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A NECESSARY CONDITION IN A DE GIORGI TYPE CONJECTURE FOR ELLIPTIC SYSTEMS IN INFINITE STRIPS

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Abstract. Given a bounded Lipschitz domain $\omega \subset \mathbb{R}^{d-1}$ and a lower semicontinuous function $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ that vanishes on a finite set and that is bounded from below by a positive constant at infinity, we show that every map $u : \mathbb{R} \times \omega \to \mathbb{R}^N$ with

$$\int_{\mathbb{R}\times\omega} \big(|\nabla u|^2 + W(u)\big)\,\mathrm{d}x_1\,\mathrm{d}x' < +\infty$$

has limits $u^\pm \in \{W = 0\}$ as $x_1 \to \pm\infty$. The convergence holds in $L^2(\omega)$ and almost everywhere in ω. We also prove a similar result for more general potentials W in the case where the considered maps u are divergence-free in Ω with ω being the $(d-1)$-torus and $N = d$.

1. Introduction

Let $N \geq 1$, $d \geq 2$ and $\Omega = \mathbb{R} \times \omega$ be an infinite cylinder in $\mathbb{R}^d$, where $\omega \subset \mathbb{R}^{d-1}$ is an open connected bounded set with Lipschitz boundary. For a lower semicontinuous potential $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$, we consider the functional

$$E(u) = \int_\Omega \Big(|\nabla u|^2 + W(u)\Big)\,\mathrm{d}x, \quad u \in \dot{H}^1(\Omega, \mathbb{R}^N), \tag{1.1}$$

where $|\cdot|$ is the Euclidean norm and

$$\dot{H}^1(\Omega, \mathbb{R}^N) = \Big\{u \in H^1_{loc}(\Omega, \mathbb{R}^N) \ : \ \nabla u = (\partial_j u_i)_{1\leq i\leq N, 1\leq j\leq d} \in L^2(\Omega, \mathbb{R}^{N\times d})\Big\}.$$

A natural problem consists in studying optimal transition layers for the functional E between two wells $u^\pm$ of W (i.e., $W(u^\pm) = 0$). In particular, motivated by the De Giorgi conjecture, one aim is to analyse under which conditions on the potential W and on the dimensions d and N, every minimizer u of E connecting $u^\pm$ as $x_1 \to \pm\infty$ is one-dimensional, i.e., depending only on x_1. Obviously, such one-dimensional transition layers u coincide with their x'-average $\overline{u} : \mathbb{R} \to \mathbb{R}^N$ defined as

$$\overline{u}(x_1) := \fint_\omega u(x_1, x')\,\mathrm{d}x', \quad x_1 \in \mathbb{R}, \tag{1.2}$$

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where $x' = (x_2, \ldots, x_d)$ denotes the $d-1$ variables in ω and the x'-average symbol is denoted by $\fint_\omega = \frac{1}{|\omega|}\int_\omega$.

1.1. **Main results.** The purpose of this note is to prove a necessary condition for finite energy configurations u provided that W satisfies the following two conditions:

(H1): W has a finite number of wells, i.e.,

$$\operatorname{card}(\{z \in \mathbb{R}^N : W(z) = 0\}) < \infty;$$

(H2): $\liminf_{|z|\to\infty} W(z) > 0.$

More precisely, we prove that under these assumptions, there exist two wells $u^\pm$ of W such that $u(x_1, \cdot)$ converges to $u^\pm$ in L^2 and a.e. in ω as $x_1 \to \pm\infty$; in particular, the x'-average $\overline{u}$ (as a continuous map in $\mathbb{R}$) of u admits the limits $\overline{u}(\pm\infty) = u^\pm$ as $x_1 \to \pm\infty$. In definition (1.2) of $\overline{u}$, $u(x_1, \cdot)$ stands for the trace of the Sobolev map $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ on the section $\{x_1\} \times \omega$ for every $x_1 \in \mathbb{R}$.

Theorem 1.1. *Let $\Omega = \mathbb{R} \times \omega$, where $\omega \subset \mathbb{R}^{d-1}$ is an open connected bounded set with Lipschitz boundary. If $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous potential satisfying* **(H1)** *and* **(H2)***, then every $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ with $E(u) < \infty$ connects two wells*[1] *$u^\pm \in \mathbb{R}^N$ of W at $x_1 = \pm\infty$ (i.e., $W(u^\pm) = 0$) in the sense that*

$$\lim_{x_1\to\pm\infty} \|u(x_1,\cdot) - u^\pm\|_{L^2(\omega,\mathbb{R}^N)} = 0 \quad \textit{and} \quad \lim_{x_1\to\pm\infty} u(x_1,\cdot) = u^\pm \quad \textit{a.e. in } \omega. \tag{1.3}$$

In particular,

$$\lim_{x_1\to\pm\infty} \fint_\omega u(x_1, x')\, \mathrm{d}x' = u^\pm.$$

Remark 1.2. i) As a consequence of the Poincaré-Wirtinger inequality[2], for $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$ with $\bar{u}(\pm\infty) = u^\pm$, there exist two sequences $(R_n^+)_{n\in\mathbb{N}}$ and $(R_n^-)_{n\in\mathbb{N}}$ such that $(R_n^\pm)_{n\in\mathbb{N}} \to \pm\infty$ and the stronger convergence holds true:

$$\|u(R_n^\pm, \cdot) - u^\pm\|_{H^1(\omega,\mathbb{R}^N)} \underset{n\to\infty}{\longrightarrow} 0 \tag{1.4}$$

(see [25, Lemma 3.2]).

ii) Theorem 1.1 also holds true if ω is a closed (i.e., compact, connected without boundary) Riemannian manifold.

iii) Theorem 1.1 also applies for maps u taking values into a closed set $\mathcal{N} \subset \mathbb{R}^N$ (e.g., $\mathcal{N}$ could be a compact manifold embedded in $\mathbb{R}^N$). More precisely, if the potential $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies **(H1)**, **(H2)** and $\mathcal{N} := \{z \in \mathbb{R}^N : W(z) < +\infty\}$ is a closed set such that $W_{|\mathcal{N}} : \mathcal{N} \to \mathbb{R}_+$ is lower semicontinuous, then Theorem 1.1 handles the case where the nonlinear constraint $u \in \mathcal{N}$ is present.

[1] u^- and u^+ could be equal.

[2] The assumption that ω is connected with Lipschitz boundary is needed for the Poincaré-Wirtinger inequality.

The result in Theorem 1.1 extends to slightly more general potentials W in the following context of divergence-free maps. For that, let $d = N$ and $\Omega = \mathbb{R} \times \omega$ with $\omega = \mathbb{T}^{d-1}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ being the flat torus. We consider maps $u \in H^1_{loc}(\Omega, \mathbb{R}^d)$ periodic in $x' \in \omega$ and divergence-free, i.e.,

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega.$$

Then the x'-average $\bar{u} : \mathbb{R} \to \mathbb{R}^d$ is continuous and its first component is constant, i.e., there is $a \in \mathbb{R}$ such that

$$\bar{u}_1(x_1) = a \quad \text{for every} \quad x_1 \in \mathbb{R}$$

(see [25, Lemma 3.1]). For such maps u, we consider potentials W satisfying the following two conditions:

$\mathbf{(H1)}_a$**:** $W(a, \cdot)$ has a finite number of wells, i.e.,

$$\operatorname{card}(\{z' \in \mathbb{R}^{d-1} : W(a, z') = 0\}) < \infty;$$

$\mathbf{(H2)}_a$**:** $\liminf_{z_1 \to a, |z'| \to \infty} W(z_1, z') > 0.$

In this context, we have proved in our previous paper [25] that the x'-average map $\bar{u}$ admits limits $u^\pm$ as $x_1 \to \pm\infty$, where $u_1^\pm = a$ and the limits $u^\pm$ are two wells of $W(a, \cdot)$, see [25, Lemma 3.7]. As in Theorem 1.1, we will prove that $u(x_1, \cdot)$ converges to $u^\pm$ in L^2 and a.e. in ω as $x_1 \to \pm\infty$.

