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SOLUTIONS OF NONHOMOGENEOUS EQUATIONS INVOLVING HARDY POTENTIALS WITH SINGULARITIES ON THE BOUNDARY

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Abstract. In this paper, we present a new distributional identity for the solutions of elliptic equations involving Hardy potentials with singularities located on the boundary of the domain. Then we use it to obtain the boundary isolated singular solutions of nonhomogeneous problems.

1. Introduction

The classical Hardy inequality is stated as following: For any smooth bounded domain $\mathcal{O}$ in $\mathbb{R}^N$ containing the origin, there holds

$$\int_{\mathcal{O}} |\nabla u|^2 dx \geq c_N \int_{\mathcal{O}} |x|^{-2}|u|^2 dx, \quad \forall\, u \in H_0^1(\mathcal{O}), \tag{1.1}$$

with the best constant $c_N = \frac{(N-2)^2}{4}$. The qualitative properties of Hardy inequality and its improved versions have been studied extensively, see for example [1, 4, 19, 21], motivated by great applications in the study of stability of solutions to semilinear elliptic and parabolic equations (cf. [5, 6, 13, 30, 31]). The isolated singular solutions of Hardy problem with absorption nonlinearity have been studied in [11, 12, 23] and the one with source nonlinearity has been done in [3, 16]. The related semilinear elliptic problem involving the inverse square potential has been studied by variational methods in [15, 14, 18] and the references therein. In a very recent work [9], we established a new distributional identity with respect to a specific weighted measure and we then classify the classical isolated singular solutions of

$$-\Delta u + \frac{\mu}{|x|^2} u = f \quad \text{in} \quad \mathcal{O} \setminus \{0\},$$

subject to the homogeneous Dirichlet boundary condition with $\mu \geq -c_N$. These results allow us to draw a complete picture of the existence, non-existence and the singularities for classical solutions for the above problems (cf. [10]).

It is of interest to consider the corresponding problem involving Hardy potential with singularity on the boundary. While the sharp constant c_N in Hardy inequality (1.1) could be replaced by $\frac{N^2}{4}$ when the origin is addressed on the boundary of the domain, see [20, Corollary 2.4], also [7, 8, 17].

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Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $0 \in \partial\Omega$. We study boundary isolated singular solutions of nonhomogeneous problems:

$$\begin{cases} \mathcal{L}_\beta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{1.2}$$

where $f \in C^\gamma_{loc}(\bar\Omega \setminus \{0\})$ with $\gamma \in (0,1)$, $g \in C(\partial\Omega \setminus \{0\})$ and $\mathcal{L}_\beta := -\Delta + \frac{\beta}{|x|^2}$ is the Hardy operator which is singular at 0 (with $N \geq 2$, $\beta \geq \beta_0 := -\frac{N^2}{4}$). Recall that for $\beta \geq \beta_0$, the problem

$$\begin{cases} \mathcal{L}_\beta u = 0 & \text{in } \mathbb{R}^N_+, \\ \quad u = 0 & \text{on } \partial\mathbb{R}^N_+ \setminus \{0\} \end{cases} \tag{1.3}$$

has two special solutions with the explicit formulas as

$$\Lambda_\beta(x) = \begin{cases} x_N |x|^{\tau_-(\beta)} & \text{if } \beta > \beta_0, \\ -x_N |x|^{\tau_-(\beta)} \ln|x| & \text{if } \beta = \beta_0 \end{cases} \quad \text{and} \quad \lambda_\beta(x) = x_N |x|^{\tau_+(\beta)}, \tag{1.4}$$

where $x = (x', x_N) \in \mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, +\infty)$, and

$$\tau_-(\beta) = -\frac{N}{2} - \sqrt{\beta - \beta_0} \quad \text{and} \quad \tau_+(\beta) = -\frac{N}{2} + \sqrt{\beta - \beta_0}, \tag{1.5}$$

are two roots of $\beta - \tau(\tau + N) = 0$.

As in [10, 9], we first find a certain distributional identity which shows that the singularity of solution Λ_β for (1.3) is associated to a Dirac mass. Let $C^{1.1}_0(\mathbb{R}^N_+)$ be the set of functions in $C^{1.1}(\overline{\mathbb{R}^N_+})$ vanishing on the boundary and having compact support in $\overline{\mathbb{R}^N_+}$. Then we have

Theorem 1.1. *Let $d\gamma_\beta := \lambda_\beta(x)dx$ and*

$$\mathcal{L}^*_\beta := -\Delta - \frac{2\tau_+(\beta)}{|x|^2} x \cdot \nabla - \frac{2}{x_N}\frac{\partial}{\partial x_N}, \quad x = (x', x_N) \in \mathbb{R}^N_+. \tag{1.6}$$

Then there holds

$$\int_{\mathbb{R}^N_+} \Lambda_\beta \mathcal{L}^*_\beta (\frac{\zeta}{x_N})\, d\gamma_\beta = c_\beta \frac{\partial \zeta}{\partial x_N}(0), \quad \forall\, \zeta \in C^{1.1}_0(\mathbb{R}^N_+), \tag{1.7}$$

where

$$c_\beta = \begin{cases} \sqrt{\beta - \beta_0}\, |\mathcal{S}^{N-1}|/N & \text{if } \beta > \beta_0, \\ |\mathcal{S}^{N-1}|/N & \text{if } \beta = \beta_0, \end{cases} \tag{1.8}$$

and $\mathcal{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$ and $|\mathcal{S}^{N-1}|$ denotes its $(N-1)$-dimensional Hausdorff measure.

From the distributional identity (1.7), Λ_β is called as a fundamental solution of (1.3). We remark that when $\beta = 0$, $\mathcal{L}^*_0 = -\Delta - \frac{2}{x_N}\frac{\partial}{\partial x_N}$, $\lambda_\beta(x) = x_N$ and (1.7) could be reduced to

$$c_0 \frac{\partial \zeta}{\partial x_N}(0) = \int_{\mathbb{R}^N_+} \Lambda_0 \mathcal{L}^*_0 (\frac{\zeta}{x_N})\, d\gamma_\beta = \int_{\mathbb{R}^N_+} \Lambda_0 (-\Delta \zeta)\, dx, \quad \forall\, \zeta \in C^{1.1}_0(\mathbb{R}^N_+),$$

which coincides with the classical distributional identity proposed in [22]. On this classical subject, it has been vastly expanded in the works [2, 26, 27, 28, 29].

For simplicity, here and in the sequel, we always assume that Ω is a bounded C^2- domain satisfying that

$$B^+_{r_0}(0) \subset \Omega \subset B^+_{R_0}(0), \tag{1.9}$$

for some $0 < r_0 < R_0 < +\infty$ where $B^+_r(0) := B_r(0) \cap \mathbb{R}^N_+$. Let $d\omega_\beta(x) := |x|^{\tau_+(\beta)} d\omega(x)$, where ω is the Hausdorff measure of $\partial\Omega$. We can state our main result as follows

Theorem 1.2. *Let $\mathcal{L}^*_\beta$ be given by (1.6), $f \in C^\theta_{loc}(\bar\Omega \setminus \{0\})$ with $\theta \in (0,1)$, $g \in C(\partial\Omega \setminus \{0\})$.*

(i) If

$$\int_\Omega |f|\, d\gamma_\beta + \int_{\partial\Omega} |g|\, d\omega_\beta < +\infty, \tag{1.10}$$

then for any $k \in \mathbb{R}$, problem (1.2) admits a unique solution $u_k \in C^2(\Omega) \cap L^1(\Omega, |x|^{-1} d\gamma_\beta)$ such that

$$\int_\Omega u_k \mathcal{L}^*_\beta\Big(\frac{\xi}{x_N}\Big)\, d\gamma_\beta = \int_\Omega \frac{f\xi}{x_N}\, d\gamma_\beta - \int_{\partial\Omega} g \frac{\partial\xi}{\partial\nu} d\omega_\beta + c_\beta k \frac{\partial\xi}{\partial x_N}(0), \quad \forall\, \xi \in C^{1.1}_0(\Omega), \tag{1.11}$$

where ν is the unit outward vector on $\partial\Omega$.

(ii) If f, g are nonnegative and

$$\lim_{r\to 0^+} \Big(\int_{\Omega\setminus B_r(0)} f\, d\gamma_\beta + \int_{\partial\Omega\setminus B_r(0)} g\, d\omega_\beta \Big) = +\infty, \tag{1.12}$$

then problem (1.2) has no nonnegative solution.

When $g = 0$ on $\partial\Omega$ and $f = 0$ in Ω, we prove in Proposition 3.2 in Section 3 that problem (1.2) admits an isolated singular solution Λ^Ω_β, which has the asymptotic behavior at the origin as the fundamental function Λ_β. More precisely, we have

$$\lim_{t\to 0^+} \sup_{z\in S^{N-1}_+} \Big(\frac{\Lambda^\Omega_\beta(tz)}{\Lambda_\beta(tz)} - 1 \Big) = 0. \tag{1.13}$$

When $g = 0$ on $\partial\Omega$ and $f \in C^\theta_{loc}(\bar\Omega \setminus \{0\}) \cap L^1(\Omega, d\gamma_\beta)$, Theorem 4.1 in Section 4 shows that problem (1.2) has a solution u_f verifying the isolated singularity (see Remark 4.2)

$$\lim_{t\to 0^+} \inf_{z\in S^{N-1}_+} \frac{u_f(tz)}{\Lambda_\beta(tz)} = 0, \tag{1.14}$$

which is less precise than (1.13) due to the lack of estimates of Green kernel of Hardy operator with singularity on the boundary. However, when $f = 0$ and $g \neq 0$, it is not convenient to use (1.14) to describe the singularity of the solution u_g, so we may distinguish this by the distributional identity

$$\int_\Omega u_g \mathcal{L}^*_\beta\Big(\frac{\xi}{x_N}\Big)\, d\gamma_\beta = -\int_{\partial\Omega} g \frac{\partial\xi}{\partial\nu} d\omega_\beta, \quad \forall\, \xi \in C^{1.1}_0(\Omega),$$

All in all, the solution u_k of (1.2) can be decomposed into three components $k\Lambda_\beta^\Omega$, u_f and u_g.

The method we use to prove the existence of solutions for problem (1.2) is different from the classical method of the boundary data problem used by Gmira-Véron in [22] due to the appearance of Hardy potential. They obtained the very weak solutions by approximating the Dirac mass at boundary. Then they considered the limit of the solutions to the corresponding problem where the convergence is guaranteed by the Poisson kernel. In this paper, we prove the existence of moderate singular solution by using the function Λ_β to construct suitable solutions of problem (1.2) with the zero Dirichlet boundary condition. While for nonzero Dirichlet boundary condition, we transform the boundary data into nonhomogeneous term. However, for $\beta > 0$, that transformation can not totally solve (1.2) with the nonzero Dirichlet boundary condition, and our idea is to cut off the boundary data and approximate the solutions.

The rest of the paper is organized as follows. In Section 2, we start from a comparison principle for $\mathcal{L}_\beta$ and show the moderate singular solution of (1.2) when $g = 0$. Section 3 is devoted to prove the distributional identity (1.7) for the fundamental solution Λ_β in $\mathbb{R}^N_+$, to consider its trace, the corresponding distributional identity in bounded smooth domain. Section 4 is to study the qualitative properties of the solutions for problem (1.2) when $g = 0$ and then we give the proof of Theorem 1.2 in the case of nonzero boundary data in Section 5. In what follows, we denote by c_i a generic positive constant in the proofs of the results.

