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RESOLVENT CONDITIONS AND GROWTH OF POWERS OF OPERATORS ON L^p SPACES

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Abstract. Let T be a bounded linear operator on L^p. We study the rate of growth of the norms of the powers of T under resolvent conditions or Cesàro boundedness assumptions. Actually the relevant properties of L^p spaces in our study are their type and cotype, and for $1 < p < \infty$, the fact that they are UMD. Some of the proofs make use of Fourier multipliers on Banach spaces, which explains why UMD spaces come into play.

1. Introduction

We study the rate of growth of $\|T^n\|$ for a bounded operator T on a Banach space X under various conditions, continuing recent works of Bermúdez, Bonilla, Müller and Peris [4], Bonilla and Müller [5] and Cohen, Cuny, Eisner and Lin [6]. In particular, we extend several results of [6], obtained when $X = H$ is a Hilbert space, to L^p spaces and more generally to spaces with non trivial type and/or finite cotype.

Let us recall the conditions that are relevant to our study. We refer to [4] and [6] for more information as well as historical background concerning those conditions.

Let T be a bounded operator on a Banach space X. For simplicity we shall assume that X is a *complex* Banach space, while all the meaningful statements (i.e. the statements that do not require a complex Banach space in their formulation) hold true also for real spaces.

We say that T is *Kreiss bounded* if there exists $C > 0$ such that, with $R(\lambda, T) := (\lambda I - T)^{-1}$,

$$\|R(\lambda, T)\| \le \frac{C}{|\lambda| - 1} \qquad \forall |\lambda| > 1\,. \tag{1.1}$$

We say that T is *uniformly Kreiss bounded* if there exists $C > 0$ such that

$$\sup_{n\ge 1} \Big\| \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} \Big\| \le \frac{C}{|\lambda| - 1} \qquad \forall |\lambda| > 1. \tag{1.2}$$

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We say that T is *absolutely Cesàro bounded* if there exists $C > 0$ such that

$$\sup_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x\| \leq C\|x\| \qquad \forall\, x \in X. \tag{1.3}$$

We say that T *is strongly Cesàro bounded* if there exists $C > 0$ such that

$$\sup_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n-1} |\langle x^*, T^k x\rangle| \leq C\|x^*\| \cdot \|x\| \quad \forall (x, x^*) \in X \times X^*. \tag{1.4}$$

It was proved in [6] that (1.4) is equivalent to the existence of $C > 0$ such that

$$\sup_{n\geq 1} \sup_{|\gamma_0|=1,\ldots,|\gamma_{n-1}|=1} \frac{1}{n} \| \sum_{k=0}^{n-1} \gamma_k T^k x\| \leq C\|x\| \quad \forall x \in X\,. \tag{1.5}$$

Let us mention the following implications concerning those conditions. First of all any power bounded operator T, i.e. such that $\sup_{n\geq 0} \|T^n\| < \infty$, satisfies all of the above conditions. If T is uniformly Kreiss bounded it is also Kreiss bounded; if T is absolutely Cesàro bounded it is strongly Cesàro bounded, hence [6] uniformly Kreiss bounded. The converse of the above implications do not hold in general.

Let T be absolutely Cesàro bounded on a Banach space X. Then, see [6, Proposition 3.1], $\|T^n\| = O(n^{1-\varepsilon})$ for some $\varepsilon \in (0, 1)$ and this estimate is best possible in general Banach spaces. Earlier, the estimate $\|T^n\| = o(n)$ was proved in [4]. If X is a Hilbert space then, by Theorem 4.4 of [6], $\|T^n\| = O(n^{1/2-\varepsilon})$ for some $\varepsilon \in (0, 1/2)$ and this is best possible.

Let T be Kreiss bounded on a Banach space. Then, by [15], $\|T^n\| = O(n)$ and this is best possible in general Banach spaces by an example of Shields [19]. If X is a Hilbert space then, see [6, Theorem 4.1] or [5], $\|T^n\| = O(n/\sqrt{\log n})$. We do not know whether this is optimal. As far as we know the only result in that direction is that for every $\varepsilon \in (0, 1)$, there exists T on a Hilbert space that is uniformly Kreiss bounded and such that $\|T^n\| \asymp n^{1-\varepsilon}$. This is proved in [5], where in fact T is even strongly Cesàro bounded [6].

In this paper we obtain estimates of $\|T^n\|$ for absolutely Cesàro bounded, strongly Cesàro bounded or Kreiss bounded operators, according to the type and/or cotype of X. Some results only hold on UMD spaces, see later for the definitions. Estimates when $X = L^p(\Omega, \mu)$, $1 < p < \infty$, are obtained as corollaries.

2. Growth of the powers for absolutey Cesàro bounded operators

In this section we study the growth rate of $\|T^n\|$ when T is an absolutely Cesàro bounded operator on a Banach space of type p and cotype q, and then apply the results to L^p spaces. We recall the definitions [1, p. 151].

Definition. A Banach space X is said to be *of type* $p \in [1, 2]$ if there exists $K > 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$, one has

$$\mathbb{E}(\|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n\|^p) \leq K(\|x\|^p + \ldots + \|x_n\|^p\,)\,,$$

where $(\varepsilon_1,\ldots,\varepsilon_n)$ are iid Rademacher random variables (defined on $[0,1]$ with Lebesgue's measure λ; the expectation $\mathbb{E}$ is integration, see [1, p. 145-6]).

