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HADAMARD SEMIDIFFERENTIAL OF FUNCTIONS ON AN UNSTRUCTURED SUBSET OF A TVS

MICHEL C. DELFOUR

Abstract. This paper extends the notion of Hadamard semidifferentiability to unstructured subsets of an *ambiant* topological vector space without introducing local bases or coordinate spaces and in such a way that all the rules of the differential calculus are preserved including the chain rule. This work is motivated by problems where the independent variable is a family of subsets of the Euclidean space and the issue is to introduce notions of differential such as, for instance, the *shape derivative* via groups of diffeomorphisms or of semidifferential such as the *topological derivative*, a generalization of the *set derivative* of Lebesgue, for variations with respect to the *topology* by using Radon measures.

1. Introduction

This paper extends the notion of Hadamard semidifferentiability to unstructured subsets of an *ambiant* topological vector space (TVS) without introducing local bases or coordinate spaces in such a way that all the rules of the classical differential calculus are preserved including the chain rule. This work is motivated by problems where the independent variable is a family of subsets of the Euclidean space $\mathbb{R}^n$ and the issue is to introduce notions of differential such as, for instance, the *set derivative* of Lebesgue. In the literature two types of derivatives are considered: the *shape derivative* for variations with respect to the *shape* of a set through a family of *diffeomorphisms* of $\mathbb{R}^n$ and the *topological derivative* for variations with respect to the *topology* of the set such as the introduction of holes, cracks or connected components by using Radon measures. The *shape derivative* turns out to be a *differential* associated with the group of diffeomorphisms constructed by Micheletti [26] in 1972 and the *topological derivative* is generally only a s*emidifferential* associated with the group of characteristic functions of Lebesgue measurable subsets of $\mathbb{R}^n$ (cf. [5, 6, 7]) where the tangent space is a cone of Radon measures. In that context the non-differentiability arises from the underlying geometry and not from the function itself.

In 1923 Hadamard [19] gave a *geometric interpretation* of the *differential* of a function f of several variables at a point x by considering the function $t \mapsto f(h(t))$ along *paths* or *trajectories* $t \mapsto h(t)$ passing through the point $h(0) = x$ with a

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tangent $h'(0)$ at x instead of using linear increments $f(x + \Delta x)$ or $f(x + tv)$ for Δx or t small. A function f is differentiable at x if and only if there exists a linear function $Df(x)$ such that for all such trajectories $(f \circ h)'(0) = Df(x)h'(0)$. For instance, this suggests to define the differential of a function defined on an embedded submanifold M via trajectories in M.

The *analytical* generalization of the notion of differential to functions defined on a normed vector space was given by Fréchet [14] in 1925. In 1937 Fréchet [15] extends the geometrical definition of Hadamard to functions of functions and points out that Hadamard's approach is more general than his own since it works in a TVS that is not a normed space. A nice compact account of and comparison between the numerous notions of differentials in a TVS that followed can be found in Averbuh-Smoljanov [2], but we concentrate on what is directly pertinent to this paper. In 1938 and 1939 Michal [21, 22] introduced his *M-differentiability* that relaxes Fréchet to functions defined on a Hausdorff TVS (an *analytic notion* stronger than the one of Hadamard-Fréchet).

It was known in the early moments of the *Calculus of variations* by Volterra [36] and Gateaux [17, 18] in 1913 that the directional derivative of a function is not necessarily a linear mapping with respect to the direction. It is less known that, in his 1937 paper, Fréchet [15] suggested to relax the condition that the differential be linear and gave an example of such a function. It seems that it is only in 1978 that Penot [29] introduced the notion of *M-semidifferential*[1] (one sided directional derivative) for a TVS, which does not require the linearity while preserving all the operations of the classical differential calculus including the chain rule. However, his use of the term *M-differentiability* is misleading since his definition is definitely original compared to Michal's complicated notion. His M-semidifferentiability implies the *Hadamard semidifferentiability*, which also preserves all the operations of the classical differential calculus including the chain rule. It will be shown that Hadamard semidifferentiability is equivalent to the weaker notion of *sequential M-semidifferentiability*. In a Fréchet space all those notions coincide. A large class of nondifferentiable functions are Hadamard semidifferentiable such as, for instance, all continuous convex functions in a locally convex TVS and all norms both important in Optimization.

The M-semidifferentiability does not readily extend to functions defined on embedded submanifolds of the Euclidean space or, more generally, on an unstructured subset of a TVS, but, in view of its geometric nature, the Hadamard semidifferentiability does without introducing local bases or coordinate spaces while preserving all the rules of the finite dimensional differential calculus. In this context, the natural set of semitangents is the *intermediary or adjacent tangent cone* as defined by Aubin-Frankowska [1].

[1]In order to avoid a potential confusion between a *one-sided directional derivative* and a *directional derivative* at x in the direction v

$$\lim_{t \searrow 0} \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad \lim_{t \to 0} \frac{f(x+tv)-f(x)}{t},$$

we adopt the terminology *semidifferential* for a one-sided directional derivative and *directional derivative* for the second one. The term *differential* will be used for a one-sided directional derivative which is linear with respect to the direction (see, for instance, [8]).

Section 2 revisits the notion of differential in TVS. Section 3 revisits and expands the notion of semidifferential in TVS. Section 4 develops the Hadamard semidifferential for an unstructured subset of a TVS. Section 5 illustrates the application of this work to the groups of diffeomorphisms of Micheletti [26] associated with the *shape derivative* and the group of Lebesgue measurable characteristic functions [5, 6] associated with the *topological derivative*.

1.1. **Some Notation and Properties for TVS.** Recall that in a *topological vector space* there is a *fundamental system* $\mathscr{R}$ of neighborhoods of the origin for which (i) every V in $\mathscr{R}$ is absorbing and balanced, and (ii) for every $V \in \mathscr{R}$, there exists $U \in \mathscr{R}$ such that $U + U \subset V$. In this paper we always assume that the *neighborhoods* are elements of $\mathscr{R}$. A set A is bounded if, for all $V \in \mathscr{R}$, there exists $\alpha > 0$ such that $A \subset \lambda V$ for all $\lambda \geq \alpha$ ([20, Dfn. 1, p. 108]). A Fréchet space is a complete, metrizable, locally convex space and its metric is translation invariant (cf., for instance, [20, Dfn. pp. 79–80, Thm. 1, p. 81, Dfn. 4, p, 136]).

2. Differentials in vector spaces

2.1. **Differentials in the Euclidean Space.** The right definition of the differential for functions of several variables can be found in O. Stolz [32, p. 133][2] in 1893, J. Pierpont [31, p. 268][3] in 1905, W. H. Young [37, p. 157] in 1909 and [38, p. 21][4] in 1910, and M. Fréchet [12] in 1911 and [13] in 1912. But, according to V. M. Tihomirov [35], K. Weierstrass is most likely the first to have given a correct definition of the differential of a function of several variables:

> "Fréchet wrote, that it was the "*différentielle à mon sens*". But this was not quite right, because the correct definitions of derivative and differential of a function of many variables were given by K. Weierstrass in his lectures in the eighties of the 19th century. These lectures were published in the thirties of our century (20th)."

Indeed, Stolz, Pierpont, and Young through his wife Grace Chisholm had had direct or close contacts with Karl Weierstrass (1815–1897) or his work.

2.2. **Differentials in Topological Vector Spaces.** The proofs are omitted since they will be given in section 3 for the general semidifferential case.

2.2.1. *Hadamard, Fréchet, and Michal.* An equivalent definition of a geometric nature was introduced by Hadamard [19] in 1923 using trajectories in the Euclidean space. His definition was extended from the Euclidean space to function spaces by Fréchet [15] in 1937.

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[2]Otto Stolz (1842–1905).
[3]James Pierpont (1866–1938).
[4]William Henry Young (1863–1942).

Definition 2.1 (Admissible trajectory)**.** An *admissible trajectory*[5] at x in a topological vector space X is a function $h : (-\tau, \tau) \to X$, for some $\tau > 0$, such that

$$h(0) = x \quad \text{and} \quad h'(0) \overset{\text{def}}{=} \lim_{t \to 0} \frac{h(t) - h(0)}{t} \text{ exists,} \tag{2.1}$$

where $h'(0)$ is the *tangent* to the trajectory at $h(0) = x$.

Definition 2.2 (Hadamard [19] in 1923, Fréchet [15, p. 244] in 1937)**.** Let X and Y be topological vector spaces. The function $f : X \to Y$ is *Hadamard differentiable* at $x \in X$ if there exists a linear mapping[6] $Df(x) : X \to Y$ such that for all admissible trajectories at x

$$(f \circ h)'(0) \overset{\text{def}}{=} \lim_{t \to 0} \frac{f(h(t)) - f(h(0))}{t} \text{ exists and } (f \circ h)'(0) = Df(x)\, h'(0).$$

The mapping $v \mapsto Df(x)v : X \to Y$ is the *differential* of f at x.

Michal [21, pp. 341–342] in 1938 and [22, pp. 534–535] in 1939 introduced a slightly different notion that he called *HM-differentiabilty* where the function $h : (-\tau,\tau) \to X$ is differentiable everywhere instead of just at 0. That notion is equivalent to the above Hadamard-Fréchet definition (cf. [2, p. 80]).

In 1925 Fréchet extended his 1911 definition in [12] for functions of several variables to functions of functions.

Definition 2.3 (Fréchet [14] in 1925)**.** Let X be a normed space and Y a topological vector space. The function $f : X \to Y$ is *Fréchet differentiable* at $x \in X$ if there exists a *continuous* linear mapping $Df(x) : X \to Y$ such that

$$\lim_{\|v\| \to 0} \frac{f(x + v) - f(x) - Df(x)v}{\|v\|} = 0 \text{ in } Y. \tag{2.2}$$

The linear mapping $v \mapsto Df(x)v : X \to Y$ is the *differential* of f at x.

Following the extension of the Hadamard differentiability to function spaces by Fréchet [15, p. 244] in 1937, Michal in a note [21, pp. 340–341] in 1938 followed by a paper [22, pp. 532–533] in 1939 introduced a notion of *topological differential* for functions $f : X \to Y$ between two Hausdorff topological vector spaces and showed that it enjoys all the nice properties of the finite dimensional differential calculus [21, Thms. 1, 2, 3, p. 341].

Definition 2.4 (Michal [21, pp. 340–341], [22, p. 532])**.** Let X and Y be Hausdorff TVS. The function $f : X \to Y$ is *M-differentiable* at $x \in X$ if there exists a *continuous* linear mapping $Df(x) : X \to Y$ and a function $(x_1,x_2) \mapsto \varepsilon(x; x_1,x_2) : X \times X \to Y$ such that

(i) for all $x_2 \in X$, $\varepsilon(x; 0, x_2) = 0$,
(ii) there exists a neighborhood $V(0)$ of 0 such that for all $\lambda > 0$, $x_2 \in X$, and $x_1 \in V(0)$, $\varepsilon(x; x_1, \lambda x_2) = \lambda\, \varepsilon(x; x_1, x_2)$,
(iii) the function $(x_1,x_2) \mapsto \varepsilon(x; x_1,x_2) : X \times X \to Y$ is continuous at $(0,x_2)$ for all $x_2 \in X$,

[5]Since the term *path* is often understood as a continuous function, the term *trajectory* will be preferred for a function that can undergo jumps.
[6]It is not a priori assumed that $Df(x)$ be continuous as for Fréchet and Michal below.

(iv) there exists some neighborhood $V(0)$ of 0 in X such that

$$\forall w \in V(0), \quad f(x+w) - f(x) - Df(x)w = \varepsilon(x; w,w). \tag{2.3}$$

This is a generalization of Fréchet without using a norm.

Both Fréchet and M-differentiability lead to a simpler and weaker notion that will be introduced under the same name of M-differentiability by Penot [29] in 1978 (Definition 3.4 (iii)). Let $t > 0$ going to 0 and $w \to v$ for some $v \in X$. By replacing w by tw in Definition 2.4, we get $tw \to 0$ and

$$\frac{f(x+tw)-f(x)}{t} - Df(x)v = \varepsilon(x,tw,w) - Df(x)(w-v) \to \varepsilon(x,0,v) = 0$$
$$\Rightarrow \lim_{\substack{w \to v \\ t \searrow 0}} \frac{f(x+tw)-f(x)}{t} = Df(x)v,$$

since $Df(x)$ is continuous. Similarly, for the Definition 2.3 of Fréchet,

$$\frac{f(x+tw)-f(x)}{t} - Df(x)v$$
$$= \|w\| \begin{Bmatrix} \dfrac{f(x+tw)-f(x)-Df(x)tw}{\|tw\|}, & w \neq 0 \\ 0, & w = 0 \end{Bmatrix} + Df(x)(w-v)$$
$$\Rightarrow \lim_{\substack{w \to v \\ t \searrow 0}} \frac{f(x+tw)-f(x)}{t} = Df(x)v, \tag{2.4}$$

by continuity of $Df(x)$. Since Michal's definition is a little complicated and not really fundamental, we introduce the following weaker and simpler definition that will coincide with the Hadamard differentiability.

Definition 2.5. Let X and Y be topological vector spaces. The function $f : X \to Y$ is *MS-differentiable* at $x \in X$ if there exists a linear mapping $Df(x) : X \to Y$ such that for all $v \in X$ and all sequences $\{v_n\}$ converging to v,

$$\lim_{\substack{v_n \to v \\ t \searrow 0}} \frac{f(x+tv_n)-f(x)}{t} = Df(x)v. \tag{2.5}$$

Again $v \mapsto Df(x)v : X \to Y$ is the *differential* of f at x.

