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NEW APPLICATIONS OF MONOTONICITY METHODS TO A CLASS OF NON-MONOTONE PARABOLIC QUASILINEAR SUB-HOMOGENEOUS PROBLEMS

JESÚS ILDEFONSO DÍAZ

Abstract. The main goal of this survey is to show how some monotonicity methods related with the subdifferential of suitable convex functions lead to new and unexpected results showing the continuous and monotone dependence of solutions with respect to the data (and coefficients) of the problem. In this way, this paper offers 'a common roof' to several methods and results concerning monotone and non-monotone frameworks. Besides to present here some new results, this paper offers also a peculiar review to some topics which attracted the attention of many specialists in elliptic and parabolic nonlinear partial differential equations in the last years under the important influence of Haïm Brezis. To be more precise, the model problem under consideration concerns to positive solutions of a class of doubly nonlinear diffusion parabolic equations with some sub-homogeneous non-monotone forcing terms.

1. Introduction

This survey offers *a common roof* to several methods and results concerning the continuous dependence of solutions with respect to the data in monotone and non-monotone frameworks. So, besides to present here some new results, this paper offers also a peculiar survey to some topics which attracted the attention of many specialists in elliptic and parabolic nonlinear partial differential equations in the last years. We will show how some monotonicity methods (as in Brezis [48] and Lions [114]), related with the subdifferential of suitable convex functions, lead to new results concerning the monotone and continuous dependence of solutions on an unexpected framework for the problem under consideration. Our main goal here is not exactly the existence of solutions but the continuous and monotone dependence of solutions with respect to the data (and coefficients) of the problem in L^2 when the expected space for it is reduced to L^1. Most of the result of this paper will deal with positive solutions of the following class of doubly nonlinear diffusion parabolic equations (in divergence form) with a sub-homogeneous non-monotone forcing term

$$(P)\qquad \begin{cases} \partial_t(u^{2q-1}) - \Delta_p u = f(x,u) + h(t,x)u^{q-1} & \text{in } Q_T := (0,T)\times\Omega, \\ u = 0 & \text{on } \Sigma := (0,T)\times\partial\Omega, \\ u(0,.) = u_0(.) & \text{on } \Omega, \end{cases}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, $T > 0$ and with $\Delta_p u$ the usual $p-$Laplacian operator, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$. We emphasize that probably the interest of our results is not for the applications to the above doubly nonlinear equations but by its method of proof. Moreover, they are new even for the case of a linear diffusion as (P) with $p = 2$. We assume in (P) a possible nonlinear inertia term (i.e. in the time derivative), for some

$$q \in (1, p] \tag{1.1}$$

and a *sub-homogeneous* forcing term $f(x,u) + h(t,x)u^{q-1}$, where

$$h \in L^1(0, T : L^2(\Omega)), \tag{1.2}$$

and with the non-homogeneous perturbation term $f(x,u)$ satisfying the following structural assumptions:

(f1) $f(x,u)$ is a continuous function on $u \in (0,+\infty)$, for a.e. $x \in \Omega$ and $x \to f(x,u)$ belongs to $L^2(\Omega)$, for any $u \in (0,+\infty)$,

(f2) $f(x,u) = f_1(x,u) + f_2(x,u)$ with $\dfrac{f_1(x,u)}{u^{q-1}}$ non increasing and $\dfrac{f_2(x,u)}{u^{q-1}}$ is globally Lipschitz continuous in $u \in (0,+\infty)$, of Lipschitz constant $K \geq 0$, for a.e. $x \in \Omega$,

(f3) $\lim_{r\downarrow 0} \dfrac{f_1(x,r)}{r^{q-1}} = a_0(x)$ with $a_0 \in L^2(\Omega)$.

Additionally, in some cases, we shall need also the condition

(f4) for any $z > 0$ there exists $v_z \in L^\infty(\Omega)$ such that

$$z = \frac{f_1(x, v_z(x))}{|v_z(x)|^{q-1}} - a_0(x) \ a.e. x \in \Omega.$$

Notice that, since we shall not pay attention to the existence of solutions but to the continuous dependence with respect to the data, *no sign condition* is assumed on $h(t,x)$ although we are interested in positive solutions of (P). Notice also that, as in [88], condition (f2) can be simply formulated as

$$f(x,u) - f(x,\widehat{u}) \geq -K\big(u^{q-1} - \widehat{u}^{q-1}\big) \text{ for any } u > \widehat{u} \geq 0 \text{ and a.e. } x \in \Omega.$$

Condition (f4), of technical nature, will be required only when $f_1(x,r)$ is $x-$dependent and express some kind of surjectivity condition of the application $u \mapsto \frac{f_1(x,u)}{u^{q-1}}$, over $(0,+\infty)$. We also point out that assumptions (f1) and (f4), for some $q \in (1,p]$, are compatible with other assumptions, near $r = 0$ and near $r = +\infty$, which arise in the literature and that allows to consider some singular problems. For instance, in [86] it was proved that the necessary and sufficient condition for the existence of a positive solution for the stationary problem associated to (P), when $h(t,x) = K = 0$ is that

$$\lambda_1(-\Delta_p v - a_0 v^{p-1}) < 0$$

and

$$\lambda_1(-\Delta_p v - a_\infty v^{p-1}) > 0, \ a_\infty(x) = \lim_{r\uparrow+\infty} \frac{f_1(x,r)}{r^{p-1}}.$$

There are many variants in the literature: for instance, in [101] (see page 275) it is assumed (for $p = 2$) that $\lim_{r\downarrow 0} \frac{f_1(x,r)}{r^{p-1}} = +\infty$ and that $\lim_{r\uparrow+\infty} \frac{f_1(x,r)}{r^{p-1}} = 0$.

On the initial condition we will assume that

$$u_0 \in L^{2q}(\Omega) \cap W_0^{1,p}(\Omega), \; u_0 > 0 \text{ on } \Omega. \tag{1.3}$$

but some more general conditions are also possible (see Remark 3.3).

Very often the nonlinear diffusion equation is equivalently written, in terms of $W = u^{2q-1}$ with

$$m = \frac{1}{2q-1} \in \left[\frac{1}{2p-1}, 1\right)$$

as

$$(P_{m,p,q}) \qquad \begin{cases} \partial_t W - \Delta_p W^m &= f(x, W^m) + h(t,x)(W^m)^{\frac{1-2m}{2m}} & \text{in } Q_T, \\ W^m &= 0 & \text{on } \Sigma, \\ W(0,.) &= u_0^{2q-1} & \text{on } \Omega. \end{cases}$$

Since $(p-1)m = \frac{p-1}{2q-1} \in [\frac{p-1}{2q-1}, p-1)$, the diffusion operator in problem (P), i.e. $(P_{m,p,q})$, offers three different classes of diffusions, in the terminology of [81], [108], [67], [134], [103], [68], [136]:

i) *fast diffusion* (which corresponds to $(p-1)m < 1$, *i.e.* $q \in (\max(\frac{p}{2}, 1), p])$,
ii) *slow diffusion* (which corresponds to $(p-1)m > 1$, *i.e.* $p > 2$ and $q \in (1, \frac{p}{2})$), and
iii) the case $(p-1)m = 1$ (*i.e.* $q = \frac{p}{2}$), which was considered, for instance, in [65] in connection with *optimal logarithmic Sobolev inequalities*: see also [128].

Since the perturbation in the right hand side can be written as $(W^m)^{\frac{1-2m}{2m}} = W^r$ with $r := \frac{1-2m}{2}$, if we assume, for instance, $p = 2$ then $m \in [\frac{1}{3}, 1)$ and, in particular $0 < r < m < 1$: a case considered for $h = 1$ and $f = 0$ by several authors as, e.g. [119], and [107]: see also [105].

In the limit case $q = 1$ (*i.e.* $m = 1$), the problem *formally* includes a Heaviside function (a model similar to the one which appears in some climate models with the p-Laplace operator) since, roughly speaking, we can approximate the problem by other ones corresponding to a sequence of exponents $q_n \searrow 1$ as $n \to +\infty$ and thus it seems possible to extend the conclusions to the multivalued problem

$$(P_H) \qquad \begin{cases} \partial_t W - \Delta_p W &\in f(x, W) + h(t,x)H(W) & \text{in } Q_T, \\ W &= 0 & \text{on } \Sigma, \\ W(0,.) &= W_0 & \text{on } \Omega, \end{cases}$$

with $H(r)$, the Heaviside, multivalued-function, $H(r) = \{0\}$ if $r < 0$, $H(r) = \{1\}$ if $r > 0$ and $H(0) = [0,1]$. Problems similar to (P_H) appear in many contexts, and, in particular, in climate Energy Balance Models (see, e.g., [87], [38], and their references). For some comparison results concerning solutions of $(P_{m,p,q})$ corresponding to two different values of m see [33]. The continuous dependence on m (even in a more general framework than the one here considered) was studied in [29] and [32].