2. Preliminary

2.1. **Comparison principle.** We start the analysis from a comparison principle for $\mathcal{L}_\beta$. Let $\eta_0 : [0,+\infty) \to [0,\, 1]$ be a decreasing C^∞ function such that

$$\eta_0 = 1 \quad \text{in } [0,1] \quad \text{and} \quad \eta_0 = 0 \quad \text{in } [2,+\infty). \tag{2.1}$$

Lemma 2.1. *Let Ω be a bounded open set in $\mathbb{R}^N_+$, $L : \Omega \times [0,+\infty) \to [0,+\infty)$ be a continuous function satisfying that for any $x \in \Omega$,*

$$L(x, s_1) \ge L(x, s_2) \quad \text{if} \quad s_1 \ge s_2,$$

then $\mathcal{L}_\beta + L$ with $\beta \ge \beta_0$ verifies the comparison principle, that is, if u, $v \in C^{1,1}(\Omega) \cap C(\bar\Omega)$ verify that

$$\mathcal{L}_\beta u + L(x,u) \ge \mathcal{L}_\beta v + L(x,v) \quad \text{in } \Omega \quad \text{and} \quad u \ge v \quad \text{on } \partial\Omega,$$

then $u \ge v$ in Ω.

Proof. Let $w = u - v$ and then $w \ge 0$ on $\partial\Omega$. Denote $w_- = \min\{w, 0\}$, and we claim that $w_- \equiv 0$. Indeed, if $\Omega_- := \{x \in \Omega : w(x) < 0\}$ is not empty, then it is a bounded $C^{1,1}$ domain in Ω and $w_- = 0$ on $\partial\Omega$. We observe that $\Omega_- \subset \mathbb{R}^N_+$ and then from Hardy inequality [7, (1.7)] (see also [25]), it holds that

$$\begin{aligned} 0 &= \int_{\Omega_-} (-\Delta w_- + \frac{\beta}{|x|^2} w_-) w_- dx + \int_{\Omega_-} [L(x,u) - L(x,v)] w_-\, dx \\ &\ge \int_{\Omega_-} \left(|\nabla w_-|^2 + \frac{\beta}{|x|^2} w_-^2 \right) dx \ge c_1 \int_{\Omega_-} w_-^2 dx, \end{aligned}$$

then $w_- = 0$ in Ω_-, by the continuity of w_-, which is impossible with the definition of Ω_-. □

Lemma 2.2. *Assume that $\beta \geq \beta_0$, f_1, f_2 are two functions in $C^{\theta}_{loc}(\Omega)$ with $\theta \in (0,1)$, g_1, g_2 are two continuous functions on $\partial\Omega \setminus \{0\}$, and*

$$f_1 \geq f_2 \quad \text{in } \Omega \quad \text{and} \quad g_1 \geq g_2 \quad \text{on } \partial\Omega \setminus \{0\}.$$

Let u_i $(i = 1,2)$ be the classical solutions of

$$\begin{cases} \mathcal{L}_\beta u = f_i & \text{in } \Omega, \\ u = g_i & \text{on } \partial\Omega \setminus \{0\}. \end{cases}$$

If

$$\lim_{r\to 0^+} \inf_{x\in\partial_+ B_r(0)} [u_1(x) - u_2(x)]\Lambda_\beta^{-1}(x) \geq 0, \tag{2.2}$$

where $\partial_+ B_r(0) = \partial B_r(0) \cap \Omega$. Then $u_1 \geq u_2$ in $\overline{\Omega} \setminus \{0\}$.

Proof. Let $w = u_2 - u_1$, then w satisfies

$$\begin{cases} \mathcal{L}_\beta w \leq 0 & \text{in } \Omega, \\ w \leq 0 & \text{on } \partial\Omega \setminus \{0\}, \\ \displaystyle\lim_{r\to 0^+} \sup_{x\in\partial_+ B_r(0)} w(x)\Lambda_\beta^{-1}(x) \leq 0. \end{cases}$$

Thus for any $\epsilon > 0$, there exists $r_\epsilon > 0$ converging to zero as $\epsilon \to 0$ such that

$$w \leq \epsilon\Lambda_\beta \quad \text{on} \quad \partial B_{r_\epsilon}(0) \cap \Omega.$$

We observe that $w \leq 0 < \epsilon\Lambda_\beta$ on $\partial\Omega \setminus B_{r_\epsilon}(0)$, which implies by Lemma 2.1 that

$$w \leq \epsilon\Lambda_\beta \quad \text{in} \quad \overline{\Omega} \setminus \{0\}.$$

Therefore we obtain that $w \leq 0$ in $\overline{\Omega} \setminus \{0\}$ which ends the proof. □

For any $\varepsilon > 0$, denote

$$\mathcal{L}_{\beta,\varepsilon} = -\Delta + \frac{\beta}{|x|^2 + \varepsilon}. \tag{2.3}$$

We remark that $\mathcal{L}_{\beta,\varepsilon}$ is strictly elliptic operator and we have the following existence result for related nonhomogeneous problem.

Lemma 2.3. *Assume that $\varepsilon \in (0, 1)$, $\beta \geq \beta_0$, $\mathcal{L}_{\beta,\varepsilon}$ is given by (2.3) and $f \in C^{\theta}_{loc}(\Omega) \cap C(\bar{\Omega})$ with $\theta \in (0,1)$ and $g \in C(\partial\Omega)$. Then the problem*

$$\begin{cases} \mathcal{L}_{\beta,\varepsilon} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

has a unique classical solution $u_\varepsilon \in C^2(\Omega) \cap C(\bar{\Omega})$, which verifies that

$$\int_\Omega u_\varepsilon \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \int_\Omega \frac{f\xi}{x_N} d\gamma_\beta - \int_{\partial\Omega} g\frac{\partial\xi}{\partial\nu}\, d\omega_\beta + \beta\varepsilon \int_\Omega \frac{u_\varepsilon\xi}{(|x|^2+\varepsilon)|x|^2 x_N}\, d\gamma_\beta, \tag{2.5}$$

for any $\xi \in C^{1.1}_0(\Omega)$.

Assume more that $f \geq 0$ in Ω and $g \geq 0$ on $\partial\Omega$. Then the mapping $\varepsilon \mapsto u_\varepsilon$ is decreasing if $\beta > 0$, and is increasing if $\beta_0 \leq \beta < 0$.

Proof. We first prove the existence of solution to problem (2.4). We introduce Poisson kernel P_Ω of $-\Delta$ in Ω, and denote Poisson operator as

$$\mathbb{P}_\Omega[g](x) = \int_{\partial\Omega} P_\Omega(x,y)g(y)dy.$$

We observe that

$$\mathcal{L}_{\beta,\varepsilon}\mathbb{P}_\Omega[g] = \frac{\beta}{|x|^2+\varepsilon}\mathbb{P}_\Omega[g] \in C^1(\Omega)\cap C(\bar{\Omega}).$$

Then the solution of (2.4) denoted by u_ε, could be reduced to $u_\varepsilon = \mathbb{P}_\Omega[g] + u_f$, where u_f is the solution of

$$\begin{cases} \mathcal{L}_{\beta,\varepsilon}u = f - \dfrac{\beta}{|x|^2+\varepsilon}\mathbb{P}_\Omega[g] & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

For $\beta \ge \beta_0$, a solution u_f in $H_0^1(\Omega)$ of (2.6) could be derived by Ekeland's variational methods as the critical point of the functional

$$I(u) = \int_\Omega |\nabla u|^2 dx + \beta\int_\Omega \frac{u^2}{|x|^2+\varepsilon}dx - \int_\Omega \Big(f - \frac{\beta}{|x|^2+\varepsilon}\mathbb{P}_\Omega[g]\Big)u dx.$$

That is well-defined in $H_0^1(\Omega)$ since $\beta\in(\beta_0,0)$. From the Hardy's inequality in [17], we have that, for any $u\in C_0^2(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx + \beta\int_\Omega \frac{u^2}{|x|^2+\varepsilon}dx \ge (\beta-\beta_0)\int_\Omega |\nabla u|^2 dx,$$

for $\beta=\beta_0$, from the improved Hardy inequality in [17], it holds

$$\begin{aligned} c_2\int_\Omega u^2 dx &\le \int_\Omega |\nabla u|^2 dx - |\beta_0|\int_\Omega \frac{u^2}{|x|^2}dx \\ &< \int_\Omega |\nabla u|^2 dx - |\beta_0|\int_\Omega \frac{u^2}{|x|^2+\varepsilon}dx. \end{aligned}$$

Finally it is trivial for the case $\beta \ge 0$.

By the standard regularity result (e.g. [24]), we have that u_f is a classical solution of (2.6). Then problem (2.4) admits a classical solution and the uniqueness follows by comparison principle.

Finally, we prove (2.5). Multiple $\frac{\lambda_\beta\xi}{x_N}$ with $\xi\in C_0^{1.1}(\Omega)$ and integrate over Ω, we have that

$$\begin{aligned} &\int_\Omega \frac{\lambda_\beta\xi}{x_N} f\,dx = \int_\Omega \frac{\lambda_\beta\xi}{x_N}\mathcal{L}_{\beta,\varepsilon}u_\varepsilon\,dx \\ =\ &\int_\Omega u_\varepsilon(-\Delta\frac{\lambda_\beta\xi}{x_N})\,dx + \int_{\partial\Omega} g\frac{\partial(|x|^{\tau_+(\beta)}\xi)}{\partial\nu}\,d\omega(x) + \int_\Omega \frac{\beta}{|x|^2+\varepsilon}u_\varepsilon\lambda_\beta\xi\,dx \\ =\ &\int_\Omega u_\varepsilon\mathcal{L}^*_\beta(\frac{\xi}{x_N})\,d\gamma_\beta + \int_{\partial\Omega} g\frac{\partial\xi}{\partial\nu}\,d\omega_\beta - \beta\varepsilon\int_\Omega \frac{u_\varepsilon\xi}{(|x|^2+\varepsilon)|x|^2x_N}\,d\gamma_\beta. \end{aligned}$$

Note that if $f\ge 0$ in Ω and $g\ge 0$ on $\partial\Omega$, then $u_\varepsilon\ge 0$ in Ω. Let $\varepsilon_1\ge\varepsilon_2$ and u_{ε_1}, u_{ε_2} be two solutions of (2.4) respectively. If $\beta\ge\beta_0$, we observe that

$\mathcal{L}_{\beta,\varepsilon_2}u_{\varepsilon_1} \ge \mathcal{L}_{\beta,\varepsilon_1}u_{\varepsilon_1} = f$, so u_{ε_1} is a super solution of (2.4) with $\varepsilon = \varepsilon_2$ and by comparison principle, it holds $u_{\varepsilon_1} \ge u_{\varepsilon_2}$ in Ω. The proof ends. □

Now we build the distributional identity for the classical solution of nonhomogeneous problem with $g = 0$ and moderate singularity at the origin, i.e.

$$\lim_{r\to 0^+} \sup_{x\in\partial_+ B_r(0)} \frac{|u(x)|}{\Lambda_\beta(x)} = 0. \tag{2.7}$$

Proposition 2.4. *Let $\beta \ge \beta_0$, $N \ge 2$, $f \in C^\theta_{loc}(\bar\Omega)$ with $\theta \in (0,1)$, then*

$$\begin{cases} \mathcal{L}_\beta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{2.8}$$

subjecting to (2.7), has a unique solution u_β, which satisfies the distributional identity

$$\int_\Omega u_\beta \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \int_\Omega \frac{f\xi}{x_N}\, d\gamma_\beta, \quad \forall\, \xi \in C^{1.1}_0(\Omega). \tag{2.9}$$

Proof. The uniqueness follows by Lemma 2.2. Since $\mathcal{L}_\beta$ is a linear operator, we only have to deal with the case that $f \ge 0$ in Ω.

Part 1: $\beta > 0$. In this case, the mapping $\varepsilon \mapsto u_\varepsilon$ is decreasing, where $u_\varepsilon > 0$ is the solution of (2.4) with $g = 0$. Then $u_\beta := \lim_{\varepsilon\to 0^+} u_\varepsilon$ exists, and by the standard regularity theory, we have that u_β is a classical solution of

$$\begin{cases} \mathcal{L}_\beta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Part 2: $\beta \in [\beta_0, 0)$. Without loss of generality, we assume that $\Omega \subset B_{\frac12}(0)$. Denote

$$V_{t,s}(x) := \begin{cases} t x_N |x|^{-\frac N2} - s x_N^2 |x|^{\tau_+(\beta)} & \text{if } \beta \in (\beta_0, 0), \\ t x_N |x|^{-\frac N2} (-\ln|x|)^{\frac12} - s x_N^2 |x|^{-\frac N2} & \text{if } \beta = \beta_0, \end{cases}$$

where the parameters $s, t \ge 0$.