Theorem 1.3. *Let $\Omega = \mathbb{R} \times \omega$ with $\omega = \mathbb{T}^{d-1}$ the $(d-1)$-dimensional torus and $u \in \dot{H}^1(\Omega, \mathbb{R}^d)$ such that $E(u) < \infty$ and $\bar{u}_1 \equiv a$ in $\mathbb{R}$ for some $a \in \mathbb{R}$. If $W : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous potential satisfying $\mathbf{(H1)}_a$ and $\mathbf{(H2)}_a$, then there exist two wells $u^\pm \in \mathbb{R}^d$ of W such that (1.3) holds true and $u_1^\pm = a$. In particular, $\bar{u}(\pm\infty) = u^\pm$.*

Note that we do not assume that u is divergence-free in Theorem 1.3, only the weaker assumption that $\bar{u}_1$ is constant.

1.2. **Motivation.** Our main result is motivated by the well-known De Giorgi conjecture that consists in investigating the one-dimensional symmetry of critical points of the functional E, i.e., solutions $u : \Omega \to \mathbb{R}^N$ to the nonlinear elliptic system

$$\begin{cases} \Delta u = \frac{1}{2} \nabla W(u) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega = \mathbb{R} \times \partial\omega, \end{cases} \tag{1.5}$$

where W is assumed to be locally Lipschitz in (1.5) and ν is the unit outer normal vector field at $\partial\omega$. Theorem 1.1 states in particular that solutions u of finite energy satisfy the boundary condition (1.3) for two wells $u^\pm$ of W. A natural question related to the De Giorgi conjecture arises in this context:

Question: Under which assumptions on the potential W and the dimensions d and N, is it true that every global minimizer u of E connecting two wells[3] of W is one-dimensional symmetric, i.e., $u = u(x_1)$?

Link with the Gibbons and De Giorgi conjectures. i) In the scalar case $N = 1$ (d is arbitrary) and $W(u) = \frac{1}{2}(1 - u^2)^2$, the answer to the above question is positive

[3] We say that u connects two wells $u^\pm$ of W if (1.3) is satisfied.

provided that the limits (1.3) are replaced by uniform convergence (see [12, 17]); within these uniform boundary conditions, the problem is called the Gibbons conjecture. We mention that many articles have been written on Gibbons' conjecture in the case of the entire space $\Omega = \mathbb{R}^d$: more precisely, if a solution[4] $u : \mathbb{R}^d \to \mathbb{R}$ of the PDE

$$\Delta u = \frac{1}{2}\frac{dW}{du}(u) \quad \text{in} \quad \mathbb{R}^d \tag{1.6}$$

satisfies the convergence $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ uniformly in $x' \in \mathbb{R}^{d-1}$ and $|u| \leq 1$ in $\mathbb{R}^d$, then u is one-dimensional (see [5, 6, 11, 18]).

Let us now speak about the long standing De Giorgi conjecture in the scalar case $N = 1$. It predicts that any bounded solution u of (1.6) that is monotone in the x_1 variable is one-dimensional in dimension $d \leq 8$, i.e., the level sets $\{u = \lambda\}$ of u are hyperplanes. The conjecture has been solved in dimension $d = 2$ by Ghoussoub-Gui [22], using a Liouville-type theorem and monotonicity formulas. Using similar techniques, Ambrosio-Cabré [4] extended these results to dimension $d = 3$, while Ghoussoub-Gui [23] showed that the conjecture is true for $d = 4$ and $d = 5$ under some antisymmetry condition on u. The conjecture was finally proved by Savin [32] in dimension $d \leq 8$ under the additional condition $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ pointwise in $x' \in \mathbb{R}^{d-1}$, the proof being based on fine regularity results on the level sets of u. Lately, Del Pino-Kowalczyk-Wei [13] gave a counterexample to the De Giorgi conjecture in dimension $d \geq 9$, which satisfies the pointwise limit conditions $\lim_{x_1 \to \pm\infty} u(x_1, x') = \pm 1$ for a.e. $x' \in \mathbb{R}^{d-1}$. It would be interesting to investigate whether these results transfer (or not) to the context of the strip $\Omega = \mathbb{R} \times \omega$ as stated in Question. Theorem 1.1 is helpful here because it proves that the pointwise convergence as $x_1 \to \pm\infty$ is a necessary condition in the context of a strip $\mathbb{R} \times \omega$ and for finite energy configurations.

ii) Less results are available for the vector-valued case $N \geq 2$. In the case $\Omega = \mathbb{R}^d$, $N = 2$ and $W(u_1, u_2) = \frac{1}{2}(u_1^2 - 1)^2 + \frac{1}{2}(u_2^2 - 1)^2 + \Lambda u_1^2 u_2^2 - \frac{1}{2}$ with $\Lambda > 1$ (so $W \geq 0$ and W has exactly four wells $\{(0, \pm 1), (\pm 1, 0)\}$, thus, **(H1)** and **(H2)** are satisfied), the Gibbons and De Giorgi conjectures corresponding to the system (1.5) are discussed in [19]. Several other phase separation models (e.g., arising in a binary mixture of Bose-Einstein condensates) are studied in the vectorial case where W has a non-discrete set of zeros (see e.g., [7, 8, 20]).

We recall that in the study of the De Giorgi conjecture for (1.6), i.e., $N = 1$, there is a link between monotonicity of solutions (e.g., the condition $\partial_1 u > 0$), stability (i.e., the second variation of the corresponding energy at u is nonnegative), and local minimality of u (in the sense that the energy does not decrease under compactly supported perturbations of u). We refer to [2, Section 4] for a fine study of these properties. In particular, it is shown that the monotonicity condition in the De Giorgi conjecture implies that u is a local minimizer of the energy (see [2, Theorem 4.4]). Therefore, it is natural to study Question under the monotonicity condition in x_1 (instead of the global minimality condition on u).

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[4] Here, u needs not be a global minimizer of E within the boundary condition (1.3), nor monotone in x_1, i.e., $\partial_1 u > 0$. Obviously, this result applies also to global minimizers, as $|u| \leq 1$ in $\mathbb{R}^d$ by the maximum principle.

Link with micromagnetic models. We have studied Question in the context of divergence-free maps $u : \mathbb{R} \times \omega \to \mathbb{R}^N$ where $d = N$ and $\omega = \mathbb{T}^{d-1}$ is the $(d-1)$-dimensional torus, see [25]. By developing a theory of calibrations, we have succeeded to give sufficient conditions on the potential W in order that the answer to Question is positive, in particular in the case where $\mathbf{(H1)}_a$ and $\mathbf{(H2)}_a$ are satisfied, see [25, Theorem 2.11]. In that context, Question is related to some reduced model in micromagnetics in the regime where the so-called stray-field energy is strongly penalized favoring the divergence constraint $\nabla \cdot u = 0$ of the magnetization u (the unit-length constraint on u being relaxed in the system). In the theory of micromagnetics, a challenging question concerns the symmetry of domain walls. Indeed, much effort has been devoted lately to identifying on the one hand, the domain walls that have one-dimensional symmetry, such as the so-called symmetric Néel and symmetric Bloch walls (see e.g. [14, 27, 24]), and on the other hand, the domain walls involving microstructures, such as the so-called cross-tie walls (see e.g., [3, 31]), the zigzag walls (see e.g., [26, 30]) or the asymmetric Néel / Bloch walls (see e.g. [16, 15]). Thus, answering to Question would give a general approach in identifying the anisotropy potentials W for which the domain walls are one-dimensional in the elliptic system (1.5).