A Banach space X is said to be of finite *cotype* $q\geq 2$ if there exists $K>0$ such that for any $n\in\mathbb{N}$ and any $x_1,\ldots,x_n\in X$, one has

$$\|x_1\|^q+\ldots+\|x_n\|^q\leq K\mathbb{E}(\|\varepsilon_1x_1+\ldots+\varepsilon_nx_n\|^q)\,.$$

A Banach space is said to be of *cotype* ∞ if there exists $K>0$ such that for any $n\in\mathbb{N}$ and $x_1,\ldots,x_n\in X$, one has

$$\max_{1\leq i\leq n}\|x_i\|\leq K\mathbb{E}(\|\varepsilon_1x_1+\ldots+\varepsilon_nx_n\|)\,. \tag{2.1}$$

Every Banach space is of type 1 and (using for instance (*ii*) of Proposition 2.2 below) of cotype ∞. A Banach space with type $1<p\leq 2$ is said to have non-trivial type, and a Banach space with cotype $q\in[2,\infty)$ is said to have finite cotype. If X has non trivial type, it has finite cotype (see Theorem 7.3.11 page 98 of [11]) but the converse is wrong (any L^1 space has cotype 2 but has trivial type, see page 154 of [11]). If X is of type $p>1$ then X^* is of (finite) cotype $q=p/(p-1)$ (see Theorem 7.1.13 page 63 [11]). If X is of finite cotype q, X^* may be of trivial type (take $X=L^1$ again) but X^* is of non trivial type $p=q/(q-1)$ if we further assume that X has non trivial type (see Theorem 7.4.10 page 114 of [11] and recall that X is K-convex if and only if it has non trivial type). Finally, let us mention that X has non trivial type if and only if X^* does (see page 124 of [11]).

Typical examples of Banach spaces with non trivial type and finite cotype are given by the reflexive L^p-spaces. Indeed (see page 154 of [1]) when $X=L^p(\nu)$ for some σ-finite measure ν, X has type $p'=\min(p,2)$ and cotype $p''=\max(p,2)$ (and this is is best possible if the space is not finite dimensional). More generally (combine Theorem 10.1 with Propositions 10.1 and 10.2 of [18]) uniformly convexifiable Banach spaces have non trivial type and finite cotype. Those spaces are again relexive.

Now, for a general Banach space, there is no relation between the property of being reflexive and the property of having non trivial type and finite cotype. Taking $1<p_n<\infty$ with $\lim_n p_n=+\infty$ and for X the ℓ^2 direct sum of ℓ^{p_n} one obtains a reflexive Banach space with trivial type and only finite cotype (ℓ^{p_n} has type 2 but the best constant in the definition must go to ∞ as $p_n\to\infty$ since ℓ^∞ has trivial type). Notice that this example is such that X^* has cotype 2). Moreover, for every $\varepsilon>0$ there exists a non reflexive Banach space with type 2 and cotype $2+\varepsilon$ by Corollary 12.20 page 492 of [18]. However, one cannot take $\varepsilon=0$ since by a result of Kwapień (see Theorem 7.3.1 page 89 of [11]) every Banach space with type 2 and cotype 2 is isomorphic to a Hilbert space.

We shall need a somewhat direct consequence of the definition of type and cotype. By Proposition 9.11 of [14], if X is of finite cotype q then, for every independent integrable and centered (i.e. $\mathbb{E}(\xi_i)=0$) X-valued variables $\xi_1,\ldots\xi_n$, we have

$$\mathbb{E}(\|\xi_1+\ldots+\xi_n\|^q)\geq C_q\mathbb{E}(\|\xi_1\|^q)+\ldots+\mathbb{E}(\|\xi_n\|^q)\,. \tag{2.2}$$

When X has type p we have a reverse inequality

$$\mathbb{E}(\|\xi_1+\ldots+\xi_n\|^p)\le C_p\mathbb{E}(\|\xi_1\|^p)+\ldots+\mathbb{E}(\|\xi_n\|^p)\,. \tag{2.3}$$

We will need Kahane-Khintchine's inequalities [1, p. 148], which we recall for convenience in the following form.

Theorem 2.1. *Let X be a Banach space. For every $p,q>0$ there exists $C_{p,q}>0$ such that for every $x_1,\ldots,x_n\in X$,*

$$(\mathbb{E}(\|\varepsilon_1x_1+\ldots+\varepsilon_nx_n\|^p)^{1/p}\le C_{p,q}(\mathbb{E}(\|\varepsilon_1x_1+\ldots+\varepsilon_nx_n\|^q)^{1/q}\,.$$

We will need the following corollaries of Kahane's contraction principle [11, Theorem 6.1.13(ii)]. The first follows by application to $b_k\xi_k$ with $a_k=1/b_k$, and the second follows from [11] by taking $a_k=1$ for $k\in I$ and 0 for $k\notin I$.

Proposition 2.2. *Let $1\le p\le\infty$ and let $(\xi_k)_{k=1}^n$ be independent $\mathbb{R}$-symmetric X-valued random variables in $L^p(\Omega,\lambda;X)$. Then for $I\subset J\subset\{1,2,\ldots,n\}$ we have:*

(i) $\mathbb{E}(\|\sum_{k\in I}\xi_k\|^p)\le\Big(\frac{\pi}{2}\Big)^p\max_{j\in I}\Big|\frac{1}{b_j}\Big|^p\mathbb{E}(\|\sum_{k\in I}b_k\xi_k\|^p)$ *when $b_j\neq 0$ for any j.*

(ii) $\mathbb{E}(\|\sum_{k\in I}\xi_k\|^p)\le\mathbb{E}(\|\sum_{k\in J}\xi_k\|^p)$.

We are now in position to state and prove the main result of this section.

Theorem 2.3. *Let T be an absolutely Cesàro bounded operator on a Banach space X of type $1\le p\le 2$. Then $\|T^n\|=O(n^{1/p})$. Moreover, there exists $C>0$ such that for every $x\in X$, every sequence of iid Rademacher variables $(\varepsilon_n)_{n\ge 0}$ and every $n\in\mathbb{N}$,*

$$\mathbb{E}(\|\sum_{k=0}^{n-1}\varepsilon_kT^kx\|^p)\le Cn\|x\|^p\,. \tag{2.4}$$

If in addition X is of finite cotype $q\ge 2$, then $\|T^n\|=O(n^{1/p}/(\log n)^{1/q})$, and there exists $\tilde C>0$ such that for every $x\in X$ and every $n\in\mathbb{N}$,

$$\sum_{k=0}^{n-1}\|T^kx\|^q\le\tilde Cn^{q/p}\|x\|^q\,. \tag{2.5}$$

Remark 2.4. As mentionned previously, when $p>1$ then X automatically has finite cotype and the second part of the theorem applies. Both items of the theorem apply to uniformly convexifiable Banach spaces.