Then for $\beta \in (\beta_0, 0)$, we see that $V_{t,s}(x) > 0$ for $x \in \Omega$ if $t \ge s$ and

$$\mathcal{L}_\beta V_{t,s}(x) = t c_\beta(-N/2) x_N |x|^{-\frac N2 - 2} + 2s|x|^{\tau_+(\beta)} + 2s\tau_+(\beta) x_N^2 |x|^{\tau_+(\beta)-2},$$

where $c_\beta(-N/2) > 0$ and $\tau_+(\beta) < 0$. Since f is bounded in Ω, let

$$s_0 = \frac12 \sup_{x\in\Omega} \frac{|f(x)|}{|x|^{\tau_+(\beta)}}$$

and then we fix $t_0 \ge s_0$ such that

$$t_0 c_\beta(-N/2) x_N |x|^{-\frac N2 - 2} + 2s_0\tau_+(\beta) x_N^2 |x|^{\tau_+(\beta)-2} \ge 0.$$

So V_{t_0,s_0} is a positive supersolution of (2.8).

For $\beta = \beta_0$, $\tau_-(\beta) = -\frac N2$, we have that

$$\mathcal{L}_\beta V_{t,s}(x) = \frac t4 x_N |x|^{-\frac N2 - 2}(-\ln|x|)^{-\frac12} + 2s|x|^{-\frac N2} - 2sN x_N^2 |x|^{-\frac N2 - 2}.$$

We take s_0 as above where β is replaced by β_0 and we fix $t_0 \geq s_0$ such that

$$\frac{t_0}{4} x_N |x|^{-\frac{N}{2}-2}(-\ln|x|)^{-\frac{1}{2}} - 2s_0 N x_N^2 |x|^{-\frac{N}{2}-2} \geq 0.$$

So V_{t_0,s_0} is also a positive supersolution of (2.8) in this case which implies, by comparison principle, that we have

$$u_\varepsilon(x) \leq V_{t_0,s_0}(x), \quad \forall\, x \in \Omega.$$

Proof of (2.9). We need to estimate $\int_\Omega \frac{u_\varepsilon \xi}{(|x|^2+\varepsilon)|x|^2 x_N}\, d\gamma_\beta$ for $0 < \varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$ fixed. we first consider the case $\beta > 0$. We observe that

$$\begin{aligned}
& \varepsilon \int_{\Omega \setminus B_{\sqrt{\varepsilon}}(0)} \frac{u_\varepsilon \xi \lambda_\beta(x)}{(|x|^2+\varepsilon)|x|^2 x_N}\, dx \\
\leq\ & \varepsilon \|u_{\varepsilon_0}\|_{L^\infty(\Omega)} \|\xi/\rho\|_{L^\infty(\Omega)} \int_{\Omega \setminus B_{\sqrt{\varepsilon}}(0)} \frac{|x|^{\tau_+(\beta)-2}}{|x|^2+\varepsilon}\, dx \\
\leq\ & \|u_{\varepsilon_0}\|_{L^\infty(\Omega)} \|\xi/\rho\|_{L^\infty(\Omega)} \varepsilon^{\frac{N-2+\tau_+(\beta)}{2}} \int_{B_{\frac{1}{2\sqrt{\varepsilon}}}(0) \setminus B_1(0)} |y|^{\tau_+(\beta)-4}\, dy \\
\leq\ & c_3 \|u_{\varepsilon_0}\|_{L^\infty(\Omega)} \|\xi/\rho\|_{L^\infty(\Omega)} (2^{-\tau_+(\beta)+4-N} \varepsilon + \varepsilon^{\frac{N-2+\tau_+(\beta)}{2}}) \\
\to\ & 0 \quad \text{as} \quad \varepsilon \to 0^+
\end{aligned}$$

and

$$\begin{aligned}
& \varepsilon \int_{B_{\sqrt{\varepsilon}}(0)} \frac{u_\varepsilon \xi \lambda_\beta(x)}{(|x|^2+\varepsilon)|x|^2 x_N}\, dx \\
\leq\ & \|u_{\varepsilon_0}\|_{L^\infty(\Omega)} \|\xi/\rho\|_{L^\infty(\Omega)} \int_{B_{\sqrt{\varepsilon}}(0)} |x|^{\tau_+(\beta)-2}\, dx \\
\leq\ & c_4 \|u_{\varepsilon_0}\|_{L^\infty(\Omega)} \|\xi/\rho\|_{L^\infty(\Omega)} \varepsilon^{\frac{N-2+\tau_+(\beta)}{2}} \\
\to\ & 0 \quad \text{as} \quad \varepsilon \to 0^+,
\end{aligned}$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ and $\frac{N-2+\tau_+(\beta)}{2} > 0$. Therefore, passing to the limit of (2.5), we obtain (2.9).

For $\beta \in (\beta_0, 0)$, from the increasing monotonicity and the upper bound V_{s_0,t_0}, we have that

$$\lim_{\varepsilon \to 0^+} \int_\Omega u_\varepsilon \mathcal{L}_\beta^*(\frac{\xi}{x_N}) d\gamma_\beta = \int_\Omega u_\beta \mathcal{L}_\beta^*(\frac{\xi}{x_N}) d\gamma_\beta$$

and

$$\varepsilon \int_\Omega \frac{\xi u_\varepsilon \lambda_\beta(x)}{(|x|^2+\varepsilon)|x|^2 x_N} dx \leq c_5 \varepsilon \int_\Omega \frac{|x|^{-N+\sqrt{\beta-\beta_0}}}{|x|^2+\varepsilon} dx.$$

By directly compute, we have that

$$\varepsilon \int_{\Omega \setminus B_{\sqrt{\varepsilon}}(0)} \frac{|x|^{-N+\sqrt{\beta-\beta_0}}}{|x|^2+\varepsilon} dx \quad \leq \quad c_6 \varepsilon^{\frac{\sqrt{\beta-\beta_0}}{2}} \int_{B_{\frac{1}{2\sqrt{\varepsilon}}}(0) \setminus B_1(0)} |y|^{-N-2+\sqrt{\beta-\beta_0}}\, dy$$

$$\leq \quad c_7(\varepsilon + \varepsilon^{\frac{\sqrt{\beta-\beta_0}}{2}}) \to 0 \quad \text{as} \quad \varepsilon \to 0^+$$

and

$$\begin{aligned} \varepsilon \int_{B_{\sqrt{\varepsilon}}(0)} \frac{|x|^{-N+\sqrt{\beta-\beta_0}}}{|x|^2+\varepsilon} dx \quad &\leq \quad \int_{B_{\sqrt{\varepsilon}}(0)} |x|^{-N+\sqrt{\beta-\beta_0}}\, dx \\ &\leq \quad c_8 \varepsilon^{\frac{\sqrt{\beta-\beta_0}}{2}} \to 0 \quad \text{as} \quad \varepsilon \to 0^+, \end{aligned}$$

As a conclusion, passing to the limit in (2.5) as $\varepsilon \to 0^+$, we have that u_β satisfies that

$$\int_\Omega u_\beta \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \int_\Omega \frac{f\xi}{x_N}\, d\gamma_\beta, \quad \forall\, \xi \in C^{1.1}_0(\Omega). \tag{2.11}$$

Finally, we prove (2.9) with $\beta = \beta_0$, We claim that the mapping $\beta \mapsto u_\beta$ with $\beta \in (\beta_0, 0)$ is decreasing. In fact, if $\beta_0 < \beta_1 \leq \beta_2 < 0$, we know that

$$\begin{aligned} f = \mathcal{L}_{\beta_1} u_{\beta_1} \quad &= \quad -\Delta u_{\beta_1} + \frac{\beta_1}{|x|^2} u_{\beta_1} \\ &\leq \quad -\Delta u_{\beta_1} + \frac{\beta_2}{|x|^2} u_{\beta_1} = \mathcal{L}_{\beta_2} u_{\beta_1}, \end{aligned}$$

by Lemma 2.2, which implies that $u_{\beta_1} \geq u_{\beta_2}$.

We know that V_{s_0,t_0} is a super solution of (2.8) with $\beta \in (\beta_0, 0)$. So it follows by Lemma 2.2 that $\{u_\beta\}_\beta$ is uniformly bounded by the upper bound $V_{s_0,t_0} \in L^1(\Omega, \frac{1}{x_N} d\gamma_\beta)$.

For $\xi \in C^{1.1}_0(\Omega)$, we have that

$$|\mathcal{L}^*_\beta(\frac{\xi}{x_N})| \leq c_9(\|\frac{\xi}{x_N}\|_{C^{1.1}(\Omega)} + \|\frac{\xi}{x_N}\|_{C^1(\Omega)} x_N^{-1}),$$

where $c_9 > 0$ is independent of β.

From the dominate monotonicity convergence theorem and the uniqueness of the solution, we have that

$$u_\beta \to u_{\beta_0} \quad \text{a.e. in } \Omega \text{ as } \ \beta \to \beta_0^+ \quad \text{and in } \ L^1(\Omega,\, x_N^{-1} d\gamma_\beta)$$

and u_{β_0} is a classical solution of (2.8) with $\beta = \beta_0$. Passing to the limit of (2.11) as $\beta \to \beta_0^+$ to obtain that

$$\int_\Omega u_{\beta_0} \mathcal{L}^*_{\beta_0}(\frac{\xi}{x_N})\, d\gamma_{\beta_0} = \int_\Omega \frac{f\xi}{x_N} d\gamma_{\beta_0}.$$

The proof ends. □

Remark 2.5. We note that when $\beta \geq 0$ and f is bounded, the moderate singular solution of problem (2.8) is no longer singular, that means, it is a classical solution of

$$\begin{cases} \mathcal{L}_\beta u = f & \text{in } \ \Omega, \\ \quad u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{2.12}$$

Now we prove the following

Lemma 2.6. *(i) The problem*

$$\begin{cases} \mathcal{L}^*_\beta(\dfrac{u}{x_N}) = 1 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.13}$$

has a unique positive solution $w_1 \in C^2(\Omega) \cap C^{0.1}_0(\Omega)$.

(ii) The problem

$$\begin{cases} \mathcal{L}^*_\beta(\dfrac{u}{x_N}) = \dfrac{1}{x_N} & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.14}$$

has a unique positive solution $w_2 \in C^2(\Omega) \cap C^1_0(\bar{\Omega} \setminus \{0\}) \cap C^{0.1}_0(\Omega)$.

Proof. We first claim that problem (2.8) has a unique classical positive solution w_β under the constraint (2.7) when $f(x) = \lambda_\beta(x)$ or $f(x) = |x|^{\tau_+(\beta)}$.

In fact, let $f_n(x) = \lambda_\beta(x)\eta_0(n|x|)$, where $\eta_0 : [0, +\infty) \to [0, 1]$ is a decreasing C^∞ function satisfying (2.1). Then $f_n \in C^\theta(\bar{\Omega})$ with $\theta \in (0, 1), f_n \le f$, and by Proposition 2.4, let w_n be the solution of problem

$$\begin{cases} \mathcal{L}_\beta u = f_n & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{2.15}$$

subject to (2.7). We know that the mapping: $n \to w_n$ is increasing by the increasing monotone of $\{f_n\}$. So we only construct a suitable upper bound for w_n in the cases that $f(x) = \lambda_\beta(x)$ and $f(x) = |x|^{\tau_+(\beta)}$ respectively.

When $f(x) = \lambda_\beta(x)$, let $V_{t,s}(x) = t\lambda_\beta(x) - sx_N|x|^{\tau_+(\beta)+2}$ for $s, t > 0$. It is know that

$$\mathcal{L}_\beta V_{t,s} = -sc_{\tau_+(\beta)+2}\lambda_\beta(x), \quad x \in \mathbb{R}^N_+,$$

for some $c_{\tau_+(\beta)+2} < 0$. So fix $s = -1/c_{\tau_+(\beta)+2}$ and then fix $t > 0$ such that

$$V_{t,s}(x) > 0, \quad \forall x \in \Omega.$$

The limit of $\{w_n\}_n$, denoting by $w_{\beta,1}$, is a solution of (2.7) satisfying $w_{\beta,1} \le V_{t,s}(x)$.