Remark 2.6. 1) Fréchet and Michal both assume that $Df(x)$ is continuous (additive and continuous), but, for the MS-differentiability, the continuity is not necessary since the sequential continuity will directly follow from Definition 2.5 by Theorems 2.7 and 2.8. This subtlety does not occur in finite dimension.
2) Condition (2.5) in Definition 2.5 implies the equivalent condition

$$\lim_{\substack{v_n \to v \\ t \to 0}} \frac{f(x+tv_n)-f(x)}{t} = Df(x)v, \tag{2.6}$$

since $\{-v_n\}$ converges to $-v$ and, by linearity of $Df(x)$,

$$\lim_{\substack{v_n \to v \\ t \nearrow 0}} \frac{f(x+tv_n)-f(x)}{t} = \lim_{\substack{v_n \to v \\ -t \searrow 0}} -\frac{f(x+(-t)(-v_n))-f(x)}{-t}$$
$$= -Df(x)(-v) = Df(x)v.$$

The MS-differentiability and the Hadamard differentiability coincide.

Theorem 2.7. *Let X and Y be topological vector spaces, $f : X \to Y$ a function, and $x \in X$.*

(i) *f is MS-differentiable at x if and only if it is Hadamard differentiable at x.*
(ii) *In a normed space X, if $f : X$-strong $\to Y$ is Fréchet differentiable at x, then $f : X$-strong $\to Y$ is MS-differentiable at x.*

The Hadamard differentiability enjoys all the nice properties of the classical finite dimensional differential calculus including the chain rule.

Theorem 2.8. *Let X and Y be topological vector spaces and $f : X \to Y$.*

(i) *If f is Hadamard differentiable at $x \in X$, then the linear mapping $v \mapsto Df(x)v : X \to Y$ is sequentially continuous.*
(ii) *If f_1 and f_2 are Hadamard differentiable at $x \in X$, then for all α and β in $\mathbb{R}$, $\alpha f_1 + \beta f_2$ is Hadamard differentiable at x.*
(iii) *(Chain rule) Let X, Y, Z be topological vector spaces, $g : X \to Y$ and $f : Y \to Z$ be functions such as g is Hadamard differentiable at x and f is Hadamard differentiable at $g(x)$. Then $f \circ g$ is Hadamard differentiable at x and*

$$\forall v \in X, \quad D(f \circ g)(x)v = Df(g(x))\, Dg(x)v. \tag{2.7}$$

The above theorem is quite general since the topologies are not specified. For instance, in a normed space X, Definition 2.5 can be given for the strong (norm) topology or the weak topology. Since all strongly convergent sequences are weakly convergent, the *weak Hadamard differentiability* is stronger than the *strong Hadamard differentiability* of Definition 2.5. There are further simplifications when $Y = \mathbb{R}^n$.

Theorem 2.9. *Let X and Y be normed vector spaces, $f : X \to Y$ a function, and $x \in X$.*

(i) *If $f : X$-strong $\to Y$-strong is Fréchet differentiable at x, then $f : X$-weak $\to Y$-weak is MS-differentiable at x.*
(ii) *Let X be a reflexive Banach space. Then, if $f : X$-weak $\to Y$-strong is MS-differentiable at x, then $f : X$-strong $\to Y$-strong is Fréchet differentiable at x.*
(iii) *Let X be a reflexive Banach space and $Y = \mathbb{R}^n$. Then $f : X$-strong $\to \mathbb{R}^n$ is Fréchet differentiable at x if and only if $f : X$-weak $\to \mathbb{R}^n$ is MS-differentiable at x.*

Remark 2.10. 1) In finite dimension Fréchet, Hadamard, and MS-differentiabilities coincide. In infinite dimension, the Hadamard and MS-differentiabilities do not require X to be a normed space.
2) Part (iii) was given in [9, Chapter 9, sec. 2.2, p. 461, Thm. 2.1, p. 462–463] in 2011. It is interesting to observe that the widely used Fréchet differentiability $f : X$-strong $\to Y$ seems to be strictly stronger than the Hadamard differentiability of $f : X$-strong $\to Y$ for which $Df(x) : X$-strong $\to Y$ is continuous.

Proof. (i) Let f be Fréchet differentiable at x. Given $v \in X$, pick arbitrary sequences $\{v_n\}$ in X and $\{t_n > 0\}$ such that $v_n \to v$ in X-weak and $t_n \to 0$. Then $\{v_n\}$ in X

is bounded in norm and $t_n v_n \to 0$ in X-strong. Consider the differential quotient

$$\frac{f(x+t_n v_n)-f(x)}{t_n} - Df(x)\,v$$
$$= \|v_n\| \left\{ \begin{array}{ll} \dfrac{f(x+t_n v_n)-f(x)-Df(x)\,t_n v_n}{\|t_n\, v_n\|}, & v_n \neq 0 \\ 0, & v_n = 0 \end{array} \right\} + Df(x)\,(v_n - v).$$

The first term goes to zero in Y-strong and hence in Y-weak. The second term also goes to zero since the continuous linear mapping $Df(x) : X$-strong $\to Y$-strong is also weakly continuous $Df(x) : X$-weak $\to Y$-weak. Therefore, for all v

$$\lim_{\substack{w\to v \\ t\searrow 0}} \frac{f(x+tw)-f(x)}{t} = Df(x)\,v \text{ in } Y\text{-weak} \tag{2.8}$$

and, since the right-hand side is linear, $f : X$-weak $\to Y$-weak is MS-differentiable.

(ii) Since $f : X$-weak $\to Y$-strong is MS-differentiable, we have the sequential continuity of $Df(x) : X$-weak $\to Y$-strong by Theorem 2.8 (i). By contradiction, assume that $Df(x) : X$-strong $\to Y$-strong is not continuous. Then, for $\varepsilon > 0$, there exists a sequence $v_n \to v$ in X-strong such that $\|Df(x)v_n\| \geq \varepsilon$. But the sequence $v_n \to v$ is also weakly convergent, then $Df(x)v_n \to Df(x)v$ in Y-strong and we get a contradiction.

Let $\{v_n\}$ be a sequence in X such that $\|v_n\| \to 0$ and consider the differential quotient

$$q_n \stackrel{\text{def}}{=} \frac{f(x+v_n)-f(x)-Df(x)v_n}{\|v_n\|}.$$

The sequence $w_n = v_n/\|v_n\|$ on the unit sphere has a subsequence, still denoted $\{w_n\}$, that converges to some w in X-weak. Then

$$q_n = \left[\frac{f(x+t_n w_n)-f(x)}{t_n} - Df(x)w\right] + Df(x)(w_n - w).$$

The first term converges to 0 in Y-strong. Since $Df(x) : X$-weak $\to Y$-strong is sequentially continuous, by Theorem 2.8 (i) the second term also sequentially converges to 0 in Y-strong. So given a sequence $\{v_n\}$ in X such that $\|v_n\| \to 0$, there exists a subsequence $\{v_{n_k}\}$ such as $q_{n_k} \to 0$ and, hence, the whole sequence $\{q_n\}$ converges to 0 in Y-strong. Hence, $f : X$-strong $\to Y$-strong is Fréchet différentiable.

(iii) From part (ii) since X is reflexive and from part (i) since for $Y = \mathbb{R}^n$ the strong and weak topologies coincide. □

2.2.2. *Continuity of Differentiable Functions.* The next question is the continuity of a Hadamard differentiable function.

Theorem 2.11. *Let X and Y be topological vector spaces, $f : X \to Y$ a function. Assume that f is Hadamard differentiable at $x \in X$.*

(i) *If there exists a bounded neighborhood $U(0) \in \mathscr{R}$ in X, then f is sequentially continuous at x.*

(ii) *If X and Y are Fréchet spaces, then f is continuous at x and $Df(x) : X \to Y$ is linear and continuous.*

The proof of a more general version of this theorem will be given in Theorem 3.8.

There are topological vector spaces that are not metrizable, nor first countable in which continuity and sequential continuity are equivalent and in which there are bounded neighborhoods of the origin.

Example 2.12 (Horváth [20, Example 12.5, p. 164, Example 4.9, p. 90])**.** Given an open set $\Omega \subset \mathbb{R}^n$, denote by $\mathscr{K}(\Omega)$ the space of continuous functions in Ω with compact support in Ω. Given a compact K in Ω, let $\mathscr{K}(K)$ be the space of continuous functions with support in K endowed with the sup norm $\|f\|_{K,\infty} = \max_{x\in K} |f(x)|$. So the support of each function f is contained in some compact subset K of Ω. The space $\mathscr{K}(\Omega)$ is the union of the linear subspaces $\mathscr{K}(K)$ over all compact $K \subset \Omega$

$$\mathscr{K}(\Omega) = \cup_{\substack{K\subset\Omega \\ K \text{ compact}}} \mathscr{K}(K), \quad \mathscr{K}(K) \hookrightarrow \mathscr{K}(\Omega) \text{ injective.} \tag{2.9}$$

This space endowed with the finest locally convex topology making all injections $\mathscr{K}(K) \hookrightarrow \mathscr{K}(\Omega)$ continuous is a locally convex topological space which is Hausdorff and complete.

A sequence converges in $\mathscr{K}(\Omega)$ if and only if it is included in a space $\mathscr{K}(K)$ and converges there; a function $f : \mathscr{K}(\Omega) \to \mathbb{R}$ is continuous if and only if it is continuous on every $\mathscr{K}(K)$ where there is equivalence between continuity and sequential continuity. The space $\mathscr{K}(\Omega)$ is not metrizable, nor first countable, but being an inductive limit of the metric TVS $\mathscr{K}(K)$, sequential continuity of $f : \mathscr{K}(\Omega) \to \mathbb{R}$ is equivalent to continuity.

Another remark is that $\mathscr{K}(\Omega)$ has bounded neighborhoods of the origin: a set B is *bounded* in $\mathscr{K}(\Omega)$ if and only if there exists a compact subset K of Ω and a number $\mu > 0$ such that all $f \in B$ have their support in K and $|f(x)| \le \mu$ for all $x \in \Omega$ and $f \in B$ (cf. Horváth [20, p. 165]).

3. Semidifferentials in topological vector spaces

3.1. **Fréchet Drops the Linearity of the Differential.** Volterra [36] was the first to have the idea to extend the field of application of the Differential Calculus to Functional Analysis.[7] In the early moments of the *Calculus of variations*, it was already known to him and Gateaux ([17] in 1913 and [18] in his posthumous paper of 1919 after his death in 1914) that some functions have directional derivatives in all directions without being differentiable.

Definition 3.1. Let X and Y be topological vector spaces and $f : X \to Y$ a function.

(i) f is *Gateaux semidifferentiable at $x \in X$ in the direction $v \in X$* if

$$df(x; v) \overset{\text{def}}{=} \lim_{t\searrow 0} \frac{f(x + tv) - f(x)}{t} \text{ exists in } Y. \tag{3.1}$$

[7]C'est Volterra qui eut le premier l'idée d'étendre le champ d'application du Calcul différentiel à l'Analyse fonctionnelle. (quoted from Fréchet [15, p, 241]).

(ii) f is *Gateaux semidifferentiable at* $x \in X$ if it is Gateaux semidifferentiable at $x \in X$ in all directions $v \in X$.

(iii) f is *Gateaux differentiable at* $x \in X$ if f is Gateaux semidifferentiable at $x \in X$ and $v \mapsto Df(x)v \overset{\text{def}}{=} df(x;v) : X \to Y$ is linear.

In his 1937 paper Fréchet [15, p. 239] proposed to drop the condition that $v \mapsto Df(x)v : \mathbb{R}^n \to \mathbb{R}^m$ be linear in Definition 2.2 and to replace it by the existence of a function $v \mapsto g(x;v) : \mathbb{R}^n \to \mathbb{R}^m$ and he gives as an example ([15, p. 239]) the following homogeneous function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) \overset{\text{def}}{=} \begin{Bmatrix} x\sqrt{\dfrac{x^2}{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{Bmatrix}, \quad g((0,0),(v,w)) = f(v,w).$$

It is readily seen that such a function $v \mapsto g(x;v)$ must be homogeneous. If h is an admissible trajectory, then, for all $\alpha \neq 0$, $t \mapsto h_\alpha(t) = h(\alpha t)$ is also admissible since $h_\alpha(0) = x$ and $h'_\alpha(0) = \alpha h'(0)$: by definition

$$\begin{aligned} (f \circ h)' = g(x; h'(0)), \quad (f \circ h_\alpha)' &= g(x; h'_\alpha(0)) = g(x; \alpha h'(0)) \\ (f \circ h_\alpha)'(0) = \lim_{t \to 0} \frac{f(h(\alpha t)) - f(h(\alpha 0)}{t} &= \alpha \lim_{t \to 0} \frac{f(h(\alpha t)) - f(h(0))}{\alpha t} \\ \Rightarrow \forall v \in X, \forall \alpha \in \mathbb{R}, \quad g(x; \alpha h'(0)) &= \alpha\, g(x; h'(0)). \end{aligned}$$

Since $h'(0)$ ranges over all X, $v \mapsto g(x;v) : \mathbb{R}^n \to \mathbb{R}^m$ is homogeneous.