Proof. The bound $\|T^n\|=O(n^{1/p})$ will follow from (2.4) and item *(ii)* of Proposition 2.2.

Let us prove (2.4). By the contraction principle, it suffices to prove that, for every $N\ge 0$,

$$\mathbb{E}(\|\sum_{k=0}^{2^N-1}\varepsilon_kT^kx\|^p)\le C2^N\|x\|^p\,.$$

Using that X has type p, by (2.3), it suffices to prove that, for every $N \geq 1$,

$$\mathbb{E}(\|\sum_{k=2^{N-1}}^{2^N-1} \varepsilon_k T^k x\|^p) \leq C 2^N \|x\|^p \,. \tag{2.6}$$

Let $N \in \mathbb{N}$. Fix $x \in X$ with $\|x\| = 1$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be the Rademacher system on $([0,1],\lambda)$. Denote by S the Koopman operator associated with multiplication by 2 (mod 1), so that $S\varepsilon_n = \varepsilon_{n+1}$.

Define

$$y_N := \sum_{k=1}^{2^N} \frac{\varepsilon_k T^k x}{\|T^k x\| + 1}$$

and

$$u_N := \sum_{j=0}^{2^N-1} (S^j \otimes T^j) y_N = \sum_{j=1}^{2^N} \varepsilon_j T^j x \sum_{k=1}^{j} \frac{1}{\|T^k x\| + 1} + \sum_{j=2^N+1}^{2^{N+1}} \varepsilon_j T^j x \sum_{k=j-2^N}^{2^N} \frac{1}{\|T^k x\| + 1} \,.$$

Since T is absolutely Cesàro bounded, by the computations in (17) of [6] with $\varepsilon = 1$ and N instead of 2^{N-1} (the computations in [6] are done in the Hilbert case but just make use of the norm, hence apply equally in general Banach spaces), we obtain

$$\sum_{k=1}^{N} \frac{1}{\|T^k x\| + 1} \geq \tilde{C} N \,, \tag{2.7}$$

for some $\tilde{C} > 0$ (independent of N and of x with norm 1).

We now use the contraction principle twice, first item (ii), then item (i), and we use (2.7) for the last inequality, to obtain

$$\frac{\pi}{2}\|u_N\|_{L^p(X)} \geq \frac{\pi}{2}\Big\| \sum_{j=2^{N-1}+1}^{2^N} \varepsilon_j T^j x (\sum_{k=1}^{j} \frac{1}{\|T^k x\| + 1})\Big\|_{L^p(X)} \geq$$

$$\min_{2^{N-1}+1 \leq \ell \leq 2^N} \Big(\sum_{k=1}^{\ell} \frac{1}{\|T^k x\| + 1} \Big) \ \Big\| \sum_{j=2^{N-1}+1}^{2^N} \varepsilon_j T^j x \Big\|_{L^p(X)} =$$

$$\Big(\sum_{k=1}^{2^{N-1}+1} \frac{1}{\|T^k x\| + 1} \Big) \ \Big\| \sum_{j=2^{N-1}+1}^{2^N} \varepsilon_j T^j x \Big\|_{L^p(X)} \geq \frac{\tilde{C}}{2} 2^N \Big\| \sum_{j=2^{N-1}+1}^{2^N} \varepsilon_j T^j x \Big\|_{L^p(X)} \,.$$

Hence

$$\Big\| \sum_{j=2^{N-1}+1}^{2^N} \varepsilon_j T^j x \Big\|_{L^p(X)}^p \leq C' 2^{-Np} \|u_N\|_{L^p(X)}^p \,. \tag{2.8}$$

Since X is of type p,

$$\mathbb{E}(\|y_N\|^p) \leq K_p \sum_{k=1}^{2^N} \Big(\frac{\|T^k x\|}{\|T^k x\| + 1} \Big)^p \leq K_p 2^N \,.$$

Hence $\mathbb{E}(\|y_N\|) \le \mathbb{E}(\|y_N\|^p)^{1/p} \le K_p^{1/p} 2^{N/p}$. By stationarity of the Rademacher system, $\mathbb{E}(\|(S^j \otimes T^j)y_N\|) = \mathbb{E}(\|T^j y_N\|)$ and by the absolute Cesàro boundedness of T, we obtain

$$\mathbb{E}(\|u_N\|) \le \sum_{j=0}^{2^N-1} \mathbb{E}(\|(S^j \otimes T^j)y_N\| \le C\mathbb{E}(2^N\|y_N\|) \le D2^{N+N/p} \,. \tag{2.9}$$

Applying Kahane-Khintchine's inequality to the decomposition of u_N, we obtain

$$\|u_N\|_{L^p(X)} = (E(\|u_N\|^p)^{1/p} \le C_{p,1}\mathbb{E}(\|u_N\|) \le C_{p,1}D2^{N+N/p} \ .$$

Combining the last estimate with (2.8) we obtain

$$\mathbb{E}\Big(\Big\| \sum_{j=2^{N-1}+1}^{2^N} \varepsilon_j T^j x\Big\|^p\Big) \le \tilde{K}2^{-Np}\mathbb{E}(\|u_N\|^p) \le C2^N \ .$$

We now assume that X has finite cotype q.