Link with heteroclinic connections. One-dimensional[5] solutions $u = u(x_1)$ of the system (1.5) are called heteroclinic connections. Given two wells $u^\pm$ of a potential W satisfying **(H1)** and **(H2)**, it is known that there exists a heteroclinic connection $\gamma : \mathbb{R} \to \mathbb{R}^N$ obtained by minimizing $\int_{\mathbb{R}} |\frac{d}{dx_1}\gamma|^2 + W(\gamma)\, dx_1$ under the condition $\gamma(\pm\infty) = u^\pm$ (see [28, 34, 35]). In the vectorial case $N \geq 2$, this connection may not be unique in the sense that there could exist two (minimizing) heteroclinic connections γ_1, γ_2 such that $\gamma_i(\pm\infty) = u^\pm$ for $i = 1, 2$ but $\gamma_1(\cdot)$ and $\gamma_2(\cdot - \tau)$ are distinct for every $\tau \in \mathbb{R}$. If this is the case, at least in dimension $d = 2$ and $\Omega = \mathbb{R}^2$, there also exists a solution u to $\Delta u = \frac{1}{2}\nabla W(u)$ which realizes an interpolation between γ_1 and γ_2 in the following sense (see [33, 1, 29]):

$$\begin{cases} u(x_1, x_2) \to u^\pm & \text{as } x_1 \to \pm\infty \text{ uniformly in } x_2, \\ u(x_1, x_2) \to \gamma_1(x_1) & \text{as } x_2 \to -\infty \text{ uniformly in } x_1, \\ u(x_1, x_2) \to \gamma_2(x_1) & \text{as } x_2 \to +\infty \text{ uniformly in } x_1. \end{cases}$$

Moreover, this solution u is energy local minimizing, i.e., the energy cannot decrease by compactly supported perturbations of u. Solutions to the system $\Delta u = \frac{1}{2}\nabla W(u)$ naturally arise when looking at the local behavior of a transition layer near a point at the interface between two wells $u^\pm$; solutions satisfying the preceding boundary conditions correspond to the case of an interface point where the 1D connection passes from γ_1 to γ_2. The existence of such stable entire solutions to the Allen-Cahn system makes a significative difference with the scalar case, i.e. $N = 1$, where only 1D solutions are present by the De Giorgi conjecture.

[5]If $u = u(x_1)$, the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ is automatically satisfied.

2. Pointwise convergence and convergence of the x'-average

In this section we prove that under the assumptions in Theorem 1.1, the x'-average $\overline{u}$ (as a continuous map in $\mathbb{R}$) has limits $\overline{u}(\pm\infty) = u^\pm$ as $x_1 \to \pm\infty$ corresponding to two wells of W. For that, we will follow the strategy that we developed in our previous paper (see [25, Section 3.1]). The idea consists in introducing an "averaged" potential V in $\mathbb{R}^N$ with $W \geq V \geq 0$ and $\{V = 0\} = \{W = 0\}$ (see Lemma 2.1), and a new functional E_V associated to the x'-average $\overline{u}$ of a map u such that $\frac{1}{|\omega|}E(u) \geq E_V(\bar{u})$. This can be seen as a dimension reduction technique since the new map $\bar{u}$ has only one variable. We will prove that every transition layer $\bar{u}$ connecting two wells $u^\pm$ has the energy $E_V(\bar{u})$ bounded from below by the geodesic pseudo-distance geod_V between the wells $u^\pm$ (see Lemma 2.3). As the Euclidean distance in $\mathbb{R}^N$ is absolutely continuous with respect to geod_V (see Lemma 2.2), we will conclude that $\bar{u}$ admits limits at $\pm\infty$ given by two wells of W (see Lemma 2.4). Note that in Section 3, we will give a second proof of the claim $\bar{u}(\pm\infty) = u^\pm$ without using the geodesic pseudo-distance geod_V.

We first introduce the energy functional E (defined in (1.1)) restricted to appropriate subsets $A \subset \Omega$ (e.g., A can be a subset of the form $I \times \omega$ for an interval $I \subset \mathbb{R}$): for every map $u \in \dot{H}^1(A, \mathbb{R}^N)$, we set

$$E(u, A) := \int_A |\nabla u|^2 + W(u)\,\mathrm{d}x,$$

so that for $A = \Omega$, we have $E(u) = E(u, A)$. For any interval $I \subset \mathbb{R}$, the Jensen inequality yields

$$\begin{aligned} E(u, I \times \omega) &= \int_I \int_\omega \big(|\partial_1 u|^2 + |\nabla' u|^2 + W(u)\big)\,\mathrm{d}x'\,\mathrm{d}x_1 \\ &\geq |\omega| \int_I \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + e(u(x_1, \cdot))\,\mathrm{d}x_1, \end{aligned}$$

where $\nabla' = (\partial_2, \ldots, \partial_d)$, $\bar{u}$ is the x'-average of u given in (1.2) and the x'-average energy e is defined by

$$e(v) := \fint_\omega \big(|\nabla' v|^2 + W(v)\big)\,\mathrm{d}x' \quad \text{ for all } v \in H^1(\omega, \mathbb{R}^N).$$

Introducing the averaged potential $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defined for all $z \in \mathbb{R}^N$ by

$$V(z) := \inf\left\{ e(v) \;:\; v \in H^1(\omega, \mathbb{R}^N),\ \fint_\omega v\,\mathrm{d}x' = z \right\} \geq 0, \tag{2.1}$$

we have

$$E(u, I \times \omega) \geq |\omega| \int_I \left(\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + V(\overline{u}(x_1)) \right) \mathrm{d}x_1. \tag{2.2}$$

This observation is the starting point in the proof of the following lemma:

Lemma 2.1. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H2)**. *Then the averaged potential $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defined in* (2.1) *satisfies the following:*

(1) *V is lower semicontinuous in $\mathbb{R}^N$,*

(2) *for all* $z \in \mathbb{R}^N$, $V(z) \le W(z)$, *the infimum in* (2.1) *is achieved and*[6] $\Big[V(z) = 0 \Leftrightarrow W(z) = 0\Big]$,

(3) $V_\infty := \liminf_{|z|\to\infty} V(z) > 0$,

(4) *for every interval* $I \subset \mathbb{R}$ *and for every* $u \in \dot{H}^1(I \times \omega, \mathbb{R}^N)$, *one has*

$$\frac{1}{|\omega|} E(u, I \times \omega) \ge E_V(\overline{u}, I), \quad E_V(\overline{u}, I) := \int_I \Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + V(\overline{u}(x_1))\,\mathrm{d}x_1.$$

The new energy $E_V(\bar{u}) := E_V(\overline{u}, \mathbb{R})$ associated to the x'-average $\bar{u}$ will play an important role for proving the existence of the two limits $\bar{u}(\pm\infty)$.

Proof of Lemma 2.1. The claim *4* follows from (2.2). We divide the rest of the proof in three steps.

STEP 1: PROOF OF CLAIM *2*. Clearly, for all $z \in \mathbb{R}^N$, one has $V(z) \le e(z) = W(z)$. By the compact embedding $H^1(\omega) \hookrightarrow L^1(\omega)$, the lower semicontinuity of W, Fatou's lemma and the lower semicontinuity of the L^2 norm in the weak L^2-topology (see [9]), we deduce that e is lower semicontinuous in the weak $H^1(\omega, \mathbb{R}^N)$-topology. Then the direct method in the calculus of variations implies that the infimum is achieved in (2.1) (infimum that could be equal to $+\infty$ as W can take the value $+\infty$).

If $W(z) = 0$, then $V(z) = 0$ (as $0 \le V \le W$ in $\mathbb{R}^N$). Conversely, if $V(z) = 0$ with $z \in \mathbb{R}^N$, then a minimizer $v \in H^1(\omega, \mathbb{R}^N)$ in (2.1) satisfies $V(z) = e(v) = 0$ so that $v \equiv z$ and $W(z) = 0$.

STEP 2: V IS LOWER SEMICONTINUOUS IN $\mathbb{R}^N$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence converging to z in $\mathbb{R}^N$. We need to show that

$$V(z) \le \liminf_{n\to\infty} V(z_n).$$

Without loss of generality, one can assume that $(V(z_n))_{n\in\mathbb{N}}$ is a bounded sequence that converges to $\liminf_{n\to\infty} V(z_n)$. By Step 1, for each $n \in \mathbb{N}$, there exists $v_n \in H^1(\omega, \mathbb{R}^N)$ such that

$$\fint_\omega v_n\,\mathrm{d}x' = z_n \quad \text{and} \quad e(v_n) = V(z_n).$$

Since $(z_n)_{n\in\mathbb{N}}$ and $(e(v_n))_{n\in\mathbb{N}}$ are bounded, we deduce that $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\omega, \mathbb{R}^N)$ by the Poincaré-Wirtinger inequality. Thus, up to extraction, one can assume that $(v_n)_{n\in\mathbb{N}}$ converges weakly in H^1, strongly in L^1 and a.e. in ω to a limit $v \in H^1(\omega, \mathbb{R}^N)$. In particular, $\fint_\omega v\,\mathrm{d}x' = z$. Since e is lower semicontinuous in weak $H^1(\omega, \mathbb{R}^N)$-topology (by Step 1), we conclude

$$V(z) \le e(v) \le \liminf_{n\to\infty} e(v_n) = \liminf_{n\to\infty} V(z_n).$$

STEP 3: PROOF OF CLAIM *3*. Assume by contradiction that there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $|z_n| \to \infty$ and $V(z_n) \to 0$ as $n \to \infty$. Then, there exists a