It is well known (see, *e.g.*, the exposition made in [48], [28], [88], [68]) that the theory of maximal monotone operators on Hilbert spaces [or, more in general, the theory of m-accretive operators in Banach spaces: see, *e.g.*, [18], [31] and the surveys [93] and [37]] can be applied to the above class of problems in the absence of the forcing term or when it is assumed to be globally Lipschitz continuous on the corresponding functional space. But it seems that the applicability of the abstract theory of such type of operators is not well known in the literature when the forcing term is merely *sublinear* (if $p = 2$) or, more generally, *sub-homogeneous* ($q \leq p$ if $p \neq 2$). For some pioneering results we send the reader to [111], [98], [109], [110], [6], [41], [113], [112] and the book [129].

As said before, the main goal of this paper is to show how the above mentioned *monotonicity methods* can be suitably applied also to this class of non-monotone problems, leading to a general framework (specially concerning the x-dependence of coefficients) in which it is possible to show the *continuous and monotone dependence with respect to the data* (the initial datum and the *potential type* coefficient $h(t,x)$) even if there are non-monotone terms in the right hand side.

As a matter of fact, in contrast with the previous literature, we will show that it is possible to give a sense to the solvability of the equation even for time dependent coefficients $h(x,t)$ satisfying merely (1.2) (see some comments on the difficulties arising when using a more classical variational approach in [43], [12], [120]) and, what it is more important, *without prescribing any sign* on $h(x,t)$, which corresponds to the so-called *indefinite perturbed problems* arising, for instance, in population dynamics: see [117], [17], [15] and [11], among many other possible references.

As we will see, it is useful to start our program by considering the sub-homogeneous *simpler* problem corresponding to $f(x,u) \equiv 0$, *i.e.* the problem

$$(P_q) \qquad \begin{cases} \partial_t(u^{2q-1}) - \Delta_p u &= h(t,x)u^{q-1} \quad \text{in } Q_T, \\ u &= 0 \quad \text{on } \Sigma, \\ u(0,.) &= u_0(.) \quad \text{on } \Omega. \end{cases}$$

The existence and uniqueness of a L^1-mild positive solution when $h(t,x) \leq 0$ is a consequence of the well-known m-T-accretivity results of the associated operator (see, e.g. [26], [88] and [136]). Nevertheless, since the right hand side is non-Lipschiz continuous, problem (P_q) (and also problem (P)) may have more than one solution (in particular when $h(t,x)$ is changing sign and negative near Σ and we assume $p > 2$ and $q \in (1, \frac{p}{2})$). Nevertheless we can introduce a method to select only one L^1-mild positive solution by means of some monotonicity arguments. Indeed, we will select the L^1-mild positive solution u of (P_q) such that $w(\frac{q}{2q-1}t) = u(t)^q$ coincides with the unique L^2-mild positive solution of the problem

$$\begin{cases} \dfrac{dw}{dt} + \partial J_{0,q}(w) \ni h(t) \quad \text{in } L^2(\Omega), \\ w(0) = w_0, \end{cases} \tag{1.4}$$

where $J_{0,q}$ is the functional in $L^2(\Omega)$ given by

$$J_{0,q}(w) = \begin{cases} \dfrac{q}{p}\displaystyle\int_\Omega |\nabla w^{\frac{1}{q}}|^p \mathrm{d}x & \text{if } w \in D(J_{0,q}), \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$D(J_{0,q}) := \{w \in L^2(\Omega) \text{ such that } w \geq 0 \text{ and } w^{\frac{1}{q}} \in W_0^{1,p}(\Omega)\}.$$

Developing an idea of Díaz and Saá [86] (for $p \neq 2$) we will see that $J_{0,q}$ is a convex, lower semicontinuous functional and thus its subdifferential $\partial J_{0,q}(w)$ is well defined and the uniqueness of a L^2-mild positive solution w of (1.4) is well-known. In that case we say that $u(t)$ is the *selected* L^1-*mild positive solution* of (P_q) (and so it is unique). Of course that if, under some additional assumptions, it can be shown the uniqueness of a positive weak solution of the equation then necessarily it must coincides with the selected L^1-mild positive solution (see, *e.g.*, [119], [107], [105], [59], [90] and [61], among others).

In Section 2 of this paper we will study the subdifferential $\partial J_{0,q}(w)$. We we will prove that, given $\mu > 0$ and $h \in L^2(\Omega)$, the resolvent equation

$$w + \mu \partial J_{0,q}(w) \ni h \tag{1.5}$$

is connected, through the relation $w = u^q$, with the auxiliary variational problem

$$\min_{v \in K} J_{h,q}(v)$$

where

$$K := \left\{ v \in W_0^{1,p}(\Omega) \cap L^{2q}(\Omega), v \geq 0 \text{ on } \Omega \right\}$$

and

$$J_{h,q}(v) := \frac{\mu}{p} \int_\Omega |\nabla v|^p \mathrm{d}x + \frac{1}{2q} \int_\Omega |v|^{2q} \mathrm{d}x - \frac{1}{q} \int_\Omega h(x)|v|^q \mathrm{d}x.$$

Since the problem is sub-homogeneous ($q \in (1,p]$) the different terms of $J_{h,q}(v)$ satisfy good growth conditions and the existence and uniqueness of a minimum $v_{h,q} \in K$ can be obtained by standard direct methods of the Calculus of Variations (see, *e.g.*, Lemma 5 of [23] for the case $p = 2$ and [132] for $p > 1$ and $q \in [1,p]$). Once again, the Euler-Lagrange equation

$$-\mu \Delta_p v + v^{2q-1} = h(x) v^{q-1} \text{ in } \Omega, \tag{1.6}$$

may have other weak solutions $\widehat{v} \in W_0^{1,p}(\Omega)$ different to the minimum v of $J_{h,q}$ (specially if the sign of $h(x)$ is not prescribed, $h(x)$ is negative near Σ and we assume $p > 2$ and $q \in (1, \frac{p}{2}))$) but the relation $w = u^q$ allows to select only v when we assume that w is the solution of (1.5).

As we shall show in Section 3, the definition of a unique $u(t)$ *selected* L^1-*mild positive solution* of (P) can be also obtained for the general case of $f \neq 0$ as indicated before by following a similar process to the indicated above. The main result of this paper is the following:

Theorem 1.1. *Let $q \in (1,p]$ and $h \in L^1(0,T : L^2(\Omega))$. Let u_0, f satisfying (1.3) and (f1)-(f3). Assume that $f_1(x,u) = f_1(u)$ independent of x or $f_1(x,u)$ satisfying also (f4). Then for any $T > 0$, there exists a unique selected positive L^1-mild solution u to problem* (P) *and $u^q \in C([0,T] : L^2(\Omega))$. In addition, if $h \in L^\infty(0,T : L^\infty(\Omega))$ and $u_0 \in L^\infty(\Omega)$ then $u \in L^\infty(0,T : L^\infty(\Omega))$. Moreover, if v_0 and g satisfy the same conditions than u_0 and h, and if v is the respective selected positive L^1-mild solution of problem* (P), *then, for any $t \in [0,T]$ we have the monotone* *continuous dependence estimate*

$$\begin{aligned} \|(u^q(t)-v^q(t))^+\|_{L^2(\Omega)} &\le e^{Kt}\|(u_0^q-v_0^q)^+\|_{L^2(\Omega)} \\ &+\int_0^t e^{K(t-s)}\,\|[h(s)-g(s)]_+\|_{L^2(\Omega)}\,ds, \end{aligned} \tag{1.7}$$

where $K \ge 0$ is the constant indicated in (f2).