Using the definition of cotype, Kahane-Khintchine's inequalities and (2.4), we obtain

$$\sum_{k=0}^{n-1}\|T^k x\|^q \le K\mathbb{E}(\|\sum_{k=0}^{n-1}\varepsilon_k T^k x\|^q) \le KC_{q,p}^q\mathbb{E}(\|\sum_{k=0}^{n-1}\varepsilon_k T^k x\|^p)^{q/p} \le \tilde{C}n^{q/p}\|x\|^q \,,$$

which proves (2.5). However, this yields only $\|T^n\| = O(n^{1/p})$.

Denote by $r = q/(q-1)$ the dual index of q. Since T is absolutely Cesàro bounded, it is strongly Cesàro bounded, and by [6, Corollary 3.7] so is T^*. By Kahane's inequalities and [6, Proposition 3.6], for $x^* \in X^*$ and $Q > P \ge 0$ we have

$$\Big(\mathbb{E}\big(\|\sum_{k=P}^{Q-1}\varepsilon_k T^{*k}x^*\|^r\big)\Big)^{1/r} \le C_{r,1}\mathbb{E}(\|\sum_{k=P}^{Q-1}\varepsilon_k T^{*k}x^*\|) \le 2Q\cdot C_{r,1}K_{scb}\|x^*\|.$$

For every $(x,x^*) \in X \times X^*$ and integers $N \ge Q > P \ge 0$, we have

$$\begin{aligned}(Q-P)|\langle x^*, T^N x\rangle| = \Big|\mathbb{E}\Big(\sum_{\ell=P}^{Q-1}\sum_{k=P}^{Q-1}\langle\varepsilon_k T^{*k}x^*, \varepsilon_\ell T^{N-\ell}x\rangle\Big)\Big| \le \\ \mathbb{E}\Big(\Big|\langle\sum_{\ell=P}^{Q-1}\varepsilon_k T^{*k}x^*, \sum_{k=P}^{Q-1}\varepsilon_\ell T^{N-\ell}x\rangle\Big|\Big) \le \\ \Big(\mathbb{E}\Big(\|\sum_{k=P}^{Q-1}\varepsilon_k T^{*k}x^*\|^r\Big)\Big)^{1/r}\Big(\mathbb{E}\Big(\|\sum_{\ell=P}^{Q-1}\varepsilon_\ell T^{N-\ell}x\|^q\Big)\Big)^{1/q} \le \\ CQ\|x^*\|\cdot\Big(\mathbb{E}\Big(\|\sum_{\ell=P}^{Q-1}\varepsilon_\ell T^{N-\ell}x\|^q\Big)\Big)^{1/q},\end{aligned}$$

with $C = 2C_{r,1}K_{scb}$. Taking the supremum over $\{\|x^*\|=1\}$ we conclude that

$$\frac{(Q-P)^q}{Q^q}\|T^N x\|^q \le C^q\mathbb{E}\Big(\Big\|\sum_{\ell=P}^{Q-1}\varepsilon_\ell T^{N-\ell}x\Big\|^q\Big) \qquad 0 \le P < Q \le N \,. \tag{2.10}$$

Fix $N \in \mathbb{N}$ and put $L := \log(N/2)/\log 2$. It follows from (2.10) that for every $0 \le \ell \le L$,

$$\mathbb{E}\Big(\|\sum_{k=N+1-2^{\ell+1}}^{N-2^\ell} \varepsilon_k T^k x\|^q\Big) \ge 2^q \|T^N x\|^q / C^q \,. \tag{2.11}$$

Denote $Z_\ell := \sum_{k=N+1-2^{\ell+1}}^{N-2^\ell} \varepsilon_k T^k x$. Then (z_ℓ) are independent on $\Omega = ([0,1],\lambda)$.

Since X has cotype q, by (2.2),

$$\sum_{l=0}^{L} \mathbb{E}(\|z_l\|^q) \le K\mathbb{E}(\|\sum_{l=0}^{L} \varepsilon_\ell z_\ell\|^q)\,.$$

Using Item (ii) of Proposition 2.2,

$$\mathbb{E}(\|\sum_{l=0}^{L} y_l\|^q) \le \mathbb{E}(\|\sum_{k=0}^{N-1} \varepsilon_k T^k x\|^q)\,.$$

Combining with (2.11), we obtain

$$2^q(L+1)\|T^N x\|^q/C^q \le \sum_{\ell=0}^{L} \mathbb{E}(\|z_\ell\|^q) \le K \cdot \mathbb{E}(\|\sum_{k=0}^{N-1} \varepsilon_k T^k x\|^q)\,.$$

Using Kahane's inequalities and (2.4), we conclude that $\|T^N x\| \le C' N^{1/p}\|x\|/L^{1/q}$. □

Remark 2.5. 1. When X is of type $p = 1$, the Theorem yields for T absolutely Cesàro bounded that $\|T^n\| = O(n/(\log n)^{1/q})$; the estimate $\|T^n\| = O(n^{1-\varepsilon})$ in [6, Proposition 3.1] is better.

2. When X has type $p > 1$, the estimate $\|T^n\| = O(n^{1/p})$ for T absolutely Cesàro bounded should be compared with $O(n^{1-1/K_{ac}})$ (with K_{ac} the best constant in the definition of absolute Cesàro boundedness) given in [6, Proposition 3.1] (with $p = 1$ there). The estimate of the theorem is better when $K_{ac} > p/(p-1)$.