[6] In particular, if W satisfies **(H1)**, then V satisfies **(H1)**, too.

sequence of maps $(w_n)_{n\in\mathbb{N}}$ in $H^1(\omega,\mathbb{R}^N)$ satisfying

$$\int_\omega w_n(x')\,\mathrm{d}x' = 0 \quad \text{for each } n\in\mathbb{N} \quad \text{and} \quad e(z_n+w_n) \underset{n\to\infty}{\longrightarrow} 0.$$

By the Poincaré-Wirtinger inequality, we have that $(w_n)_{n\in\mathbb{N}}$ is bounded in H^1. Thus, up to extraction, one can assume that it converges weakly in H^1, strongly in L^1 and a.e. to a map $w\in H^1(\omega,\mathbb{R}^N)$. We claim that w is constant since

$$⨍_\omega |\nabla' w|^2\,\mathrm{d}x' \le \liminf_{n\to\infty} ⨍_\omega |\nabla' w_n|^2\,\mathrm{d}x' \le \liminf_{n\to\infty} e(z_n+w_n) = 0.$$

We deduce $w\equiv 0$ since $\int_\omega w = \lim_{n\to\infty}\int_\omega w_n = 0$. Thus $w_n\to 0$ a.e and **(H2)** implies that for a.e. $x'\in\omega$,

$$\liminf_{n\to\infty} W(z_n+w_n(x')) \ge \liminf_{|z|\to\infty} W(z) > 0,$$

which contradicts the fact that $e(z_n+w_n)\to 0$. $\square$

For every lower semicontinuous function $W:\mathbb{R}^N\to\mathbb{R}_+\cup\{+\infty\}$ satisfying **(H1)** and **(H2)**, we introduce the geodesic pseudo-distance geod_W in $\mathbb{R}^N$ endowed with the singular pseudo-metric $4Wg_0$, g_0 being the standard Euclidean metric in $\mathbb{R}^N$; this geodesic pseudo-distance (that can take the value $+\infty$) is defined for every $x,y\in\mathbb{R}^N$ by

$$\begin{aligned}\text{(2.3)}\qquad \mathrm{geod}_W(x,y) := \inf\Big\{ &\int_{-1}^{1} 2\sqrt{W(\sigma(t))}|\dot\sigma|(t)\,\mathrm{d}t \ : \\ &\sigma\in \mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N),\ \sigma(-1)=x,\ \sigma(1)=y\Big\},\end{aligned}$$

where $\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N)$ is the set of continuous and **piecewise locally Lipschitz** curves [7] on $[-1,1]$:

$$\begin{aligned}\mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N) :=\Big\{ &\sigma\in\mathcal{C}^0([-1,1],\mathbb{R}^N) \ : \\ &\text{there is a partition } -1=t_1<\cdots<t_{k+1}=1, \\ &\text{with } \sigma\in \mathrm{Lip}_{loc}((t_i,t_{i+1})) \text{ for every } 1\le i\le k\Big\}.\end{aligned}$$

By *pseudo-distance*, we mean that geod_W satisfies all the axioms of a distance; the only difference with respect to the standard definition is that a pseudo-distance can take the value $+\infty$. We will prove that geod_W yields a lower bound for the energy E (see Lemma 2.3); this plays an important role in the proof of our claim $\overline{u}(\pm\infty)=u^\pm$.

We start by proving some elementary facts about the pseudo-metric structure induced by geod_W on $\mathbb{R}^N$:

[7] In general, we cannot hope that a minimizing sequence in (2.3) is better than piecewise locally Lipschitz because W is not assumed locally bounded ($\dot\sigma$ is the derivative of σ). However, in the case of a locally bounded W, we could use a regularization procedure in order to restrict to Lipschitz curves σ.

Lemma 2.2. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H1)** *and* **(H2)***. Then the function* $\mathrm{geod}_W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ *defines a pseudo-distance over $\mathbb{R}^N$ and the Euclidean distance is absolutely continuous with respect to* geod_W*, i.e., for every $\delta > 0$, there exists $\varepsilon > 0$ such that for every $x, y \in \mathbb{R}^N$ with* $\mathrm{geod}_W(x, y) < \varepsilon$*, we have $|x - y| < \delta$.*

Proof of Lemma 2.2. In proving that $\mathrm{geod}_W : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ defines a pseudo-distance over $\mathbb{R}^N$, the only non-trivial axiom to check is the non-degeneracy, i.e., $\mathrm{geod}_W(x, y) > 0$ whenever $x \neq y$. In fact, we prove the stronger property that for every $\delta > 0$, there exists $\varepsilon > 0$ such that for every $x, y \in \mathbb{R}^N$, $|x-y| \geq \delta$ implies $\mathrm{geod}_W(x, y) \geq \varepsilon$ which also yields the absolute continuity of the Euclidean distance with respect to geod_W. For that, we recall that the set $\{W = 0\}$ is finite (by **(H1)**); therefore, w.l.o.g. we can assume that $\delta > 0$ is small enough so that the open balls $B(p, \delta/2)$ with respect to the Euclidean distance for $p \in \{W = 0\}$, are disjoint. We consider the following disjoint union of balls

$$\Sigma_\delta := \bigsqcup_{p\in\{W=0\}} B(p, \frac{\delta}{4}),$$

the Euclidean distance between each ball being larger than $\delta/2$. We now take two points $x, y \in \mathbb{R}^N$ with $|x - y| \geq \delta$. In order to obtain a lower bound on $\mathrm{geod}_W(x, y)$, we take an arbitrary continuous and piecewise locally Lipschitz curve $\sigma : [-1, 1] \to \mathbb{R}^N$ such that $\sigma(-1) = x$ and $\sigma(1) = y$. As $|x - y| \geq \delta$ (so no ball in Σ_δ can contain both x and y), by connectedness, the image $\sigma([-1, 1])$ cannot be contained in Σ_δ. Thus, there exists $t_0 \in [-1, 1]$ with $\sigma(t_0) \notin \Sigma_\delta$. It implies that $B(\sigma(t_0), \delta/8)\cap\Sigma_{\delta/2} = \emptyset$. Moreover, since $|x-y| \geq \delta$, we have either $|\sigma(t_0)-x| \geq \delta/2$ or $|\sigma(t_0) - y| \geq \delta/2$; w.l.o.g., we may assume that $|\sigma(t_0) - y| \geq \delta/2$. Then the (continuous) curve $\sigma\big|_{[t_0,1]}$ has to get out of the ball $B(\sigma(t_0), \delta/8)$; in particular, it has length larger than $\delta/8$ and

$$\int_{-1}^{1} 2\sqrt{W(\sigma(t))}|\dot\sigma|(t)\,\mathrm{d}t \geq \frac{\delta}{4} \inf_{z\in B(\sigma(t_0),\delta/8)} \sqrt{W(z)} \geq \frac{\delta}{4} \inf_{z\in\mathbb{R}^N\setminus\Sigma_{\delta/2}} \sqrt{W(z)}.$$

Since W is lower semicontinuous and bounded from below at infinity (by **(H2)**), we deduce that W is bounded from below by a constant $c_\delta > 0$ on $\mathbb{R}^N \setminus \Sigma_{\delta/2}$. Taking the infimum over curves $\sigma \in \mathrm{Lip}_{ploc}([-1, 1], \mathbb{R}^N)$ connecting x to y, we deduce from the preceding lower bound that

$$\mathrm{geod}_W(x, y) \geq \frac{\delta\sqrt{c_\delta}}{4} > 0.$$

This finishes the proof of the result. □

We now use a regularization argument to derive the following lower bound on the energy:

Lemma 2.3. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function. Then, for every interval $I \subset \mathbb{R}$ and every map $\sigma \in \dot H^1(I, \mathbb{R}^N)$ having limits $\sigma(\inf I)$ and $\sigma(\sup I)$ at the endpoints of I, we have*

$$E_W(\sigma, I) := \int_I \Big(|\dot\sigma(t)|^2 + W(\sigma(t))\Big)\,\mathrm{d}t \geq \mathrm{geod}_W\big(\sigma(\inf I), \sigma(\sup I)\big). \tag{2.4}$$