Unfortunately, he does not pursue but he was very close to catch the Euclidean norm $n(x) = \|x\|$ in $\mathbb{R}^n$ which is not differentiable at the origin $x = 0$ and all the continuous convex functions as we shall see later!

In spite of this, his new definition fails for the Euclidean norm $n(x)$ at the origin: given a trajectory h through 0 such that $h'(0)$ exists and $t \neq 0$

$$\frac{n(h(t)) - n(h(0))}{t} = \frac{\|h(t)\| - \|h(0)\|)}{t} = \frac{\|h(t) - h(0)\|}{t} = \frac{|t|}{t} \underbrace{\left\| \frac{h(t) - h(0)}{t} \right\|}_{\to \|h'(0)\|}$$

since $|t|/t$ does not converge as t goes to 0. But, the idea of relaxing the linearity was the right one. If, instead of using an admissible trajectory, he had used a *semitrajectory* $h : [0,\tau) \to \mathbb{R}^n$ such that

$$h(0) = x \text{ and } h'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{h(t) - h(0)}{t} \text{ exists,}$$

then, since $h(0) = 0$ and $t > 0$, the differential quotient becomes

$$\frac{n(h(t)) - n(h(0))}{t} = \frac{\|h(t)\| - \|h(0)\|}{t} = \frac{\|h(t) - h(0)\|}{t} = \left\| \frac{h(t) - h(0)}{t} \right\|,$$

that goes to $\|h'(0^+)\|$ as $t \searrow 0$. So, the linear mapping $Df(0)$ is replaced by the *positively homogeneous continuous* mapping $v \mapsto g(0;v) = \|v\|$.

3.2. **From Differentials to Semidifferentials.** In that spirit, we introduce the following notion of Hadamard semidifferentiabilty that was missing to Fréchet in 1937. All the theorems of the previous section extend from differentiabilty to semidifferentiability.

Definition 3.2. An *admissible semitrajectory*[8] at x in a topological vector space X is a function $h : [0,\tau) \to X$, for some $\tau > 0$, such that

$$h(0) = x \quad \text{and} \quad h'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{h(t) - h(0)}{t} \text{ exists in } X, \tag{3.2}$$

where $h'(0^+)$ is the *semitangent* to the trajectory h at $h(0) = x$.

Definition 3.3. Let X and Y be topological vector spaces and $f : X \to Y$ a function.

(i) f is *Hadamard semidifferentiable at* $x \in X$ *in the direction* $v \in X$ if there exists $d_H f(x;v) \in Y$ such that for all admissible semitrajectories h in X at x such that $h'(0^+) = v$, we have

$$(f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{(f \circ h)(t) - (f \circ h)(0)}{t} = d_H f(x;v). \tag{3.3}$$

(ii) f is *Hadamard semidifferentiable at* $x \in X$ if there exists a function

$$v \mapsto d_H f(x;v) : X \to Y \tag{3.4}$$

such that for each *admissible semitrajectory* h in X at x,

$$(f \circ h)'(0^+) \text{ exists and } (f \circ h)'(0^+) = d_H f(x; h'(0^+)).$$

(iii) f is *Hadamard differentiable at* $x \in X$ if f is Hadamard semidifferentiable at x and the function $v \mapsto Df(x)v \overset{\text{def}}{=} d_H f(x;v) : X \to Y$ is linear.

Coming back to the Euclidean norm $f(x) = \|x\|$

$$(f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{\|h(t)\| - \|h(0)\|}{t} = \left\{ \begin{matrix} \dfrac{x}{\|x\|} \cdot h'(0^+)), & x \neq 0 \\ \|h'(0^+)\|, & x = 0 \end{matrix} \right\} = g(x; h'(0^+)),$$

where $x \cdot y$ denotes the inner product in $\mathbb{R}^n$. The linear mapping is replaced by the positively homogeneous and continuous function

$$v \mapsto g(x;v) \overset{\text{def}}{=} \begin{cases} \dfrac{x}{\|x\|} \cdot v, & x \neq 0, \\ \|v\|, & x = 0. \end{cases} \tag{3.5}$$

In 1978 Penot [29] introduces the following definition which generalizes the notion obtained in (2.4) to the semidifferential case for functions $f : X \to Z$ between topological vector spaces X and Z, where Z is completely ordered.

Definition 3.4 (Penot [29, p. 250], 1978)**.** Let X and Y be topological vector spaces and $f : X \to Y$ a function.

[8] In 1973 Durdil [10] uses semitrajectories in his definition of Hadamard differentiability in normed vector spaces, that is, he does not drop the linearity.

(i) f is *M-semidifferentiable*[9] *at $x \in X$ in the direction $v \in X$* if

$$d_M f(x;v) \overset{\text{def}}{=} \lim_{\substack{w\to v\\ t\searrow 0}} \frac{f(x+tw)-f(x)}{t} \text{ exists in } Y. \tag{3.6}$$

(ii) f is *M-semidifferentiable at $x \in X$* if it is M-semidifferentiable at $x \in X$ in all directions $v \in X$.

(iii) f is *M-differentiable at $x \in X$* if f is M-semidifferentiable at $x \in X$ and the function $v \mapsto Df(x)v \overset{\text{def}}{=} d_M f(x;v) : X \to Y$ is linear.

As in the differentiable case, we introduce the weaker sequential version.

Definition 3.5. Let X and Y be topological vector spaces, $f : X \to Y$, and $x \in X$.

(i) f is *MS-semidifferentiable at x in the direction $v \in X$* if there exists $d^s_M f(x;v) \in Y$ such that for each sequence $\{v_n\}$ converging to v,

$$\lim_{\substack{v_n\to v\\ t\searrow 0}} \frac{f(x+tv_n)-f(x)}{t} = d^s_M f(x;v) \tag{3.7}$$

(ii) f is *MS-semidifferentiable at x* if it is MS-semidifferentiable at $x \in X$ in all directions $v \in X$.

(iii) f is *MS-differentiable at x* if f is MS-semidifferentiable at x and the function $v \mapsto Df(x)v \overset{\text{def}}{=} d^s_M f(x;v) : X \to Y$ is linear.

Theorem 3.6. *Let X and Y be topological vector spaces, $f : X \to Y$ a function, and $x \in X$.*

(i) *The function f is MS-semidifferentiable at x in the direction v if and only if it is Hadamard semidifferentiable at x in the direction v. In particular, $d_H f(x;v) = d^s_M f(x;v)$.*

(ii) *If, in addition, X is a Fréchet space, then the notions of Hadamard, MS-, and M-semidifferentiability coincide.*

Proof. (i) ($\Rightarrow$) Let h be an arbitrary admissible semitrajectory at x in the direction $h'(0^+) = v$. We want to prove that for any sequence $\{t_n > 0\}$ converging to 0

$$\frac{f(h(t_n)) - f(h(0))}{t_n} \to d^s_M f(x; h'(0^+)) = d^s_M f(x;v).$$

Associate with $\{t_n\}$ the sequence

$$v_n \overset{\text{def}}{=} \frac{h(t_n)-h(0)}{t_n} \to h'(0^+) = v.$$

Since f is M-semidifferentiable at x, then

$$\frac{f(h(t_n)) - f(h(0))}{t_n} = \frac{f(x+t_n v_n) - f(x)}{t_n} \to d^s_M f(x;v),$$

f is Hadamard semidifferentiable at x, and $d_H f(x;v) = d^s_M f(x;v)$.

[9]He uses the terms *M-semi-dérivable* and *M-dérivable* in [29]. However, in his book [30, Chapter 2] in 2013 he uses the terms *directional derivative* at x in the direction $v \in X$ and *directionally differentiable* at x if it has a directional derivative at x in all directions $v \in X$ for a function $f : X \to Y$ between two normed spaces.

Conversely, let $\{v_n\}$ be a sequence converging to v and $\{t_n > 0\}$ a strictly decreasing sequence converging to 0. Define the admissible semitrajectory

$$h(t) \overset{\text{def}}{=} x + tv_n,\ t_{n+1} < t \le t_n, \quad h(0) = x$$
$$\Rightarrow \frac{h(t)-h(0)}{t} = v_n,\ t_{n+1} < t \le t_n \quad \Rightarrow h'(0^+) = v.$$

Then for $t_{n+1} < t \le t_n$

$$\frac{f(x+t_n v_n)-f(x)}{t} = \frac{f(h(t_n))-f(h(0))}{t} \to d_H f(x;v)$$

and f is MS-semidifferentiable at x in the direction v.

(ii) By contradiction. If f is not M-semidifferentiable, there exists $W(0) \in \mathscr{R}$ such that for each $n \ge 1$, there exist t_n, $0 < t_n < 1/n$, and v_n, $\rho(v_n,v) < 1/n$, such that

$$\frac{f(x+t_n v_n)-f(x)}{t_n} - d_M f(x;v) \notin W(0),$$

where ρ is the metric in X. Since the sequences $\{t_n\}$ and $\{v_n\}$ are convergent to 0 and v, this contradicts the MS-differentiability. □

The Hadamard semidifferentiability enjoys all the nice properties of the classical finite dimensional differential calculus including the chain rule.

Theorem 3.7. *Let X and Y be topological vector spaces and $f : X \to Y$ a function.*

(i) *If f is Hadamard semidifferentiable at x, then $v \mapsto d_H f(x;v) : X \to Y$ is positively homogeneous and sequentially continuous.*

(ii) *If f_1 and f_2 are Hadamard semidifferentiable at $x \in X$ in the direction $v \in X$, then for all α and β in $\mathbb{R}$,*

$$d_H(\alpha f_1 + \beta f_2)(x;v) = \alpha\, d_H f_1(x;v) + \beta\, d_H f_2(x;v). \tag{3.8}$$

(iii) *(Chain rule) Let X, Y, Z be topological vector spaces, $g : X \to Y$ and $f : Y \to Z$ be functions such as g is Hadamard semidifferentiable at x in the direction $v \in X$ and f is Hadamard semidifferentiable at $g(x)$ in the direction $d_H g(x;v)$. Then $f \circ g$ is Hadamard semidifferentiable at x in the direction $v \in X$ and*

$$d_H(f \circ g)(x;v) = d_H f(g(x); d_H g(x;v)). \tag{3.9}$$

Proof. (i) By definition of $d_H f(x;v)$, we have the positive homogeneity. As for the sequential continuity, we use the equivalence between $d_H f$ and $d_M^s f$ from Theorem 3.6(i). Let $W(0) \in \mathscr{R}$ in Y. There exists a closed neighbourhood $F(0) \in \mathscr{R}$ contained in $W(0)$ (cf. [20, Prop. 3, p. 84]). Let $\{v_n\}$ be a sequence converging to v. By definition of $d_M^s f(x;v)$, there exists $\delta > 0$ and N such that

$$\forall t, 0 < t < \delta,\ \forall n > N, \quad \frac{f(x+tv_n)-f(x)}{t} \in d_M^s f(x;v) + F(0).$$

Letting t go to zero

$$\forall n > N, \quad d_M^s f(x;v_n) \in d_M^s f(x;v) + F(0) \subset d_M^s f(x;v) + W(0). \tag{3.10}$$

This proves the sequential continuity of $v \mapsto d_M^s f(x;v) : X \to Y$.

(ii) Let h be an admissible semitrajectory such that $h'(0^+) = v$. For any neighborhood $W(0) \in \mathscr{R}$ in Y, there exists a neighbourhood $U(0) \in \mathscr{R}$ such that $U(0) + U(0) \subset W(0)$. If f_1 and f_2 are Hadamard semidifferentiable at x, there exists $\delta > 0$ such that

$$\forall t, 0 < t < \delta, \quad \frac{f_i(h(t)) - f_i(h(0))}{t} \in d_H f_i(x; v) + U(0), \quad i = 1, 2.$$

So, adding the two, for all t, $0 < t < \delta$,

$$\frac{(f_1 + f_2)(h(t)) - (f_1 + f_2)(h(0))}{t} \in d_H f_1(x; v) + d_H f_2(x; v) + W(0)$$
$$\Rightarrow\; d_H(f_1 + f_2)(x; v) = d_H f_1(x; v) + d_H f_2(x; v).$$

For $\alpha \neq 0$,

$$\frac{(\alpha f)(h(t)) - (\alpha f)(h(0))}{t} = \alpha \, \frac{f(h(t)) - f(h(0))}{t}$$
$$\Rightarrow\; d_H(\alpha f)(x; v) = \alpha \, d_H f(x; v).$$

Finally, the mapping $f \mapsto d_H f(x; v)$ is linear.

(iii) Let h be an admissible semitrajectory in X such that $h'(0^+) = v$. Since $d_H g(x; v)$ exists,

$$\lim_{t \searrow 0} \frac{g(h(t)) - g(h(0))}{t} = d_H g(x; v) \tag{3.11}$$

and $g \circ h$ is an admissible semitrajectory in Y at $g(x)$ such that $(g \circ h)'(0^+) = d_H g(x; v)$. By repeating the same argument, since $d_H f(g(x); d_H g(x; v))$ exists

$$\begin{aligned} &\lim_{t \searrow 0} \frac{(f \circ g)(h(t)) - (f \circ g)(h(0))}{t} \\ &= \lim_{t \searrow 0} \frac{f((g \circ h)(t))) - f((g \circ h)(0))}{t} = d_H f(g(x); d_H g(x; v)) \end{aligned} \tag{3.12}$$

and $d_H(f \circ g)(x; v) = d_H f(g(x); d_H g(x; v))$. □

The next question is the continuity of a Hadamard semidifferentiable function.