Corollary 2.6. *Let T be absolutely Cesàro bounded on $L^p(\Omega,\mu)$, $1 \le p < \infty$, with μ σ-finite. Then, $\|T^n\|_p = O(n^{1/p'}/(\log n)^{1/p''})$ where $p' = \min(p,2)$ and $p'' = \max(p,2)$. Moreover, there exists $C_p > 0$ such that*

$$\sum_{k=0}^{N-1} \|T^k x\|_p^{p''} \le C_p N^{p''/p'} \|x\|_p^{p''}\,, \tag{2.12}$$

Proof. It is well known [1, p. 154] that $L^p(\Omega,\mu)$ is of type p' and of cotype p''. □

Remark 2.7. 1. When $1 \le p \le 2$, the bound (2.12) reads $\sum_{k=0}^{N-1} \|T^k x\|_p^2 \le C_p N^{2/p}\|x\|_p^2$, which is implied by the bound $\sum_{k=0}^{N-1} \|T^k x\|_p^p \le C_p^{p/2} N \|x\|_p^p$. When $p \ge 2$, the bound (2.12) reads $\sum_{k=0}^{N-1} \|T^k x\|_p^p \le C_p N^{p/2}\|x\|_p^p$, which is implied by the bound $\sum_{k=0}^{N-1} \|T^k x\|_p^2 \le C_p^{2/p} N \|x\|_p^p$.

2. For $p = 2$, (2.12) shows that T is Cesàro square bounded, as was also shown in [6, Theorem 4.3]. However, the bound on $\|T^n\|$ in [6, Theorem 4.3], obtained from [6, Proposition 3.1], is better than that of Corollary 2.6.

3. Comparing the examples of [6, Theorem 3.3] with Corollary 2.6, we see that when $p \in [1, 2]$, the corollary gives the correct bound for $\|T^n\|$, up to some $\varepsilon > 0$ in the exponent. One may wonder whether the bound $\|T^n\| = O(n^{1/2})$, provided by the corollary when $p > 2$, is the right one; in the examples of [6, Theorem 3.3], $\|T^n\| \le (n+1)^{1/p} < (n+1)^{1/2}$. We provide below another class of examples, which show that for $p > 2$, the bound in Corollary 2.6 is close to optimal.

Let $\mathbb{T} := \{\gamma \in \mathbb{C} : |\gamma| = 1\}$ and let λ be that Haar measure on $\mathbb{T}$.

Proposition 2.8. *Let $2 < p < \infty$. For $\varepsilon \in (0, 1/2)$ there exists an absolutely Cesàro bounded T on $L^p(\mathbb{T}, \lambda)$ such that $\|T^n\| \asymp n^{1/2-\varepsilon}$*

Proof. We first define a projection $Q : L^2(\mathbb{T}, \lambda) \to L^2(\mathbb{T}, \lambda)$ as follows. For $f = \sum_{n\in\mathbb{Z}} c_n \gamma^n \in L^2(\mathbb{T}, \lambda)$ set $Qf := \sum_{m\ge1} c_{2^m} \gamma^{2^m}$. Notice that Q is the operator obtained by multiplying the Fourier coefficients term by term with the sequence $(a_n)_{n\in\mathbb{Z}}$ given by $a_{2^m} = 1$ for every $m \ge 1$ and $a_n = 0$ otherwise. Since the sequence $(a_n)_{n\in\mathbb{Z}}$ has bounded dyadic variation, by the Marcinkiewicz multiplier theorem [21, Theorem XV(4.14)] it defines a bounded Fourier multiplier on $L^p(\mathbb{T}, \lambda)$ (i.e. $\| \sum_{n\in\mathbb{Z}} a_n c_n \gamma^n\|_p \le A_p \| \sum_{n\in\mathbb{Z}} c_n \gamma^n\|_p$ for $\sum_{n\in\mathbb{Z}} c_n \gamma^n$ in L^p). Thus Q extends to a bounded operator on $L^p(\mathbb{T})$, and for every $f \in L^p$ we have, by [21, Theorem V(8.20)]

$$\frac{1}{C_p}\Big(\sum_{m\ge1} |c_{2^m}|^2\Big)^{1/2} \le \|Qf\|_p \le C_p\Big(\sum_{m\ge1} |c_{2^m}|^2\Big)^{1/2}, \tag{2.13}$$

where C_p depends only on p.

By (2.13), Q actually takes values in $M := \{g = \sum_{m\ge1} b_m \gamma^{2^m} : (b_m)_{m\in\mathbb{N}} \in \ell^2(\mathbb{N})\}$, which is closed in L^p, and is clearly isomorphic to $\ell^2(\mathbb{N})$. For $g \in M$, (2.13) yields that $\|g\|_p \sim \|g\|_2$.

We now define $R : M \to M$ as follows. For $g = \sum_{m\ge1} b_m \gamma^{2^m} \in M$ set $Rg := \sum_{m\ge1} d_m \gamma^{2^m}$, where $d_m = \left(\frac{m+1}{m}\right)^{1/2-\varepsilon} b_{m+1}$.

Finally, we set $T := RQ$. Since Q is a projection of L^p onto M and R takes values in M, we see that $T^n = R^n Q$ for every $n \ge 1$, so T is absolutely Cesàro bounded on L^p whenever R is (on M), and the desired estimate on $\|T^n\|$ follows from the same estimate for R.

But, since M and $\ell^2(\mathbb{N})$ are isomorphic, we see that R is similar to the weighted backward shift on $\ell^2(\mathbb{N})$ defined in Theorem 2.1 of [4], so the estimates of [4] finish the proof. □

In view of the above remarks and the known examples of absolutely Cesàro bounded operators on L^p spaces, and in view of the results in the Hilbert case, the following question seems natural. The notion of p-absolute Cesàro boundedness was defined in [6]; for $p = 2$ see also [4].

Question. Let T be an absolutely Cesàro bounded operator on an L^p-space with $1 \leq p < 2$ (resp. with $p > 2$); is T p-absolutely Cesàro bounded (resp. 2-absolutely Cesàro bounded)?

3. Growth of the powers for Kreiss bounded operators

Montes-Rodríguez et al. [16] and Aleman-Suciu [2] asked whether any uniformly Kreiss bounded operator T satisfies $\|T^n\| = o(n)$ (hence is mean ergodic whenever X is reflexive). In this section, we prove that every Kreiss bounded operator T on a UMD space, satisfies $\|T^n\| = o(n)$, with even a logarithmic rate.