When $f(x) = |x|^{\tau_+(\beta)}$, let

$$W_{t,s,l}(x) = t\lambda_\beta(x) - s(x_N|x|^{\tau_+(\beta)+2} + lx_N^2|x|^{\tau_+(\beta)+2}),$$

where $s, t, l > 0$. We observe that

$$\mathcal{L}_\beta W_{t,s,l}(x) = s[-c_{\tau_+(\beta)+2}\lambda_\beta(x) + 2l|x|^{\tau_+(\beta)} + 2l\tau_+(\beta)x_N^2|x|^{\tau_+(\beta)}], \ x \in \mathbb{R}^N_+,$$

with the same constant $c_{\tau_+(\beta)+2} < 0$ as above. Then we choose $l > 0$ such that $-2c_{\tau_+(\beta)+2}l\tau_+(\beta)x_N > 0$ for $x \in \Omega$, $s = \frac{1}{2l}$ and we take $t > 0$ such that $W_{t,s,l} > 0$ in Ω and

$$\mathcal{L}_\beta W_{t,s,l}(x) \ge |x|^{\tau_+(\beta)}.$$

Thus, the limit of $\{w_n\}_n$, denoting by $w_{\beta,2}$, is a solution of (2.7) such that

$$w_{\beta,2}(x) \le W_{t,s,l}(x).$$

As a conclusion, for $i = 1, 2$,

$$w_{\beta,i} \le t\lambda_\beta \quad \text{in} \quad \Omega. \tag{2.16}$$

Denote $w_i = w_{\beta,i}x_N/\lambda_\beta$, we observe that

$$1 = \lambda_\beta^{-1}\mathcal{L}_\beta w_{\beta,1} = \lambda_\beta^{-1}\mathcal{L}_\beta(\lambda_\beta w_1/x_N) = \mathcal{L}^*_\beta(w_1/x_N)$$

and

$$1/x_N = \lambda_\beta^{-1}\mathcal{L}_\beta w_{\beta,2} = \lambda_\beta^{-1}\mathcal{L}_\beta(\lambda_\beta w_2/x_N) = \mathcal{L}^*_\beta(w_2/x_N).$$

Moreover, by (2.16), it follow that $w_i \le tx_N$. Then we have that $w_i \in C^2(\Omega)\cap C_0^{0.1}(\Omega)$ for $i = 1, 2$. Away from the origin, Hardy's operator is uniform elliptic, thus $u \in C_0^1(\bar\Omega \setminus \{0\})$ and then $u \in C^2(\Omega) \cap C_0^1(\bar\Omega\setminus\{0\}) \cap C_0^{0.1}(\Omega)$. □

Although $C^2(\Omega)\cap C_0^1(\bar\Omega\setminus\{0\})\cap C_0^{0.1}(\Omega)$ is not suitable as test function space for problem (1.2), w_1, w_2 are still valid as test functions for formula (1.11) with $k = 0$ in the distributional sense.

For given $f \in C^1(\bar\Omega)$, a direct consequence of Lemma 2.6 can be stated as follows

Corollary 2.7. *Assume that $f \in C^1(\bar\Omega \setminus \{0\})$ satisfying for some $c_{10} > 0$*

$$|f(x)| \le \frac{c_{10}}{x_N}.$$

Then there exists a unique solution of $w_f \in C^2(\Omega) \cap C_0^{0.1}(\Omega)$ of

$$\begin{cases} \mathcal{L}^*_\beta(\dfrac{u}{x_N}) = f & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

3. Fundamental solution

3.1. **In half space.** In this subsection, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. For any $\xi \in C_0^{1.1}(\mathbb{R}^N_+)$, we know there exists a unique $\zeta \in C_c^{1.1}(\mathbb{R}^N)$ such that $\xi(x) = x_N\zeta(x)$ for $x \in \overline{\mathbb{R}^N_+}$. Moreover, we have that $\frac{\partial\xi}{\partial x_N}(0) = \zeta(0)$.

Take $\zeta \in C_c^{1.1}(\mathbb{R}^N)$, multiplying $\lambda_\beta\zeta$ in (1.3) and integrating over $\mathbb{R}^N_+ \setminus \overline{B_r(0)}$, then we have that

$$\begin{aligned} 0 \;&=\; \int_{\mathbb{R}^N_+\setminus\overline{B_r(0)}} \mathcal{L}_\beta(\Lambda_\beta)\lambda_\beta\zeta\, dx = \int_{\mathbb{R}^N_+\setminus\overline{B_r(0)}} \Lambda_\beta\mathcal{L}^*_\beta(\zeta)\, d\gamma_\beta \\ &+ \int_{\partial_+B_r(0)} \Big(-\nabla\Lambda_\beta\cdot\frac{x}{|x|}\lambda_\beta + \nabla\lambda_\beta\cdot\frac{x}{|x|}\Lambda_\beta\Big)\zeta\, d\omega \\ &+ \int_{\partial_+B_r(0)} \Lambda_\beta\lambda_\beta\Big(\nabla\zeta\cdot\frac{x}{|x|}\Big)\, d\omega, \end{aligned}$$

where $\partial_+B_r(0) = \partial B_r(0)\cap\mathbb{R}^N_+$. For $\beta \ge \beta_0$, we see that for $r = |x| > 0$ small,

$$\begin{aligned} &-\nabla\Lambda_\beta(x)\cdot\frac{x}{|x|}\lambda_\beta(x) + \nabla\lambda_\beta(x)\cdot\frac{x}{|x|}\Lambda_\beta(x) \\ &= \begin{cases} 2\sqrt{\beta-\beta_0}\, x_N^2 r^{-N-1} & \text{if } \beta > \beta_0, \\ x_N^2 r^{-N-1} & \text{if } \beta = \beta_0 \end{cases} \end{aligned}$$

and

$$|\zeta(x)-\zeta(0)|\le c_{11}r,$$

then

$$\begin{aligned}
&\int_{\partial_+B_r(0)}\sqrt{\beta-\beta_0}\,x_N^2r^{-N-1}\zeta(0)x_Nd\omega(x)\\
&= \begin{cases}\sqrt{\beta-\beta_0}\displaystyle\int_{\partial_+B_1(0)}x_N^2d\omega(x)\,\zeta(0) & \text{if } \ \beta>\beta_0,\\ \displaystyle\int_{\partial_+B_1(0)}x_N^2d\omega(x)\,\zeta(0) & \text{if } \ \beta=\beta_0\end{cases}\\
&= c_\beta\zeta(0)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\int_{\partial_+B_r(0)}\Big(-\nabla\Lambda_\beta\cdot\frac{x}{|x|}\lambda_\beta+\nabla\lambda_\beta\cdot\frac{x}{|x|}\Lambda_\beta\Big)\zeta\,d\omega-c_\beta\zeta(0)\Big|\\
\le\ & c_{12}(\sqrt{\beta-\beta_0}+1)\,r\int_{\partial_+B_1(0)}x_N^2d\omega(x)\\
\to\ & 0 \quad \text{as} \quad r\to0^+,
\end{aligned}$$

that is,

$$\lim_{r\to0}\Big(\int_{\partial_+B_r(0)}-\nabla\Lambda_\beta\cdot\frac{x}{|x|}\lambda_\beta\zeta\,d\omega+\int_{\partial B_r(0)}\nabla\lambda_\beta\cdot\frac{x}{|x|}\Lambda_\beta\zeta\,d\omega\Big)=c_\beta\zeta(0).$$

Moreover, we see that

$$\Big|\int_{\partial_+B_r(0)}\Lambda_\beta\lambda_\beta\Big(\nabla\zeta\cdot\frac{x}{|x|}\Big)\,d\omega\Big|\le\|\zeta\|_{C^1}\,r\int_{\partial_+B_1(0)}x_N^2d\omega\to0 \quad\text{as}\quad r\to0^+.$$

Therefore, we have that

$$\lim_{r\to0^+}\int_{\mathbb{R}^N\setminus\overline{B_r(0)}}\Lambda_\beta\mathcal{L}^*_\beta(\zeta)d\gamma_\beta=c_\beta\zeta(0),$$

which implies (1.7). The proof ends. □

3.2. **Trace of Λ_β.** The following theorem shows the trace of Λ_β.

Theorem 3.1. *Let $d\omega_\beta(x')=|x'|^{\tau_+(\beta)}dx'$ for $x'\in\mathbb{R}^{N-1}$, then for any $\zeta\in C_c(\mathbb{R}^{N-1})$,*

$$\lim_{t\to0^+}\int_{\mathbb{R}^{N-1}}\Lambda_\beta(x',t)\zeta(x')d\omega_\beta(x')=b_N\zeta(0),\tag{3.1}$$

where

$$b_N=\int_{\mathbb{R}^{N-1}}(1+|y'|^2)^{-\frac{N}{2}}dy'>0.$$

This is to say that the trace of Λ_β is δ_0 in the $d\gamma_\beta$-distributional sense.

Proof. For any $\zeta \in C_c(\mathbb{R}^{N-1})$, there exists $R > 0$ such that $\operatorname{supp}\zeta \subset B'_R(0)$, here and in the sequel, denoting by $B'_R(0)$ the ball in $\mathbb{R}^{N-1}$. By direct computations, we have that

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} \Lambda_\beta(x',t)\zeta(x')\,d\omega_\beta(x') &= \int_{B'_R(0)} \Lambda_\beta(x',t)\zeta(x')\,d\omega_\beta(x')\\ &= \int_{B'_{R/t}(0)} (|y'|^2+1)^{\frac{\tau_-(\beta)}{2}}|y'|^{\tau_+(\beta)}\zeta(ty')dy'.\end{aligned}$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 1$ such that

$$\begin{aligned}&\int_{B'_{R/t}(0)\setminus B'_{R_\varepsilon}(0)} (|y'|^2+1)^{\frac{\tau_-(\beta)}{2}}|y'|^{\tau_+(\beta)}\zeta(ty')dy'\\ \le\ & \|\zeta\|_{L^\infty(\mathbb{R}^{N-1})}\int_{\mathbb{R}^{N-1}\setminus B'_{R_\varepsilon}(0)} |y'|^{-N}dy'\\ \le\ & \|\zeta\|_{L^\infty(\mathbb{R}^{N-1})}|\mathcal{S}^{N-2}|\varepsilon,\end{aligned}$$

where $R_\varepsilon \le \frac{1}{\varepsilon}$. Let

$$A := \int_{B'_{R_\varepsilon}(0)} (|y'|^2+1)^{\frac{\tau_-(\beta)}{2}}|y'|^{\tau_+(\beta)}\zeta(ty')dy' - \int_{\mathbb{R}^{N-1}} (|y'|^2+1)^{\frac{\tau_-(\beta)}{2}}|y'|^{\tau_+(\beta)}\zeta(0)dy',$$

we have that

$$\begin{aligned}|A| &\le \int_{B'_{R_\varepsilon}(0)} (|y'|^2+1)^{\frac{\tau_-(\beta)}{2}}|y'|^{\tau_+(\beta)}\left|\zeta(ty')-\zeta(0)\right|\,dy' + \varepsilon|\zeta(0)||\mathcal{S}^{N-2}|\\ &\le t\|\zeta\|_{C^1(\mathbb{R}^{N-1})}\int_{B'_{R_\varepsilon}(0)} (|y'|^2+1)^{\frac{\tau_-(\beta)}{2}}|y'|^{\tau_+(\beta)}dy' + \varepsilon|\zeta(0)||\mathcal{S}^{N-2}|\\ &= R_\varepsilon t\|\zeta\|_{C^1(\mathbb{R}^{N-1})} + \varepsilon|\zeta(0)||\mathcal{S}^{N-2}|\\ &\le \Big(\|\zeta\|_{C^1(\mathbb{R}^{N-1})} + |\zeta(0)||\mathcal{S}^{N-2}|\Big)\,\varepsilon,\end{aligned}$$

if we take $t = \varepsilon^2$. Passing to the limit as $\varepsilon \to 0$, we derive (3.1). □

3.3. **Fundamental solution in bounded domain.** In this subsection, we do an approximation of the isolated singular solution.

