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COMPACTNESS AND LIOUVILLE EQUATIONS

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Abstract. This paper surveys some results regarding compactness of *Liouville equations*, which were principally initiated in the seminal paper [6]. We will begin showing an alternative between compactness and blow-up depending on concentration properties of exponential terms, and after turn to a more precise quantization property. We will also treat more recent developments concerning higher-order equations in conformal geometry.

1. Introduction

Liouville equations (elliptic, with exponential nonlinear terms) appear in different contexts in models from mathematical physics or in differential geometry. Concerning the former motivations, they arise as mean field equations of Euler flows or in the enumerate Chern-Simons-Higgs models, see for example [42], [44] and references therein. In geometry they are related to the prescription of Gaussian curvature by conformal change of the metric. Indeed on a surface Σ, for $\tilde{g} = e^{2w} g$, one has

$$\Delta_{\tilde{g}} = e^{-2w} \Delta_g; \qquad -\Delta_g w + K_g = K_{\tilde{g}} e^{2w}, \tag{1.1}$$

where K_g and $K_{\tilde{g}}$ are the Gauss curvatures of (Σ, g) and of $(\Sigma, \tilde{g})$, see [2].

One fundamental tool for studying Liouville equations analytically is *compactness*. With such a property at hand, one can then apply general techniques such as continuity methods, degree-theoretical arguments of min-max principles. In the introduction of this survey paper we will describe part of the work in [6] (all the statements of this section are from there, where sometimes a different notation on the exponents is used), which paved the way for the above question, allowing a substantial development of the subject.

Consider a bounded domain Ω of $\mathbb{R}^2$, and the following problem with measurable right-hand side

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

By means of the next result, one can then derive exponential integrability on the solution depending on the L^1 norm of the datum.

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Proposition 1.1. *Suppose u is a solution of* (1.2)*. Then for all $\delta \in (0, 4\pi)$ one has that*

$$\int_\Omega e^{\frac{(4\pi-\delta)|u|}{\|f\|_{L^1(\Omega)}}} \leq \frac{4\pi^2}{\delta}(diam(\Omega))^2.$$

Proof. Since Ω is bounded, there exists $x_0 \in \mathbb{R}^2$ such that $\Omega \subseteq B_R(x_0)$, where $R = \frac{1}{2}\mathrm{diam}(\Omega)$: without loss of generality we can assume that $x_0 = 0$. The function f can be extended to zero outside Ω: define then the following function

$$\bar{u}(x) = \frac{1}{2\pi}\int_{B_R} \log\left(\frac{2R}{|x-y|}\right)|f(y)|dy.$$

From the maximum principle it follows that $|u| \leq \bar{u}$ in Ω, and therefore we get

$$\int_\Omega e^{\frac{(4\pi-\delta)|u|}{\|f\|_{L^1(\Omega)}}}dx \leq \int_\Omega e^{\frac{(4\pi-\delta)\bar{u}}{\|f\|_{L^1(\Omega)}}}dx.$$

The latter term can be estimated via Jensen's inequality as

$$\exp\left(\int_\Omega w(y)\varphi(y)dy\right) \leq \int_\Omega w(y)\exp(\varphi(y))dy,$$

with

$$w(y) = \frac{|f(y)}{\|f\|_{L^1(\Omega)}}; \qquad \varphi(y) = \frac{4\pi-\delta}{2\pi}\log\left(\frac{2R}{|x-y|}\right).$$

In this way, via Fubini's theorem one finds that

$$\int_\Omega e^{\frac{(4\pi-\delta)\bar{u}}{\|f\|_{L^1(\Omega)}}}dx \leq \int_{B_R}\frac{|f(y)|}{\|f\|_{L^1(\Omega)}}dy\left[\int_{B_R}\left(\frac{2R}{|x-y|}\right)^{2-\frac{\delta}{2\pi}}dx\right].$$