Definition. We say that a Banach space X is *UMD (Unconditional Martingale Differences property)* if for some (every) $p > 1$, there exists $C_p > 0$ such that for every sequence $(d_n)_{1\leq n\leq N}$ of martingale differences in some $L^p(\Omega, X, \nu)$ and every sequence $(\varepsilon_n)_{1\leq n\leq N} \in \{-1, 1\}^N$, we have

$$\|\sum_{n=1}^{N} \varepsilon_n d_n\|_{L^p(\Omega,X)} \leq C_p \|\sum_{n=1}^{N} d_n\|_{L^p(\Omega,X)} .$$

We will refer to the book of Hytönen, van Nerven, Vervaar and Weis [10] for the definitions and results about UMD spaces and Fourier multipliers, as well as to the paper of Zimmermann [20].

Let us recall some important features of UMD spaces. UMD spaces are reflexive [10, p. 306], with non-trivial type and finite cotype [10, p. 313], but the converse is not true (there exist reflexive Banach spaces with non-trivial type and finite cotype which are not UMD [10, p. 311]). Moreover, a UMD space has an equivalent uniformly convex norm (via super-reflexivity [10, pp. 308 and 363]), but the converse is false [10, p. 354]. The class of UMD spaces contains all L^p-spaces with $1 < p < \infty$, and if X is UMD, so is $L^p(X)$. Finally, let us mention that X is UMD if and only if X^* is [10, p. 292]

The class of UMD spaces is the right one to work with Fourier Multipliers. In our context, the relevance of UMD spaces is that those spaces are precisely the ones for which the Riesz property (see below) holds, see for instance Theorem 5.2.10 page 398 of [10]. In particular the Marcinkiewicz theorem cannot hold on non-UMD spaces.

Definition. We say that $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is an $L^p(\mathbb{T}, X)$-*Fourier multiplier* if there exists $C_p > 0$, such that whenever $(c_n)_{n\in\mathbb{Z}} \in X^{\mathbb{Z}}$ and the series $\sum_{n\in\mathbb{Z}} \gamma^n c_n$ converges in $L^p(\mathbb{T}, X)$, we have convergence of $\sum_{n\in\mathbb{Z}} a_n \gamma^n c_n$, and

$$\int_{\mathbb{T}} \|\sum_{n\in\mathbb{Z}} a_n \gamma^n c_n\|^p \, d\gamma \leq C_p^p \int_{\mathbb{T}} \|\sum_{n\in\mathbb{Z}} \gamma^n c_n\|^p \, d\gamma . \tag{3.1}$$

Then, we denote by $\|(a_n)_{n\in\mathbb{Z}}\|_{\mathcal{M}_p(X)}$ the best constant C_p for which (3.1) holds.

Set $I_0 = \{0\}$ and for every $n \in \mathbb{N}$, put $I_n = \{2^n, \dots, 2^{n+1} - 1\}$ and $I_{-n} = \{1 - 2^{n+1}, \dots, -2^n\}$. Given a sequence $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$ of complex numbers and an interval $I = [\alpha, \beta]$ of integers, we define the *variation of* $\mathbf{a}$ *on* I by $V(\mathbf{a}, I) := \sum_{k=\alpha}^{\beta-1} |a_{k+1} -$

$a_k|$. We define *the dyadic variation of* $\mathbf{a}$ by $V_d(\mathbf{a}) := \sup_{n\in\mathbb{Z}} |a_n| + \sup_{n\in\mathbb{Z}} V(\mathbf{a}, I_n)$ and we say that $\mathbf{a}$ has bounded dyadic variation if $V_d(\mathbf{a}) < \infty$.

In the following multiplier theorems, X is (necessarily) a UMD space.

Zimmermann [20] proved that any sequence with bounded dyadic variation is an $L^p(\mathbb{T}, X)$-Fourier multiplier, thus extending the Marcinkiewicz theorem which states the same result with $X = \mathbb{C}$. Moreover, there exists $C_p(X) > 0$ such that $\|\mathbf{a}\|_{\mathcal{M}_p(X)} \le C_p(X)V_d(\mathbf{a})$.

In particular, for any interval $I \subset \mathbb{Z}$, $(\delta_n(I))_{n\in\mathbb{Z}}$ is an $L^p(\mathbb{T}, X)$-Fourier multiplier and the norm $\|(\delta_n(I))_{n\in\mathbb{Z}}\|_{\mathcal{M}_p(X)}$ is bounded independently of I. We will call this result the *Riesz theorem*.

Moreover, any bounded monotone sequence of real numbers is an $L^p(\mathbb{T}, X)$-Fourier multiplier. We will call that result the *Stechkin theorem*.

Finally, any sequence with values in $\{-1, 1\}$ that is constant on each dyadic interval is an $L^p(\mathbb{T}, X)$-Fourier multiplier. We will call that result the *Littlewood-Paley theorem.*

We are now in position to prove the main result of this section.

Theorem 3.1. *Let X be a UMD Banach space. Let q and q^* be the (finite) cotypes of X and X^* respectively, and put $s = \min(q, q^*)$. Let T be a Kreiss bounded operator on X. Then $\|T^n\| = O(n/(\log n)^{1/s})$; in particular $\|T^n\| = o(n)$.*

Proof. Assume the theorem is proved when $s = q$. When $s = q^*$, we apply the result to T^* on X^*, noting that T^* is also Kreiss bounded, and by reflexivity $X^{**} = X$; we obtain that $\|T^{*n}\| = O(n/(\log n)^{1/q^*})$, and use $\|T^n\| = \|T^{*n}\|$.

Hence, we just have to prove the case where $s = q$.