Notice that, in particular, for the case of a slow diffusion, $p > 2$ and $q \in (1, \frac{p}{2})$, the above conclusions hold for 'flat solutions' (*i.e.* positive solutions such that $u = \frac{\partial u}{\partial n} = 0$ on Σ). Notice that even for the special case $h = g$ estimate (1.7) is new for the doubly nonlinear problem (P): indeed, as indicated before the accretivity results of the doubly nonlinear diffusion operator leads only to $L^1(\Omega)-$monotone continuous dependence estimates (if $p = 2$ such estimates also hold on $H^{-1}(\Omega)$ [48]), but not in $L^2(\Omega)$ (see, e.g., Bénilan [27]) as it is expressed in (1.7).

We point out that, obviously, the function $u_\infty(x) \equiv 0$ in Ω is a trivial solution of the stationary problem associated to (P). Here we are interested on positive solutions of problem (P). We will prove (see Theorem 3.9) that, in fact, if $q \in (1, \frac{p}{2})$ and $p > 2, f(x,u) \equiv 0,\ h \in L^1(0,T : L^2(\Omega))$, $h \ge 0$ and $u_0 \gneqq 0$ then there is no extinction in finite time, so that $\|u^{2q-1}(t)\|_{L^2(\Omega)} > 0$ for any $t > 0$. The situation is different if $q \in (\frac{p}{2}, p]$ since, at least for $f(x,u) \equiv 0$ and $h \le 0$, there is a finite extinction time $T_e > 0$, such that $w(t) \equiv 0$, in Ω, for any $t \ge T_e$. In that case, we understand that the L^1-mild solution $u(t)$ of (P) also extinguishes in Ω after T_e.

In the Section 3 we will study of the auxiliary simplified problem (P_q) through the study of the subdifferential operator $\partial J_{0,q}(v)$ in $L^2(\Omega)$. This will allow to get the proof of Theorem 1.1 by application of some abstract results on monotone operators on Hilbert spaces. Many other variants, commented in form of a series of Remarks, opening the application of this view point to many other different formulations, will be presented. This is the case, for instance when the p-Laplacian is replaced by an homogeneous diffusion operator of the form $\operatorname{div}(a(x,\nabla u))$ with the homogeneity condition

$$A(x,t\xi) = |t|^p\, A(x,\xi) \text{ for all } t \in \mathbb{R} \text{ and all } (x,\xi) \in \Omega \times \mathbb{R}^N,$$

where $a(x,\xi) = \frac{1}{p}\partial_\xi A(x,\xi)$.

2. On the subdifferential of $J_{0,q}$

The proof of the main results will be obtained through the study of the Cauchy problem

$$\begin{cases} \dfrac{dw}{dt} + \partial J_{0,q}(w) \ni h(t) & \text{in } L^2(\Omega) \\ w(0) = w_0, \end{cases}$$

with $J_{0,q}$ the functional presented in the Introduction. The convexity of $J_{0,q}$ will play a crucial role in the rest of the paper.

Lemma 2.1. *Given $q \in (1, p]$, the functional $J_{0,q}$ is convex, lower semicontinuous and proper on $L^2(\Omega)$.*

Proof. The proof for the case $q = p$ was given in Lemma 1 of [86], and the proof for the case $q \in (1, p)$ was obtained in [132] (see Lemma 4 and Example 5.2). A different proof of this last case can be obtained from Proposition 2.6 of [46]. To prove that $J_{0,q}$ is lower semicontinuous in $L^2(\Omega)$ it suffices to prove that if we have a sequence $\rho_n \to \rho$ in $L^2(\Omega)$ such that $J_{0,q}(\rho_n) \leq \lambda$ then $J_{0,q}(\rho) \leq \lambda$. But since $\rho_n^{1/q}$ is bounded in $W_0^{1,p}(\Omega)$ there exists a subsequence, still labeled as $\rho_n^{1/q}$, such that $\rho_n^{1/q}$ converges weakly in $W_0^{1,p}(\Omega)$, so that $\nabla \rho_n^{1/q}$ converges weakly in $L^p(\Omega)^N$ and since the norm is lower semicontinuous we obtain that $\liminf_n J_{0,q}(\rho_n) \geq J_{0,q}(\rho)$, and hence $J_{0,q}(\rho) \leq \lambda$. □

Remark 2.2. As indicated in [86], the main results of [86] were presented in September 1985 in [85]. Its Lemma 1 extends and develops to the case $p \neq 2$ Remark 2 of Brezis and Oswald [55] which was inspired in the paper Benguria, Brezis and Lieb [23] where some previous results of Rafael Benguria's Ph.D. thesis [22] were presented together with some newer results. So, in contrast to what is indicated in [46], the consideration of the case $p \neq 2$ was not carried for the first time in [21] but in [85], [86] seventeen years before. The extension to the case of $\mathbb{R}^N$ was carried out in [60] (for an extension to weaker solutions see [64]).

Remark 2.3. It seems, that the connection between Lemma 1 of [86] (called by some authors *Díaz-Saá inequality* when $q = p$, [60], [133]) and the generalization of the 1910 Picone inequality [121] (concerning originally with ordinary differential equations and much more later extended to partial differential equations in [2]; see, also the survey [91]) was pointed out for the first time in Chaib [60]. As a matter of fact, it was proved in Section 3.2 of [46] that the convexity of $J_{0,q}$ (for any $q \in (1, p]$) is equivalent to the generalized Picone inequality

$$\frac{1}{p} |\nabla u|^{p-2} \langle \nabla u, \nabla \left(\frac{z^q}{u^{q-1}} \right) \rangle \leq \frac{q}{p} |\nabla z|^p + \frac{p-q}{p} |\nabla u|^p \text{ a.e. on } \Omega$$

if $u, z \in W_{loc}^{1,p}(\Omega),\ u > 0,\ z \geq 0$ on Ω.

We recall that given a convex, l.s.c. function $\phi : H \to (-\infty, +\infty]$, ϕ proper, over a Hilbert space H, a pair $(w, z) \in H \times H$ is such that $z \in \partial\phi(w)$ if $\forall \xi \in H$, $\phi(\xi) \geq \phi(w) + (z, \xi - w)$. We say that $w \in D(\phi) := \{v \in H \text{ such that } \phi(v) < +\infty\}$ is such that $w \in D(\partial\phi)$ if the set of $z \in \partial\phi(w)$ is not empty. We have

$$D(\partial J_{0,q}) \subset D(J_{0,q}) \subset \overline{D(J_{0,q})}^{L^2} = \overline{D(\partial J_{0,q})}^{L^2}$$

(see Proposition 2.11 of Brezis [49]). The following result proves that the operator $\partial J_{0,q}$ satisfies an additional property to the mere monotonicity: it is a $T-$monotone operator in $L^2(\Omega)$ in the sense of Brezis-Stampacchia ([56]). This will explain later the comparison of solutions of problem (P) with respect to different data $h(t, x)$ for solutions.

Lemma 2.4. *Let $\tau(s) = s_+$. Then for any $w, \widehat{w} \in L^2(\Omega)$*

$$J_{0,q}(w - \tau(w - \widehat{w})) + J_{0,q}(\widehat{w} + \tau(w - \widehat{w})) \leq J_{0,q}(w) + J_{0,q}(\widehat{w}). \tag{2.1}$$

In particular $\partial J_{0,q}$ is a $T-$monotone operator in $L^2(\Omega)$, i.e. for any $w, \widehat{w} \in D(\partial J_{0,q})$ and $z \in \partial J_{0,q}(w)$, $\widehat{z} \in \partial J_{0,q}(\widehat{w})$,

$$\int_\Omega (z - \widehat{z})[w - \widehat{w}]_+ \mathrm{d}x \geq 0, \tag{2.2}$$

and given $h, \widehat{h} \in L^2(\Omega)$, if for $\mu > 0$, $w, \widehat{w} \in L^2(\Omega)$ are such that

$$w + \mu\partial J_{0,q}(w) \ni h \text{ and } \widehat{w} + \mu\partial J_{0,q}(\widehat{w}) \ni \widehat{h}, \tag{2.3}$$

then

$$\|[w - \widehat{w}]_+\|_{L^2(\Omega)} \leq \left\|[h - \widehat{h}]_+\right\|_{L^2(\Omega)}. \tag{2.4}$$

Proof. Property (2.1) is equivalent to the inequality

$$J_{0,q}(\min(w, \widehat{w})) + J_{0,q}(\max(w, \widehat{w})) \leq J_{0,q}(w) + J_{0,q}(\widehat{w}). \tag{2.5}$$