Concerning the last integral, we can use polar coordinates to find that

$$\int_{B_R}\left(\frac{2R}{|x-y|}\right)^{2-\frac{\delta}{2\pi}}dx \leq \int_{B_R}\left(\frac{2R}{|x|}\right)^{2-\frac{\delta}{2\pi}}dx = \frac{4\pi^2}{\delta}(\mathrm{diam}\Omega)^2,$$

which concludes the proof. □

Corollary 1.2. *Suppose $(u_n)_n$ solves*

$$-\Delta u_n = V_n(x)e^{2u_n} \qquad in\ \Omega. \tag{1.3}$$

Assume that $\|V_n\|_{L^\infty(\Omega)} \leq C_1$, $\|u_n^+\|_{L^1(\Omega)} \leq C_2$ for some $C_1, C_2 > 0$ and that

$$\int_\Omega |V_n|e^{2u_n}dx \leq \varepsilon_0 < 2\pi.$$

Then (u_n^+) is bounded in $L^\infty_{loc}(\Omega)$.

Proof. We can assume that Ω is a ball $B_R := B_R(0)$, replacing it possibly a smaller domain. Let us write $u_n = u_{1,n} + u_{2,n}$, where $u_{1,n}$ solves

$$\begin{cases} -\Delta u_{1,n} = V_n e^{2u_n} & \text{in } \Omega; \\ u_{1,n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and where $u_{2,n}$ is harmonic in Ω. The mean value theorem then implies

$$\|u_{2,n}^+\|_{L^\infty(B_{R/2})} \leq C\|u_{2,n}^+\|_{L^1(B_{R/2})} \leq C\left[\|u_n^+\|_{L^1(B_R)} + \|u_{1,n}^+\|_{L^1(B_R)}\right] \leq C.$$

Using the latter boundary value problem and the previous theorem we have that $(e^{2u_{1,n}})$ is bounded in $L^{1+\delta}(B_R)$ for some $\delta > 0$. This implies in turn that $(V_n e^{2u_n})$ is bounded in $L^q(B_{R/2})$ for some $q > 1$. Using the boundary value problem once more we obtain that $(u_{1,n})_n$ is bounded in $L^\infty(B_{R/4})$, and therefore $(u_n)_n$ is bounded in $L^\infty(B_{R/4})$. □

We define the *blow-up set* S of $(u_n)_n$ as

$$S = \{x \in \Omega \,:\, \exists\, x_n \to x \text{ with } u_n(x_n) \to +\infty\}\,.$$

With the help of the above result it is possible to obtain the next theorem from [6], of which we only state a particular case.

Theorem 1.3 ([6]). *Suppose Ω is a bounded domain of $\mathbb{R}^2$ and consider a sequence of solutions to* (1.3). *Suppose that for some $C_1 > 0$ $(V_n)_n$ satisfies*

$$V_n \geq 0; \qquad\qquad \|V_n\|_{L^\infty(\Omega)} \leq C_1,$$

and that $(u_n)_n$ is such that

$$\int_\Omega e^{2u_n} dx \leq C_1.$$

Then, up to a subsequence, we have one among the following alternatives

(i) *$(u_{n_k})_k$ is bounded in $L^\infty_{loc}(\Omega)$;*
(ii) *$u_{n_k} \to -\infty$ uniformly on compact sets of Ω;*
(iii) *the blow-up set S of u_{n_k} is finite, non-empty and $u_{n_k} \to -\infty$ uniformly on compact sets of $\Omega \setminus S$. Moreover, $(V_{n_k} e^{2u_{n_k}})_k$ converges weakly in the sense of measures to a sum of Dirac masses $\sum_i \alpha_i \delta_{a_i}$, with $\alpha_i \geq 2\pi$ and $S = \cup_i \{a_i\}$.*

Proof. Since $(V_n e^{2u_n})_n$ is bounded in $L^1(\Omega)$ by our assumptions, this sequence converges in the sense of measures to some non-negative and bounded measure μ, i.e.

$$\int_\Omega V_n e^{2u_n} \psi\, dx \to \int_\Omega \psi\, d\mu \qquad\qquad \text{for all } \psi \in C_c(\Omega).$$

A point $x_0 \in \Omega$ is called *regular* is there exists $\psi \in C_c(\Omega)$, $\psi(x) \in [0, 1]$ for all x, and ψ identically equal to 1 in a neighborhood of x_0 such that

$$\int_\Omega \psi\, d\mu < 2\pi. \tag{1.4}$$

The above corollary implies that if x_0 is a regular point then there is some R_0 small such that $(u_n)_n$ is bounded in $L^\infty(B_{R_0}(x_0))$.