Proof of Lemma 2.3. W.l.o.g. we assume that I is an open interval. Since $\dot{H}^1(I,\mathbb{R}^N) \subset W^{1,1}_{loc}(I,\mathbb{R}^N)$, we can define the arc-length $s : I \to J := s(I) \subset \mathbb{R}$ by

$$s(t) := \int_{t_0}^{t} |\dot\sigma|(x_1)\,\mathrm{d}x_1, \quad t \in I,$$

where $t_0 \in I$ is fixed. Thus s is a nondecreasing continuous function with $\dot s = |\dot\sigma|$ a.e. in I. Then the arc-length reparametrization of σ, i.e.

$$\tilde\sigma(s(t)) := \sigma(t), \quad t \in I,$$

is well-defined and provides a Lipschitz curve $\tilde\sigma : J \to \mathbb{R}^N$ with constant speed on the interval J, i.e. $|\dot{\tilde\sigma}| = 1$ a.e., and such that $\tilde\sigma(\inf J) = \sigma(\inf I)$ and $\tilde\sigma(\sup J) = \sigma(\sup I)$. W.l.o.g. we may assume that σ is not constant, so J has a nonempty interior. Then we consider an arbitrary function $\varphi \in \mathrm{Lip}_{loc}((-1,1), \mathrm{int}J)$ which is nondecreasing and surjective onto the interior of the interval J and we set

$$\gamma(t) := \tilde\sigma(\varphi(t)), \quad t \in (-1,1).$$

So γ is a locally Lipschitz map that is continuous on $[-1,1]$ as $\tilde\sigma$ admits limits at $\inf J$ and $\sup J$; thus, $\gamma \in \mathrm{Lip}_{ploc}([-1,1],\mathbb{R}^N)$. The changes of variable $s := \varphi(t)$, resp. $s := s(t)$, yield

$$\int_{-1}^{1} 2\sqrt{W(\gamma(t))}|\dot\gamma|(t)\,\mathrm{d}t = \int_J 2\sqrt{W(\tilde\sigma(s))}|\dot{\tilde\sigma}|(s)\,\mathrm{d}s = \int_I 2\sqrt{W(\sigma(t))}\,|\dot\sigma|(t)\,\mathrm{d}t.$$

Combined with $\gamma(-1) = \sigma(\inf I)$ and $\gamma(1) = \sigma(\sup I)$, the definition of geod_W and the Young inequality imply

$$\begin{aligned} E_W(\sigma, I) &\geq \int_I 2\sqrt{W(\sigma(t))}\,|\dot\sigma|(t)\,\mathrm{d}t \\ &= \int_{-1}^{1} 2\sqrt{W(\gamma(t))}|\dot\gamma|(t)\,\mathrm{d}t \geq \mathrm{geod}_W\big(\sigma(\inf I), \sigma(\sup I)\big). \end{aligned}$$

This completes the proof. □

The convergence of the x'-average in Theorem 1.1 stating that $\overline{u}(\pm\infty) = u^\pm$ is a consequence of the following lemma:

Lemma 2.4. *Let $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ be a lower semicontinuous function satisfying* **(H1)** *and* **(H2)**. *Then for every map $\sigma \in \dot{H}^1(\mathbb{R},\mathbb{R}^N)$ such that $E_W(\sigma,\mathbb{R}) < +\infty$ with E_W defined at* (2.4)*, there exist two wells u^-, $u^+ \in \{W = 0\}$ such that $\lim_{t\to\pm\infty} \sigma(t) = u^\pm$.*

Proof of Lemma 2.4. We use the fact that the energy bound $E_W(\sigma,\mathbb{R}) < +\infty$ yields a bound on the total variation of $\sigma : \mathbb{R} \to \mathbb{R}^N$ where $\mathbb{R}^N$ is endowed with the pseudo-metric geod_W. More precisely, for every sequence $t_1 < \cdots < t_k$ in $\mathbb{R}$, we have by Lemma 2.3:

$$\sum_{i=1}^{k} \mathrm{geod}_W(\sigma(t_{i+1}), \sigma(t_i)) \leq \sum_{i=1}^{k} E_W(\sigma, [t_i, t_{i+1}]) \leq E_W(\sigma, \mathbb{R}) < +\infty.$$

In particular, for every $\varepsilon > 0$, there exists $R > 0$ such that for all $t, s \in \mathbb{R}$ with $t, s \geq R$ or $t, s \leq -R$, one has $\mathrm{geod}_W(\sigma(t), \sigma(s)) < \varepsilon$. Since by Lemma 2.2, smallness of $\mathrm{geod}_W(x, y)$ implies smallness of $|x-y|$, we deduce that σ has a limit $u^\pm \in \mathbb{R}^N$ at $\pm\infty$. Since $W(\sigma(\cdot))$ is integrable in $\mathbb{R}$, we have furthermore that $W(u^\pm) = 0$. □

Now we can prove the convergence of the x'-average $\bar{u}$ at $\pm\infty$ as stated in Theorem 1.1:

Proof of the convergence in x'-average in Theorem 1.1. By Lemma 2.1, we have $E_V(\overline{u}, \mathbb{R}) < +\infty$ for the lower semicontinuous function $V : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying **(H1)** and **(H2)**. By Lemma 2.4 applied to E_V, we deduce that there exists $u^\pm \in \{V = 0\} = \{W = 0\}$ such that $\lim_{t\to\pm\infty} \overline{u}(t) = u^\pm$. □

The pointwise convergence of $u(x_1, \cdot)$ as $x_1 \to \pm\infty$ stated in Theorem 1.1 is proved in the following:

Proof of the pointwise convergence in Theorem 1.1. We prove that $u(x_1, \cdot)$ converges a.e. in ω to $u^\pm \in \{W = 0\}$ as $x_1 \to \pm\infty$, where $u^\pm$ are the limits $\bar{u}(\pm\infty)$ of the x'-average $\bar{u}$ proved above. For that, we have by Fubini's theorem:

$$E(u) \geq \int_\Omega |\partial_1 u|^2 + W(u)\, \mathrm{d}x \geq \int_\omega E_W(u(\cdot, x'), \mathbb{R})\, \mathrm{d}x'$$

with the usual notation

$$E_W(\sigma, \mathbb{R}) = \int_\mathbb{R} |\dot{\sigma}|^2 + W(\sigma)\, \mathrm{d}x_1, \quad \sigma \in \dot{H}^1(\mathbb{R}, \mathbb{R}^N).$$

As $E(u) < \infty$, we deduce that $E_W(u(\cdot, x'), \mathbb{R}) < \infty$ for a.e. $x' \in \omega$. By Lemma 2.4, we deduce that for a.e. $x' \in \omega$, there exist two wells $u^\pm(x')$ of W such that

$$\lim_{x_1\to\pm\infty} u(x_1, x') = u^\pm(x'). \tag{2.5}$$

By (1.4), as $\bar{u}(\pm\infty) = u^\pm$, we know that $\|u(R_n^\pm, \cdot) - u^\pm\|_{L^2(\omega,\mathbb{R}^N)} \to 0$ as $n \to \infty$ for two sequences $R_n^\pm \to \pm\infty$. Up to a subsequence, we deduce that $u(R_n^\pm, \cdot) \to u^\pm$ a.e. in ω as $n \to \infty$. By (2.5), we conclude that $u^\pm(x') = u^\pm$ for a.e. $x' \in \omega$. □

3. The L^2 convergence

In this section, we prove that $u(x_1, \cdot)$ converges in $L^2(\omega, \mathbb{R}^N)$ to $u^\pm$ as $x_1 \to \pm\infty$. The idea is to go beyond the averaging procedure in Section 2 and keep the full information given by the x'-average energy e introduced at Section 2 over the set $H^1(\omega, \mathbb{R}^N)$. More precisely, we extend e to the space $L^2(\omega, \mathbb{R}^N)$ as follows

$$e(v) = \begin{cases} \displaystyle\fint_\omega \Big(|\nabla' v|^2 + W(v)\Big)\, \mathrm{d}x' & \text{if } v \in H^1(\omega, \mathbb{R}^N), \\ +\infty & \text{if } v \in L^2(\omega, \mathbb{R}^N) \setminus H^1(\omega, \mathbb{R}^N). \end{cases} \tag{3.1}$$