By assumption, for every $r > 1$ and every $\gamma \in \mathbb{T}$, we have for every $x \in X$ with $\|x\| = 1$,

$$\|\sum_{n\ge 0} \frac{\gamma^n T^n x}{r^{n+1}}\| = \|R(\bar{\gamma} r, T)\| \le \frac{C}{(r-1)}\,.$$

Let $p > 1$. Let $N \in \mathbb{N}$ and take $r = 1 + 1/N$. By the Riesz theorem (in $L^p(\mathbb{T}, X)$) there exists $C_p > 0$ such that

$$\int_{\mathbb{T}} \|\sum_{n=0}^{N-1} \frac{\gamma^n T^n x}{(1+1/N)^{n+1}}\|^p d\gamma \le C_p^p \int_{\mathbb{T}} \|\sum_{n\ge 0} \frac{\gamma^n T^n x}{(1+1/N)^{n+1}}\|^p\, d\gamma \le (C_p C)^p N^p\,.$$

Define a sequence $(a_n)_{n\in\mathbb{Z}}$ as follows: $a_n = 1 + 1/N$, if $n \le 0$, $a_n = (1 + 1/N)^{n+1}$ if $1 \le n \le N$ and, $a_n = (1 + 1/N)^{N+1}$ if $n \ge N + 1$. Then, $(a_n)_{n\in\mathbb{Z}}$ is bounded and monotone, hence, by the Stechkin theorem, there exists $C_p > 0$ such that

$$\int_{\mathbb{T}} \Big\| \sum_{n=0}^{N-1} \gamma^n T^n x \Big\|^p d\gamma \le C_p^p \int_{\mathbb{T}} \|\sum_{n=0}^{N-1} \frac{\gamma^n T^n x}{(1+1/N)^{n+1}}\|^p d\gamma \le C' N^p\,.$$

Using the Riesz theorem again, for every $0 \le M \le N - 1$, we have

$$\int_{\mathbb{T}} \Big\| \sum_{n=M}^{N-1} \gamma^n T^n x \Big\|^p\, d\gamma \le C'' N^p\,. \tag{3.2}$$

Let $x^* \in X^*$ with $\|x^*\| = 1$. Let $0 \le P < Q \le N$ be integers. We have, writing $q' := q/(q-1)$ and using orthogonality,

$$(Q-P)|\langle x^*, T^N x\rangle| = \Big| \int_{\mathbb{T}} \sum_{k=P}^{Q-1} \sum_{\ell=P}^{Q-1} \langle \bar\gamma^k T^{*k} x^*, \gamma^\ell T^{N-\ell} x\rangle d\gamma \Big|$$
$$\le \int_{\mathbb{T}} \Big\| \sum_{k=P}^{Q-1} \bar\gamma^k T^{*k} x^* \Big\| \Big\| \sum_{\ell=P}^{Q-1} \gamma^\ell T^{N-\ell} x \Big\| d\gamma$$
$$\le \Big(\int_{\mathbb{T}} \Big\| \sum_{k=P}^{Q-1} \bar\gamma^k T^{*k} x^* \Big\|^{q'} d\gamma \Big)^{1/q'} \Big(\int_{|\mathbb{T}} \Big\| \sum_{\ell=P}^{Q-1} \gamma^\ell T^{N-\ell} x \Big\|^{q} d\gamma \Big)^{1/q}$$

Since, T^* is also Kreiss bounded on X^* (which is also UMD), (3.2) holds for T^* with $p = q'$ and $N = Q$. Hence, taking the supremum over $\{\|x^*\| = 1\}$, we see that

$$\int_{\mathbb{T}} \Big\| \sum_{\ell=P}^{Q-1} \gamma^\ell T^{N-\ell} x \Big\|^q d\gamma \ge C \frac{(Q-P)^q}{Q^q} \|T^N x\|^q . \tag{3.3}$$

Let $N \in \mathbb{N}$. Let $L := \log(N/2)/\log 2$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be Rademacher variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Using the Littlewood-Paley theorem, there exists $C_q > 0$ such that

$$\int_{\mathbb{T}} \Big\| \sum_{n=N+1-2^L}^{N-1} \gamma^n T^n x \Big\|^q d\gamma = \int_{\mathbb{T}} \Big\| \sum_{n=1}^{2^L-1} \gamma^n T^{N-n} x \Big\|^q d\gamma$$
$$\ge C_q \int_{\mathbb{T}} \Big\| \sum_{k=0}^{L-1} \varepsilon_k \sum_{\ell=2^k}^{2^{k+1}-1} \gamma^\ell T^{N-\ell} x \Big\|^q d\gamma .$$

Using (2.2), we have

$$\int_{\Omega} \Big\| \sum_{k=0}^{L-1} \varepsilon_k \sum_{\ell=2^k}^{2^{k+1}-1} \gamma^\ell T^{N-\ell} x \Big\|^q d\mathbb{P} \ge \sum_{k=0}^{L-1} \| \sum_{\ell=2^k}^{2^{k+1}-1} \gamma^\ell T^{N-\ell} x \Big\|^q$$

Hence,

$$\int_{\mathbb{T}} \Big\| \sum_{n=N+1-2^L}^{N-1} \gamma^n T^n x \Big\|^q d\gamma \ge C_q \sum_{k=0}^{L-1} \int_{\mathbb{T}} \| \sum_{\ell=2^k}^{2^{k+1}-1} \gamma^\ell T^{N-\ell} x \Big\|^q d\gamma .$$

In particular, using (3.2) and (3.3), we obtain that

$$C_q N^q \ge L \|T^N x\|^q ,$$

and the result follows. □

Corollary 3.2. *Let T be a Kreiss bounded operator on $L^p(\Omega, \mu)$, $1 < p < \infty$. Then $\|T^n\| = O(n/\sqrt{\log n})$.*

Proof. For $1 < p < \infty$, [1, p. 154] yields $s = 2$. □

Remark 3.3. The corollary extends the result proved for Hilbert spaces in [6, Theorem 4.1] and in [5].