Obviously we can assume $w, \widehat{w}, \min(w, (\widehat{w}-k)), \max((w-k), \widehat{w}) \in D(J_{0,q}) := \{v \geq 0$ and $v^{\frac{1}{q}} \in W_0^{1,p}(\Omega) \cap L^{\frac{2}{q}}(\Omega)\}$ and then, by Stampacchia's truncation results, we can write

$$\int_\Omega |\nabla \min(w, \widehat{w})^{\frac{1}{q}}|^p \mathrm{d}x = \int_{\{w \leq \widehat{w}\}} |\nabla w^{\frac{1}{q}}|^p \mathrm{d}x + \int_{\{w > \widehat{w}\}} |\nabla \widehat{w}^{\frac{1}{q}}|^p \mathrm{d}x$$

and

$$\int_\Omega |\nabla \max(w, \widehat{w})^{\frac{1}{q}}|^p \mathrm{d}x = \int_{\{w > \widehat{w}\}} |\nabla w^{\frac{1}{q}}|^p \mathrm{d}x + \int_{\{w \leq \widehat{w}\}} |\nabla \widehat{w}^{\frac{1}{q}}|^p \mathrm{d}x.$$

Adding both expressions we get inequality (2.5). To show that (2.5) implies that $\partial J_{0,q}$ is a $T-$monotone operator in $L^2(\Omega)$, i.e. (2.2) we shall develop a suggestion made by H. Brezis in Remark 1.10 of [47]. Since $z \in \partial J_{0,q}(w)$ and $\widehat{z} \in \partial J_{0,q}(\widehat{w})$ we know that

$$J_{0,q}(v) - J_{0,q}(w) \geq \int_\Omega z[v - w]\mathrm{d}x \geq 0 \quad \text{for any } v \in L^2(\Omega),$$
$$J_{0,q}(v) - J_{0,q}(\widehat{w}) \geq \int_\Omega \widehat{z}[v - \widehat{w}]\mathrm{d}x \geq 0 \quad \text{for any } v \in L^2(\Omega).$$

By taking $v = \min(w, \widehat{w}) = w - [w - \widehat{w}]_+$ in the first of the two inequalities, and $v = \max(w, \widehat{w}) = \widehat{w} + [w - \widehat{w}]_+$ in the second one, using that

$$\min(w, \widehat{w}) - w = -[w - \widehat{w}]_+ \text{ and } \max(w, \widehat{w}) - \widehat{w} = [w - \widehat{w}]_+,$$

by adding the results we get

$$J_{0,q}(\min(w, \widehat{w})) + J_{0,q}(\max(w, \widehat{w})) - J_{0,q}(w) - J_{0,q}(\widehat{w}) \leq -\int_\Omega (z - \widehat{z})[w - \widehat{w}]_+ \mathrm{d}x,$$

and thus inequality (2.5) implies property (2.2). By well-known results (see Section IV.4 of Brezis [49]) we get conclusion (2.4). □

Remark 2.5. For some convex functionals J a stronger property than (2.1) holds:

$$J(\min(w, (\widehat{w} - k))) + J(\max((w - k), \widehat{w})) \leq J(w) + J(\widehat{w}) \tag{2.6}$$

for any $k > 0$. This property is equivalent ([35]) to inequality (2.1) for any $\tau : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous with $0 \leq \tau' \leq 1$ and $\tau(0) = 0$ and for any $k > 0$. This property (2.1) implies several important properties for the realization of the operator

$w \to \partial J(w)$ over the Banach spaces $L^s(\Omega)$, $1 \leq s \leq +\infty$ (see Lemma 3 of [57] and its generalization in a series of papers (Théorème 1.2 and Remark 1.4 of [35], [24], [25]) and ([35]). Property (2.1) holds for the class of the, so called, *normal convex functionals* (see the above mentioned references) but to check it for the special case of the functional $J_{0,q}$ remains as an open problem (some partial results can be obtained in this direction: see Remark 3.11).

An uneasy task is to identify the operator $\partial J_{0,q}$ involved in the *resolvent equation* (2.3) in terms of the Euler-Lagrange equation associated to the functional $J_{0,q}$. When trying to do that directly, using merely the functional $J_{0,q}$, we see that, if we assume that $w > 0$ on Ω, given a *direction test function* $\zeta \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ the Gâteaux derivative of $J_{0,q}$ in w in the direction ζ is given formally by

$$J'_{0,q}(w;\zeta) = -\int_\Omega \frac{\Delta_p(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}}\zeta \mathrm{d}x. \tag{2.7}$$

Thus, at least formally, the convexity of $J_{0,q}$ implies the monotonicity in $L^2(\Omega)$ of its subdifferential and so

$$\int_\Omega \left(-\frac{\Delta_p(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} + \frac{\Delta_p(\widehat{w}^{\frac{1}{q}})}{\widehat{w}^{\frac{q-1}{q}}} \right)(w - \widehat{w})\mathrm{d}x \geq 0. \tag{2.8}$$

In [86] it was shown that expression (2.7) is well justified if we assume $w \in \mathcal{D}(J_{0,q})$ and $w, \Delta_p(w^{\frac{1}{q}}) \in L^\infty(\Omega)$. A different justification was made in Remark 3.3 of Takač [131], this time under the additional condition that $w > 0$ on any compact subset $M \subset \Omega$,

$$\frac{\Delta_p(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} \in \mathcal{D}'(\Omega),$$

and $w \in C^0(\Omega)$. Nevertheless, it is possible to get some more general justifications when instead of analyzing separately $J'_{0,q}(w;\zeta)$ we consider the *resolvent equation* (2.3). The following result is inspired by Lemma 6 of [23] concerning a related problem in which $p = q = 2$ and $N = 3$.

Lemma 2.6. *Given $q \in (1,p]$, $h \in L^2(\Omega)$ and $\mu > 0$, assume that $w \in D(\partial J_{0,q})$, $w \geq 0$, satisfies the resolvent equation (1.5). Then function $v := w^{\frac{1}{q}}$ satisfies that $v \in W_0^{1,p}(\Omega) \cap L^{2q}(\Omega)$, $\Delta_p v, h(x)v^{q-1} \in L^1(\Omega)$, v is positive in the sense that*

$$\left|\left\{x \in \Omega : v(x) = 0\right\}\right| = 0, \tag{2.9}$$

and v satisfies the sub-homogeneous equation (1.6) in the sense of distributions. Moreover,

i) if $1 < q < p$ and $0 < h_-(x) = \max(-h(x), 0) \leq C_{h_-}$ near $\partial\Omega$

$$v(x) \geq Cd(x,\partial\Omega)^{\frac{p}{p-q}} \text{ a.e. } x \in \Omega, \text{ for some } C > 0 \text{ dependent of } C_{h_-}, \tag{2.10}$$

ii) if $h_-(x) \equiv 0$ near $\partial\Omega$ and $p > 2$ with $q \in (1, \frac{p}{2})$ then

$$v(x) \geq Cd(x,\partial\Omega)^{\frac{p}{p-2q}} \text{ a.e. } x \in \Omega, \text{ for some } C > 0 \text{ independent on } h, \tag{2.11}$$

iii) if $h_-(x) \equiv 0$ *near* $\partial\Omega$ *and* $q \in [\frac{p}{2}, p)$ *if* $p > 2$, *or* $q \in (\max(1, \frac{p}{2}), p)$ *if* $p \leq 2$, *then*

$$v(x) \geq Cd(x, \partial\Omega) \text{ a.e. } x \in \Omega, \text{ for some } C > 0 \text{ independent on } h, \tag{2.12}$$

iv) if $q = p$ *then*

$$v(x) \geq Cd(x, \partial\Omega) \text{ a.e. } x \in \Omega, \text{ for some } C > 0 \text{ independent on } h. \tag{2.13}$$

Proof. Since $D(\partial J_{0,q}) \subset D(J_{0,q})$ we know that $v = w^{\frac{1}{q}} \in W_0^{1,p}(\Omega) \cap L^{2q}(\Omega)$. Moreover $h(x)v^{q-1} \in L^1(\Omega)$ since $v \in L^{2q-2}(\Omega)$ and $h \in L^2(\Omega)$. Therefore the equation (1.6) has a meaning in the sense of distributions. Let $\eta \in \widetilde{D} := W_0^{1,p}(\Omega) \cap L^{2q}(\Omega)$ (i.e. without the sign condition $\eta \geq 0$). Define the functional

$$J_{h,q}(\eta) = \frac{\mu}{p} \int_\Omega |\nabla\eta|^p \mathrm{d}x + \frac{1}{2q} \int_\Omega |\eta|^{2q} \mathrm{d}x - \frac{1}{q} \int_\Omega h(x)|\eta|^q \mathrm{d}x.$$