In particular, we have for every $u \in \dot{H}^1(\Omega, \mathbb{R}^N)$,

$$E(u) = \int_\mathbb{R} \Big(\|\partial_1 u(x_1, \cdot)\|^2_{L^2(\omega,\mathbb{R}^N)} + |\omega| e(u(x_1, \cdot))\Big)\, \mathrm{d}x_1. \tag{3.2}$$

In the sequel, we will also need the following properties of the energy e:

Lemma 3.1. *If $W : \mathbb{R}^N \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous function satisfying* **(H2)**, *then*

(1) *e is lower semicontinuous in $L^2(\omega,\mathbb{R}^N)$,*
(2) *the sets of zeros of e and W coincide; moreover $\Sigma := \{e=0\} = \{W=0\} \subset \mathbb{R}^N$ is compact,*
(3) *for every $\varepsilon > 0$, we have*

$$k_\varepsilon := \inf\left\{e(v) \;:\; v \in L^2(\omega,\mathbb{R}^N) \text{ with } d_{L^2}(v,\Sigma) \geq \varepsilon\right\} > 0.$$

Proof. We divide the proof in several steps:

Step 1. Lower semicontinuity of e in $L^2(\omega,\mathbb{R}^N)$. Indeed, let $v_n \to v$ in $L^2(\omega,\mathbb{R}^N)$. W.l.o.g., we may assume that $(e(v_n))_n$ is bounded, in particular, $(v_n)_n$ is bounded in $H^1(\omega,\mathbb{R}^N)$; thus, $(v_n)_n$ converges to v weakly in $H^1(\omega,\mathbb{R}^N)$. By Step 1 in the proof of Lemma 2.1, we know that $e\big|_{H^1(\omega,\mathbb{R}^N)}$ is lower semicontinuous w.r.t. the weak H^1 topology and the conclusion follows.

Step 2. Zeros of e. The equality of the zero sets of e and W is straightforward thanks to the connectedness of ω. Thanks to the assumption **(H2)**, the set of zeros Σ of W is bounded and by the lower semicontinuity and non-negativity of W, the set of zeros Σ of W is closed; thus, Σ is compact in $\mathbb{R}^N$.

Step 3. We prove that $k_\varepsilon > 0$. Assume by contradiction that $k_\varepsilon = 0$ for some $\varepsilon > 0$. Then there exists a minimizing sequence $v_n \in L^2(\omega,\mathbb{R}^N)$ such that $d_{L^2}(v_n,\Sigma) \geq \varepsilon$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} e(v_n) = 0$. W.l.o.g., we may assume that $v_n \in H^1(\omega,\mathbb{R}^N)$ for every n as $\|v_n\|_{\dot{H}^1} \to 0$. Denoting $\overline{v_n}$ the (x'-)average of v_n, the Poincaré-Wirtinger inequality implies that the sequence $(w_n := v_n - \overline{v_n})_n$ converges in $H^1(\omega,\mathbb{R}^N)$ to 0. Up to extracting a subsequence, we may assume that $w_n \to 0$ for a.e. $x' \in \omega$.

Claim: The sequence $(\overline{v_n})_n$ is bounded in $\mathbb{R}^N$.

Indeed, assume by contradiction that there exists a subsequence of $(\overline{v_n})_n$ (still denoted by $(\overline{v_n})_n$) such that $|\overline{v_n}| \to \infty$ as $n \to \infty$. As W is l.s.c. and $w_n \to 0$ for a.e. $x' \in \omega$, the assumption (**H2**) implies

$$\liminf_{n\to\infty} W(v_n(x')) = \liminf_{n\to\infty} W(w_n(x') + \overline{v_n}) \geq \liminf_{|z|\to\infty} W(z) > 0 \quad \text{for a.e. } x' \in \omega$$

which by integration over $x' \in \omega$ contradicts the assumption $e(v_n) \to 0$. This finishes the proof of the claim.

As a consequence of the claim, we deduce that $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\omega,\mathbb{R}^N)$. In particular, $(v_n)_{n\in\mathbb{N}}$ has a subsequence that converges in $L^2(\omega,\mathbb{R}^N)$ to a map $v \in H^1(\omega,\mathbb{R}^N)$ and we deduce $d_{L^2}(v,\Sigma) \geq \varepsilon$, in particular, v is not a zero of e, i.e., $e(v) > 0$. As e is l.s.c. in $L^2(\omega,\mathbb{R}^N)$, we have $0 = \lim_{n\to\infty} e(v_n) \geq e(v)$, which contradicts that $e(v) > 0$. □

Now we prove the L^2-convergence of $u(x_1,\cdot)$ to $u^\pm$ as $x_1 \to \pm\infty$:

Proof of the L^2-convergence in Theorem 1.1. Take $u \in H^1_{loc}(\Omega,\mathbb{R}^N)$ such that $E(u) < +\infty$ and set $\sigma(t) := u(t,\cdot) \in H^1(\omega,\mathbb{R}^N)$ for a.e. $t \in \mathbb{R}$. We prove that $\sigma(t)$ converges in $L^2(\omega,\mathbb{R}^N)$ to a limit that is a zero in Σ as $t \to +\infty$ (the proof of the convergence as $t \to -\infty$ is similar). Moreover, we will see that these limits are in fact the zeros $u^\pm$ of W given by the x'-average $\bar{u}$ and the a.e. convergence of $u(x_1,\cdot)$ as $x_1 \to \pm\infty$.

Step 1: Continuity. We prove that $t \in \mathbb{R} \mapsto \sigma(t) \in L^2(\omega,\mathbb{R}^N)$ is continuous in $\mathbb{R}$, and moreover, it is a $\frac{1}{2}$-Hölder map. Indeed, for a.e. $t, s \in \mathbb{R}$, we have

$$d_{L^2}(\sigma(t),\sigma(s))^2 = \int_\omega \Big| \int_t^s \partial_{x_1} u(x_1,x')\,\mathrm{d}x_1 \Big|^2 \mathrm{d}x' \le |t-s| \|\partial_{x_1} u\|^2_{L^2(\Omega,\mathbb{R}^N)}.$$

Step 2: Convergence of a subsequence $(\sigma(t_n))_n$ to some $u^+ \in \Sigma$. Since $e(\sigma(\cdot)) \in L^1(\mathbb{R})$ by (3.2), there is a sequence $(t_n)_{n\in\mathbb{N}} \to +\infty$ such that $\lim_{n\to\infty} e(\sigma(t_n)) = 0$. Exactly like in Step 3 in the proof of Lemma 3.1, we deduce that $(\sigma(t_n))_{n\in\mathbb{N}}$ has a subsequence that converges strongly in $L^2(\omega,\mathbb{R}^N)$ to some map $\sigma_\infty \in L^2(\omega,\mathbb{R}^N)$ (the assumption **(H2)** is essential here). Since e is l.s.c. in L^2 and $e \ge 0$ in L^2, we deduce that $e(\sigma_\infty) = 0$ and so, there exists $u^+ \in \Sigma$ such that $\sigma_\infty \equiv u^+$.

Step 3: Convergence to u^+ in L^2 as $t \to +\infty$. Assume by contradiction that $\sigma(t)$ does not converge in $L^2(\omega,\mathbb{R}^N)$ to u^+ as $t \to \infty$. Then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{L^2}(\sigma(s_n),u^+) > 0$. Now, by Step 1, the curve $t \in [s_n,+\infty) \mapsto \sigma(t) \in L^2(\omega,\mathbb{R}^N)$ is continuous. Moreover, $\sigma(s_n)$ doesn't belong to the L^2-ball centered at u^+ with radius $\frac{3\varepsilon}{4}$. By Step 2, it has to enter (at some time $t > s_n$) in the L^2-ball centered at u^+ with radius $\frac{\varepsilon}{4}$. Therefore, the curve $\sigma_{|(s_n,+\infty)}$ has to cross the ring $\mathcal{R} := B_{L^2}(u^+,\frac{3\varepsilon}{4}) \setminus B_{L^2}(u^+,\frac{\varepsilon}{4})$, so it has L^2-length larger than $\frac{\varepsilon}{2}$, i.e.,

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\ \mathrm{d}t \\ = \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \|\dot{\sigma}\|_{L^2(\omega,\mathbb{R}^N)}\ \mathrm{d}t \ge \frac{\varepsilon}{2}.$$