Letting S' denote the set of non-regular points in Ω, for every $x_0 \in S'$ we must have $\mu(\{x_0\}) \geq 2\pi$. This implies that S' is finite and that

$$\mathrm{card}(S') \leq \frac{C_1 C_2}{2\pi},$$

where the C_i's are as above. The proof is then divided into three steps.

Step 1. $S' = S$. By Corollary 1.2 we have that $S \subseteq S'$. Suppose that $x_0 \in S'$: then one has for all $R > 0$ $\lim \|u_n^+\|_{L^\infty(B_R(x_0))} = +\infty$. Otherwise, there would exist $R_0 > 0$ and a subsequence u_{n_k} such that $\|u_n^+\|_{L^\infty(B_R(x_0))} \leq C$. This would then imply

$$\int_{B_R(x_0)} V_{n_k} e^{2u_{n_k}} dx \leq C\, C_1 R \qquad \text{for all } R < R_0,$$

yielding (1.4) for some $\psi \in C_c(\Omega)$ and contradicting the above assumption.

Once that the above claim is established, we can choose $R > 0$ small so that $\bar{B}_R(x_0)$ does not contain other points of S. Suppose $(x_n)_n \subseteq B_R(x_0)$ is such that

$$u_n^+(x_n) = \max_{\bar{B}_R(x_0)} u_n^+ \to +\infty.$$

Then it must be $x_n \to x_0$: if this were not true, x_n would converge to a regular point, which is impossible by the above equation, concluding the proof of the first step.

Step 2. If $S = \emptyset$, either (i) or (ii) must hold true. In fact, by (32) u_n^+ is bounded in $L^\infty_{loc}(\Omega)$, and hence $V_n e^{2u_n}$ would be bounded in $L^\infty_{loc}(\Omega)$. Let v_n solve

$$\begin{cases} -\Delta v_n = f_n & \text{in } \Omega; \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $v_n \to v$ uniformly on the compact sets of Ω, where v satisfies

$$\begin{cases} -\Delta v = \mu & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $w_n - u_n - v_n$: this function is clearly harmonic and w_n^+ is bounded in $L^\infty_{loc}(\Omega)$. By Harnack's theorem either a subsequence w_{n_k} is bounded in $L^\infty_{loc}(\Omega)$ or w_n converges to $-\infty$ uniformly on compact sets of Ω. (i) corresponds to the first possibility, while (ii) to the second one.

Step 3. If $S \neq \emptyset$, then (iii) holds. To see this, use the fact that u_n^+ is bounded in $L^\infty_{loc}(\Omega \setminus S)$ to get that $V_n e^{2u_n}$ is also bounded in $L^\infty_{loc}(\Omega \setminus S)$. From this we deduce that μ is a bounded measure in Ω with $\mu \in L^p_{loc}(\Omega \setminus \Sigma)$. Let v_n and w_n be defined as in the previous step: then v_n converges to some function v uniformly on compact sets of $\Omega \setminus S$. By Harnack's theorem we have that either a subsequence w_{n_k} is bounded in $L^\infty_{loc}(\Omega \setminus S)$ or w_n converges to $-\infty$ on compact sets of $\Omega \setminus S$. We will show that the first alternative cannot hold.

Fixing some point $x_0 \in \Sigma$ and $R > 0$ small enough so that $\bar{B}_R(x_0) \cap S = \{x_0\}$, suppose the first alternative holds, and hence also $(v_n)_n$ is bounded in $L^\infty(\partial B_R(x_0))$. This implies that $|u_{n_k}| \leq C$ for some fixed $C > 0$. Consider the solutions of

$$\begin{cases} -\Delta z_{n_k} = f_{n_k} & \text{in } B_R(x_0); \\ z_{n_k} = -C & \text{on } \partial B_R(x_0). \end{cases}$$