Therefore, for every $\eta \in \widetilde{D}$

$$J_{h,q}(v) \leq J_{h,q}(\eta)$$

so, v is a minimum of $J_{h,q}$ on $\widetilde{D}$. Now, for $\zeta \in C_0^\infty(\Omega)$, using that $d(J_{h,q}(v + \epsilon\zeta))/d\epsilon = 0$ we conclude easily that

$$\mu \int_\Omega |\nabla v|^{p-2} \nabla v \nabla\eta \mathrm{d}x + \int_\Omega v^{2q-1}\eta \mathrm{d}x = \int_\Omega h(x)v^q \eta \mathrm{d}x,$$

which proves v satisfies (1.6) and $\Delta_p v \in L^2(\Omega)$. On the other hand,

$$-\frac{\Delta_p(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} = h(x) - w \in L^2(\Omega),$$

so, necessarily, w is positive (in the sense of (2.9)). Moreover, using the decomposition $h(x) = h_+(x) - h_-(x)$, with

$$h_+(x) = \max(h(x), 0), \; h_-(x) = \max(-h(x), 0),$$

we can write (1.6) as

$$-\mu\Delta_p v + v^{2q-1} + h_-(x)v^{q-1} = h_+(x)v^{q-1} \text{ in } \Omega.$$

The proof of iii) and v) is consequence of the strong maximum principle ([135], [124]) once that $v \geq 0$ on Ω, $-\mu\Delta_p v + v^{2q-1} + h_-(x)v^{q-1} \geq 0$ and since the zero order terms in the above inequality are super-homogeneous ($2q - 1 \geq p - 1$ and $h_-(x) = 0$ near $\partial\Omega$ if $q \in [\frac{p}{2}, p)$).

To prove i) and ii) notice that in both cases there is a *strong absorption with respect to the diffusion* once we write

$$-\mu\Delta_p v + v^{2q-1} + h_-(x)v^{q-1} = h_+(x)v^{q-1}.$$

In the case ii), if $h_-(x) = 0$ on a neighborhood D_δ of $\partial\Omega$, with $D_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$, for some $\delta > 0$, then $-\mu\Delta_p v + v^{2q-1} \geq 0$ in D_δ. Given $M > 0$ and $\epsilon > 0$, small enough, the set

$$\Omega_{\epsilon,M} = \{x \in \Omega : \epsilon \leq v(x) \leq M\}$$

is a neighborhood of $\partial\Omega$ contained in D_δ (i.e. $\Omega_{\epsilon,M} \subset D$). Then, for any $x_0 \in \partial\Omega_{\epsilon,M}$, we can use a local barrier function $\underline{V}(x)$ based on the expression

$c\,|x-x_0|^{\frac{p}{p-2q}}$ over the set $\Omega_{\epsilon,M}\cap B_\delta(x_0)$, for some $c>0$. As in the proof of Theorem 2.3 of [5], it is possible to chose $c>0$ (independent of h) such that $\underline{V}(x)$ is a local subsolution, in the sense that

$$\begin{cases} -\mu\Delta_p\underline{V}+\underline{V}^{2q-1}\le 0 & \text{in } \Omega_{\epsilon,M}\cap B_\delta(x_0),\\ \underline{V}\le v & \text{on } \partial(\Omega_{\epsilon,M}\cap B_\delta(x_0)).\end{cases}$$

Thus, by the weak comparison principle $v(x)\ge\underline{V}(x)$ on $\Omega_{\epsilon,M}\cap B_\delta(x_0)$, which implies (2.11) since Ω is bounded (see an alternative direct proof, for $N=1$, in Proposition 1.5 of [70]).

The proof of i) follows also those type of arguments. Since $q<p$ and $h_-(x)\le\overline{h}_-$ on a neighborhood D_δ of $\partial\Omega$ we can built a local subsolution $\underline{V}^*(x)$ on the set $\Omega_{\epsilon,M}$ (a neighborhood D_δ of $\partial\Omega$) such that

$$-\mu\Delta_p\underline{V}^*+\overline{h}_-\underline{V}^{*q-1}\le 0 \text{ in } \Omega_{\epsilon,M}\cap B_\delta(x_0),$$

and the same above arguments apply (leading to the estimate (2.11) since Ω is bounded) but now building the subsolution by modifying the function $c\,|x-x_0|^{\frac{p}{p-q}}$ with c depending on $\overline{h}_-$. □

It is useful to study some additional properties satisfied by the subdifferential $\partial J_{0,q}$.

Lemma 2.7. i) *$\partial J_{0,q}$ generates a compact semigroup over $L^2(\Omega)$.* ii) *the resolvent operator $(I+\mu\partial J_{0,q})^{-1}$ leaves invariant the subspace $L^\infty(\Omega)$; i.e. if $h\in L^\infty(\Omega)$ and if $w\in D(\partial J_{0,q})$, $w\ge 0$, satisfies (1.5) then $w\in L^\infty(\Omega)$, for any $\mu>0$.*

Proof. i) Let $\{h_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\Omega)$,

$$\|h_n\|_{L^2(\Omega)}\le M.$$

In particular, $h_n\rightharpoonup h$ in $L^2(\Omega)$ to some $h\in L^2(\Omega)$. Let $w_n\in D(\partial J_{0,q})$, $w_n\ge 0$ be the associated solution of (1.5) for any given $\mu>0$. Then, by Lemma 2.6 $v_n:=w_n^{\frac{1}{q}}$ satisfies that $v_n\in W_0^{1,p}(\Omega)\cap L^{2q}(\Omega)$, $\Delta_p v_n, h_n(x)v_n^{q-1}\in L^1(\Omega)$, v_n is positive and satisfies the sub-homogeneous equation

$$-\mu\Delta_p v_n+v_n^{2q-1}=h_n(x)v_n^{q-1} \text{ in } \Omega, \tag{2.14}$$

in the sense of distributions. By multiplying the equation $w_n+\mu\partial J_{0,q}(w_n)\ni h_n$ by w_n, from the monotonicity of $\partial J_{0,q}$ we get

$$\|w_n\|_{L^2(\Omega)}\le M$$

and so

$$\|v_n\|_{L^{2q}(\Omega)}\le M.$$

Thus

$$\left\|-\mu\Delta_p v_n+v_n^{2q-1}\right\|_{L^1(\Omega)}\le M'$$

for some $M'>0$ (independent on n) and thus there exists a subsequence such that $v_n\to v$ strongly in $L^1(\Omega)$ and weakly in $W^{1,s}(\Omega)$ for any $1\le s\le N(p-1)/(N-$

1) (see, e.g., [67] Chapter 4 and its references). By the dominated convergence Lebesgue theorem $v_n^q \to v^q$ strongly in $L^1(\Omega)$. Moreover, integrating by parts

$$\mu \int_\Omega |\nabla v|^p \mathrm{d}x + \int_\Omega v^{2q} \mathrm{d}x \leq M''$$

for some $M'' > 0$ and then $v \in W_0^{1,p}(\Omega) \cap L^{2q}(\Omega)$, $\Delta_p v$, $h(x)v^{q-1} \in L^1(\Omega)$ (see, e.g. [43]) and so $w_n \to w$ in $L^2(\Omega)$. Applying the results of [50] (see also Theorem 2.2.2 of [141]) we get the conclusion.