Corollary 3.4. *Let T be a uniformly Kreiss bounded operator on a UMD space. Then γT is mean ergodic for every $\gamma \in \mathbb{T}$.*

Proof. By uniform Kreiss boundedness γT is Cesàro bounded (see [16]), and $\frac{1}{n}\|(\gamma T)^n\| = \frac{1}{n}\|T^n\| \to 0$ by Theorem 3.1. □

If we strengthen the Kreiss boundedness to strong Cesàro boundedness, we may drop the assumption that X be UMD, assuming only finite cotype for X or X^*. Recall that there exist Banach spaces with finite cotype that are not UMD and even not reflexive; for instance, any L^1 space has cotype 2 [1, p. 154].

Proposition 3.5. *Let T be a strongly Cesàro bounded operator on a Banach space X. Let q and q^* be the cotypes of X and X^* respectively, and put $s = \min(q, q^*)$. Then $\|T^n\| = O(n/(\log n)^{1/s})$; in particular, if s is finite, $\|T^n\| = o(n)$.*

Remark 3.6. 1. Of course the proposition is relevant only if $s < \infty$. A positive Cesàro bounded operator on a Banach lattice is strongly Cesàro bounded [6, Proposition 5.13]. Our result seems to be also new for positive operators.

2. Assume that X has type $p > 1$, hence X has cotype $p/(p-1)$. Then, as already mentionned X^* has non trivial type, say $p^* > 1$ and X^{**} has cotype $p^*/(p^* - 1)$. Now, if T is strongly Cesàro bounded on X so is T^* on X^* and the proposition gives $\|T^n\| = \|(T^*)^n\| = O(n/(\log n)^{1/s})$ with $s := \min(p/(p - 1), p^*/(p^* - 1))$.

Proof. It follows from (1.5) that T is strongly Cesàro bounded if and only if T^* is. Hence, as in the previous proof we may and DO assume that $s = q$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be Rademacher variables. By (1.5) and the contraction principle, for every $p \in [1, \infty)$, there exists $C_p > 0$ such that, for every $0 \le M \le N - 1$ and every $x \in X$, we have

$$\mathbb{E}(\| \sum_{n=M}^{N-1} \varepsilon_n T^n x\|^p) \le C_p N^p \|x\|^p \,. \tag{3.4}$$

We have a similar estimate for T^*.

Assume that $s = q < \infty$ otherwise there is nothing to prove and set $q' := q/(q-1)$. Notice that for every $(x, x^*) \in X \times X^*$ and every $0 \le P < Q \le N$, we have

$$(Q - P)|\langle x^*, T^N x\rangle| = \Big|\mathbb{E}\Big(\sum_{k=P}^{Q-1}\sum_{\ell=P}^{Q-1} \langle \varepsilon_k T^{*k}, \varepsilon_\ell T^{N-\ell}\rangle\Big)\Big| \,.$$

Proceeding as in the previous proof, in particular taking supremum over $\{x^* : \|x^*\| = 1\}$, we infer that for every $0 \le P < Q \le N$ and every $x \in X$,

$$\mathbb{E}\Big(\Big\| \sum_{\ell=P}^{Q-1} \varepsilon_\ell T^{N-\ell} x\Big\|^q\Big) \ge C\frac{(Q - P)^q}{Q^q} \|T^N x\|^q \,. \tag{3.5}$$

Setting $L := \log(N/2)\log 2$ and using (2.2), we infer that

$$\mathbb{E}\Big(\Big\|\sum_{n=N+1-2^L}^{N-1}\varepsilon_n T^n x\Big\|^q\Big) \geq C_q \sum_{k=0}^{L-1}\mathbb{E}\Big(\Big\|\sum_{\ell=2^k}^{2^{k+1}-1}\varepsilon_\ell T^{N-\ell}\Big\|^q\Big).$$

Then the result follows by applying (3.4) with $p = q$ and $M = N+1-2^L$, combined with (3.5) (as in the previous proof). □

Corollary 3.7. *Let T be a strongly Cesàro bounded operator on $L^1(\Omega,\mu)$. Then $\|T^n\| = O(n/\sqrt{\log n})$.*

Remark 3.8. 1. The corollary improves the case $p = 1$ of Corollary 2.6, which yields the same result under the stronger assumption of absolute Cesàro boundedness. However, under this stronger assumption [6, Proposition 3.1] yields the better estimate $\|T^n\| = O(n^{1-\epsilon})$.

2. The corollary shows that the operator of Kosek [12] is not strongly Cesàro bounded.

Corollary 3.9. *Let T be a positive Cesàro bounded operator on a Banach lattice with finite cotype q. Then, $\|T^n\| = O(n/(\log n)^{1/q})$. If $X = L^p(\Omega,\mu)$, with $1 \leq p < \infty$, then $\|T^n\| = O(n/\sqrt{\log n})$.*

Proof. By [6, Prop. 5.13], T is strongly Cesàro bounded and we apply Proposition 3.5. When $X = L^p$ and $p \leq 2$ we use the fact that X has cotype 2. When $p > 2$, we apply the previous case to T^*, which is positive and Cesàro bounded. □

Remark 3.10. As far as we know the corollary is new. The only result we are aware of in this direction is due to Emilion [8] and says that a positive Cesàro bounded operator on a reflexive Banach lattice X satisfies $\|T^n x\| = o(n)$ for every $x \in X$.

Corollary 3.11. *Let T be a strongly Cesàro bounded operator on a reflexive Banach space X such that X or X^* has finite cotype. Then γT is mean ergodic for every $\gamma \in \mathbb{T}$.*

Remark 3.12. If X is uniformly convexifiable norm, then it is reflexive with finite cotype and the corollary applies to any strongly Cesàro bounded operator on X.

In view of the above results the following question seems natural.
Question. Is every strongly Cesàro bounded operator on a reflexive Banach space mean ergodic ?

Notice that we even do not know whether a strongly Cesàro bounded operator on a reflexive Banach space is *weakly* mean ergodic.

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