By the maximum principle we have that $u_{n_k} \geq z_{n_k}$ in $B_R(x_0)$, and hence that

$$\int_{B_R(x_0)} e^{2z_{n_k}} dx \leq \int_{B_R(x_0)} e^{2u_{n_k}} dx \leq C_2^2.$$

We also have that $z_{n_k} \to z$ a.e., where z solves

$$\begin{cases} -\Delta z = \mu & \text{in } B_R(x_0); \\ z = -C & \text{on } \partial B_R(x_0). \end{cases}$$

As x_0 is a singular point, we must have $\mu \geq \mu|_{\{x_0\}} \geq 2\pi\delta_{x_0}$, which implies that

$$z(x) \geq \log \frac{1}{|x - x_0|} + O(1) \qquad \text{as } x \to x_0.$$

This forces $e^{2z(x)} \geq \frac{C}{|x-x_0|^2}$ for some $C > 0$, and hence $\int_{B_R(x_0)} e^{2z} dx = +\infty$, contradicting the fact that $\int_{B_R(x_0)} e^{2z} dx \leq C_2^2$ by Fatou's lemma.

We proved that the first alternative above cannot hold, and hence $(u_n)_n$ converges to $-\infty$ on compact subsets of $\Omega \setminus S$. This implies that $V_n e^{2u_n} \to 0$ in $L^p_{loc}(\Omega \setminus \Sigma)$ and therefore μ is supported on S. Hence we have that $\mu = \sum_i \alpha_i \delta_{a_i}$ with $\alpha_i \geq 2\pi$, as desired. $\square$

In the next section we will discuss more precise blow-up results, assuming stronger bounds on the functions $(V_n)_n$, and some of their consequences. We will next discuss some counterpart of Theorem 1.3 for fourth-order equations, which are motivated from conformal geometry. Finally, we aim to describe some more recent compactness result for Liouville equations with principal terms of mixed orders, motivated by questions in spectral theory.

2. Quantization

In this section we will describe a quantization result for solutions of Liouville's equations from [26]. If α_i is as in Theorem 1.3, then it is proved under more regularity on the V_n's that this coefficient is an integer multiple of 4π. We will then mention some applications.

Theorem 2.1 ([26]). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded subset. Suppose $(V_n)_n$ is a sequence of continuous functions on Ω, with $V_n \geq 0$ and such that $V_n \to V$ in $C^0(\bar{\Omega})$. Suppose $(u_n)_n$ is a sequence of solutions of* (1.3) *such that $\int_\Omega e^{2u_n} dx \leq C_1$. Then in the third alternative of the above theorem one has that α_i is an integer multiple of 4π for all i.*

It is sufficient to localize the above result on a small ball B_R, which will be proved using a number of lemmas. First, one has a sharper lower bound on α_i.

Lemma 2.2. *If α_i is a before and if $V(0) > 0$, then $\alpha_i \geq 4\pi$.*

Proof. Suppose that $x_n \in B_R$ is such that $u_n(x_n) = \max_{B_R} u_n$. Then we have that $x_n \to 0$ and $u_n(x_n) \to +\infty$. Calling $\delta_n = e^{-u_n(x_n)}$, we have clearly that $\delta_n \to 0$. Consider now the new sequence of functions

$$\tilde{u}_n(x) = u_n(\delta_n x + x_n) + \log \delta_n.$$

By the continuity of V this new sequence satisfies

$$-\Delta \tilde{u}_n = V_n(\delta_n x + x_n) e^{2\tilde{u}_n}; \qquad \tilde{u}_n(0) = 0, \quad \tilde{u}_n \leq 0, \quad \int_{B_{\frac{R}{2\delta_n}}} e^{2\tilde{u}_n} dx \leq C_0.$$

By the previous theorem then $\tilde{u}_n$ is bounded in $L^\infty_{loc}(B_r)$ for any $r > 0$ and for n large, and therefore u_n would locally converge to an entire solution $\tilde{u}$ of

$$-\Delta \tilde{u} = V(0) e^{2\tilde{u}}; \qquad \tilde{u}(0) = 0, \quad \tilde{u} \leq 0, \quad \int_{B_{\frac{R}{2\delta_n}}} e^{2\tilde{u}} dx \leq C_0.$$