Moreover, by the third claim in Lemma 3.1, we know that $e(\sigma(t)) \ge k_{\varepsilon/4}$ if $\sigma(t) \in \mathcal{R}$ (up to lowering ε, we may assume that the other zeros of Σ are placed at L^2-distance larger than 2ε from u^+, the assumption **(H1)** is essential here). For every large n, we obtain

$$\int_{s_n}^{+\infty} \sqrt{e(u(t,\cdot))}\ \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\ \mathrm{d}t \\ \ge \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}\}} \sqrt{e(u(t,\cdot))}\ \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\ \mathrm{d}t \ge \frac{\varepsilon}{2}\sqrt{k_{\varepsilon/4}}. \tag{3.3}$$

This is a contradiction with the assumption $E(u) < +\infty$ implying by (3.2):

$$2|\omega|^{\frac{1}{2}} \int_{s_n}^{+\infty} \sqrt{e(u(t,\cdot))}\ \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^N)}\,\mathrm{d}t \\ \le \int_{s_n}^{+\infty} \Big(|\omega| e(u(t,\cdot)) + \|\partial_{x_1} u(t,\cdot)\|^2_{L^2(\omega,\mathbb{R}^N)} \Big)\,\mathrm{d}t \underset{n\to\infty}{\longrightarrow} 0.$$

Step 4: The L^2 limits $u^\pm$ coincide with the average limits $\bar{u}(\pm\infty)$. This is clear as L^2 convergence implies convergence in average. □

Remark 3.2. i) The above proof does not use (so, it is independent of) the almost everywhere convergence of $u(x_1,\cdot)$ as $x_1 \to \pm\infty$ or the convergence of the x'-average $\bar{u}$. Therefore, thanks to this proof, one can obtain as a direct consequence the convergence of the x'-average $\bar{u}$ as well as the almost everywhere convergence of $u(x_1,\cdot)$ as $x_1 \to \pm\infty$.[8]

ii) Also, the above proof applies to Lemma 2.4 leading to a second method that does not use the geodesic distance geod_W.

iii) Behind the above proof, the notion of geodesic distance over $L^2(\omega,\mathbb{R}^N)$ with the degenerate weight $\sqrt{e}$ is hidden (see (3.3)). Therefore, one could repeat the arguments in the first proof of Theorem 1.1 based on this geodesic distance.

The above argument can also be used directly to obtain a second proof for the existence of limits of $\bar{u}$ at $\pm\infty$ without using the geodesic pseudo-distance geod_W (as presented in the proof in Section 2). For completeness, we redo the proof in the sequel:

Second proof of the convergence in x'-average in Theorem 1.1. Let $u \in \dot{H}^1(\Omega,\mathbb{R}^N)$ such that $E(u) < \infty$. We want to prove that the x'-average $\bar{u}$ admits a limit u^+ as $x_1 \to \infty$ and $W(u^+) = 0$ (the proof of the convergence as $x_1 \to -\infty$ is similar). Let V and E_V given by Lemma 2.1. Recall that $\Sigma := \{V = 0\} = \{W = 0\}$ and $E_V(\bar{u}) \le \frac{1}{|\omega|}E(u) < \infty$.

Step 1. We prove that for every $\varepsilon > 0$,

$$\kappa_\varepsilon := \inf\big\{V(z) \ : \ z \in \mathbb{R}^N,\, d_{\mathbb{R}^N}(z,\Sigma) \ge \varepsilon\big\} > 0.$$

Assume by contradiction that there exists a sequence $(z_n)_n$ such that $V(z_n) \to 0$ and $d_{\mathbb{R}^N}(z_n,\Sigma) \ge \varepsilon$. By the third claim in Lemma 2.1, we deduce that $(z_n)_n$ is bounded, so that, up to a subsequence, $z_n \to z$ for some $z \in \mathbb{R}^N$ yielding $d_{\mathbb{R}^N}(z,\Sigma) \ge \varepsilon$ and $V(z) = 0$, i.e., $z \in \Sigma$ (since V is l.s.c. and $V \ge 0$) which is a contradiction.

Step 2. There exists a sequence $(\bar{u}(t_n))_n$ converging to a well $u^+ \in \Sigma$. Indeed, as $V(\bar{u}) \in L^1(\mathbb{R})$, there exists a sequence $t_n \to \infty$ with $V(\bar{u}(t_n)) \to 0$. By **(H2)**, $(\bar{u}(t_n))_n$ is bounded, so that up to a subsequence, $\bar{u}(t_n) \to u^+$ as $n \to \infty$ for some point $u^+ \in \mathbb{R}^N$. As V is l.s.c. and $V \ge 0$, we deduce that $V(u^+) = 0$, i.e., $u^+ \in \Sigma$.

Step 3: Convergence of $\bar{u}$ to u^+ as $x_1 \to +\infty$. Assume by contradiction that $\bar{u}(x_1)$ does not converge to u^+ as $x_1 \to \infty$. Then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{\mathbb{R}^N}(\bar{u}(s_n),u^+) > 0$. As $\bar{u} : [s_n,+\infty) \to \mathbb{R}^N$ is continuous, by Step 2, the curve has to get out of the Euclidean ball $B(\bar{u}(s_n),\varepsilon/4)$ and it has to enter in the ball $B(u^+,\varepsilon/4)$. Therefore, $\bar{u}$ has to cross the ring $\mathcal{R} := B(u^+,\frac{3\varepsilon}{4}) \setminus B(u^+,\frac{\varepsilon}{4}) \subset \mathbb{R}^N$. Moreover, by Step 1, we know that $V(\bar{u}(x_1)) \ge \kappa_{\varepsilon/4}$ if $\bar{u}(x_1) \in \mathcal{R}$ (where we assumed w.l.o.g. that $\varepsilon > 0$ is small enough so that the other zeros of Σ

[8]As the L^2-convergence implies almost everywhere convergence of $u(x_1,\cdot)$ only up to a subsequence, one should repeat the argument in the proof of the a.e. convergence in Theorem 1.1 at page 991.

are placed at distance larger than 2ε from u^+). We obtain

$$\int_{s_n}^{+\infty} \sqrt{V(\bar{u}(x_1))}\,\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|\,\mathrm{d}x_1 \geq \int_{\{x_1\in(s_n,+\infty)\,:\,\bar{u}(x_1)\in\mathcal{R}\}} \sqrt{V(\bar{u}(x_1))}\,\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|\,\mathrm{d}x_1 \geq \frac{\varepsilon}{2}\sqrt{\kappa_{\varepsilon/4}}.$$

This is a contradiction with the assumption $E_V(\bar{u}) < +\infty$ implying

$$2\int_{s_n}^{+\infty} \sqrt{V(\bar{u}(x_1))}\,\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|\,\mathrm{d}x_1 \leq \int_{s_n}^{+\infty} \Big(\Big|\frac{\mathrm{d}}{\mathrm{d}x_1}\overline{u}(x_1)\Big|^2 + V(\bar{u}(x_1))\Big)\,\mathrm{d}x_1 \underset{n\to\infty}{\longrightarrow} 0.$$

□

4. Proof of Theorem 1.3

In this section, we consider $d = N$, $\Omega = \mathbb{R}\times\omega$ with $\omega = \mathbb{T}^{d-1}$ and $u \in H^1_{loc}(\Omega,\mathbb{R}^d)$ periodic in $x' \in \omega$ with $\bar{u}_1 = a$ in $\mathbb{R}$ for some constant $a \in \mathbb{R}$ (recall that $\bar{u}$ is the x'-average of u). Note that $|\omega| = 1$. We set

$$L^2_a(\omega,\mathbb{R}^d) := \Big\{ v = (v_1,\dots,v_d) \in L^2(\omega,\mathbb{R}^d)\ :\ \int_\omega v_1\,dx' = a \Big\}$$

and $H^1_a(\omega,\mathbb{R}^d) := H^1 \cap L^2_a(\omega,\mathbb{R}^d)$. Note that for a.e. $x_1 \in \mathbb{R}$, $u(x_1,\cdot) \in H^1_a(\omega,\mathbb{R}^d)$. We define the following energy e_a on the convex closed subset $L^2_a(\omega,\mathbb{R}^d)$ of $L^2(\omega,\mathbb{R}^d)$:

$$e_a(v) = \begin{cases} \displaystyle\int_\omega \Big(|\nabla' v|^2 + W(v)\Big)\,\mathrm{d}x' & \text{if } v \in H^1_a(\omega,\mathbb{R}^d),\\ +\infty & \text{if } v \in L^2_a(\omega,\mathbb{R}^d)\setminus H^1(\omega,\mathbb{R}^d). \end{cases} \tag{4.1}$$