The proof of ii) follows by the Stampacchia iteration method and it is an obvious modification of Theorem 5.5 of ([43]) (notice that their arguments, for the case $1 < q < p$, apply also for this special purpose to the limit case $q = p$). □

Remark 2.8. Notice that the functional $J_{h,q}$ may have other stationary points different to $w^{1/q}$, with w solution of the resolvent equation (1.5). What the above lemma shows is that the relation $v = w^{1/q}$ gives a uniqueness criterion for positive solutions of (1.6). The positivity of v is fundamental since it is known that if $|\{x \in \Omega : v(x) = 0\}| > 0$ (which arise, in particular, when $h(x) \leq -\overline{h}_- < 0$ in a neighborhood of $\partial\Omega$ and $q < p$ ([130])) there is multiplicity of nonnegative solutions of (1.6) (see also [17]). Nevertheless, if $q < p$, the uniqueness result applies to 'flat solutions' (*i.e.* positive solutions such that $u = \frac{\partial u}{\partial n} = 0$ on Σ) (see [77]).When the set $\{x \in \Omega\colon\ h(x) < 0\}$ is *big enough* (or if $\{x \in \Omega\colon\ h(x) = 0\}$ is *big enough* and $q \in (1, p)$) there are some nonnegative solutions v of (1.6) which may vanish on some positively measured subset of Ω (and so their support is strictly included in $\overline{\Omega}$). This property (which does not holds when $v = w^{1/q}$ with w solution of (1.5)) can be obtained by comparison methods: through a refined version of [39] (see [67], [69]), by local energy type methods ([10]), etc.

Remark 2.9. It is clear that it is possible to consider equations like (1.6) with some different balances between the nonlinear absorption (v^{2q-1}) and forcing (v^{q-1}) terms. Our special case is motivated by the application of the semigroup theory to the operator $\partial J_{0,q}(w)$ in $L^2(\Omega)$.

Remark 2.10. Lemma 2.6 admits many generalizations dealing with $h \notin L^2(\Omega)$ but still with solutions $v \in W_0^{1,p}(\Omega) \cap L^{2q}(\Omega)$**.** It seems possible to complement inequality (2.4) by other inequalities involving different exponents on the norms of the data and the solutions (see, *e.g.*, [43] and [120] in the parabolic framework and Remark 3.11).

Remark 2.11. It is possible to extend the above approach by replacing the p-Laplace operator by more general quasilinear homogeneous operators of the form $\mathrm{div}(a(x, \nabla u))$ with

$$A(x, t\xi) = |t|^p\, A(x, \xi) \text{ for all } t \in \mathbb{R},\ \xi \in \mathbb{R}^N \text{and a.e. } x \in \Omega,$$

where

$$a(x, \xi) = \frac{1}{p} \partial_\xi A(x, \xi)$$

(see [131] and [104]). We point out that the application of the abstract results of the accretive operators theory allows also the consideration of this type of diffusion operators (see, *e.g.*, [26]).

Remark 2.12. A crucial property of the functional $J_{0,q}(w)$ is its strict *ray-convexity*: it means that $J_{0,q}(w)$ is strictly convex except for any couple of colinear points w, $\widehat{w}$ with $\widehat{w} = \alpha w$ for some $\alpha \in (0, +\infty)$. That was used in [7], [132] and [131] to get the uniqueness of nonnegative solutions when $\frac{f_1(x,u)}{u^{q-1}}$ in (f2) is not strictly decreasing (as it is the case of the first eigenfunction of the p-Laplacian).

Remark 2.13. The limit case $p = \infty$ (defined in a suitable way) can be also considered since, curiously enough, it is an homogeneous operator of exponent 3 (see, *e.g.*, [66]). It is well-known that the other limit case $p = 1$ can be also treated as a subdifferential of a convex function (see *e.g.*, [8]) but the unique choice to apply the reasoning of this paper seems to be $q = p = 1$ and then the results reduce to the well-known case of monotone perturbations. It would be interesting to know if it is possible to get the uniqueness of nonnegative solutions of equations involving some different kind of non-monotone sub-homogeneity nonlinear term.

3. Selected L^s−mild solutions, proof of the main theorem and further remarks

It is useful to unify the application of abstract results on the associated Cauchy Problem to the case of the Banach spaces $L^s(\Omega)$, for any $s \in [1, +\infty]$. For instance, we can define the realizations of the operator $\partial J_{0,q}$ over the spaces $L^s(\Omega)$, for any $s \in [1, +\infty]$ as $A_s = \overline{\partial J_{0,q}}^{L^s}$ in the sense of graphs over $L^s(\Omega) \times L^s(\Omega)$: *i.e.*, $A_s : D(A_s) \to \mathcal{P}(L^s(\Omega))$ and $z \in A_s(w)$ if and only if there exists $z_n \in \partial J_{0,q}(w_n)$ such that $w_n \to w$ and $z_n \to z$ in $L^s(\Omega)$, so that $D(A_s) = \Big\{ w \in L^s(\Omega) : \exists w_n \in L^2(\Omega)$, with $w_n^{\frac{1}{q}} \in W_0^{1,p}(\Omega) \cap L^{2q}(\Omega)$ such that $w_n \to w$ in $L^s(\Omega) \Big\}$.

Then, we consider the Cauchy problem

$$\begin{cases} \dfrac{dw}{dt} + A_s w \ni F(t) \quad \text{in } L^s(\Omega) \\ w(0) = w_0, \end{cases} \tag{3.1}$$

where $w_0 \in \overline{D(A_s)}$ and $F \in L^1(0, T : L^s(\Omega))$. In our case, two relevant examples are $A_2 = \partial J_{0,q}$ and the $L^1(\Omega)$ operator

$$\begin{cases} AW = -\Delta_p W^m, \text{ for } W \in D(A), \text{ with} \\ D(A) = \{ W \in L^1(\Omega),\, W^m \in W_0^{1,1}(\Omega), \Delta_p W^m \in L^1(\Omega) \}, \end{cases}$$

given $m > 0$ and $p > 1$.

We start by recalling the definition of *mild solution* of (3.1) by particularizing the abstract framework to the case of the Banach space $X = L^s(\Omega)$. The good class of operators to solve (3.1) is the class of *accretive operators* (resp. *T-accretive operators*) over a Banach space X : i.e. $A : D(A) \to \mathcal{P}(X)$ such that

$$\begin{aligned} \|x - \widehat{x}\| &\leq \|x - \widehat{x} + \mu(y - \widehat{y})\| \\ \text{(resp. } \big\|[x - \widehat{x}]_+\big\| &\leq \big\|[x - \widehat{x} + \mu(y - \widehat{y})]_+\big\| \text{)} \\ \text{whenever } \mu &> 0 \text{ and } (x, y), (\widehat{x}, \widehat{y}) \in A. \end{aligned}$$

The operator is called *m-accretive* if in addition $R(I + A) = X$. For many results and definitions about mild solutions of the Cauchy Problem for accretive operators in Banach spaces see, e.g., [18], [19], [31], [67], [138], [93] and [37]. We recall that over any Hilbert space (as $L^2(\Omega)$) the class of m-T-accetive operators coincides with the class of maximal T-monotone operators and thus it is possible to apply the abstract theory presented in Brezis [49]) to problem (1.4). The notion of mild solution below is well defined in both cases: Hilbert and Banach spaces.

Definition 3.1. A function $w \in C([0,T] : L^s(\Omega))$ is a L^s-*mild solution* of (3.1) if for any $\epsilon > 0$, there exists a partition $\{0 = t_0 < t_1 < ... \, t_n\}$ of $[0, t_n]$ and there exist two finite sequences $\{w_i\}_{i=0}^n$, $\{F_i\}_{i=0}^n$ in $L^s(\Omega)$ such that

$$\begin{cases} \text{(i)} \ \dfrac{w_{i+1} - w_i}{t_{i+1} - t_i} + A_s w_{i+1} \ni F_{i+1}, \qquad i = 0, 1, ..., n-1 \\ \text{(ii)} \ t_{i+1} - t_i < \epsilon \\ \text{(iii)} \ 0 \leq T - t_n < \epsilon \\ \text{(iv)} \ \displaystyle\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \|F_i - F(t)\|_{L^s(\Omega)} \, dt < \epsilon, \end{cases}$$

and

$$\|w_\epsilon(t) - w(t)\|_{L^s(\Omega)} \leq \epsilon \text{ on } [0, t_n],$$

where

$$w_\epsilon(t) = w_i \text{ for } t_i \leq t < t_{i+1}, \ i = 0, 1, ..., n-1.$$

Definition 3.2. The piecewise constant function $w_\epsilon(t)$ defined before is called an ϵ-L^s-*approximate solution* of (3.1).