Theorem 3.8. *Let X and Y be topological vector spaces, $f : X \to Y$ a function. Assume that f is Hadamard semidifferentiable at $x \in X$.*

(i) *If there exists a bounded neighborhood $U(0) \in \mathscr{R}$ in X, then f is sequentially continuous at x.*
(ii) *If X is a Fréchet space, then $v \mapsto d_H f(x; v) : X \to Y$ is positively homogeneous and continuous. If X and Y are Fréchet spaces, then f is continuous at x.*

Proof. (i) Let $U \in \mathscr{R}$ be the bounded neighborhood of 0. We proceed by contradiction. Assume that there is a convergent sequence $x_n \to x$ in X such that $f(x_n)$ does not converge to $f(x)$ in Y. So, there exists a neighborhood W of 0 in Y such that for each $k \geq 1$ there exists $n_k > \max\{k, n_{k-1}\}$ such that

$$x_{n_k} - x \in \frac{1}{k^2} U \quad \text{and} \quad f(x_{n_k}) - f(x) \notin W. \tag{3.13}$$

Consider the sequence

$$v_k \stackrel{\text{def}}{=} \frac{x_{n_k} - x}{1/k} \in \frac{1}{k} U.$$

This sequence converges to 0. Since U is bounded, for each $V \in \mathscr{R}$, there exists $\alpha_V > 0$ such that $U \subset kV$ for all $k \geq \alpha_V$. Therefore, for each V,

$$\exists \alpha_V > 0, \forall k > \alpha_V, \quad v_k = \frac{x_{n_k} - x}{1/k} \in V \quad \Rightarrow v_k \to 0.$$

Since f is MS-semidifferentiable at x, for the above neighborhood W, there exists δ, $0 < \delta < 1$, and $K \geq 1$ such that

$$\forall t,\, 0 < t < \delta,\, \forall k > K, \quad \frac{f(x + tv_k) - f(x)}{t} - d_H f(x; 0) \in W.$$

Since $d_H f(x; 0) = 0$, for $\bar{K} > \max\{K, 1/\delta\}$,

$$\forall k > \bar{K}, \quad \frac{f(x_{n_k}) - f(x)}{1/k} = \frac{f\left(x + \frac{1}{k} v_k\right) - f(x)}{1/k} \in W$$
$$\Rightarrow \forall k > \bar{K}, \quad f(x_{n_k}) - f(x) \in \frac{1}{k} W \subset W$$

and this contradicts our initial conjecture.

(ii) In a Fréchet space, continuity and sequential continuity coincide and there is a bounded neighborhood of the origin. So, the continuity of $v \mapsto d_M f(x; v)$ follows from its sequentially continuity of Theorem 3.7 (i) and the continuity of f at x follows from part (i). ☐

3.3. **Lipschitz Functions and Gateaux Semidifferentiability.** Lipschitz continuous functions enjoy the nice property that, if they are Gateaux semidifferentiable, they are M-semidifferentiable.

Definition 3.9. Let X and Y be normed spaces. A function $f : X \to Y$ is *Lipschitz continuous* at $x \in X$ if there exists a constant $c(x) > 0$ and a ball $B_r(x)$ such that

$$\forall y, z \in B_r(x), \quad \|f(y) - f(z)\|_Y \leq c(x)\, \|y - z\|_X. \tag{3.14}$$

A function $f : X \to Y$ is *Lipschitz continuous* in a subset U of X if there exists a constant $c(U) > 0$ such that

$$\forall y, z \in U, \quad \|f(y) - f(z)\|_Y \leq c(U)\, \|y - z\|_X. \tag{3.15}$$

Theorem 3.10. *Let X and Y be normed spaces, $f : X \to Y$ be a function which is Lipschitz continuous at $x \in X$. If f is Gateaux semidifferentiable at x in the direction v (that is, $df(x; v)$ exists), then f is M-semidifferentiable*[10] *at x in the direction v and $d_M f(x; v) = df(x; v)$.*

Proof. Let $c(x)$ be the Lipschitz constant of f at x for the ball $B_r(x)$ of radius r. For $w \to v$ and $t \searrow 0$, $tw \to 0$, that is, there exists $\delta > 0$ such that

$$\forall t,\, 0 < t < \delta, \quad \forall w \in B_\delta(v), \quad x + tw \in B_r(x).$$

[10]Hence, Hadamard semidifferentiable at x in the direction v and $d_H f(x; v) = df(x; v)$.

Therefore,

$$\begin{aligned}
&\frac{f(x+tw)-f(x)}{t} - df(x;v) \\
&= \frac{f(x+tv)-f(x)}{t} - df(x;v) + \frac{f(x+tw)-f(x+tv)}{t} \\
&\Rightarrow \left\| \frac{f(x+tw)-f(x)}{t} - df(x;v) \right\| \\
&\qquad \le \left\| \frac{f(x+tv)-f(x)}{t} - df(x;v) \right\| + c(x)\, \|w-v\|.
\end{aligned}$$

As $w \to v$ and $t \searrow 0$, we get $d_M f(x;v) = df(x;v)$. □

3.4. **Convex Functions.** In this section we show that, in a locally convex topological vector space, all convex functions continous (resp. sequentially continuous) at x are M- (resp. Hadamard) semidifferentiable at x. In particular, the norm is M-semidifferentiable.

Definition 3.11. Let U be a convex subset of a locally convex topological vector space X. A function $f : U \to \mathbb{R}$ is *convex* if

$$\forall x,y \in U,\ \forall \lambda \in (0,1), \quad f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y). \tag{3.16}$$

Theorem 3.12. *Let X be a locally convex topological vector space and U a convex open subset of X. A function $f : U \to \mathbb{R}$ is convex if and only if*

(i) *for all $y \in U$, f is Gateaux semidifferentiable at y, that is, $df(y;v)$ exists in all directions $v \in X$ at all points $y \in U$,*

$$\forall y \in U,\ \forall v \in X, \quad df(y;v) + df(y;-v) \ge 0, \tag{3.17}$$

$$\forall x,y \in U, \quad f(y) \ge f(x) + df(x;y-x), \tag{3.18}$$

(ii) *and for each $y \in U$, the function*

$$v \mapsto df(y;v) : X \to \mathbb{R} \tag{3.19}$$

is positively homogeneous, convex, and subadditive, that is,

$$\forall v,w \in X, \quad df(y;v+w) \le df(y;v) + df(y;w). \tag{3.20}$$

Corollary 3.13. *In a normed space, the norm is M-semidifferentiable.*

Proof. (⇒) (i) U is a convex neighbourhood of each point $x \in U$.
(a) *Existence.* Given $v \in U$, there exists α_0, $0 < \alpha_0 < 1$, such that $x - \alpha v \in U$, $0 < \alpha \le \alpha_0$, and there exists θ_0, $0 < \theta_0 < 1$, such that $x + \theta v \in U$, $0 < \theta \le \theta_0$. Fix α, $0 < \alpha \le \alpha_0$. We first show that

$$\forall \theta, \quad 0 < \theta < \theta_0, \quad \frac{f(x)-f(x-\alpha v)}{\alpha} \le \frac{f(x+\theta v)-f(x)}{\theta}. \tag{3.21}$$

Indeed, x can be written as

$$x = \frac{\alpha}{\alpha+\theta}(x+\theta v) + \frac{\theta}{\alpha+\theta}(x-\alpha v)$$

and, by convexity,

$$f(x) \le \frac{\alpha}{\alpha+\theta} f(x+\theta v) + \frac{\theta}{\alpha+\theta} f(x-\alpha v)$$

or, by rearranging,

$$\frac{\theta}{\theta+\alpha}[f(x)-f(x-\alpha v)] \le \frac{\alpha}{\theta+\alpha}[f(x+\theta v)-f(x)],$$

and hence we get (3.21). Define

$$\varphi(\theta) \overset{\text{def}}{=} \frac{f(x+\theta v)-f(x)}{\theta}, \quad 0<\theta<\theta_0,$$

and show that φ is monotone increasing. For all θ_1 and θ_2, $0<\theta_1<\theta_2<\theta_0$,

$$\begin{aligned} &f(x+\theta_1 v)-f(x) = f\left(\frac{\theta_1}{\theta_2}(x+\theta_2 v)+\left(1-\frac{\theta_1}{\theta_2}\right)x\right)-f(x) \\ &\le \frac{\theta_1}{\theta_2}f(x+\theta_2 v)+\left(1-\frac{\theta_1}{\theta_2}\right)f(x)-f(x) \le \frac{\theta_1}{\theta_2}[f(x+\theta_2 v)-f(x)] \\ &\Rightarrow\ \varphi(\theta_1)\le\varphi(\theta_2). \end{aligned}$$

Since the function $\varphi(\theta)$ is decreasing with θ and bounded below for $\theta \in (0,\theta_0)$, the limit as θ goes to 0 exists. By definition, it coincides with the semidifferential $df(x;v)$.

(b) Given $v \in X$, there exists α_0, $0<\alpha_0<1$, such that

$$\frac{f(x)-f(x-\alpha v)}{\alpha} \le df(x;v),\ 0<\alpha\le\alpha_0.$$

From part (i) for all $v\in X$, $df(x;v)$ and $df(x;-v)$ exist. Letting α go to 0, we get

$$-df(x;-v) = -\lim_{\alpha\searrow 0}\frac{f(x-\alpha v)-f(x)}{\alpha} \le df(x;v)$$

and the inequality $df(x;-v)+df(x;v)\ge 0$.

As f is convex on U, for $x,y\in U$ and $\theta\in(0,1]$

$$\begin{aligned} &f(\theta y+(1-\theta)x) \le \theta f(y)+(1-\theta)f(x) \\ \Rightarrow\ &f(x+\theta(y-x))-f(x)\le\theta\,[f(y)-f(x)]. \end{aligned}$$

By dividing by θ and going to the limit as θ goes to 0, we get

$$df(x;y-x)\le f(y)-f(x).$$

(ii) By definition, $v\mapsto df(x;v)$ is clearly positively homogeneous. We next show that it is convex:, that is, for all α, $0\le\alpha\le 1$, and v,$w\in\mathbb{R}^n$,

$$df(x;\alpha v+(1-\alpha)w)\le\alpha df(x;v)+(1-\alpha)df(x;w).$$

Since U is open and convex, for each $x\in U$

$$\begin{aligned} &\exists\theta_0,\ 0<\theta_0<1, \text{ such that } \forall\theta,\ 0<\theta\le\theta_0,\quad x+\theta v\in U \text{ and } x+\theta w\in U \\ &\Rightarrow\ \forall 0\le\alpha\le 1,\quad x+\theta(\alpha v+(1-\alpha)w)=\alpha(x+\theta v)+(1-\alpha)(x+\theta w)\in U, \end{aligned}$$

and by convexity of f,

$$\begin{aligned} f(x+\theta(\alpha v+(1-\alpha)w)) &= f(\alpha[x+\theta v]+(1-\alpha)[x+\theta w]) \\ &\le \alpha f(x+\theta v)+(1-\alpha)f(x+\theta w) \\ \Rightarrow [f(x+\theta(\alpha v&+(1-\alpha)w))-f(x)] \\ \le \alpha\ [f(x+\theta v)&-f(x)]+(1-\alpha)\ [f(x+\theta w)-f(x)]\,. \end{aligned}$$

Dividing by θ and going to the limit as θ goes to 0, we get the convexity

$$df(x;\alpha v+(1-\alpha)w)\leq \alpha df(x;v)+(1-\alpha)df(x;w).$$

Combining the positive homogeneity and the convexity,

$$\begin{aligned} df(x;v+w) &= df\left(x;\frac{1}{2}2v+\frac{1}{2}2w\right)\\ &\leq \frac{1}{2}df(x;2v)+\frac{1}{2}df(x;2w)=df(x;v)+df(x;w), \end{aligned}$$

we get the subadditivity.

($\Leftarrow$) Conversely, for $\lambda\in(0,1)$ and $x,y\in U$

$$\begin{aligned} f(x)-f(\lambda x+(1-\lambda)y)\geq\ & df(\lambda x+(1-\lambda)y;(1-\lambda)(y-x))\\ &=(1-\lambda)df(\lambda x+(1-\lambda)y;y-x)\\ f(y)-f(\lambda x+(1-\lambda)y)\geq\ & df(\lambda x+(1-\lambda)y;\lambda(x-y))\\ &=\lambda df(\lambda x+(1-\lambda)y;x-y). \end{aligned}$$

Multiply the first inequality by λ and the second by $1-\lambda$ and sum up:

$$\begin{aligned} &\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)\\ &\geq \lambda\,(1-\lambda)\ [df(\lambda x+(1-\lambda)y;x-y)+df(\lambda x+(1-\lambda)y;y-x)]\geq 0 \end{aligned}$$

and f is convex. □

Proof of Corollary 3.13. By using the triangle inequality,

$$|\|y\|-\|x\||\leq\|y-x\|,$$

the norm $n(x)=\|x\|$ is uniformly Lipschitz continuous in X with constant one. It is also convex since, for $\lambda\in(0,1)$,

$$\|\lambda x+(1-\lambda)y\|\leq\|\lambda x\|+\|(1-\lambda)y\|=\lambda\|x\|+(1-\lambda)\|y\|.$$

So, by Theorem 3.12, $dn(x;v)$ exists for all x and v and, by Theorem 3.10, it is M-semidifferentiable. □

The next theorem connects continuity and semidifferentiability.

