lim_{n\to\infty} \frac{1}{2}r(x_n) + \frac{1}{2}r(x) - r\left(\frac{x_n + x}{2}\right) = 0.$$

Because r is locally uniformly convex, this implies $\|x_n - x\| \to 0$. This shows g is locally uniformly convex, as desired.

The case for strictly convex functions is similar using $r = \|\cdot\|^2$ where $\|\cdot\|$ is strictly convex. Then, in place of (3.1) one would suppose

$$\frac{1}{2}g(x) + \frac{1}{2}g(y) - g\left(\frac{x + y}{2}\right) = 0.$$

and deduce $\frac{1}{2}r(x)+\frac{1}{2}r(y)-r\left(\frac{x+y}{2}\right) = 0$, and use the strict convexity of r to conclude $x = y$, and thus show that g is strictly convex. $\square$

Finally, we should note that a large motivation for the paper [9] was to use the Baire category theorem to provide Asplund Averaging results for norms, that is, to combine rotundity and smoothness properties. The same can be done for convex functions thanks to some foundational work of Beer. Namely, let $\Gamma^*(X^*)$ denote the proper weak*-lower semicontinuous convex functions on X^*. Then the mapping $f \mapsto f^*$ is a homeomorphism between $(\Gamma(X), \tau_{AW})$ and $(\Gamma^*(X^*), \tau_{AW})$, see [3, Theorem 7.2.11]. In particular the mapping $f \mapsto f^*$, or its inverse, will preserve residual sets. As a consequence of this and the proof of Theorem 3.3, we have the following.

Theorem 3.4. *Suppose the Banach space X admits a norm $\|\cdot\|$ whose dual norm on X^* is locally uniformly convex (resp. strictly convex). Let*

$$S = \{f \in \Gamma(X) : f^* \textit{ is locally uniformly convex (resp. striclty convex)}\}.$$

Then S is residual in $(\Gamma(X), \tau_{AW})$.

Moreover, the various classes of strict convexity, local uniform convexity and uniform convexity possess nice duality properties with smoothness. Indeed, see Theorem 1.3(b) and [5, Proposition 5.3.6] which shows for $f \in \Gamma(X)$ that f is Gâteaux differentiable (resp. Fréchet differentiable) on the interior of its domain if f^* is strictly convex (resp. locally uniformly convex). Therefore, combining [3, Theorem 7.2.11] with the residuality results for the various class of strictly convex functions along with their corresponding duality with smoothess one can establish the following averaging results.

Corollary 3.5. (a) *Suppose X admits a locally uniformly convex norm, and X^* admits a dual locally uniformly convex norm. Then the set S of functions in $\Gamma(X)$ that are locally uniformly convex and Fréchet differentiable is residual in $(\Gamma(X), \tau_{AW})$.*

(b) *Suppose X admits a strictly convex norm, and X^* admits a dual strictly convex norm. Then the set S of functions in $\Gamma(X)$ that are strictly convex and Gâteaux differentiable is residual in $(\Gamma(X), \tau_{AW})$.*

(c) *Suppose X admits a uniformly convex norm. Then the set S of functions in $\Gamma(X)$ that are uniformly smooth on bounded sets and uniformly convex on bounded sets is residual in $(\Gamma(X), \tau_{AW})$.*

Note that strong coercivity plays a crucial role in this result, because when f^* is strongly coercive, then f has full domain, so the results of [5, Proposition 5.3.6] apply to ensure differentiability on the whole space.

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