Proof of Theorem 1.1. Let us start by considering the simpler problem $f(x, v) \equiv 0$. Since $w_0 = u_0^q \in D(J_{0,q}) \subset \overline{D(J_{0,q})}^{L^2} = \overline{D(\partial J_{0,q})}^{L^2}$, the existence and uniqueness of a mild solution $w \in C([0, \widetilde{T}] : L^2(\Omega))$, for any arbitrary $T > 0$, is a direct consequence of the application of the abstract theory (Brezis [49]) on maximal T-monotone operators in $L^2(\Omega)$. Moreover, we know that w is a weak solution (in the sense of Definition 3.1 of [49]): i.e. if we assume $w_{0,n} \in D(\partial J_{0,q})$ and $h_n \in W^{1,1}(0, \widetilde{T} : L^2(\Omega))$ such that $w_{0,n} \to w_0$ in $L^2(\Omega)$ and $h_n \to h$ in $L^1(0, \widetilde{T} : L^2(\Omega))$ then the respective solutions w_n satisfy that $w_n \to w$ in $C([0, \widetilde{T}] : L^2(\Omega))$ (see, Theorem 3.4 of [49]). By applying Theorem 3.7 of [49] we know that, in fact, w_n is a strong solution in the sense that $w_n(t)$ is Lipschitz continuous on $[\delta, \widetilde{T}]$ for any $\delta \in (0, \widetilde{T})$ and thus differentiable. Then the associate problem (1.4) can be written as

$$\frac{dw_n}{d\tau}(\tau) - \frac{\Delta_p(w_n(\tau)^{\frac{1}{q}})}{w_n(\tau)^{\frac{q-1}{q}}} = h_n(\tau),$$

i.e.,

$$w_n(\tau)^{\frac{q-1}{q}} \frac{dw_n}{d\tau}(\tau) - \Delta_p(w_n(\tau)^{\frac{1}{q}}) = h_n(\tau) w_n(\tau)^{\frac{q-1}{q}}.$$

If we define $w_n(\tau) = u_n(t)^q$ then

$$w_n(\tau)^{\frac{q-1}{q}} \frac{dw_n}{d\tau}(\tau) = \frac{q}{2q-1} \frac{d(w_n^{(2q-1)/q})}{d\tau}(\tau) = \frac{d(u_n^{2q-1})}{dt}(t)$$

if

$$\tau = \frac{q}{2q-1}t.$$

Obviously we take now $\widetilde{T} = \frac{q}{2q-1}T$. Notice that $w_n \in C([0,\widetilde{T}] : L^2(\Omega))$ implies that $u_n^q \in C([0,T] : L^2(\Omega))$ and thus $u_n^{2q-1} \in C([0,T] : L^2(\Omega))$ since $(2q-1)/q > 1$ (remember that $q > 1$). In addition, for those regular data

$$\frac{d(u_n^{2q-1})}{dt} \in [\widehat{\delta}, T] \text{ for any } \widehat{\delta} \in (0,T].$$

Thus, we conclude that $u_n(t) := w_n(\frac{q}{2q-1}t)^{1/q}$ is a L^1-*mild positive solution* of (P_q) on $[0,T]$, associated to $u_{0,n} := {w_{0,n}}^{1/q}$ and h_n (which the corresponding unique selected L^1-*mild positive solution* of (P_q). Finally, as $w_n \to w$ in $C([0,\widetilde{T}] : L^2(\Omega))$ we get that $u(t) := w(\frac{q}{2q-1}t)^{1/q}$ is a L^1-*mild positive solution* of (P_q) on $[0,T]$, associated to $u_0 := {w_0}^{1/q}$ and h since the notion of mild solution is stable by approximations of the data (see, e.g. Theorem 11.1 of [31]). The rest of conclusions of Theorem 1.1, when $f(x,v) \equiv 0$ are a consequence of Lemma 2.7 and the T-monotocity of operator $\partial J_{0,q}$ (Lemmas 2.4 and 2.6).

We consider now the parabolic problem (P) in the general case, i.e., with a non-homogeneous term $f(x,u)$ satisfying the structural assumptions (f1)-(f3). We consider now the operator on $L^2(\Omega)$

$$Cw = \partial J_{0,q}(w) - \frac{f_1(x,w)}{w^{q-1}} \tag{3.2}$$

with $D(C) = D(\partial J_{0,q})$. Since (f1)-(f3) hold and $f_1(x,w) = f_1(w)$, independent of x, or $f_1(x,w)$ satisfies also (f4), then the function $E : \Omega \times [0,+\infty) \to \mathbb{R}$, given by

$$E(x,w) = -\frac{f_1(x,w)}{w^{p-1}} - a_0(x)$$

generates a m–T-accretive operator $L^2(\Omega)$ with $E(x,0) = 0$. Then, the operator C is m-T-accretive on $L^2(\Omega)$. Moreover, the Lipschitz function

$$G(x,w) = -\frac{f_2(x,w)}{w^{q-1}} + a_0(x)$$

(of constant $K_G > 0$) generates a Lipschitz operator on $L^2(\Omega)$ (of constant K for some $K > 0$). Then the operator $C + KI$ is a m-T-accretive in $L^2(\Omega)$ (see, *e.g.*, Chapter 2, Example 2.2 of [31]), i.e., C is a K-m-T-accretive in $L^2(\Omega)$. So, by the Crandall-Ligget theorem (see, *e.g.*, [18], and [31]), for any $w_0 \in \overline{D(\partial J_{0,q})}$ and $h \in L^1(0,T : L^2(\Omega))$ there exists a unique positive L^2-mild solution $w \in C([0,T] : L^2(\Omega))$ of the Cauchy Problem

$$\begin{cases} \dfrac{dw}{dt} + \partial J_{0,q}(w) - \dfrac{f_1(x,w)}{w^{q-1}} - \dfrac{f_2(x,w)}{w^{q-1}} \ni h(t) \quad \text{in } L^2(\Omega) \\ w(0) = w_0, \end{cases} \tag{3.3}$$

and if $\widehat{w} \in C([0,T] : L^2(\Omega))$ is the L^2-mild solution corresponding to the data $\widehat{w}_0 \in \overline{D(\partial J_{0,q})}$ and $\widehat{h} \in L^1(0,T : L^2(\Omega))$ then for any $t \in [0,T]$

$$\begin{aligned} \|[w(t) - \widehat{w}(t)]_+\|_{L^2(\Omega)} &\leq e^{Kt}\|(w_0 - \widehat{w}_0)_+\|_{L^2(\Omega)} \\ &+ \int_0^t e^{K(t-s)} \left\|[h(s) - \widehat{h}(s)]_+\right\|_{L^2(\Omega)} ds, \end{aligned}$$

(see, *e.g.*, [19] Proposition 4.1 or Theorem 13.1 of [31]). Arguing as before $u(t) := w(\frac{q}{2q-1}t)^{1/q}$ is a L^1-*mild positive solution* of (P). The proof that $u \in L^\infty(0,T : L^\infty(\Omega))$ once we assume $h \in L^\infty(0,T : L^\infty(\Omega))$ and $u_0 \in L^\infty(\Omega)$ is a consequence of Lemma 2.7 (which implies the compactness of the semigroup generated by operator $\partial J_{0,q}(w) - \dfrac{f_1(x,w)}{w^{q-1}} - \dfrac{f_2(x,w)}{w^{q-1}})$ and the abstract invariant results presented in Theorem 2.4.1 of Vrabie [141] (see also [89]), which ends the proof of Theorem 1.1. □

Remark 3.3. In fact, the existence and uniqueness of a L^2-mild positive solution of problem (1.4) can be assured in the more general case of $w_0 \in \overline{\mathcal{D}(\partial J_{0,q})}$. Notice that if $w_0 \in \overline{\mathcal{D}(\partial J_{0,q})}$ the selected L^1-mild positive solution u of (P_q) such that $w(\frac{q}{2q-1}t) = u(t)^q$, with $w(t)$ the corresponding L^2-mild positive solution of problem (1.4) satisfies (in some sense) the decay estimates given in Lemma 2.6 since they are obtained through the implicit Euler scheme given in the definition of mild solution. As a matter of fact, if $w(t_0) \in \mathcal{D}(\partial J_{0,q})$ for some $t_0 \in [0,T]$, i.e. $\partial J_{0,q}(w(t_0)) \ni h(t_0)$ for some $h(t_0) \in L^2(\Omega)$ then $-\Delta_p v(t_0) + h(t_0)_-(x) v(t_0)^{q-1} = h(t_0)_+(x) v(t_0)^{q-1}$ and necessarily we get the estimates iii) and iv) of 2.6 for $v(t_0)$. We also point out that some uniqueness results for suitable sublinear parabolic problems, when $u_0(x) \geq Cd(x,\partial\Omega)$, can be found in [59], [102], [63], [78], [73] (see also their references to previous works in this direction). Curiously enough such type of assumptions also lead to the uniqueness of solutions in the case of equations with multivalued right hand side terms as problem (P_H) (see [94], [87]) which until now required completely different ideas.