Theorem 3.14. *Let X be a locally convex topological vector space, $x\in X$, and $f:V(x)\to\mathbb{R}$ a convex function in a convex neighborhood $V(x)$ of $x\in X$.*

(i) *If f is continuous at x, then f is M-semidifferentiable at x.*
(ii) *If f is sequentially continuous at x, then f is Hadamard semidifferentiable at x.*
(iii) *If X is a Fréchet space, then f is continuous at x if and only if f is Hadamard semidifferentiable at x.*

Proof. (i) Since, f is continuous at x, for each $\varepsilon>0$, there exists a convex neighborhood $V(x)$ of x such that

$$\forall y\in V(x),\quad |f(y)-f(x)|<\varepsilon/2.$$

Let $V(0)=V(x)-x$ be the corresponding neighborhood of 0 in $\mathscr{R}$. The symmetric $W(0)=V(0)\cap(-V(0))$ is also a neighborhood and there exists a symmetric convex neighborhood $U(0)$ such that $U(0)+U(0)\subset W(0)$.

From Theorem 3.12, for all $v \in U(0)$ and $y \in x + U(0)$

$$df(y;v) \le f(y+v) - f(y) \le |f(y+v) - f(x)| + |f(y) - f(x)|$$
$$-df(y;v) \le df(y;-v) \le f(y-v) - f(y) \le |f(y-v) - f(x)| + |f(y) - f(x)|.$$

Since $y \pm v - x = y - x \pm v \in U(0) + U(0) \subset W(0)$

$$\begin{aligned} &|df(y;v)| < \varepsilon/2 + \varepsilon/2 < \varepsilon \\ &\Rightarrow \forall y \in x + U(0),\ \forall v \in U(0), \quad |df(y;v)| < \varepsilon \end{aligned} \tag{3.22}$$

and $(y,v) \mapsto df(x;v) : (x + W(0)) \times X \to \mathbb{R}$ is continuous at $(y,v) = (x,0)$.

Given $v \in X$, for all $w \in v + U(0)$

$$\begin{aligned} \frac{f(x+tw) - f(x)}{t} - df(x;v) &\ge df(x;w) - df(x;v) \\ &\ge df(x;w) - (df(x;w) + df(x;v-w)) \\ &\ge -|df(x;v-w)|. \end{aligned}$$

Therefore, since $df(x;0) = 0$, by continuity of $v \mapsto df(x;v)$ at $v = 0$,

$$\liminf_{\substack{w \to v \\ t \searrow 0}} \frac{f(x+tw) - f(x)}{t} - df(x;v) \ge \lim_{w \to v} -df(x;v-w) = 0.$$

In the other direction,

$$\begin{aligned} &\frac{f(x+tw) - f(x)}{t} - df(x;v) \\ &= \frac{f(x+tv) - f(x)}{t} - df(x;v) + \frac{f(x+tw) - f(x+tv)}{t} \\ &\le \left[\frac{f(x+tv) - f(x)}{t} - df(x;v)\right] - df(x+tw;v-w). \end{aligned}$$

So, there exists δ, $0 < \delta < 1$, such that

$$\forall t, 0 < t < \delta, \quad \left|\frac{f(x+tv) - f(x)}{t} - df(x;v)\right| < \varepsilon/2.$$

For the second term, use the continuity (3.22) at $(x,0)$ and choose $N(0) \in \mathscr{R}$ such that $N(0) + N(0) \subset U(0)$. For $w - v \in N(0)$ and $0 < t < \delta < 1$, $tw = tv + t(w-v) \in tv + \delta N(0) \subset tv + N(0)$ and, since $N(0)$ is absorbing, there exists $\bar\delta$, $0 < \bar\delta < \delta$ such that for all $0 < t < \bar\delta$, $tv \in N(0)$. Therefore for $t < \bar\delta$ and $w \in v + N(0)$, $tw \in U(0)$ and

$$\begin{aligned} &\frac{f(x+tw) - f(x)}{t} - df(x;v) \le \frac{\varepsilon}{2} + \varepsilon = \frac{3}{2}\varepsilon \\ &\Rightarrow \limsup_{\substack{w \to v \\ t \searrow 0}} \frac{f(x+tw) - f(x)}{t} - df(x;v) \le \frac{3}{2}\varepsilon. \end{aligned}$$

Finally,

$$df(x;v) \le \liminf_{\substack{w \to v \\ t \searrow 0}} \frac{f(x+tw) - f(x)}{t} \le \limsup_{\substack{w \to v \\ t \searrow 0}} \frac{f(x+tw) - f(x)}{t} \le df(x;v),$$

$d_H f(x;v) = df(x;v)$, and f is M-semidifferentiable at x.

(ii) Same proof as in part (i) with a sequence $\{v_n\}$ converging to v. Then use the equivalence of sequential M-differentiability and Hadamard semidifferentiability.

(iii) In a Fréchet space all the previous notions of semidifferentiability coincide. From Theorem 3.8 (ii), if f is Hadamard semidifferentiable, then f is continuous at x. □

4. Semidifferentials for functions on unstructured sets

For functions defined on a smooth embedded submanifold of $\mathbb{R}^n$ of dimension $d < n$ or an unstructured subset A of a TVS X, the Hadamard semidifferential is the natural choice over the M-semidifferential since it uses semitrajectories that do not require some algebraic structure on A. For a subset A of X, the tangent space at interior points of A is the whole space X, but at the boundary ∂A the tangent space will generally be only a cone. For instance, for a smooth embedded submanifold of dimension $d < n$, $A = \partial A$ and all points of A are boundary points where the tangent space is $\mathbb{R}^d$.

The following extension of the notion of Hadamard semidifferentiability to unstructured subsets of an *ambiant* topological vector space without introducing local bases or coordinate spaces is different from the approach of Michal ([22, 1939], [23, 1940], [24, 1945], [25, 1947]) and Fréchet [16, 1948] who independently extended their notions to functions defined on topological Abelian groups.

Definition 4.1. Let A be a non-empty subset of a topological vector space X. An *admissible semitrajectory* at $x \in \overline{A}$ in A is a function $h : [0,\tau) \to A$ such that

$$h(0) = x \quad \text{and} \quad h'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{h(t) - h(0)}{t} \text{ exists in } X, \tag{4.1}$$

where $h'(0^+)$ is the *semitangent* to the trajectory h in A at $h(0) = x$.

For $x \in \operatorname{int} A$ and t small, $h(t) = x + tv$ is an admissible semitrajectory such that $h'(0^+) = v$ and all directions in X are admissible. For $x \in \partial A$, the tangent space to A might not be the whole space X. Several tangent cones are available in the literature such as the following one associated with the *Viability Theorem* of Nagumo [27] in 1942.

Definition 4.2 (Bouligand [3], 1930)**.** Let A be a non-empty subset of a Fréchet space X. The *Bouligand contingent cone* to A at $x \in \overline{A}$ is defined as

$$T_A(x) \overset{\text{def}}{=} \left\{ v \in X : \exists \{t_n \searrow 0\}, \exists \{x_n\} \subset A \text{ such that } \lim_{n\to\infty} \frac{x_n - x}{t_n} = v \right\}.$$

$T_A(x)$ is a closed cone[11] at 0, $T_{\overline{A}}(x) = T_A(x)$, and

$$T_A(x) = \left\{ v \in X : \liminf_{t \searrow 0} \frac{d_A(x + tv)}{t} = 0 \right\}.$$

$T_X(x) = X$. If A is convex, $T_A(x) = \overline{\{\lambda(A - x) : \lambda \geq 0\}}$ is a closed convex cone at 0.

For a closed sufficiently smooth embedded submanifold A of $X = \mathbb{R}^n$ of dimension $d < n$, $\overline{\mathbb{R}^n \backslash A} = \mathbb{R}^n$, $A = \partial A$, and the smoothness insures that, at each point of

[11]A cone C at 0 in X is a subset of X such that for all $\lambda > 0$ and all $x \in C$, $\lambda x \in C$.

A, the tangent space is a d-dimensional linear subspace. This is illustrated below in Figure 1 for a smooth curve A in $\mathbb{R}^2$. But, the linearity of $T_A(x)$ puts a severe

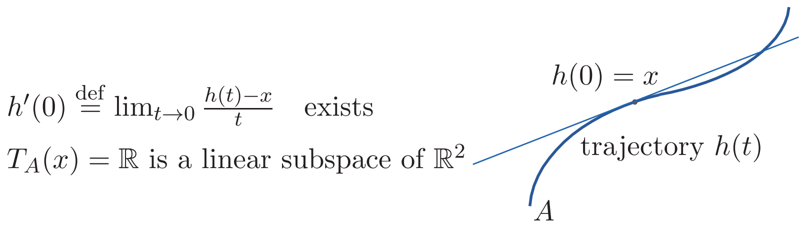


Figure 1. Tangent $h'(0)$ to the trajectory h in A at the point $h(0) = x$.

restriction on the choice of sets A. For instance, the requirement that $T_A(x)$ be linear rules out a curve in $\mathbb{R}^2$ with a kink at x as shown in the Figure 2.

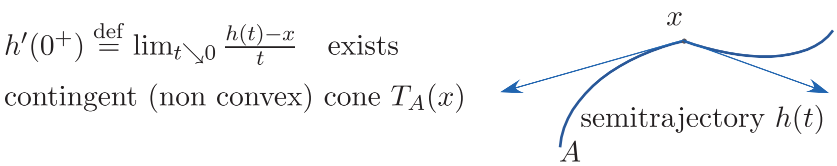


Figure 2. Half-tangent $h'(0^+)$ to the semitrajectory h in A at the point $h(0) = x$.

It turns out that the following tangent cone is more relevant than the Bouligand contingent cone $T_A(a)$ for the Hadamard semidifferentiability.

Definition 4.3. Let A be a non-empty subset of a topological vector space X. The *adjacent or intermediary tangent cone*[12] to A at $x \in \overline{A}$ is defined as

$$T^\flat_A(x) \stackrel{\text{def}}{=} \left\{ v \in X : \forall \{t_n \searrow 0\},\ \exists \{x_n\} \subset A \text{ such that } \lim_{n\to\infty} \frac{x_n - x}{t_n} = v \right\}.$$

$T^\flat_X(x) = X$. If A is convex, $T_A(x) = T^\flat_A(x) = \overline{\{\lambda(A - x) : \lambda \geq 0\}}$.[13] If X is a Fréchet space, $T^\flat_A(x)$ is closed, $T^\flat_{\overline{A}}(x) = T^\flat_A(x)$, and

$$T^\flat_A(x) = \left\{ v \in X : \lim_{t\searrow 0} \frac{d_A(x + tv)}{t} = 0 \right\};$$

$T^\flat_A(x)$ is directly related to the notion of admissible semitrajectories in A.

Theorem 4.4. *Let A be a subset of a topological vector space X. For $x \in \overline{A}$,*

$$T^\flat_A(x) = \left\{ h'(0^+) : h \text{ an admissible semitrajectory in } A \text{ at } x \right\}. \tag{4.2}$$

[12]We use the terminology of Aubin and Frankowska [1, Definition 4.1.5, pp. 126–129]. See [1, Figure 4.4, p. 161] for an example in dimension two where $T_A(x) \neq T^\flat_A(x)$.

[13]Aubin and Frankowska [1, Thm. 4.2.1, p. 138].

Proof. Let h be an admissible semitrajectory at x in $\overline{A}$ such that $h'(0^+) = v$. For any sequence $\{t_n \searrow 0\}$, choose $x_n = h(t_n) \in A$. Then

$$\frac{x_n - x}{t_n} = \frac{h(t_n) - h(0)}{t_n} \to v$$

and $v \in T_A^\flat(x)$. Conversely, by definition of an element $v \in T_A^\flat(x)$, for the sequence $t_n = 1/n$, $n \geq 1$, there exists a sequence $\{x_n\} \subset A$ such that

$$\lim_{n\to\infty} \frac{x_n - x}{t_n} = v.$$

Define the semitrajectory $h : [0,1] \to A$ as follows:

$$h(t) \overset{\text{def}}{=} x_n, \quad t_{n+1} < t \leq t_n, \quad n \geq 1, \quad h(0) \overset{\text{def}}{=} x. \tag{4.3}$$

As $t \searrow 0$, $n \to \infty$, $x_n \to x$, and $h(t) \to x$. For $t_{n+1} < t \leq t_n$

$$\frac{h(t) - x}{t} - v = \frac{x_n - x}{t} - v = \frac{t_n}{t}\left[\frac{x_n - x}{t_n} - v\right] + \left(\frac{t_n}{t} - 1\right) v,$$

where

$$1 \leq \frac{t_n}{t} < 1 + \frac{1}{n}, \quad 0 < n\left(\frac{t_n}{t} - 1\right) < 1.$$

For any $V(0) \in \mathscr{R}$, there exists $U(0) \in \mathscr{R}$ such that $U(0) + U(0) + U(0) \subset V(0)$. There exists N such that for all $n > N$

$$\frac{x_n - x}{t_n} - v \in U(0).$$

Since $U(0)$ is absorbing, there exists $\alpha > 0$ such that $\lambda v \in U(0)$ for all $|\lambda| \geq \alpha$. So, for $\bar{N} \geq \max\{\alpha, N\}$, $n \geq \bar{N}$, $t_{n+1} < t \leq t_n$,

$$\frac{t_n}{t}\left[\frac{x_n - x}{t_n} - v\right] \in \left(1 + \frac{1}{n}\right) U(0), \quad \left(\frac{t_n}{t} - 1\right) v \in \left(\frac{t_n}{t} - 1\right) nU(0) \subset U(0)$$

$$\Rightarrow \forall t,\ 0 < t \leq t_{\bar{N}}, \quad \frac{h(t) - x}{t} - v \in U(0) + U(0) + U(0) \subset V(0).$$

Hence, h is an admissible semitrajectory in A at x such that $h'(0^+) = v$. □

In that context it is natural to introduce the following notions.