If $V(0) = 0$ we would have that $\tilde{u}$ has to be entire harmonic and hence constant by Liouville's theorem, violating the integrability of its exponential. Therefore it must be $V(0) > 0$. Solutions of the above equation were classified in [9], in the form

$$\tilde{u}(x) = \frac{1}{2} \log \frac{1}{(1 + \gamma^2 |x|^2)^2}; \qquad \gamma = (V(0)/8)^{\frac{1}{2}}.$$

In particular one has $V(0) \int_{\mathbb{R}^2} e^{2\tilde{u}} dx = 4\pi$, proving that

$$\alpha = \lim_n \int_{B_R} V_n e^{u_n} dx \geq \lim_n \int_{B_{r\delta_n}} V_n e^{u_n} dx = V(0) \int_{B_r} e^{2\tilde{u}} dx.$$

Sending $r \to +\infty$, we have the desired conclusion. $\square$

The next lemma produces iteratively a finite number of bubbling profiles that are *relatively separated* one from another.

Lemma 2.3. *Let $(V_n)_n$ be as in Theorem 2.1. Let $(u_n)_n$ be solutions of* (1.3) *in B_R that are blowing up such that $\int_{B_R} e^{2u_n} dx \leq C$. Then, passing to a subsequence there exists an integer $m \leq V(0)\frac{C_0}{8\pi}$, sequences of points $x_n^{(j)}$ and sequences $k_n^{(j)}$, $j = 0, \dots, m-1$ with the following properties*

$$u_n(x_n^{(j)}) = \max_{|x - x_n^{(j)}| \leq k_n^{(j)} \delta_n^{(j)}} u_n(x) \to +\infty \qquad \text{for all } j, \tag{2.1}$$

where $\delta_n^{(j)} = e^{-u_n(x_n^{(j)})}$,

$$B_{2k_n^{(i)}\delta_n^{(i)}}(x_n^{(i)}) \cap B_{2k_n^{(j)}\delta_n^{(j)}}(x_n^{(j)}) = \emptyset \qquad \text{for } i \neq j; \tag{2.2}$$

$$\frac{\partial}{\partial t} u_n(ty + x_n^{(j)})|_{t=1} < 0 \qquad \text{for all } \delta_n^{(j)} \leq |y| \leq 2k_n^{(j)}\delta_n^{(j)} \qquad \text{and for all } j; \tag{2.3}$$

$$\lim_n \int_{B_{2k_n^{(i)}\delta_n^{(i)}}(x_n^{(i)})} V_n e^{2u_n} dx = \lim_n \int_{B_{k_n^{(i)}\delta_n^{(i)}}(x_n^{(i)})} V_n e^{2u_n} dx = 4\pi \qquad \text{for all } j; \tag{2.4}$$

$$\max_{x \in \bar{B}_R} \left\{ u_n(x) + \log \min_{0 \leq j \leq m-1} |x - x_n^{(j)}| \right\} \leq C \qquad \text{for all } j. \tag{2.5}$$

Proof. Let $x_n^{(0)}$ be such that $u_n(x_n^{(0)}) = \max_{\bar{B}_R} u_n$. Set also $\tilde{u}_n^{(0)}(x) = u_n(\delta_n^{(0)} x + x_n^{(0)}) + \log \delta_n^{(0)}$, with $\delta_n^{(0)} = e^{-u_n(x_n^{(0)})}$. Reasoning as for the previous case, we can find an entire solution $\tilde{u}$ to

$$-\Delta \tilde{u} = V(0) e^{2\tilde{u}}; \qquad \tilde{u}(0) = 0, \qquad \tilde{u} \leq 0$$

verifying

$$\|\tilde{u}_n^{(0)} - \tilde{u}\|_{C^{1,\alpha}(B_{2k_n^{(0)}})} \to 0,$$

ans such that

$$\int_{B_{2k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} V_n e^{2u_n} dx = \int_{B_{2k_n^{(0)}}} V_n(\delta_n^{(0)} + x_n^{(0)}) e^{2u_n^{(0)}} dx \to 4\pi;$$

$$\int_{B_{k_n^{(0)}\delta_n^{(0)}}(x_n^{(0)})} V_n e^{2u_n} dx = \int_{B_{k_n^{(0)}}} V_n(\delta_n^{(0)} + x_n^{(0)}) e^{2u_n^{(0)}} dx \to 4\pi;$$

$$\frac{\partial}{\partial t} u_n(ty + x_n^{(0)})|_{t=1} < 0 \qquad \text{for } \delta_n^{(0)} \leq |y| \leq 2k_n^{(0)}\delta_n^{(0)}.$$