Proposition 3.2. *Let Ω be a C^2 domain verifying (1.9). Then the problem*

$$\begin{cases}\mathcal{L}_\beta u = 0 \quad \text{in } \Omega,\\ \quad u = 0 \quad \text{on } \partial\Omega\setminus\{0\},\\ \lim\limits_{r\to 0^+}\sup\limits_{x\in B_r^+(0)} \dfrac{|u(x)-\Lambda_\beta(x)|}{\Lambda_\beta(x)} = 0\end{cases} \tag{3.2}$$

admits a unique solution Λ_β^Ω satisfying the following distributional identity:

$$\int_\Omega \Lambda_\beta^\Omega \mathcal{L}_\beta^*(\frac{\xi}{x_N})\,d\gamma_\beta = c_\beta\frac{\partial\xi}{\partial x_N}(0), \quad \forall\,\xi \in C_0^{1.1}(\Omega). \tag{3.3}$$

Proof. Let $\eta_{r_0}(t) = \eta_0(\frac{2}{r_0}t)$, which satisfies that

$$\eta_{r_0} = 1 \quad \text{in} \quad [0, r_0/2] \quad \text{and} \quad \eta_{r_0} = 0 \quad \text{in} \quad [r_0, +\infty). \tag{3.4}$$

For $i = 1, 2$ the problem

$$\begin{cases} \mathcal{L}_\beta w_i = -\nabla \eta_{r_0} \cdot \nabla \Lambda_\beta - \Lambda_\beta \Delta \eta_{r_0} \quad \text{in } \Omega, \\ \quad w_i = 0 \quad \text{on } \partial\Omega \setminus \{0\}, \\ \quad \lim\limits_{e \in \mathcal{S}^N_+,\, t \to 0^+} w_i(te)\Lambda_\beta^{-1}(te) = 2 - i, \end{cases} \tag{3.5}$$

admits a unique solutions w_1 and w_2 respectively. Obviously,

$$w_1 = \Lambda_\beta \eta_{r_0}$$

and $-\nabla \eta_{r_0} \cdot \nabla \Lambda_\beta - \Lambda_\beta \Delta \eta_{r_0}$ has compact set in $\Omega \cap (\overline{B_{r_0}(0) \setminus B_{\frac{r_0}{2}}(0)})$ and then $-\nabla \eta_{r_0} \cdot \nabla \Lambda_\beta - \Lambda_\beta \Delta \eta_{r_0}$ is smooth and bounded, it follows by the proof of Proposition 2.4 that there exist $s_0, t_0 > 0$ such that $|w_2| \le V_{s_0,t_0}$.

For $i = 1$, following the proof of Theorem 1.1, we get then for any $\xi \in C_0^{1.1}(\Omega)$,

$$\int_\Omega w_1 \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \int_\Omega \Big(-\nabla \eta_{r_0} \cdot \nabla \Lambda_\beta - \Lambda_\beta \Delta \eta_{r_0} \Big) \frac{\xi}{x_N}\, d\gamma_\beta + c_\beta \frac{\partial \xi}{\partial x_N}(0). \tag{3.6}$$

For $i = 2$, it follows by Proposition 2.4 that for any $\xi \in C_0^{1.1}(\Omega)$,

$$\int_\Omega w_2 \mathcal{L}^*_\beta(\frac{\xi}{x_N}) d\gamma_\beta = \int_\Omega \Big(-\nabla \eta_{r_0} \cdot \nabla \Lambda_\beta - \Lambda_\beta \Delta \eta_{r_0} \Big) \frac{\xi}{x_N}\, d\gamma_\beta. \tag{3.7}$$

Let $\Lambda^\Omega_\beta = \Lambda_\beta \eta_{r_0} - w_2$, it follows by (3.6) and (3.7) that

$$\int_\Omega \Lambda^\Omega_\beta \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = c_\beta \frac{\partial \xi}{\partial x_N}(0), \quad \forall\, \xi \in C_0^{1.1}(\Omega).$$

Finally, it's clear that if u_1 and u_2 are two solutions of (3.2), then $w := u_1 - u_2$ satisfies

$$\lim_{r \to 0^+} \sup_{x \in B^+_r(0)} \frac{|w(x)|}{\Lambda_\beta(x)} = 0.$$

Combining with the fact that

$$\mathcal{L}_\beta w = 0 \text{ in } \Omega \quad \text{and} \quad w = 0 \text{ on } \partial\Omega \setminus \{0\},$$

and Lemma 2.2, we have that $w \equiv 0$. Thus the uniqueness is proved. □

4. Existence

4.1. **Zero Dirichlet boundary.** Our purpose in this section is to clarify the isolated singularities of the nonhomogeneous problem

$$\begin{cases} \mathcal{L}_\beta u = f \quad \text{in } \Omega, \\ \quad u = 0 \quad \text{on } \partial\Omega \setminus \{0\}, \end{cases} \tag{4.1}$$

where $f \in C^\theta_{loc}(\bar{\Omega} \setminus \{0\})$ with $\theta \in (0, 1)$. Recall that $\mathcal{L}^*_\beta$ is given by (1.6) and $d\gamma_\beta(x) = \lambda_\beta(x) dx$. We prove the following

Theorem 4.1. *(i) Assume that $f \in L^1(\Omega,\, d\gamma_\beta)$ and $u \in L^1(\Omega,\, \frac{1}{|x|}d\gamma_\beta)$ is a classical solution of problem (4.1), then there exists some $k \in \mathbb{R}$ such that there holds*

$$\int_\Omega u\,\mathcal{L}^*_\beta(\frac{\xi}{x_N})\,d\gamma_\beta = \int_\Omega \frac{f\xi}{x_N}\,d\gamma_\beta + k\frac{\partial\xi}{\partial x_N}(0), \quad \forall\,\xi \in C^{1.1}_0(\Omega). \tag{4.2}$$

(ii) Inversely, assume that $f \in L^1(\Omega,\, d\gamma_\beta)$, then for any $k \in \mathbb{R}$, problem (4.1) has a unique solution $u_k \in L^1(\Omega,\, \frac{1}{|x|}d\gamma_\beta)$ verifying (4.2) with such k.

Proof. (i) Let $\tilde{\Omega}$ be the interior set of $\bar{\Omega} \cup \overline{\{(x', -x_N) : (x', x_N) \in \Omega\}}$ and extend u (resp. f) by the x_N-odd extension to $\tilde{u}$ (resp. $\tilde{f}$) in $\tilde{\Omega}$, then $\mathcal{L}_\beta\tilde{u} = \tilde{f}$. Our aim is to see the distributional property at the origin. Denote by L the operator related to $\mathcal{L}_\beta\tilde{u} - \tilde{f}$ in the distribution sense, i.e.

$$L(\zeta) = \int_{\tilde{\Omega}} \Big(\tilde{u}\mathcal{L}^*_\beta(\zeta) - \tilde{f}\zeta\Big)|x_N||x|^{\tau_+(\beta)}\,dx, \quad \forall \zeta \in C^\infty_c(\tilde{\Omega}). \tag{4.3}$$

For any $\zeta \in C^\infty_c(\tilde{\Omega} \setminus \{0\})$, we have that $L(\zeta) = 0$. In fact, there exists $\varepsilon > 0$ such that $\mathrm{supp}(\zeta) \subset \tilde{\Omega} \setminus B_\varepsilon(0)$ and then

$$\begin{aligned} 0 &= 2\int_\Omega \zeta(\mathcal{L}_\beta u - f)\,d\gamma_\beta = \int_{\tilde{\Omega}} \zeta(\mathcal{L}_\beta\tilde{u} - \tilde{f})\,d\tilde{\gamma}_\beta \\ &= -\int_{\tilde{\Omega}} \tilde{f}\zeta\,d\tilde{\gamma}_\beta + \int_{\Omega\setminus B_\varepsilon(0)} u\mathcal{L}^*_\beta\zeta d\gamma_\beta + \int_{\partial(\Omega\setminus B_\varepsilon(0))\cap(\mathbb{R}^{N-1}\times\{0\})} \frac{\partial u}{\partial x_N}\zeta d\omega_\beta \\ &\quad + \int_{(-\Omega)\setminus B_\varepsilon(0)} (-u)\mathcal{L}^*_\beta\zeta d\tilde{\gamma}_\beta + \int_{\partial(-\Omega\setminus B_\varepsilon(0))\cap(\mathbb{R}^{N-1}\times\{0\})} \frac{\partial\tilde{u}}{\partial(-x_N)}\zeta d\omega_\beta \\ &= \int_{\tilde{\Omega}\setminus B_\varepsilon(0)} (\tilde{u}\mathcal{L}^*_\beta\zeta - \tilde{f}\zeta)\,d\tilde{\gamma}_\beta \\ &= \int_{\tilde{\Omega}} (\tilde{u}\mathcal{L}^*_\beta\zeta - \tilde{f}\zeta)\,d\tilde{\gamma}_\beta, \end{aligned}$$

where $d\tilde{\gamma}_\beta = |\tilde{\lambda}_\beta(x)|dx$, $\tilde{\lambda}_\beta$ is the odd extension of λ_β and

$$\int_{\partial(\Omega\setminus B_\varepsilon(0))\cap(\mathbb{R}^{N-1}\times\{0\})} \frac{\partial u}{\partial x_N}\zeta d\omega_\beta = -\int_{\partial(-\Omega\setminus B_\varepsilon(0))\cap(\mathbb{R}^{N-1}\times\{0\})} \frac{\partial\tilde{u}}{\partial(-x_N)}\zeta d\omega_\beta.$$

By Theorem XXXV in [33] (see also Theorem 6.25 in [32]), it implies that

$$L = \sum_{|a|=0}^{p} k_a D^a \delta_0, \tag{4.4}$$

where $p \in \mathbb{N}$, $a = (a_1, \cdots, a_N)$ is a multiple index with $a_i \in \mathbb{N}$, $|a| = \sum_{i=1}^N a_i$ and in particular, $D^0\delta_0 = \delta_0$. Then we have that

$$L(\zeta) = \int_{\tilde{\Omega}} \Big(\tilde{u}\mathcal{L}^*_\beta\zeta - f\zeta\Big)\,d\tilde{\gamma}_\beta = \sum_{|a|=0}^{\infty} k_a D^a\zeta(0), \qquad \forall\zeta \in C^\infty_c(\tilde{\Omega}). \tag{4.5}$$

For any multiple index $a = (a_1, \cdots, a_N)$, let ζ_a be a C^∞ function such that

$$\text{supp}(\zeta_a) \subset \overline{B_2(0)} \quad \text{and} \quad \zeta_a(x) = k_a \prod_{i=1}^{N} x_i^{a_i} \quad \text{for} \quad x \in B_1(0). \tag{4.6}$$

Now we use the test function $\zeta_{\varepsilon,a}(x) := \zeta_a(\varepsilon^{-1}x)$ for $x \in \tilde{\Omega}$ in (4.5), we have that

$$\sum_{|a| \le q} k_a D^a \zeta_{\varepsilon,a}(0) = \frac{k_a^2}{\varepsilon^{|a|}} \prod_{i=1}^{N} a_i!,$$

where $a_i! = a_i \cdot (a_i - 1) \cdots 1 > 0$ and $a_i! = 1$ if $a_i = 0$.