Definition 4.5. (i) A subset A of X is *tangentially semiregular*[14] at a point $x \in \overline{A}$ if $T_A(x) = T_A^\flat(x) \neq \{0\}$.

(ii) A subset A of X is *tangentially regular* at a point $x \in \overline{A}$ if it is tangentially semiregular at x and $T_A^\flat(x)$ is a linear subspace of X.

Remark 4.6. Any nonempty convex subset of $\mathbb{R}^n$ is tangentially semiregular. An open set A with a boundary ∂A of class $C^{(1)}$ is tangentially regular. Similarly, a closed sufficiently smooth embedded submanifold A of $\mathbb{R}^n$ of dimension $d < n$ is tangentially regular. Indeed, $\overline{\mathbb{R}^n \backslash A} = \mathbb{R}^n$, $A = \partial A$ only contains boundary points, and the smoothness insures that, at each point of A, the tangent space is a d-dimensional linear subspace of $\mathbb{R}^n$.

[14]For an example where the two cones are different see [1, Figure 4.4, p. 161].

We now have all the elements to extend the definition of the Hadamard semidifferential to a subset of a TVS.

Definition 4.7. Let X and Y be topological vector spaces, A, $\varnothing \neq A \subset X$, and $f : A \to Y$.

(i) The function f is *Hadamard semidifferentiable at* $x \in A$ *in the direction* $v \in T^\flat_A(x)$ if there exists $g(x,v) \in Y$ such that, for all admissible semitrajectories h in A at x such that $h'(0^+) = v$,

$$(f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{f(h(t)) - f(h(0))}{t} = g(x,v). \tag{4.4}$$

The element $g(x,v)$ will be denoted $d_H f(x; v)$.

(ii) f is *Hadamard semidifferentiable at* $x \in A$ if f is Hadamard semidifferentiable at x in all directions $v \in T^\flat_A(x)$.

(iii) f is *Hadamard differentiable at* $x \in A$ if $T^\flat_A(x)$ is a linear subspace, f is Hadamard semidifferentiable at $x \in A$, and the function $v \mapsto d_H f(x; v) : T^\flat_A(x) \to Y$ is linear in which case it will be denoted $Df(x)$.

The Hadamard semidifferentiability enjoys all the nice properties of the classical finite dimensional differential calculus.

Theorem 4.8. *Let X and Y be topological vector spaces and A, $\varnothing \neq A \subset X$.*

(i) *If $f : A \to Y$ is Hadamard semidifferentiable at $x \in A$ in the direction $v \in T^\flat_A(x)$, then for all admissible semitrajectory h in A such that $h'(0^+) = v$, $f \circ h$ is an admissible trajectory in $f(A)$ such that $(f \circ h)'(0^+) = d_H f(x; v) \in T^\flat_{f(A)}(f(x))$. The mapping*

$$v \mapsto d_H f(x; v) : T^\flat_A(x) \to T^\flat_{f(A)}(f(x)) \subset Y \tag{4.5}$$

is sequentially continuous for the induced topologies.

(ii) *If $f_1 : A \to Y$ and $f_2 : A \to Y$ are Hadamard semidifferentiable at $x \in A$ in the direction $v \in T^\flat_A(x)$, then for all α and β in $\mathbb{R}$,*

$$d_H(\alpha f_1 + \beta f_2)(x; v) = \alpha\, d_H f_1(x; v) + (1 - \alpha)\, d_H f_2(x; v), \tag{4.6}$$

and $\alpha f_1 + \beta f_2$ is Hadamard semidifferentiable at x in the direction v.

(iii) *(Chain rule) Let X, Y, Z be topological vector spaces, $A \subset X$, $g : A \to Y$, and $f : g(A) \to Z$ be functions such as g is Hadamard semidifferentiable at x in the direction $v \in T^\flat_A(x)$ and f is Hadamard semidifferentiable at $g(x)$ in $g(A)$ in the direction $d_H g(x; v)$. Then $d_H g(x; v) \in T^\flat_{g(A)}(x)$, $f \circ g$ is Hadamard semidifferentiable at x in the direction $v \in T^\flat_A(x)$, and*

$$d_H(f \circ g)(x; v) = d_H f(g(x); d_H g(x; v)). \tag{4.7}$$

Remark 4.9. It is remarkable to obtain properties from classical differential geometry without introducing *coordinate spaces* in the terminology of Michal [22], charts, local bases, or Christoffel symbols. X and Y play the role of *ambiant spaces* where the cones of semitangents live.

Proof. (i) We cannot reproduce the proof of Theorem 3.7 (i) since there is no structure on A. We must work with semitrajectories. For $h : [0,\tau] \to A$, then $f \circ h : [0,\tau] \to f(A)$ and, by definition, $v \mapsto d_H f(x;v) : T^\flat_A(x) \to T^\flat_{f(A)}(x)$. Let $W_0 \in \mathscr{R}$ and $V_0 \in \mathscr{R}$ in X be neighborhoods of the origin in Y and X. There exists $W \in \mathscr{R}$ and $V \in \mathscr{R}$ such that $W + W \subset W_0$ and $V + V \subset V_0$. Let $v_n \to v$ be a converging sequence in $T^\flat_A(x)$. Associate with each v_n the admissible trajectories h_n in A at x such that $h'_n(0^+) = v_n$.

In a first step, we construct a sequence $t_n \to 0$ such that $0 < t_{n+1} < t_n$:
-$\exists t_1 > 0$ such that for all $0 < t \leq t_1$

$$\frac{h_1(t) - h_1(0)}{t} - h'_1(0^+) \in V, \quad \frac{f(h_1(t)) - f(h_1(0))}{t} - d_H f(x;v_1) \in W;$$

-$\exists t_n > 0$, $t_n < \min\{t_{n-1}, 1/n\}$ such that for all $0 < t \leq t_n$

$$\frac{h_n(t) - h_n(0)}{t} - h'_n(0^+) \in V, \quad \frac{f(h_n(t)) - f(h_n(0))}{t} - d_H f(x;v_n) \in W.$$

In a second step, we construct an admissible trajectory $h : [0,t_1] \to A$ at x such that $h'(0^+) = v$ as follows:

$$h(t) \overset{\text{def}}{=} h_n(t), \quad t_{n+1} < t \leq t_n, \quad h(0) = x.$$

There exists N such that for all $n > N$, $v_n - v \in V$ and, hence,

$$t_n < t \leq t_{n-1}, \ \frac{h(t) - h(0)}{t} - v = \frac{h_n(t) - h_n(0)}{t} - v_n + v_n - v \in V + V \subset V_0.$$

Therefore, for each V_0, there exists $t_N > 0$ such that

$$0 < t \leq t_N, \quad \frac{h(t) - h(0)}{t} - v \in V_0 \quad \Rightarrow h'(0^+) = v.$$

Since h is an admissible trajectory in A at x, there exists δ such that

$$\forall t, \ 0 < t < \delta, \quad \frac{f(h(t)) - f(h(0))}{t} - d_H f(x;v) \in W.$$

There exists $N > 1$ such that $t_N < \delta$ and for all $n > N$, $t_n < \delta$. But, by construction, for each $n > N$,

$$\frac{f(h_n(t_n)) - f(h_n(0))}{t_n} - d_H f(x;v_n) \in W$$

and, for each W_0, there exists N such that for all $n > N$

$$d_H f(x;v_n) - d_H f(x;v) \in W + W \subset W_0.$$

This proves the sequential continuity of the semidifferential.

(ii) Let h be an admissible trajectory in A such that $h'(0^+) = v \in T^\flat_A(x)$. For any neighborhood $W(0) \in \mathscr{R}$ in Y, there exists a neighbourhood $U(0) \in \mathscr{R}$ such that $U(0) + U(0) \subset W(0)$. If f_1 and f_2 are Hadamard semidifferentiable at x, there exists $\delta > 0$ such that

$$\forall t, 0 < t < \delta, \quad \frac{f_i(h(t)) - f_i(h(0))}{t} \in d_H f_i(x;v) + U(0).$$

So, adding the two, for all t, $0 < t < \delta$,

$$\frac{(f_1+f_2)(h(t))-(f_1+f_2)(h(0))}{t} \in d_H f_1(x;v)+d_H f_2(x;v)+W(0)$$
$$\Rightarrow\; d_H(f_1+f_2)(x;v)=d_H f_1(x;v)+d_H f_2(x;v).$$

For $\alpha \neq 0$,

$$\frac{(\alpha f)(h(t))-(\alpha f)(h(0))}{t}=\alpha\,\frac{f(h(t))-f(h(0))}{t}$$
$$\Rightarrow\; d_H(\alpha f)(xv)=\alpha\, d_H f(x;v).$$

Finally, the mapping $f \mapsto d_H f(x;v)$ is linear.

(iii) Let h be an admissible semitrajectory in A such that $h'(0^+)=v\in T_A^\flat(x)$. Since $d_H g(x;v)$ exists,

$$\lim_{t\searrow 0}\frac{g(h(t))-g(h(0))}{t}=d_H g(x;v) \tag{4.8}$$

and $g\circ h$ is an admissible semitrajectory in $g(A)$ at $g(x)$ such that $(g\circ h)'(0^+)=d_H g(x;v)\in T^\flat_{g(A)}(g(x))$. By repeating the same argument, since $d_H f(g(x);d_H g(x;v))$ exists

$$\begin{aligned}&\lim_{t\searrow 0}\frac{(f\circ g)(h(t))-(f\circ g)(h(0))}{t}\\ &=\lim_{t\searrow 0}\frac{f((g\circ h)(t)))-f((g\circ h)(0))}{t}=d_H f(g(x);d_H g(x;v))\end{aligned} \tag{4.9}$$

and $d_H(f\circ g)(x;v)=d_H f(g(x);d_H g(x;v))$. □

The next question is the continuity from the semidifferentiability.

Theorem 4.10. *Let X and Y be topological vector spaces, A a non-empty subset of X, and $f: A\to Y$ a function. Assume that f is Hadamard semidifferentiable at $x\in A$.*

(i) *If there exists a bounded neighborhood $U(0)\in \mathscr{R}$ in X, then f is sequentially continuous*[15] *at x in A for the induced topology on A.*

(ii) *If X is a Fréchet space, then $v\mapsto d_H f(x;v): T_A^\flat(x)\to T^\flat_{f(A)}(f(x))$ is positively homogeneous and continuous for the induced topologies. If X and Y are Fréchet spaces, then f is continuous at x.*

Proof. (i) Let $U\in\mathscr{R}$ be the bounded neighborhood of 0. By contradiction. Assume that there is a convergent sequence $x_n\to x$ in A such that $f(x_n)$ does not converge to $f(x)$ in Y. So, there exists a neighborhood W of 0 in Y such that for each $k\geq 1$ there exists $n_k>\max\{k,n_{k-1}\}$ such that

$$x_{n_k}-x\in\frac{1}{k^2}U \quad\text{and}\quad f(x_{n_k})-f(x)\notin W. \tag{4.11}$$

[15]Note the following natural equivalence for the semicontinuity in terms of semitrajectories. Let X and Y be topological spaces and A a subset of X. A function $f: A\to Y$ is *sequentially continuous* at $a\in A$ if and only if for all *semitrajectories* $h:[0,\tau)\to A$

$$\lim_{t\searrow 0}h(t)=a \quad\Rightarrow\quad \lim_{t\searrow 0}f(h(t))=f(a), \tag{4.10}$$

where A is endowed with the topology induced by X.

Consider the sequence

$$\frac{x_{n_k} - x}{1/k} \in \frac{1}{k} U.$$

To simplify the notation, relabel $\{x_n\}$ the sequence $\{x_{n_k}\}$ and set $t_n = 1/n$.