Remark 3.4. We point out that selected L^1-mild positive solution u satisfies some extra regularity properties due to the subdifferential of $J_{0,q}$ involved in the equation. See also some variational type techniques applied to the case $p = 2$ in [120] and the general approach (also for $p = 2$) presented to some related problems in [51], [52].

Remark 3.5. It seems possible to make a sharper study of the regularity of the solution of the equation $-\mu\Delta_p v + v^{2q-1} = h(x)v^{q-1}$, but we shall not enter into the maximum of its generality here. For instance, when $p = 2$ such equation becomes a Schrödinger equation with a potential $h(x)$ (and a nonlinear perturbation term v^3) and so it is possible to consider potentials $h(x)$ with a singular behavior near $\partial\Omega$ (and in other subregions of Ω) which goes beyond $L^1(\Omega)$ (see, *e.g.*, [36], [123], [74],[118], [76] and its many references). For the special case of $q = p \neq 2$ singular potentials were considered in [115], [122], [79] and in many other papers.

Remark 3.6. The main result of this paper may be also proved when we replace the open bounded set Ω by the whole space $\mathbb{R}^N$. The *Diaz-Saá inequality* (and the generalized Picone inequality) was obtained in [60] (respectively in [64]). We do no want to enter into details here but the arguments of truncating the domain, generate the associate problems on an expansive sequence of domains Ω_n and then to get the solution as limit of the solutions of the corresponding problems on Ω_n can be applied as in Brezis and Kamin [54] (see also [83]). The assumptions made on functions f_i allow to get some similar estimates to (1.7) to solutions of several quasilinear formulations (see, [119], and [107]) and, in particular, to solutions of the associated to the KPP equation as in the papers [54], [82], [13] and [14]).

Remark 3.7. As mentioned before, the assumptions on $f_1(x,u)$ allow the consideration of some singular terms: see, *e.g.*, [16], [45], [99], [73] and the surveys [106] and [100]. The assumption of the type $\frac{f_2(x,u)}{u^{q-1}}$ globally Lipschitz continuous in $u \in (0,+\infty)$ was used for other purposes in previous works in the literature (see, *e.g.*, [62]).

Remark 3.8. It seems possible to get similar results to positive solutions of Neumann type boundary conditions once that the homogeneity of the boundary condition is compatible with the one of the doubly nonlinear problem (P) (see, *e.g.*, [26], [4], [17] and [9] among many other possible references).

We point out that, obviously, the function $u_\infty(x) \equiv 0$ in Ω is a trivial solution of the stationary problem. Here we are interested on nonnegative solutions of problem (P) (and its implicit time discretization). The following result shows that the asymptotic behavior, as $t \to +\infty$, is very different according $q \in (1, \frac{p}{2})$ and $p > 2$ than in the case $q \in (\frac{p}{2}, p]$.

We will prove that, in fact, if $q \in (1, \frac{p}{2})$ and $p > 2, f(x,u) \equiv 0,\ h \in L^1(0,T : L^2(\Omega)),\ h \geq 0$ and $u_0 \gneqq 0$ then there is no extinction in finite time, so that $\|u^{2q-1}(t)\|_{L^2(\Omega)} > 0$ for any $t > 0$. The situation is different if $q \in (\frac{p}{2}, p]$ since, at least for $f(x,u) \equiv 0$ and $h \leq 0$, there is a finite extinction time $T_e > 0$, such that $w(t) \equiv 0$, in Ω, for any $t \geq T_e$. In that case, we understand that the selected solution $v(t)$ of (P) also extinguishes in Ω after T_e.

Theorem 3.9. *a) Assume* $q \in (1, \frac{p}{2})$ *and* $p > 2, f(x,u) \equiv 0,\ h \in L^1_{loc}(0,+\infty : L^2(\Omega)),\ h \geq 0$ *and* $u_0 \gneqq 0$ *satisfying (1.3). Then the selected* L^1-*mild positive solution* u *of* (P) *satisfies that*

$$\|u^q(t)\|_{L^2(\Omega)} \geq \frac{1}{(c_1 t + c_2)^{(q-1)/(p+q-2)}}$$

for any $t > 0$, *for some positive constants* c_1 *and* c_2.

b) Assume $q \in (\frac{p}{2}, p],\ f(x,v) \equiv 0$ *and* $h \in L^1_{loc}(0,+\infty : L^2(\Omega))$ *such that* $h \leq 0$. *Then there is a finite extinction time* $T_e > 0$, *such that the selected solution* $u(t)$ *of* (P) *extinguishes in* Ω *after* T_e, *i.e.,* $u(t) = u_\infty(x) \equiv 0$, *in* Ω, *for any* $t \geq T_e$.

Proof. Since $h \geq 0$, from the comparison estimate (1.7) we deduce that $u \geq \underline{U}$ with $\underline{U}$ the unique solution of the problem

$$(P_0) \qquad \begin{cases} \partial_t(\underline{U}^{2q-1}) - \Delta_p \underline{U} &= 0 \qquad \text{in } Q_T, \\ \underline{U} &= 0 \qquad \text{on } \Sigma, \\ \underline{U}^q(0,.) &= u_0^q(.) \quad \text{on } \Omega. \end{cases}$$

Moreover, as indicated in Theorem 1.1, we know that if $\underline{U}(t) := \underline{W}(\frac{q}{2q-1}t)^{1/q}$ then $\underline{W}$ satisfies of the problem

$$\left\{ \begin{array}{l} \dfrac{d\underline{W}}{dt} + \partial J_{0,q}(\underline{W}) \ni 0 \quad \text{in } L^2(\Omega) \\ \underline{W}(0) = u_0. \end{array} \right. \tag{3.4}$$

In addition, the operator $\partial J_{0,q}(\underline{W})$ is formally given by $\frac{\Delta_p(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}}$ and thus it is homogeneous of exponent $\theta = (p-q)/q$, in the sense that

$$\partial J_{0,q}(r\underline{W}) = r^{\theta}\partial J_{0,q}(r\underline{W}) \text{ for any } r \geq 0 \text{ and } \underline{W} \in D(\partial J_{0,q}).$$

Then, since $q \in (1, \frac{p}{2})$ and $p > 2$ implies that $\theta > 1$, applying Theorem 1.1 of [1] we get that

$$\|\underline{U}^q(t)\|_{L^2(\Omega)} \geq \frac{1}{(c_1 t + c_2)^{(q-1)/(p+q-2)}} \text{ for any } t > 0,$$

for some positive constants c_1 and c_2, and then the conclusion holds since $U \geq \underline{U}$.
b) We consider, again the solution $\underline{U}$ of (P_0). Now $0 \leq u \leq \underline{U}$ and since, in this case, the homogeneity exponent of $\partial J_{0,q}(\underline{W})$ is $\theta < 1$ the conclusion results of the application of Corollary 1 of [20]. □

Remark 3.10. Systems involving sub-homogeneous terms have been extensively considered in the literature: see, *e.g.*, [96], [97], [60] and its references. It would be interesting to apply the assumptions of the general framework in this paper to the case of systems. In the case of *higher order equations with sub-homogeneous* terms the T-accretivity in L^p fails but I conjecture that the L^2-contraction continuous dependence still holds for certain homogeneous higher order operators (as for instance those considered in [42] and [3]).

Remark 3.11. As mentioned before (see Remark 2.5) a stronger property on the convex funcional J may lead to the accretivity in L^1 and in L^∞ of the realization over these spaces of the subdifferential operator ∂J. Although we are not able to check the stronger property (2.6) in the special case of functional $J_{0,q}$ it is possible to get some continuity dependence inequalities for solutions of the equation $w + \mu\partial J_{0,q}(w) \ni h$, for any $\mu > 0$, which keep some resemblances with the inequalities expressing the L^1 and L^∞ T-accretivity for the realization of the operator $\partial J_{0,q}(w)$ over those spaces (some related techniques can be found in Brezis and Kamin [54] and [61]).

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J. I. Díaz
Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain
E-mail address: jidiaz@ucm.es