Let $r > 0$, we obtain that

$$\begin{aligned}
\Big| \int_{\tilde{\Omega}} \tilde{u} \mathcal{L}_\beta^* \zeta_\varepsilon \, d\tilde{\gamma}_\beta \Big| &= \Big| \int_{B_{2\varepsilon}(0)} \tilde{u} \mathcal{L}_\beta^* \zeta_\varepsilon \, d\tilde{\gamma}_\beta \Big| \\
&\le \frac{1}{\varepsilon^2} \Big| \int_{B_{2\varepsilon}(0)} \tilde{u}(x)(-\Delta)\zeta_a(\varepsilon^{-1}x) \, d\tilde{\gamma}_\beta \Big| \\
&\quad + \frac{2|\tau_+(\beta)|}{\varepsilon} \Big| \int_{B_{2\varepsilon}(0)} \tilde{u}(x) \frac{x}{|x|^2} \cdot \nabla \zeta_a(\varepsilon^{-1}x) \, d\tilde{\gamma}_\beta \Big| \\
&\le c_{13} \left[\frac{1}{\varepsilon^2} \int_{B_{2\varepsilon}(0)} |\tilde{u}(x)| \, d\tilde{\gamma}_\beta + \frac{1}{\varepsilon} \int_{B_{2\varepsilon}(0)} \frac{|\tilde{u}(x)|}{|x|} \, d\tilde{\gamma}_\beta \right] \\
&\le \frac{c_{14}}{\varepsilon} \int_{B_{2\varepsilon}(0)} \frac{|\tilde{u}(x)|}{|x|} \, d\tilde{\gamma}_\beta,
\end{aligned}$$

then, by the fact that $u \in L^1(\Omega, \frac{1}{|x|} d\gamma_\beta)$, it follows that

$$\lim_{\varepsilon \to 0^+} \int_{B_{2\varepsilon}(0)} \frac{|\tilde{u}(x)|}{|x|} \, d\tilde{\gamma}_\beta = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \varepsilon \Big| \int_{\tilde{\Omega}} \tilde{u} \mathcal{L}_\beta^* \zeta_\varepsilon \, d\tilde{\gamma}_\beta \Big| = 0. \tag{4.7}$$

For $|a| \ge 1$, we have that

$$k_a^2 \le c_{15} \varepsilon^{|a|-1} \Big| \int_{\tilde{\Omega}} \tilde{u} \mathcal{L}_\beta^* \zeta_\varepsilon \, d\tilde{\gamma}_\beta \Big| \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

then we have $k_a = 0$ by arbitrary of $\varepsilon > 0$ in (4.5) with $|a| \ge 1$, thus,

$$L(\zeta) = \int_{\tilde{\Omega}} \left[\tilde{u} \mathcal{L}_\beta^* \zeta - \tilde{f} \zeta \right] \, d\tilde{\gamma}_\beta = k_0 \zeta(0), \qquad \forall \, \xi \in C_c^\infty(\tilde{\Omega}). \tag{4.8}$$

For any $\zeta \in C_c^{1.1}(\tilde{\Omega})$, by taking a sequence $\zeta_n \in C_c^\infty(\tilde{\Omega})$ converging to ζ, we obtain that (4.8) holds for any $\zeta \in C_c^{1.1}(\tilde{\Omega})$.

Now we fix $\xi \in C_0^{1.1}(\Omega)$ with compact support in $\Omega \cup \{(x', 0) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < r_0\}$, then $\xi / x_N \in C^{1.1}(\bar{\Omega})$ and we may do x_N-even extension of ξ / x_N in $\tilde{\Omega}$, denoting by $\tilde{\xi}$, then $\tilde{\xi} \in C_c^{1.1}(\tilde{\Omega})$, by the x_N-even extension, we have that

$$\tilde{\xi}(0) = \frac{\partial \xi}{\partial x_N}(0).$$

So it follows from (4.8) that

$$\int_\Omega \Big(\tilde{u}\mathcal{L}^*_\beta(\frac{\xi}{x_N}) - \tilde{f}\frac{\xi}{x_N}\Big)\, d\gamma_\beta = k_0\frac{\partial \xi}{\partial x_N}(0), \qquad \forall\, \xi \in C^\infty_c(\tilde{\Omega}), \tag{4.9}$$

so (4.2) holds.

(ii) By the linearity of $\mathcal{L}_\beta$, we may assume that $f \ge 0$. Let $f_n = f\eta_n$, where $\eta_n(r) = 1 - \eta_0(nr)$ for $r \ge 0$, where η_0 satisfies (2.1) and let v_n be solution of (2.8) where f is replaced by f_n. We see that f_n is bounded and for any $\xi \in C^{1.1}_0(\Omega)$,

$$\int_\Omega v_n\, \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \int_\Omega f_n \frac{\xi}{x_N}\, d\gamma_\beta. \tag{4.10}$$

Then taking $\xi = w_2$ in Lemma 2.6, we have that v_n is uniformly bounded in $L^1(\Omega,\, d\gamma_\beta)$ and in $L^1(\Omega,\, x_N^{-1} d\gamma_\beta)$, that is,

$$\|v_n\|_{L^1(\Omega,\, x_N^{-1} d\gamma_\beta)} \le \|\frac{\xi}{x_N}\|_{L^\infty(\Omega)} \|f_n\|_{L^1(\Omega, d\gamma_\beta)} \le \|\frac{\xi}{x_N}\|_{L^\infty(\Omega)} \|f\|_{L^1(\Omega, d\gamma_\beta)}.$$

Moreover, $\{v_n\}$ is increasing, and then there exists v_f such that

$$v_n \to v_f \quad \text{a.e. in } \ \Omega \quad \text{and in } \ L^1(\Omega,\, x_N^{-1} d\gamma_\beta).$$

Then we have that

$$\int_\Omega v_f \mathcal{L}^*_\beta(\xi)\, d\gamma_\beta = \int_\Omega f\xi\, d\gamma_\beta, \quad \forall\, \xi \in C^{1.1}_0(\Omega).$$

Since $f \in C^\gamma(\overline{\Omega} \setminus \{0\})$, then it follows by the standard regularity theory that $v_f \in C^2(\Omega)$.

We claim that v_f is a classical solution of (4.1). From Corollary 2.8 in [31] with $L^* = \mathcal{L}^*_\beta$, which is strictly elliptic in $\Omega \setminus B_r(0)$, we have that for $q < \frac{N}{N-1}$,

$$\begin{aligned} \|v_n\lambda_\beta\|_{W^{1,q}(\Omega_{2r})} &\le c_{16}\|f\lambda_\beta\|_{L^1(\Omega\setminus B_r(0))} + c_{16}\|v_n\lambda_\beta\|_{L^1(\Omega\setminus B_r(0))} \\ &\le c_{17}\|f\|_{L^1(\Omega,\, d\gamma_\beta)}, \end{aligned} \tag{4.11}$$

where $\Omega_{2r} = \{x \in \Omega \setminus B_{2r}(0) :\ \rho(x) > 2r\}$. We see that

$$-\Delta v_n = -\frac{\beta}{|x|^2} v_n + f.$$

For any compact set K in Ω, it is standard to improve the regularity v_n

$$\|v_n\|_{C^{2,\lambda}(K)} \le c_{18}[\|f\|_{L^1(\Omega,\, d\gamma_\beta)} + \|f\|_{C^\lambda(K)}]$$

where $c_{18} > 0$ is independent of n. Then v_f is a classical solution of (4.1) verifying the identity

$$\int_\Omega v_f\, \mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \int_\Omega \frac{f\xi}{x_N}\, d\gamma_\beta, \quad \forall\, \xi \in C^{1.1}_0(\Omega). \tag{4.12}$$

Together with the fact that $u_{k,f} = k\Lambda^\Omega_\beta + v_f$, we conclude that the function $u_{k,f}$ is a solution of (4.1), verifying the identity (4.2) by (4.12).

Finally, we prove the uniqueness. In fact, let $w_{k,f}$ be a solution of (4.1) verifying the identity (4.2).

$$\int_\Omega (u_{k,f} - w_{k,f})\mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = 0.$$

For any Borel subset O of Ω, Corollary 2.7 implies that problem

$$\begin{cases} \mathcal{L}^*_\beta(\dfrac{u}{x_N}) = \zeta_n & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

has a solution $\eta_{\omega,n} \in C^2(\Omega)\cap C^{0.1}_0(\Omega)$, where $\zeta_n : \bar\Omega \mapsto [0,1]$ is a $C^1(\bar\Omega)$ function such that $\zeta_n \to \chi_O$ in $L^\infty(\Omega)$ as $n \to \infty$. Therefore by passing to the limit as $n \to \infty$, we have that

$$\int_O (u_{k,f} - w_{k,f}) d\gamma_\beta = 0,$$

which implies that $u_{k,f} = w_{k,f}$ a.e. in Ω and then the uniqueness holds true. □

Remark 4.2. Let u_f be the solution of (4.1) verifying the identity (4.2) with $k = 0$, then u_f satisfies the isolated singular behavior (1.14). In fact, letting $f \geq 0$, then $u_f \geq 0$ in Ω. So if (1.14) fails, it implies by the positivity of u_f, that $\liminf_{t\to 0^+} \inf_{z\in S^{N-1}_+} \frac{u_f(tz)}{\Lambda_\beta(tz)} = l_0 > 0$ and $\tilde u_f := u_f - l_0\Lambda^\Omega_\beta$ is a solution of (4.1). By Lemma 2.2, we have that $\tilde u_f \geq 0$ in Ω, By the approximating procedure, $\tilde u_f$ verifies the identity (4.2) with $k = 0$, which is impossible with the fact that $u_f - \tilde u_f = l_0\Lambda^\Omega_\beta$, which satisfies

$$\int_\Omega (u_f - \tilde u_f)\mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = l_0 c_\beta \frac{\partial \xi}{\partial x_N}(0), \quad \forall\, \xi \in C^{1.1}_0(\Omega).$$

4.2. **Nonzero Dirichlet boundary.** Recall that P_Ω is Poisson's Kernel of $-\Delta$ in Ω and $\mathbb{P}_\Omega[g](x) = \displaystyle\int_{\partial\Omega} P_\Omega(x,y)g(y)d\omega(y)$. It is known that if g is continuous, $\mathbb{P}_\Omega[g]$ is a solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

Multiply $\frac{\xi\lambda_\beta}{x_N}$ where $\xi \in C^{1.1}_0(\Omega)$ and integrate over Ω, then we have that

$$\begin{aligned} 0 &= \int_\Omega (-\Delta \mathbb{P}_\Omega[g])\frac{\xi\lambda_\beta}{x_N} dx \\ &= \int_{\partial\Omega} \mathbb{P}_\Omega[g]\nabla(\frac{\xi\lambda_\beta}{x_N})\cdot \nu d\omega + \int_\Omega \mathbb{P}_\Omega[g]\Big(-\Delta(\frac{\xi\lambda_\beta}{x_N})\Big)dx \\ &= \int_{\partial\Omega} g\frac{\partial\xi}{\partial\nu} d\omega_\beta + \int_\Omega \mathbb{P}_\Omega[g]\mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta - \beta\int_\Omega \frac{\mathbb{P}_\Omega[g]}{|x|^2}\frac{\xi}{x_N} d\gamma_\beta, \end{aligned}$$

that is, for any $\xi \in C^{1.1}_0(\Omega)$, there holds

$$\int_\Omega \mathbb{P}_\Omega[g]\mathcal{L}^*_\beta(\frac{\xi}{x_N})\, d\gamma_\beta = \beta\int_\Omega \frac{\mathbb{P}_\Omega[g]}{|x|^2}\frac{\xi}{x_N} d\gamma_\beta - \int_{\partial\Omega} g\frac{\partial\xi}{\partial\nu} d\omega_\beta. \tag{4.15}$$

Lemma 4.3. *Let $\beta \in [\beta_0, +\infty) \setminus \{0\}$, $d\tilde{\omega}_\beta = (1 + |x|^{\tau_+(\beta)})d\omega(x)$ and $g \geq 0$. We have that*

(*i*) *If $g \in C(\partial\Omega \setminus \{0\}) \cap L^1(\partial\Omega, d\tilde{\omega}_\beta)$, then $\frac{1}{|\cdot|^2}\mathbb{P}_\Omega[g] \in L^1(\Omega, d\gamma_\beta)$.*

(*ii*) *If $g \in C(\partial\Omega \setminus \{0\})$ and*

$$\lim_{r\to 0^+} \int_{\partial\Omega\setminus B_r(0)} g\, d\tilde{\omega}_\beta = +\infty, \tag{4.16}$$

then

$$\lim_{r\to 0^+} \int_{\Omega\setminus B_r(0)} \frac{1}{|x|^2}\mathbb{P}_\Omega[g](x) d\gamma_\beta = +\infty.$$