We can now construct a semitrajectory as in the second part of the proof of Theorem 4.4. Define the semitrajectory $h : [0,1] \to A$ as follows:

$$h(t) \stackrel{\text{def}}{=} x_n, \quad t_{n+1} < t \le t_n, \quad n \ge 1, \quad h(0) \stackrel{\text{def}}{=} x. \tag{4.12}$$

As $t \searrow 0$, $n \to \infty$, $x_n \to x$, and $h(t) \to x$. For $t_{n+1} < t \le t_n$

$$\frac{h(t) - x}{t} = \frac{x_n - x}{t} = \frac{x_n - x}{t_n} + t_n \left[\frac{1}{t} - \frac{1}{t_n}\right] \frac{x_n - x}{t_n},$$

where for $t_{n+1} < t \le t_n$

$$\begin{aligned} 0 \le t_n \left[\frac{1}{t} - \frac{1}{t_n}\right] &= \frac{t_n - t}{t} \le \frac{t_n - t_{n+1}}{t_{n+1}} = \frac{1}{n} \\ \Rightarrow \frac{h(t) - x}{t} &= \frac{x_n - x}{t} \in \frac{1}{n} U + \frac{1}{n}\frac{1}{n} U \subset \frac{1}{n} U + \frac{1}{n} U \\ \Rightarrow \forall t, 0 < t \le t_n, &\quad \frac{h(t) - x}{t} \in \frac{1}{n} U + \frac{1}{n} U. \end{aligned}$$

For each $V_0 \in \mathscr{R}$, there exists $V \in \mathscr{R}$ such that $V + V \subset V_0$. Since U is bounded, there exists $\alpha_V > 0$ such that for all $n \ge \alpha_V$, $U \subset nV$ and

$$\forall t, 0 < t \le t_n, \quad \frac{h(t) - x}{t} \in \frac{1}{n} U + \frac{1}{n} U \subset V + V \subset V_0.$$

Therefore, for each V_0, there exists N such that $N \ge \alpha_V \ge 1$ and

$$\forall t, 0 < t \le t_N, \quad \frac{h(t) - h(0)}{t} \in V_0 \quad \Rightarrow h'(0^+) = 0.$$

Since f is MS-semidifferentiable at x, for the neighborhood W, there exists δ, $0 < \delta < 1$, such that

$$\forall t,\ 0 < t < \delta, \quad \frac{f(h(t)) - f(h(0))}{t} - d_H f(x; 0) \in W.$$

Since $d_H f(x; 0) = 0$, for $N > 1/\delta$,

$$\begin{aligned} \forall n > N, \quad \frac{f(x_n) - f(x)}{1/n} &= \frac{f(h(1/n)) - f(h(0))}{1/n} \in W \\ \Rightarrow \forall n > N, \quad f(x_n) - f(x) &\in \frac{1}{n} W \subset W \end{aligned}$$

and this contradicts our initial conjecture.

(ii) In a Fréchet space X, continuity and sequential continuity coincide. So $v \mapsto d_H f(x; v) : T_A^\flat(x) \to T_{f(A)}^\flat(f(x))$ is continuous from Theorem 4.8 (ii). Also, there is a bounded neighborhood of the origin and from part (i) we have the continuity of f at $x \in A$ for the induced topology on A. $\square$

5. Applications

5.1. **Shape Derivative.** When a diffeomorphism T of $\mathbb{R}^n$ is applied to a subset Ω of $\mathbb{R}^n$, the image $T(\Omega)$ is a set *topologically similar* to Ω: it changes its *shape* but it cannot create holes or new connected components. One way to obtain a notion of *Shape Derivative* is to use the images of a set Ω by a family of diffeomorphisms that tend to the identity mapping I on $\mathbb{R}^n$. Such families can be conveniently generated from the solutions of an ordinary differential equation $(dx/dt)(t) = V(t, x(t))$ where the right-hand side is interpreted as a time-dependent velocity $V(t,x)$ at each point x of the space $\mathbb{R}^n$ as if the points were particules in a moving fluid medium. This is the idea behind the *Velocity Method* of Zolésio [39] in 1979.

Example 5.1. Denote by $\mathscr{L}(\mathbb{R}^n)$ the vector space of linear functions $M : \mathbb{R}^n \to \mathbb{R}^n$ ($n\times n$ matrices) and by GL_n the group of invertible linear functions ($n\times n$ invertible matrices) which can be endowed with a complete metric topology. Moreover,

$$\forall F \in GL_n, \quad T^\flat_{GL_n}(F) = T_{GL_n}(F) = \mathscr{L}(\mathbb{R}^n), \tag{5.1}$$

where $T^\flat_{GL_n}(F)$ denotes the tangent space to GL_n at the point F. If $J : GL_n \to \mathbb{R}$ is a real valued function, we can expect a semidifferential or a differential in the directions contained in the tangent space $\mathscr{L}(\mathbb{R}^n)$ at F since it is a linear vector space.

In 1972 Micheletti [26] introduced what may be one of the first complete metric topologies on a family of domains of class C^k that are the images of a fixed open C^k domain Ω_0 through a family of C^k-diffeomorphisms of $\mathbb{R}^n$. There, the natural underlying algebraic structure is the *group structure* of the composition of transformations with the identity as the neutral element. Her analysis culminates with the construction of a complete metric on the quotient of the group by a closed subgroup of diffeomorphisms F reshuffling the points but keeping $F(\Omega_0) = \Omega_0$. She called it the *Courant metric* because it is proved in the book of Courant and Hilbert [4, p. 420] that the n-th eigenvalue of the Laplace operator depends continuously on the domain Ω, where $\Omega = (I + f)\Omega_0$ is the image of a fixed domain Ω_0 by $I + f$ and $f \in C^k_0(\mathbb{R}^n,\mathbb{R}^n)$. But there is no notion of a metric in that book. Her constructions naturally extend to other families of transformations of $\mathbb{R}^n$ or of fixed hold-alls D.

Specifically, her group of invertible functions $F : \mathbb{R}^n \to \mathbb{R}^n$ is

$$\mathscr{F}(C^k_0(\mathbb{R}^n,\mathbb{R}^n)) \overset{\text{def}}{=} \left\{ F : \mathbb{R}^n \to \mathbb{R}^n \,\middle|\, F \text{ bijective}, \begin{array}{c} F - I \in C^k_0(\mathbb{R}^n,\mathbb{R}^n) \\ F^{-1} - I \in C^k_0(\mathbb{R}^n,\mathbb{R}^n) \end{array} \right\}$$

a subset of the *ambiant* Fréchet space $C^k(\mathbb{R}^n,\mathbb{R}^n)$ of k-times continuously differentiable functions. Her construction of the *Courant metric* is generic and extends to several Banach spaces Θ of functions $\theta : \mathbb{R}^n \to \mathbb{R}^n$:

$$\mathscr{F}(\Theta) \overset{\text{def}}{=} \left\{F : \mathbb{R}^n \to \mathbb{R}^n \,|\, F \text{ bijective}, F - I \in \Theta, F^{-1} - I \in \Theta\right\}, \tag{5.2}$$

where Θ is the tangent space at each point as for invertible matrices

$$\forall F \in \mathscr{F}(\Theta), \quad T_{\mathscr{F}(\Theta)}(F) = \Theta. \tag{5.3}$$

The group $\mathscr{F}(\Theta)$ is an example of an infinite dimensional Finsler manifold.

It is now well-established that the *Velocity Method* developed in the thesis of Zolésio [39] in 1979 is naturally associated with the construction of the special groups of C^k or $C^{k,1}$-diffeomorphisms,[16] where the velocities belong to the tangent space at each point of that group.

Example 5.2. Given a family of velocities $V(t)(x) \overset{\text{def}}{=} V(t,x) : \mathbb{R}^n \to \mathbb{R}^n$, $0 \le t \le \tau$, in $C_0^k(\mathbb{R}^n;\mathbb{R}^n)$ such that $V(t) \to V(0)$, consider the family of diffeomorphisms $\{T_t\}$ of $\mathbb{R}^n$, $t \ge 0$, generated by the solutions of the differential equation

$$\frac{dx}{dt}(t;X) = V(t,x(t;X)),\ x(0;X) = X, \quad T_t(X) \overset{\text{def}}{=} x(t;X),\ t \ge 0,\ X \in \mathbb{R}^n,$$
$$\Rightarrow \frac{dT_t}{dt} = V(t) \circ T_t, \quad T_0 = I.$$

Assuming that $V(t) \to V(0)$ in $C_0^k(\mathbb{R}^n,\mathbb{R}^n)$,

$$t \mapsto h(t) \overset{\text{def}}{=} T_t : [0,\tau] \to \mathscr{F}(C_0^k(\mathbb{R}^n,\mathbb{R}^n)) \tag{5.4}$$

is a trajectory in the group and

$$h'(0^+) = \left.\frac{dT_t}{dt}\right|_{t=0^+} = V(0) \in C_0^k(\mathbb{R}^n;\mathbb{R}^n). \tag{5.5}$$

Given a bounded C^1 domain Ω with compact boundary Γ and a function $f \in C_0^1(\mathbb{R}^n)$, consider the functionals for $t \ge 0$

$$J(\Omega) \overset{\text{def}}{=} \int_\Omega f\,dx \quad \text{and for } t \ge 0, \quad J(T_t(\Omega)) = \int_{T_t(\Omega)} f\,dx \tag{5.6}$$

We want to compute

$$dJ(\Omega;V(0)) = \lim_{t\searrow 0} \frac{J(T_t(\Omega)) - J(\Omega)}{t} = \lim_{t\searrow 0} \frac{1}{t}\left[\int_{T_t(\Omega)} f\,dx - \int_\Omega f\,dx\right].$$

Make a change of variable using T_t

$$\begin{aligned}
\frac{J(T_t(\Omega)) - J(\Omega)}{t} &= \frac{1}{t}\left[\int_{T_t(\Omega)} f\,dx - \int_\Omega f\,dx\right] \\
&= \frac{1}{t}\left[\int_\Omega [f \circ T_t \det DT_t - f]\ dx\right] \\
dJ(\Omega;V(0)) = \lim_{t\searrow 0} \frac{J(T_t(\Omega)) - J(\Omega)}{t} &= \int_\Omega \nabla f \cdot V(0) + f \operatorname{div} V(0)\,dx \\
&= \int_\Omega \operatorname{div}(f\,V(0))\,dx \\
&= \int_{\partial\Omega} f\,V(0) \cdot n_\Omega\,d\Gamma
\end{aligned}$$

$$dJ(\Omega;V(0)) = \int_\Omega \operatorname{div}(f\,V(0))\,dx = \int_{\partial\Omega} f\,V(0) \cdot n_\Omega\,d\Gamma.$$

[16]Delfour-Zolesio [9, Chapter 3].

5.2. **Topological Derivative.** The rigorous introduction of the *topological derivative* in 1999 by Sokołowski and Zòchowski [33])[17] provided a broader spectrum of notions of *derivatives with respect to a set*. Initially, topological perturbations were induced by creating a hole corresponding to removing a small closed ball of radius r and center $e \in \Omega$ from an open domain Ω.

The following approach via the Minkowski content has been initiated by Delfour [5, 2017] and [6, 2018]. Identify the set of equivalence classes of Lebesgue measurable subsets Ω of $\mathbb{R}^n$ with the set of their *characteristic functions* $\chi_\Omega : \mathbb{R}^n \to \{0,1\}$:

$$\mathrm{X}(\mathbb{R}^n) \overset{\text{def}}{=} \{\chi_\Omega : \Omega \subset \mathbb{R}^n \text{ Lebesgue measurable}\} \subset L^\infty(\mathbb{R}^n).$$

The *symmetric difference* operation induces an *Abelian group* structure:

$$\Omega_2 \Delta \Omega_1 \overset{\text{def}}{=} (\Omega_2 \backslash \Omega_1) \cup (\Omega_1 \backslash \Omega_2) \quad \Rightarrow \chi_{\Omega_2 \Delta \Omega_1}(x) = |\chi_{\Omega_2}(x) - \chi_{\Omega_1}(x)|,$$

where $\chi_\varnothing = 0$ is the neutral element and χ_Ω is its own *inverse*.

The group $\mathrm{X}(\mathbb{R}^n)$ is a closed subset without interior of the Banach space $L^\infty(\mathbb{R}^n)$ with the associated metric on equivalence classes of measurable susbsets of $\mathbb{R}^n$:

$$\rho([\Omega_2],[\Omega_1]) \overset{\text{def}}{=} \|\chi_{\Omega_2} - \chi_{\Omega_1}\|_{L^\infty(\mathbb{R}^n)} = \|\chi_{\Omega_2 \Delta \Omega_1}\|_{L^\infty(\mathbb{R}^n)},$$

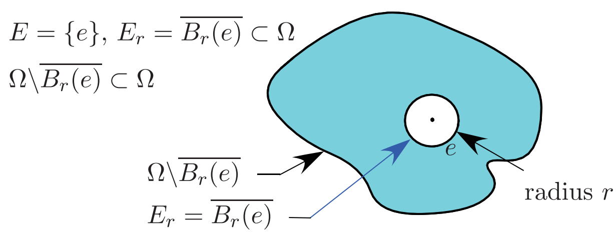
where the operation Δ is continuous. It is also a closed subset without interior of the Fréchet spaces $L^p_{\text{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$, endowed with the family of seminorms on bounded open subsets $D \subset \mathbb{R}^n$

$$\rho_D([\Omega_2],[\Omega_1]) \overset{\text{def}}{=} \|\chi_{\Omega_2 \Delta \Omega_1}\|_{L^p(D)} = \|\chi_{\Omega_2} - \chi_{\Omega_1}\|_{L^p(D)}.$$

Notation. For a closed subset E of $\mathbb{R}^n$ and $r \ge 0$, define

$$\begin{aligned} &\text{distance function from } x \text{ to } E: \quad d_E(x) \overset{\text{def}}{=} \inf_{y \in E} \|x - y\|, \\ &r\text{-dilatation of } E: \quad E_r \overset{\text{def}}{=} \{x \in \mathbb{R}^n : d_E(x) \le r\}. \end{aligned} \tag{5.7}$$

Example 5.3. Let Ω be an open subset of $\mathbb{R}^n$ and $e \in \Omega$. The closed ball $E_r = \overline{B_r(e)}$



is an r-dilatation of the set $E = \{e\}$, $\dim E = 0$. Consider the family of perturbed sets $\{\Omega \backslash E_r : 0 \le r \le R\}$. This generates a trajectory in $\mathrm{X}(\mathbb{R}^n)$:

$$r \mapsto \chi_{\Omega \backslash E_r} : [0,\tau] \to \mathrm{X}(\mathbb{R}^n), \quad \chi_{\Omega \backslash E_r} \to \chi_\Omega \text{ as } r \to 0. \tag{5.8}$$

[17] See also the book by Novotny-Sokołowski [28] and its bibliography for a review of past contributions.