The sequences $x_n^{(0)}, k_n^{(0)}$ satisfy the above properties with $m = 1$.

Suppose next we have sequences $x_n^{(j)}$ and $k_n^{(j)}$ satisfying the above properties for $j = 0, \dots, l-1$ with $m = l$.

If $\max_{x \in B_R} \left[u_n(x) + \log \min_{j=0,\dots,l-1} |x - x_n^{(j)}| \right] \leq C$ for all n we stop the procedure and define $m = l$.

Otherwise, let $\bar{x}_n^{(l)}$ be a point attaining

$$M_n := \max_{x \in B_R} \left[u_n(x) + \log \min_{j=0,\dots,l-1} |x - x_n^{(j)}| \right] \to +\infty :$$

this implies in particular that $u_n(\bar{x}_n^{(l)}) \to +\infty$. Setting also $\bar{\delta}_n^{(l)} = e^{-u_n(\bar{x}_n^{(l)})}$, from $M_n \to +\infty$ we find that $\min_{j=0,\dots,l-1} \frac{|\bar{x}_n^{(l)} - x_n^{(j)}|}{\bar{\delta}_n^{(l)}} \to +\infty$. Notice that for $|x| \leq \frac{1}{2} \min_{j=0,\dots,l-1} \frac{|\bar{x}_n^{(l)} - x_n^{(j)}|}{\bar{\delta}_n^{(l)}}$ one has

$$\begin{aligned} &\min_{j \in \{0,\dots,l-1\}} |\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} x - x_n^{(j)}| \\ &\geq \min_{j \in \{0,\dots,l-1\}} |\bar{x}_n^{(l)} - x_n^{(j)}| - \bar{\delta}_n^{(l)} |x| \geq \frac{1}{2} \min_{j \in \{0,\dots,l-1\}} |\bar{x}_n^{(l)} - x_n^{(j)}|. \end{aligned}$$

Setting $\tilde{u}_n = u_n(\bar{x}_n^{(l)} + \bar{\delta}_n^{(l)} x) + 2 \log \bar{\delta}_n^{(l)}$, by the latter formula and the choices of the latter scales and points we deduce that

$$\begin{cases} -\Delta \tilde{u}_n(x) = V_n(\bar{\delta}_n^{(l)} x + x_n^{(l)}) e^{\tilde{u}_n(x)} & \text{for } |x| \leq \frac{1}{2} \min_{j=0,\dots,l-1} \frac{|\bar{x}_n^{(l)} - x_n^{(j)}|}{\bar{\delta}_n^{(l)}}; \\ \tilde{u}_n(0) = 0; \\ \tilde{u}_n(x) \leq 2 \log 2 & \text{for } |x| \leq \frac{1}{2} \min_{j=0,\dots,l-1} \frac{|\bar{x}_n^{(l)} - x_n^{(j)}|}{\bar{\delta}_n^{(l)}}. \end{cases}$$

A subsequence of $\tilde{u}_n$ then converges in $C^{1,\alpha}_{loc}$ to an entire solution $\bar{u}$ of the Liouville equation $-\Delta \bar{u} = V(0)e^{2\bar{u}}$, and similarly to before one deduces that

$$\|\tilde{u}_n - \bar{u}\|_{C^{1,\alpha}(B_{4k_n^{(l)}})} \to 0,$$

and such that for $c > 0$ small

$$\int_{B_{ck_n^{(l)}}} V_n(\bar{\delta}_n^{(l)} x + \bar{x}_n^{(l)}) e^{\tilde{u}_n} dx \to\to 4\pi;$$

$$\int_{B_{4k_n^{(l)}}} V_n(\bar{\delta}_n^{(l)} x + \bar{x}_n^{(l)}) e^{2\tilde{u}_n} dx \to 4\pi;$$

$$\frac{\partial}{\partial t} \tilde{u}_n(ty + \bar{x})|_{t=1} < 0 \qquad \text{for } 1 \leq |y| \leq 4k_n^{(l)} \delta_n^{(0)}.$$