Proof. From Proposition 2.1 in [2] that

$$c_{19}\rho(x)|x-y|^{-N} \leq P_\Omega(x,y) \leq c_{20}\rho(x)|x-y|^{-N}, \quad x \in \Omega,\ y \in \partial\Omega, \tag{4.17}$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$. Since g is continuous in $\partial\Omega \setminus \{0\}$ and Ω is flat near the origin, we can only consider the integrability of $\frac{1}{|\cdot|^2}\mathbb{P}_\Omega[g]$ near the origin. Fix $r = r_0/2$, let $B'_r(0) = \{x' \in \mathbb{R}^{N-1} : |x'| < r\}$ and $e_{(y',0)} = (\frac{y'}{|y'|}, 0)$ for $y' \neq 0$, then

$$\begin{aligned}
&\int_{B_r^+(0)} \frac{1}{|x|^2}\mathbb{P}_\Omega[g] d\gamma_\beta \\
\geq\ & c_{21} \int_{B_r^+(0)} \int_{B'_r(0)\setminus\{0\}} g(y')|x-(y',0)|^{-N} \frac{x_N^2}{|x|^2}|x|^{\tau_+(\beta)}\, dy'dx \\
=\ & c_{22} \int_{B'_r(0)\setminus\{0\}} g(y')|y'|^{\tau_+(\beta)} \int_{B^+_{r/|y'|}(0)} |z-e_{(y',0)}|^{-N} \frac{z_N^2}{|z|^2}|z|^{\tau_+(\beta)} dz dy'
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B_r^+(0)} \frac{1}{|x|^2}\mathbb{P}_\Omega[g] d\gamma_\beta \\
\leq\ & c_{23} \int_{B_r^+(0)} \int_{B'_r(0)\setminus\{0\}} g(y')|x-(y',0)|^{-N} \frac{x_N^2}{|x|^2}|x|^{\tau_+(\beta)}\, dy'dx \\
=\ & c_{24} \int_{B'_r(0)\setminus\{0\}} g(y')|y'|^{\tau_+(\beta)} \int_{B^+_{r/|y'|}(0)} |z-e_{(y',0)}|^{-N} \frac{z_N^2}{|z|^2}|z|^{\tau_+(\beta)} dz dy'.
\end{aligned}$$

Now we do estimates for

$$\int_{B^+_{r/|y'|}(0)} I(z)dz := \int_{B^+_{r/|y'|}(0)} |z-e_{(y',0)}|^{-N} \frac{z_N^2}{|z|^2}|z|^{\tau_+(\beta)} dz,$$

we have

$$0 < \int_{B^+_{\frac{1}{2}}(0)} I(z)\, dz \leq 2^N \int_{B^+_{\frac{1}{2}}(0)} |z|^{\tau_+(\beta)} dz,$$

$$0 < \int_{B^+_{\frac{1}{2}}(e_{(y',0)})} I(z)\, dz \quad \leq \quad 2^{|\tau_+(\beta)|+2} \int_{B^+_{\frac{1}{2}}(e_{(y',0)})} |z-e_{(y',0)}|^{-N} z_N^2 dz$$

$$\leq \quad 2^{|\tau_+(\beta)|+2}\int_{B^+_{\frac12}(0)}|z|^{2-N}dz,$$

and

$$\begin{aligned}&\int_{B^+_{r/|y'|}(0)\setminus\left(B^+_{\frac12}(0)\cup B^+_{\frac12}(e_{(y',0)})\right)} I(z)dz\\ \leq\;& c_{25}\int_{B^+_{r/|y'|}(0)\setminus B^+_{\frac12}(0)}|z|^{-N+\tau_+(\beta)}\,dz\\ \leq\;& \begin{cases} c_{26}\displaystyle\int_{\mathbb{R}^N\setminus B_{\frac12}(0)}|z|^{-N+\tau_+(\beta)}\,dz & \text{if}\quad \beta<0,\\ c_{26}|y'|^{-\tau_+(\beta)} & \text{if}\quad \beta>0\end{cases}\\ \leq\;& c_{27}(1+|y'|^{-\tau_+(\beta)})\end{aligned}$$

and

$$\begin{aligned}\int_{B^+_{r/|y'|}(0)\setminus\left(B^+_{\frac12}(0)\cup B^+_{\frac12}(e_{(y',0)})\right)} I(z)\,dz \quad&\geq\quad c_{28}\int_{B^+_{r/|y'|}(0)\setminus B^+_{\frac12}(0)}|z|^{-N+\tau_+(\beta)}\,dz\\ &\geq\quad c_{29}(1+|y'|^{-\tau_+(\beta)}).\end{aligned}$$

Thus, we have that

$$c_{30}\int_{B'_r(0)\setminus\{0\}}g(y')d\tilde\omega(y')\leq\int_{B^+_r(0)}\frac{1}{|x|^2}\mathbb{P}_\Omega[g]d\gamma_\beta\leq c_{31}\int_{B'_r(0)\setminus\{0\}}g(y')d\tilde\omega(y'),\tag{4.18}$$

which, together with the fact that $\mathbb{P}_\Omega[g]$ is nonnegative and bounded in $\Omega\setminus B^+_r(0)$, proves Lemma 4.3. □

We remark that Lemma 4.3 provides estimates for transforming the boundary data into the nonhomogeneous term. Now we are ready to prove Theorem 1.2 part (i) where we distinguish two cases $\beta\in[\beta_0,\,0]$ and $\beta>0$.

Proof of Theorem 1.2. Part (i). *The existence for* $g\in L^1(\partial\Omega,\,d\tilde\omega_\beta)$. Let $\bar f = f-\frac{\beta}{|\cdot|^2}\mathbb{P}_\Omega[g]$. Then it follows from Lemma 4.3 part (i) that $\bar f\in L^1(\Omega,\,d\gamma_\beta)$ and applying Theorem 4.1 part (i), problem (4.1) verifying (4.2) for $k\in\mathbb{R}$ and replaced f by $\bar f$ admits a unique solution of u_f. Denote $u_{f,g}:=u_f+\mathbb{P}_\Omega[g]$, then

$$\mathcal{L}_\beta u_{f,g}=f\quad\text{and}\quad u_{f,g}=g\quad\text{on}\ \ \partial\Omega\setminus\{0\}.$$

Together with (4.2) and (4.15), we have that $u_{f,g}$ verifies (1.11) and it is the unique solution of problem (4.1) verifying (4.2) for that k.

Case of $\beta\in[\beta_0,\,0]$. Then $d\tilde\omega_\beta$ is equivalent to $d\omega_\beta$, so $L^1(\partial\Omega,\,d\tilde\omega_\beta)=L^1(\partial\Omega,\,d\omega_\beta)$ and we are done.

Case of $\beta>0$. We note that

$$L^1(\partial\Omega,\,d\tilde\omega_\beta)\subsetneqq L^1(\partial\Omega,\,d\omega_\beta).$$

So for $g\in L^1(\partial\Omega,\,d\omega_\beta)\setminus L^1(\partial\Omega,\,d\tilde\omega_\beta)$, we may assume $g\geq 0$ by linearity of $\mathcal{L}_\beta$. Let

$$\eta_n(s)=1-\eta_0(ns)\quad\text{and}\quad g_n(x)=g(x)\eta_n(|x|),\tag{4.19}$$

where η_0 is defined in (2.1). Then $\{g_n\}_n \subset L^1(\partial\Omega,\, d\tilde{\omega}_\beta)$ is an increasing sequence of functions. For simplicity, we assume that $f = 0$. Then the problem

$$\begin{cases} \mathcal{L}_\beta^* u = 0 & \text{in } \Omega, \\ \quad u = g_n & \text{on } \partial\Omega \setminus \{0\} \end{cases} \tag{4.20}$$

has a unique solution of u_n verifying the identify

$$\int_\Omega u_n \mathcal{L}_\beta^*(\frac{\xi}{x_N})\, d\gamma_\beta = -\int_{\partial\Omega} g_n \frac{\partial \xi}{\partial \nu} d\omega_\beta, \quad \forall\, \xi \in C_0^{1.1}(\Omega). \tag{4.21}$$

Since $0 \le g_n \le g$ and $g \in L^1(\partial\Omega,\, d\omega_\beta)$, we may expand the text function space including w_1, w_2, which are the solutions of (2.13) and (2.14) respectively. Taking $\xi = w_1$ and then w_2 , we derive that

$$\|u_n\|_{L^1(\Omega)} \le c_{32} \|g_n\|_{L^1(\partial\Omega,\, d\omega_\beta)} \le c_{33} \|g\|_{L^1(\partial\Omega,\, d\omega_\beta)}$$

and

$$\|u_n\|_{L^1(\Omega,\, x_N^{-1} d\gamma_\beta)} \le c_{34} \|g\|_{L^1(\partial\Omega,\, d\omega_\beta)}.$$

We notice that $u_n \ge 0$ and the mapping $n \mapsto u_n$ is increasing, then by the monotone converge theorem, we have that there exists u such that u_n converging to u in $L^1(\Omega,\, \frac{1}{x_N} d\gamma_\beta)$. Since $\xi \in C_0^{1.1}(\Omega)$, we have that $|\mathcal{L}_\beta^*(\xi/x_N)| \le c x_N^{-1}$. Pass to the limit of (4.21), we have that u verifies that

$$\int_\Omega u \mathcal{L}_\beta^*(\xi/x_N)\, d\gamma_\beta = -\int_{\partial\Omega} g \frac{\partial \xi}{\partial \nu} d\omega_\beta, \quad \forall\, \xi \in C_0^{1.1}(\Omega). \tag{4.22}$$

From standard interior regularity, we have that u is a classical solution

$$\begin{cases} \mathcal{L}_\beta^* u = 0 & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega \setminus \{0\}, \end{cases}$$

which ends the proof. □

5. Nonexistence

In this subsection, we establish the approximation of the fundamental solution G_μ.

Lemma 5.1. *(i) Let $\{\delta_n\}_n$ be a sequence of nonnegative L^∞-functions defined in Ω such that* $\operatorname{supp} \delta_n \subset B_{r_n}(0) \cap \Omega$, *where $r_n \to 0$ as $n \to +\infty$ and*

$$\int_\Omega \delta_n \xi dx \to \frac{\partial \xi(0)}{\partial x_N} \quad \text{as} \quad n \to +\infty, \quad \forall \xi \in C_0^1(\Omega).$$

For any n, let w_n be the unique solution of the problem in the $d\gamma_\beta$-distributional sense

$$\begin{cases} \mathcal{L}_\beta u = \delta_n / \lambda_\beta & \text{in } \Omega \setminus \{0\}, \\ \quad u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{r \to 0^+} \sup\limits_{x \in \partial_+ B_r(0)} \dfrac{|u(x)|}{\Lambda_\beta(x)} = 0. \end{cases} \tag{5.1}$$

Then

$$\lim_{n\to+\infty} w_n(x) = \frac{1}{c_\beta}\Lambda^\Omega_\beta(x), \quad \forall\, x\in\Omega\setminus\{0\}$$

and for any compact set $K\subset\Omega\setminus\{0\}$,

$$w_n \to \frac{1}{c_\beta}\Lambda^\Omega_\beta \quad \text{as} \quad n\to+\infty \ \text{ in } \ C^2(K). \tag{5.2}$$

(ii) Let $\{\sigma_n\}_n$ *be a sequence of nonnegative* L^∞ *functions defined on* $\partial\Omega$ *such that* $\operatorname{supp}\sigma_n\subset\partial\Omega\cap B_{r_n}(0)$, *where* $r_n\to 0$ *as* $n\to+\infty$ *and*

$$\int_{\partial\Omega}\sigma_n\zeta d\omega(x)\to\zeta(0) \quad \text{as} \quad n\to+\infty, \quad \forall\zeta\in C^1(\partial\Omega).$$