By the Lebesgue differentiation theorem, for all $\phi \in C_c^0(\mathbb{R}^n)$ we get

$$\begin{aligned}&\frac{1}{|B_r(e)|}\int_{B_r(e)}\left[\chi_{\Omega\backslash E_r}-\chi_\Omega\right]\,\phi\,dx\\&=-\frac{1}{|B_r(e)|}\int_{B_r(e)}\chi_\Omega\,\phi\,dx\to-\chi_\Omega(e)\,\phi(e)=-\phi(e)\end{aligned}$$

and the delta function $-\delta_{\{e\}}$ at e.

Definition 5.4. (i) Given an open subset V of $\mathbb{R}^n$, the space

$$C^0(V)\overset{\text{def}}{=}\{f:V\to\mathbb{R}:f\text{ continuous and bounded on }V\}$$

endowed with the norm

$$\|f\|_{C^0}\overset{\text{def}}{=}\sup_{x\in V}|f(x)]\} \tag{5.9}$$

is a Banach space. Denote by $M^0(V)$ its topological dual which is also called the *space of Radon measures.*

(ii) Given an open subset V of $\mathbb{R}^n$, a function $f\in L^1(V)$ has *bounded variation* if

$$\|\nabla f\|_{M^0(V)}\overset{\text{def}}{=}\sup\left\{\int_V f\,\text{div}\,\phi\,dx:\phi\in C_c^1(V;\mathbb{R}^n),\ \|\phi(x)\|\le 1\right\}<\infty. \tag{5.10}$$

As we shall see in the following examples, the space

$$\text{BV}(V)\overset{\text{def}}{=}\{f\in L^1(V):\|\nabla f\|_{M^0(V)}<\infty\}$$

is a Banach space endowed with the norm

$$\|f\|_{\text{BV}(V)}\overset{\text{def}}{=}\|f\|_{L^1(V)}+\|\nabla f\|_{M^0(V)}. \tag{5.11}$$

(ii) Given an open subset U of $\mathbb{R}^n$, the space

$$\text{BV}_{loc}(U)\overset{\text{def}}{=}\{f:U\to\mathbb{R}:f|_V\in\text{BV}(V)\text{ for all bounded open }V\subset U\}$$

endowed with the seminorms $\|f\|_{\text{BV}(V)}$ is a Fréchet space.

What should play the role of t in the definition of our admissible trajectories is the variable $t=\alpha_n\,r^n$

$$\Omega_t\overset{\text{def}}{=}\Omega\backslash E_{(t/\alpha_n)^{1/n}},\quad t\mapsto\chi_{\Omega_t}=\chi_{\Omega\backslash E_{(t/\alpha_n)^{1/n}}}:[0,\tau]\to \text{X}(\mathbb{R}^n). \tag{5.12}$$

The trajectory $t\mapsto\chi_{\Omega_t}$ is continuous in $\text{X}(\mathbb{R}^n)$. Given $\phi\in C_c^0(\mathbb{R}^n)$, the weak limit of the differential quotient $(\chi_{\Omega_t}-\chi_\Omega)/t$ is

$$\begin{aligned}&\int_{\mathbb{R}^n}\frac{\chi_{\Omega_t}-\chi_\Omega}{t}\,\phi\,dx=\frac{1}{t}\left[\int_{\Omega_t}\phi\,dx-\int_\Omega\phi\,dx\right]\\&=-\frac{1}{|B_{(t/\alpha_n)^{1/n}}(e)|}\int_{B_{(t/\alpha_n)^{1/n}}(e)}\chi_\Omega\,\phi\,dx=-\frac{1}{\alpha_n r^n}\int_{B_r(e)}\chi_\Omega\,\phi\,dx\to-\phi(e).\end{aligned}$$

The function $\phi \mapsto -<\delta_{\{e\}},\phi>= -\phi(e) : C_c^0(\mathbb{R}^n) \to \mathbb{R}$ is a Radon measure. It generates a *semitangent* since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_{\Omega} \phi\, dx\right] \to -\rho\phi(e),$$

in the adjacent tangent cone $T^{\flat}_{\mathrm{X}(\mathbb{R}^n)}(\chi_\Omega)$, but not a full tangent. We can also introduce points $b \in \mathbb{R}^n\backslash\overline{\Omega}$ and the perturbed sets $\Omega_t = \Omega \cup B_{(t/\alpha_n)^{1/n}}(b)$ to get $+\phi(b)$. Here, the ambiant space seems to be $M^0(D)$ for some sufficiently large bounded open subset of $\mathbb{R}^n$. It is the space where the semitangents "live."

As we shall see in the following examples, the semitangent that we have constructed is directly related to the notion of *d-dimensional Minkowski content* [11] for $E \subset \mathbb{R}^n$ compact, $0 \le d < n$,

$$M^d(E) \overset{\text{def}}{=} \lim_{r\searrow 0} \frac{\mathrm{m}_n(E_r)}{\alpha_{n-d}\, r^{n-d}}, \quad \alpha_{n-d} = \text{ volume of the unit ball in } \mathbb{R}^{n-d},$$

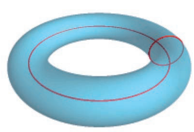
for general topological perturbations obtained by dilation of smooth submanifolds E of dimension d in $\mathbb{R}^n$. The case of $E = \{e\}$ corresponds to $d = 0$, while $d = 1$ corresponds to a curve and $d = 2$ to a surface.

It turns out that $M^d(E)$ is equal to $H^d(E)$, the d-dimensional Hausdorff measure in $\mathbb{R}^n$, for compact *d-rectifiable subsets* E of $\mathbb{R}^n$. For details and other motivating examples the reader is referred to the recent papers of Delfour [5, 6].

Definition 5.5 (Federer [11, pp. 251–252])**.** Let E be a subset of a metric space X. $E \subset X$ is *d-rectifiable* if it is the image of a compact subset K of $\mathbb{R}^d$ by a Lipschitz continuous function $f : \mathbb{R}^d \to X$.

Theorem 5.6 ([11, p. 275])**.** *If $E \subset \mathbb{R}^n$ is compact and d-rectifiable, then $M^d(E) = H^d(E)$.*

Example 5.7. Let E be a compact non-intersecting C^2-curve in $\mathbb{R}^3$ such that $H^1(E)$ is finite. Consider the r-dilatation of the curve E (a circle) where



$$\phi \mapsto \int_E \phi\, dH^1 : C_c^1(\mathbb{R}^3) \to \mathbb{R} \text{ is a Radon measure.}$$

Consider the trajectory

$$t \mapsto \Omega_t : [0,\tau] \to \mathrm{X}(\mathbb{R}^n), \quad \Omega_t \overset{\text{def}}{=} \Omega\backslash E_r = \Omega\backslash E_{(t/\alpha_2)^{1/2}}$$

The weak limit of the differential quotient $(\chi_{\Omega_t} - \chi_\Omega)/t$ is: for $\phi \in C_c^1(\mathbb{R}^3)$,

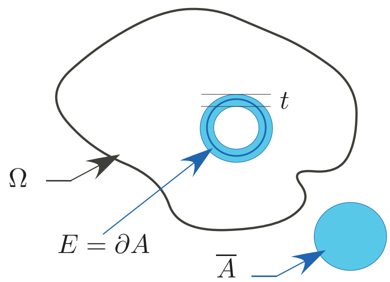
$$\frac{1}{t}\left[\int_{\Omega_t} \phi\, dx - \int_\Omega \phi\, dx\right] = -\frac{1}{t}\int_{E_{(t/\alpha_2)^{1/2}}} \chi_\Omega\, \phi\, dx$$

$$= -\frac{1}{\alpha_2\, r^2}\int_{E_r} \chi_\Omega\, \phi\, dx \to -\int_E \phi\, dH^1.$$

This Radon measure is a *semitangent* in $T^\flat_{\mathrm{X}(\mathbb{R}^n)}(\chi_\Omega)$ since for all $\rho > 0$

$$\frac{1}{t}\left[\int_{\Omega_{\rho t}} \phi\, dx - \int_\Omega \phi\, dx\right] \to -\rho \int_E \phi\, dH^1.$$

Example 5.8. When $n = 2$ and A is a ball, we can create a new connected component by dilating the boundary $E = \partial A$ of A.



So the adjacent tangent cone $T^\flat_{\mathrm{X}(\mathbb{R}^n)}(\chi_\Omega)$ to $\mathrm{X}(\mathbb{R}^n)$ at χ_Ω contains the negative of all Radon measures associated with d-dimensional rectifiable subsets E of Ω.

Given a bounded open subset Ω of $\mathbb{R}^n$, a function $f \in C^0(\overline{\Omega})$, a d-rectifiable subset E of Ω with $H^d(E) < \infty$, $t > 0$, the dilatation $r = (t/\alpha_{n-d})^{1/(n-d)}$ of E, and the functional

$$J(\Omega) = \int_\Omega f\, dx, \quad J(\Omega\backslash E_{(t/\alpha^{n-d})^{1/(n-d)}}) = \int_{\Omega\backslash E_{(t/\alpha_{n-d})^{1/(n-d)}}} f\, dx,\ t \geq 0.$$

In term of characteristic functions, with the notation $\Omega_t = \Omega\backslash E_{(t/\alpha_{n-d})^{1/(n-d)}}$

$$J(\chi_\Omega) = \int_{\mathbb{R}^n} \chi_\Omega f\, dx, \quad J(\chi_{\Omega_t}) = \int_{\mathbb{R}^n} \chi_{\Omega_t} f\, dx,\ t \geq 0.$$

Since $E_{(t/\alpha_{n-d})^{1/(n-d)}} \subset \Omega$ for t small, we have for the differential quotient

$$\frac{J(\Omega_t) - J(\Omega)}{t} = \frac{1}{t}\left[\int_{\Omega\backslash E_{(t/\alpha_{n-d})^{1/(n-d)}}} f\, dx - \int_\Omega f\, dx\right]$$

$$= -\frac{1}{t}\int_{E_{(t/\alpha_{n-d})^{1/(n-d)}}} \chi_\Omega\, f\, dx = -\frac{1}{\alpha_{n-d} r^{n-d}}\int_{E_r} \chi_\Omega\, f\, dx$$

$$\Rightarrow\ dJ(\chi_\Omega; \delta_E) = -\int_E f\, dH^d, \quad E \subset \Omega,$$

where

$$\phi \mapsto < \delta_E, \phi > \stackrel{\text{def}}{=} \int_E \phi \, dH^d : C_c(\mathbb{R}^n) \to \mathbb{R} \tag{5.13}$$

is a Radon measure.

Example 5.9. The tangent space also contains tangents generated by the velocity method. Go back to the diffeomorphisms $\{T_t : t \geq 0\}$ generated by C^1 velocities $\{V(t) : t \geq 0\}$. For $\Omega_t = T_t(\Omega)$ and $\phi \in C_0^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \left[\frac{\chi_{\Omega_t} - \chi_\Omega}{t}\right] \phi \, dx = \frac{1}{t}\left[\int_{\Omega_t} \phi \, dx - \int_\Omega \phi \, dx\right] = \frac{1}{t}\int_\Omega [\phi \circ T_t \det DT_t \, - \phi] \, dx$$

$$\begin{aligned}\lim_{t \searrow 0} \int_{\mathbb{R}^n} \left[\frac{\chi_{\Omega_t} - \chi_\Omega}{t}\right] \phi \, dx &= \int_\Omega \nabla\phi \cdot V(0) + \operatorname{div} V(0) \, \phi \, dx \\ &= \int_\Omega \operatorname{div}(V(0)\,\phi) \, dx = \int_\Gamma \phi V(0) \cdot n_\Omega \, d\Gamma.\end{aligned}$$

If Ω is a *Caccioppoli set*, that is $\chi_\Omega \in BV_{loc}(\mathbb{R}^n)^n$, and ϕ and $V(0)$ are also bounded continuous, then

$$\phi \mapsto \int_\Omega \operatorname{div}(V(0)\,\phi) \, dx = \int_{\mathbb{R}^n} \chi_\Omega \operatorname{div}(V(0)\,\phi) \, dx = - < \nabla\chi_\Omega, \phi \, V(0) >$$

is linear and continuous with respect to the sup norm of ϕ. Therefore, $\nabla\chi_\Omega \cdot V(0)$ defined as the normal component of the velocity:

$$\phi \mapsto (\nabla\chi_\Omega \cdot V(0))(\phi) \stackrel{\text{def}}{=} < \nabla\chi_\Omega, \phi \, V(0) >$$

is a Radon measure which is a tangent in $T^\flat_{\mathrm{X}(\mathbb{R}^n)}(\chi_\Omega)$. So, it extends to bounded continuous velocities $V(0)$.

Open problem: can the adjacent tangent cone to $\mathrm{X}(D)$ at χ_Ω be completely characterized?

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M. C. Delfour
Département de mathématiques et de statistique and Centre de recherhes mathématiques, Université de Montréal, CP 6128, succ. Centre-ville, Montréal (Qc), Canada H3C 3J7
E-mail address: delfour@crm.umontreal.ca, delfour@dms.umontreal.ca