Consider now a point $y_n^{(l)} \in B_{3k_n^{(l)}}$ such that $\tilde{u}_n(\bar{x} + y_n^{(l)}) = \max_{B_{4k_n^{(l)}}} \tilde{u}_n(\bar{x} + y)$, and define $x_n^{(l)} = \bar{x}_n^{(l)} + \bar{\delta}_n^{(l)}(\bar{x} + y_n^{(l)})$. From the latter equations and the expression of the limit function $\bar{u}$ one has that $y_n^{(l)} \to 0$ and that $u_n(\bar{x}_n^{(l)}) \leq u_n(x_n^{(l)}) \leq u_n(\bar{x}_n^{(l)}) + C$. Set now

$$\tilde{u}_n^{(l)}(x) = u_n(\delta_n^{(l)} x + x_n^{(l)}) + \log \delta_n^{(l)}; \qquad \delta_n^{(l)} = e^{-u_n(x_n^{(l)})}.$$

From the above inequalities one has that

$$\delta_n^{(l)} \leq \bar{\delta}_n^{(l)} \leq e^C \delta_n^{(l)}; \qquad u_n(x_n^{(l)}) = \max_{|x - x_n^{(l)}| \leq k_n^{(l)} \delta_n^{(l)}} u_n(x) \to +\infty.$$

For $\tilde{u}_n^{(l)}$ we obtain analogous conditions to (2.1)-(2.4). Therefore, the points and scales $x_n^{(j)}, k_n^{(j)}$, $j = 0, \ldots, l$ satisfy the above properties with $m = l + 1$.

We continue then in this way until (2.5) holds, since we have to stop after a finite number of steps as we accumulate 4π in mass each time. This concludes the proof. □

We next want to prove that no volume accumulates at scales greater than the ones of the bubbles, namely that one has the following result.

Lemma 2.4. *In the above notation one has that*

$$\lim_n \int_{B_R \setminus \cup_{j=1}^{m-1} B_{k_n^{(j)} \delta_n^{(j)}}(x_n^{(j)})} V_n e^{2u_n} dx = 0.$$

Proof. The argument in [26] is by induction in m, and we will describe here in detail only the case $m = 1$, giving some brief sketch about the general one.

We can assume that $x_n^{(0)} = 0$ for all n and that $r_n^{(0)} \to 0$, otherwise the result clearly holds true. As a consequence of Harnack's inequality (see Lemma 2 in [26]) there exist $\beta \in (0, 1)$ and $C > 0$ such that

$$\sup_{\partial B_r} u_n \leq C + \beta \inf_{\partial B_r} u_n + (\beta - 1) \log r; \qquad 2r_n^{(0)} \leq r \leq \frac{R}{2}.$$

As a consequence of a *sup+inf inequality* from [41] (see also [5]) one also finds that

$$\inf_{\partial B_r} u_n \leq C - \frac{1}{C_1} u_n(0) - \left(1 + \frac{1}{C_1}\right) \log r; \qquad 0 < r < R.$$

From the two inequalities one gets

$$\sup_{\partial B_r} u_n \leq C - \frac{\beta}{C_1} u_n(0) - \left(\frac{\beta}{C_1} + 1\right) \log r; \qquad 2r_n \leq r \leq \frac{R}{2},$$

which means that

$$e^{2u_n(x)} \leq C(\delta_n^{(0)})^{2\frac{\beta}{C_1}} |x|^{-2(\beta/C_1+1)}; \qquad 2r_n^{(0)} \leq |x| \leq \frac{R}{2}.$$