In particular, we have for every $u \in \dot{H}^1(\Omega,\mathbb{R}^d)$ with $\bar{u}_1 = a$:

$$E(u) = \int_{\mathbb{R}} \Big(\|\partial_1 u(x_1,\cdot)\|^2_{L^2(\omega,\mathbb{R}^d)} + e_a(u(x_1,\cdot))\Big)\,\mathrm{d}x_1. \tag{4.2}$$

The aim is to adapt the proof of Theorem 1.1 given in Section 3 to prove Theorem 1.3. We start by transfering the properties of the energy e in Lemma 3.1 to the energy e_a defined in $L^2_a(\omega,\mathbb{R}^d)$. More precisely, if $W : \mathbb{R}^d \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semicontinuous function, then e_a is lower semicontinuous in $L^2_a(\omega,\mathbb{R}^d)$ endowed with the strong L^2-norm and the sets of zeros of e_a and $W(a,\cdot)$ coincide, i.e.,

$$\Sigma^a := \{v \in L^2_a(\omega,\mathbb{R}^d)\ :\ e_a(v) = 0\} = \{z = (a,z') \in \mathbb{R}^d\ :\ W(a,z') = 0\}.$$

If in addition W satisfies $\mathbf{(H2)}_a$, then Σ^a is compact in $\mathbb{R}^d$ and for every $\varepsilon > 0$, we have

$$k^a_\varepsilon := \inf\big\{e_a(v)\ :\ v \in L^2_a(\omega,\mathbb{R}^d) \text{ with } d_{L^2}(v,\Sigma^a) \geq \varepsilon\big\} > 0$$

(the proof of these properties follows by the same arguments presented in the proof of Lemma 3.1).

Proof of Theorem 1.3. Let $u \in H^1_{loc}(\Omega, \mathbb{R}^d)$ such that $E(u) < +\infty$ and $\bar{u}_1 = a$ in $\mathbb{R}$. We set $\sigma(t) := u(t,\cdot) \in H^1_a(\omega, \mathbb{R}^d)$ for a.e. $t \in \mathbb{R}$. We prove that $\sigma(t)$ converges in $L^2(\omega, \mathbb{R}^d)$ to a limit that is a zero in Σ^a as $t \to +\infty$ (the proof of the convergence as $t \to -\infty$ is similar). As in Steps 1 and 2 in the proof of the L^2-convergence in Theorem 1.1, we have that $t \in \mathbb{R} \mapsto \sigma(t) \in L^2_a(\omega, \mathbb{R}^d)$ is a $\frac{1}{2}$-Hölder continuous map in $\mathbb{R}$ and there is a sequence $(t_n)_{n\in\mathbb{N}} \to +\infty$ such that $\sigma(t_n) \to u^+$ in $L^2(\omega, \mathbb{R}^d)$ for a well $u^+ \in \Sigma^a$ (the assumption $\mathbf{(H2)}_a$ is essential here). In order to prove the convergence of $\sigma(t)$ to u^+ in L^2 as $t \to +\infty$, we argue by contradiction. If $\sigma(t)$ does not converge in $L^2(\omega, \mathbb{R}^d)$ to u^+ as $t \to \infty$, then there is a sequence $(s_n)_{n\in\mathbb{N}} \to +\infty$ such that $\varepsilon := \inf_{n\in\mathbb{N}} d_{L^2}(\sigma(s_n), u^+) > 0$. We repeat the argument in Step 3 in the proof of the L^2-convergence in Theorem 1.1 by restricting ourselves to $L^2_a(\omega, \mathbb{R}^d)$ endowed by the strong L^2 topology. More precisely, the continuous curve $t \in [s_n, +\infty) \mapsto \sigma(t) \in L^2_a(\omega, \mathbb{R}^d)$ has to cross the ring $\mathcal{R}_a := \big(B_{L^2}(u^+, \frac{3\varepsilon}{4}) \setminus B_{L^2}(u^+, \frac{\varepsilon}{4})\big) \cap L^2_a(\omega, \mathbb{R}^d)$, so it has L^2-length larger than $\frac{\varepsilon}{2}$, i.e.,

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}_a\}} \|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t = \int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}_a\}} \|\dot\sigma\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t \ge \frac{\varepsilon}{2}.$$

As $e(\sigma(t)) \ge k^a_{\varepsilon/4}$ if $\sigma(t) \in \mathcal{R}_a$ (up to lowering ε, we may assume that the other zeros of Σ^a are placed at distance larger than 2ε from u^+, the assumption $\mathbf{(H1)}_a$ is essential here), we obtain

$$\int_{\{t\in(s_n,+\infty)\,:\,\sigma(t)\in\mathcal{R}_a\}} \sqrt{e_a(u(t,\cdot))}\,\|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t \ge \frac{\varepsilon}{2}\sqrt{k^a_{\varepsilon/4}}.$$

This is a contradiction with (4.2):

$$2\int_{s_n}^{+\infty} \sqrt{e_a(u(t,\cdot))}\,\|\partial_{x_1} u(t,\cdot)\|_{L^2(\omega,\mathbb{R}^d)}\,\mathrm{d}t \le \int_{s_n}^{+\infty} \Big(e_a(u(t,\cdot)) + \|\partial_{x_1} u(t,\cdot)\|^2_{L^2}\Big)\,\mathrm{d}t \underset{n\to\infty}{\longrightarrow} 0.$$

Clearly, the L^2 convergence implies also the convergence in average of $\sigma(t)$ over ω as $t \to \infty$ as well as the a.e. convergence $\sigma(t) \to u^+$ in ω but only up to a subsequence. For the full almost everywhere convergence of $u(x_1,\cdot) \to u^+$, we proceed as follows. First, by the Poincaré-Wirtinger inequality on $\omega = \mathbb{T}^{d-1}$, we have for a.e. $x_1 \in \mathbb{R}$,

$$\int_\omega |\nabla' u_1(x_1,x')|^2\,\mathrm{d}x' \ge 4\pi^2 \int_\omega |u_1(x_1,x') - \bar{u}_1(x_1)|^2\,\mathrm{d}x' = 4\pi^2 \int_\omega |u_1(x_1,x') - a|^2\,\mathrm{d}x'.$$

By Fubini's theorem, we deduce that

$$E(u) \ge \int_\Omega \big(|\partial_1 u|^2 + |\nabla' u_1|^2 + W(u)\big)\,\mathrm{d}x \ge \int_{\mathbb{T}^{d-1}} E_{W_a}(u(\cdot,x'),\mathbb{R})\,\mathrm{d}x',$$

where $W_a(z) := W(z) + 4\pi^2|z_1 - a|^2$ and, as usual,

$$E_{W_a}(\sigma, \mathbb{R}) = \int_{\mathbb{R}} \big(|\dot\sigma|^2 + W_a(\sigma)\big)\,\mathrm{d}x_1, \quad \sigma \in \dot H^1(\mathbb{R}, \mathbb{R}^N).$$

Hence, $E_{W_a}(u(\cdot, x'), \mathbb{R}) < \infty$ for a.e. $x' \in \omega$. Note that W_a is lower semicontinuous and satisfies assumptions **(H1)** (the set of zeros of W_a coincides with Σ^a, which is finite by $\mathbf{(H1)}_a$) and the coercivity condition **(H2)** (thanks to $\mathbf{(H2)}_a$). Thus, Lemma 2.4 implies that for a.e. $x' \in \omega$, there exist two wells $u^\pm(x')$ of W_a such that

$$\lim_{x_1 \to \pm\infty} u(x_1, x') = u^\pm(x'). \tag{4.3}$$

By (1.4), as $\bar u(\pm\infty) = u^\pm$, we know that $\|u(R_n^\pm, \cdot) - u^\pm\|_{L^2(\omega, \mathbb{R}^N)} \to 0$ as $n \to \infty$ for two sequences $(R_n^\pm)_{n\in\mathbb{N}} \to \pm\infty$. Up to a subsequence, we deduce that $u(R_n^\pm, \cdot) \to u^\pm$ a.e. in ω as $n \to \infty$. By (4.3), we conclude that $u^\pm(x') = u^\pm$ for a.e. $x' \in \omega$. $\square$

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