For any n, *let* v_n *be the unique solution of the problem*

$$\begin{cases}\mathcal{L}_\beta u=0 & \text{in } \ \Omega\setminus\{0\},\\ u=\dfrac{\sigma_n}{|\cdot|^{\tau_+(\beta)}} & \text{on } \ \partial\Omega\setminus\{0\}\end{cases} \tag{5.3}$$

subject to

$$\int_\Omega v_n\mathcal{L}^*_\beta(\xi/x_N)\,d\gamma_\beta = -\int_{\partial\Omega}\sigma_n\frac{\partial\xi}{\partial\nu}d\omega, \quad \forall\,\xi\in C^{1,1}_0(\Omega).$$

Then

$$\lim_{n\to+\infty} v_n(x) = \frac{1}{c_\beta}\Lambda^\Omega_\beta(x), \quad \forall\, x\in\Omega\setminus\{0\}$$

and for any compact set $K\subset\Omega\setminus\{0\}$, *(5.2) holds true.*

Proof. From Lemma 2.3, problems (5.1) and (5.3) have unique solutions $w_n, v_n\ge 0$ respectively and satisfying that

$$\int_\Omega w_n\mathcal{L}^*_\beta(\frac{\xi}{x_N})\,d\gamma_\beta = \int_\Omega \delta_n\xi\,dx, \quad \forall\,\xi\in C^{1,1}_0(\Omega) \tag{5.4}$$

and

$$\int_\Omega v_n\mathcal{L}^*_\beta(\frac{\xi}{x_N})\,d\gamma_\beta = -\int_{\partial\Omega}\sigma_n\frac{\partial\xi}{\partial\nu}d\omega_\beta, \quad \forall\,\xi\in C^{1,1}_0(\Omega). \tag{5.5}$$

By taking $\xi=\xi_0$, the solution of (2.13), we obtain that

$$\|w_n\|_{L^1(\Omega,\,d\gamma_\beta)}\le\|\xi_0\|_{L^\infty(\Omega)}\|\delta_n\|_{L^1(\Omega)}=\|\xi_0\|_{L^\infty(\Omega)}.$$

For any $r>0$, take ξ with the support in $\Omega\setminus B_r(0)$, then $\xi\in C^{1.1}_c(\overline{\Omega\setminus B_r(0)})$,

$$\int_{\Omega\setminus B_r(0)} w_n\mathcal{L}^*_\mu(\xi)\,d\gamma_\beta=0.$$

Take ξ the solution of (2.17) with $f(x)=\frac{1}{|x|}$, we have that

$$\int_\Omega w_n|x|^{-1}\,d\gamma_\beta=\int_\Omega\delta_n\xi\,dx\le\|\xi_0\|_{L^\infty(\Omega)} \tag{5.6}$$

and

$$\int_\Omega v_n|x|^{-1}\,d\gamma_\beta=-\int_{\partial\Omega}\sigma_n\frac{\partial\xi}{\partial\nu}d\omega_\beta\le\|\nabla\xi_0\|_{L^\infty(\Omega)}. \tag{5.7}$$

So w_n, v_n are uniform bounded in $L^1(\Omega,\,|x|^{-1}d\gamma_\beta)$.

From Corollary 2.8 in [31] with $L^* = \mathcal{L}^*_\mu$, which is strictly elliptic in $\Omega \setminus B_r(0)$, we have that for $q < \frac{N}{N-1}$,

$$\|w_n\lambda_\beta\|_{W^{1,q}(\Omega_{2r})} \le c_{35}\|\delta_n\|_{L^1(\Omega\setminus B_r(0))} + c_{36}\|w_n\|_{L^1(\Omega\setminus B_r(0),\, d\gamma_\beta)} \le c_{37}$$

and

$$\|v_n\lambda_\beta\|_{W^{1,q}(\Omega_{2r})} \le c_{38}\|\sigma_n\|_{L^1(\partial\Omega\setminus B_r(0))} + c_{39}\|v_n\|_{L^1(\Omega\setminus B_r(0),\, d\gamma_\beta)} \le c_{40},$$

where $\Omega_{2r} = \{x \in \Omega\setminus B_{2r}(0) :\, \rho(x) > 2r\}$. By the compact embedding $W^{1,q}(\Omega_{2r}) \hookrightarrow L^1(\Omega_{2r})$, up to some subsequence, there exists $w_\infty,\, v_\infty \in W^{1,q}_{loc}(\Omega)\cap L^1(\Omega,\, d\gamma_\beta)$ such that

$$w_n \to w_\infty \quad \text{as} \quad n \to +\infty \quad \text{a.e. in } \Omega \text{ and in } \quad L^1(\Omega,\, d\gamma_\beta)$$

and it follows by (5.4) and (5.5) that for $\xi \in C^{1.1}_0(\Omega)$,

$$\int_\Omega w_\infty \mathcal{L}^*_\beta(\xi)\, d\gamma_\beta = \int_\Omega v_\infty \mathcal{L}^*_\beta(\xi)\, d\gamma_\beta = \frac{\partial \xi}{\partial x_N}(0).$$

Furthermore,

$$\int_\Omega (w_\infty - \frac{1}{c_\beta} G_\beta)\mathcal{L}^*_\beta(\xi)\, d\gamma_\beta = 0.$$

From the Kato's inequality, we deduce that

$$w_\infty = v_\infty = \frac{1}{c_\beta}\Lambda^\Omega_\beta \quad a.e.\ \ \Omega.$$

Proof of (5.2). For any $x_0 \in \Omega \setminus \{0\}$, let $r_0 = \frac14\{|x_0|,\, \rho(x_0)\}$ and $\mu_n = w_n\eta$, where $\eta(x) = \eta_0(\frac{|x-x_0|}{r_0})$. There exists $n_0 > 0$ such that for $n \ge n_0$, $\text{supp}\mu_n \cap B_{r_n}(0) = \emptyset$. Then

$$\begin{aligned} -\Delta\mu_n(x) &= -\Delta w_n(x)\eta(x) - 2\nabla w_n \cdot \nabla\eta - w_n\Delta\eta \\ &= -2\nabla w_n \cdot \nabla\eta - w_n\Delta\eta, \end{aligned}$$

where $\nabla\eta$ and $\Delta\eta$ are smooth.

We observe that $w_n \in W^{1,q}(B_{2r_0}(x_0))$ and $-2\nabla w_n \cdot \nabla\eta - w_n\Delta\eta \in L^q(B_{2r_0}(x_0))$, then we have that

$$\|\mu_n\|_{W^{2,q}(B_{r_0}(x_0))} \le c\|w_n\|_{L^1(\Omega,\, d\gamma_\beta)},$$

where $c > 0$ is independent of n. Thus, $-2\nabla w_n \cdot \nabla\eta - w_n\Delta\eta \in W^{1,q}(B_{r_0}(x_0))$, repeat above process N_0 steps, for N_0 large enough, we deduce that

$$\|w_n\|_{C^{2,\gamma}(B_{\frac{r_0}{2^{N_0}}}(x_0))} \le c\|w_n\|_{L^1(\Omega,\, d\gamma_\beta)},$$

where $\gamma \in (0,1)$ and $c > 0$ is independent of n. As a conclusion, (5.2) follows by Arzelà-Ascola theorem and Heine-Borel theorem. The above process also holds for v_n. This ends the proof. □

Proof of Theorem 1.2. Part (ii). From (1.12), one of the following two cases holds true,

$$\text{case 1}: \lim_{r\to 0^+}\int_{\Omega\setminus B_r(0)} f\, d\gamma_\beta = +\infty, \quad \text{or case 2}: \quad \lim_{r\to 0^+}\int_{\partial\Omega\setminus B_r(0)} g\, d\omega_\beta = +\infty.$$

Case 1. We argue by contradiction. Assume that problem (1.2) has a nonnegative solution of u_f. Let $\{r_n\}_n$ be a sequence of strictly decreasing positive numbers converging to 0. From the fact $f \in C^{\gamma}_{loc}(\overline{\Omega} \setminus \{0\})$, for any r_n fixed, we have that

$$\lim_{r\to 0^+} \int_{(B_{r_n}(0)\setminus B_r(0))\cap\Omega} f(x) d\gamma_\beta = +\infty,$$

then there exists $R_n \in (0, r_n)$ such that

$$\int_{(B_{r_n}(0)\setminus B_{R_n}(0))\cap\Omega} f d\gamma_\beta = n.$$

Let $\delta_n = \frac{1}{n}\lambda_\beta f \chi_{B_{r_n}(0)\setminus B_{R_n}(0)}$, then the problem

$$\begin{cases} \mathcal{L}_\mu u \cdot \lambda_\beta = \delta_n & \text{in} \quad \Omega \setminus \{0\}, \\ \qquad u = 0 & \text{on} \quad \partial\Omega, \\ \lim\limits_{x\to 0} u(x)\Phi_\mu^{-1}(x) = 0 \end{cases}$$

has a unique positive solution w_n satisfying (in the usual sense)

$$\int_\Omega w_n \mathcal{L}_\mu(\lambda_\beta \xi) dx = \int_\Omega \delta_n \xi dx, \quad \forall\, \xi \in C_0^{1.1}(\Omega).$$

For any $\xi \in C_0^{1.1}(\Omega)$, we have that

$$\int_\Omega w_n \mathcal{L}^*_\mu(\xi)\, d\gamma_\beta = \int_\Omega \delta_n \xi\, dx \to \frac{\partial \xi}{\partial x_N}(0) \quad \text{as} \quad n \to +\infty.$$

Therefore, by Lemma 5.1 for any compact set $\mathcal{K} \subset \Omega \setminus \{0\}$

$$\|w_n - \Lambda^\Omega_\beta\|_{C^1(\mathcal{K})} \to 0 \quad \text{as} \quad n \to +\infty.$$

We fix a point $x_0 \in \Omega$ and let $r_0 = \frac{1}{2}\min\{|x_0|,\, \rho(x_0)\}$ and $\mathcal{K} = \overline{B_{r_0}(x_0)}$, then there exists $n_0 > 0$ such that for $n \geq n_0$,

$$w_n \geq \frac{1}{2} G_\mu \quad \text{in} \quad \mathcal{K}. \tag{5.8}$$

Let u_n be the solution (in the usual sense) of

$$\begin{cases} \mathcal{L}_\mu u \cdot \lambda_\beta = n\delta_n & \text{in} \;\; \Omega \setminus \{0\}, \\ \qquad u = 0 & \text{on} \;\; \partial\Omega, \\ \lim\limits_{r\to 0^+} \sup\limits_{x\in \partial_+ B_r(0)} \dfrac{|u(x)|}{\Lambda_\beta(x)} = 0, \end{cases}$$

then we have that $u_n \geq n w_n$ in Ω. Together with (5.8), we derive that

$$u_n \geq \frac{n}{2}\Lambda^\Omega_\mu \quad \text{in} \;\; \mathcal{K}.$$

Then by comparison principle, we have that $u_f(x_0) \geq u_n(x_0) \to +\infty \quad$ as $\;n \to +\infty$, which contradicts to the fact that u_f is classical solution of (4.1).

Case 2. Similarly for any $n \in \mathbb{N}$, we can take $r_n > R_n > 0$ such that $r_n \to 0$ as $n \to +\infty$ and

$$\int_{(B_{r_n}(0)\setminus B_{R_n}(0))\cap\partial\Omega} g d\omega_\beta = n.$$

Let $\sigma_n = \frac{1}{n} g\chi_{B_{r_n}(0)\setminus B_{R_n}(0)}$, w_n be the solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 \quad \text{in } \Omega\setminus\{0\}, \\ \qquad u = \sigma_n/|\cdot|^{\tau^+(\beta)} \qquad \text{on } \partial\Omega, \end{cases}$$

subject to

$$\int_\Omega w_n \mathcal{L}_\beta^*(\frac{\xi}{x_N})\, d\gamma_\beta = -\int_{\partial\Omega} \sigma_n \frac{\partial \xi}{\partial \nu} d\omega, \quad \forall\, \xi \in C_0^{1.1}(\Omega).$$

Repeat the procedure in Case 1, we get a contradiction which completes the proof. □

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