Since $r_n^{(0)}$ is larger than the scale of the bubble we find that

$$\int_{B_{R/2} \setminus B_{2r_n^{(0)}}} V_n e^{2u_n} dx \leq C(\delta_n^{(0)})^{2\frac{\beta}{C_1}} \int_{2r_n^{(0)}}^{\infty} r^{-2(\beta/C_1+1)} r dr = C \left(\frac{\delta_n^{(0)}}{2r_n^{(0)}}\right)^{2\beta/C_1}$$

tends to zero. This finally implies that

$$\int_{B_R} V_n e^{2u_n} dx \to \beta_0,$$

as desired. The general case follows by clustering the blow-up points properly. If some subgroup of points has comparable relative distances, one can use condition (2.5) and the Harnack inequality for u_n on (geometrically) non-degenerating multiply-connected domains of that scale to show that no residual volume accumulates on a set including all such points. One can then apply the latter reasoning to to a *thick annulus* whose inner complement includes all those points, and show that no residual volume accumulates at a larger scale. An iterative argument then gives the result in the lemma for general m. □

Remark 2.5. (i) In [11] it is proved that α_i can indeed be larger than 4π even for $V_n \equiv 1$. This is done via Liouville's local theorem for holomorphic functions.

(ii) On a compact surface the same result holds true, but in this case the α_i's are exactly 4π. This was proved in [25] using a moving plane method.

The quantization result in Theorem 2.1 allowed to compute the degree of *mean field equations* (of Liouville type) on compact surfaces or on domains of $\mathbb{R}^2$ under Dirichlet boundary conditions, see [25] and [10] It also allowed to produce solutions via min-max theory, see e.g. [13], [14]. A more detailed analysis of blowing-up solutions was also done in [12], [33] (and [10]), studying next-order terms of solutions and asymptotic expansions of the energy.

An interesting extension of Theorems 1.3 and 2.1 concerns *singular Liouville equations* or *Toda systems*, where some quantization results were obtained in [3] and [24]. For more recent developments and some applications of these results see e.g. [31] and [32].

3. Four dimensions

In this section we describe an analogous result to Theorem 1.3 in four dimensions from [1]. As we will see, the structure of solutions is less rigid, and new concentration phenomena may occur. At the end of the section we will describe some geometric motivation for the study of fourth order problems, as well as quote some results in this direction.

Consider a bounded set $\Omega \subseteq \mathbb{R}^4$, and the following sequence of equations

$$\Delta^2 u_n = V_n e^{4u_n}; \qquad \text{with } V_n \to 1 \quad \text{uniformly in } \Omega. \tag{3.1}$$

In [1] the following result was proved.

Theorem 3.1. *([1]) Let Ω be a bouned domain of $\mathbb{R}^4$ and let $(u_n)_n$ be a sequence of solutions to (3.1). Suppose thee exists $\Lambda > 0$ such that*

$$\int_\Omega V_n e^{4u_n} dx \leq \Lambda \qquad \textit{for all } n.$$

Then we have one of the following alternatives

(i) *u_n is relatively compact in $C^{3,\alpha}_{loc}(\Omega)$ up to a subsequence;*
(ii) *there exists a nowhere dense set Σ_0 of zero Lebesgue measure and at most finitely-many points $x^{(i)} \in \Omega$, $1 \leq i \leq I \leq C\Lambda$ for some constant $C > 0$, such that if*

$$\Sigma = \Sigma_0 \cup_{i=1}^I \{x^{(i)}\}$$

one has that $u_n \to -\infty$ uniformly on compact sets of $\Omega \setminus \S$ for $n \to +\infty$.

Moreover, there is $\beta_n \to +\infty$ such that

$$\frac{u_n}{\beta_n} \to \varphi \qquad \textit{in } C^{3,\alpha}_{loc}(\Omega \setminus \S),$$

where $\varphi \in C^4(\Omega \setminus \cup_{i=1}^I \{x^{(i)}\})$ such that

$$\Delta^2 \varphi = 0; \qquad \varphi \leq 0, \qquad \varphi \not\equiv 0,$$

and such that $\Sigma_0 = \left\{x \in \Omega \setminus \cup_{i=1}^I \{x^{(I)}\} \ : \ \varphi(x) = 0\right\}$.

Furthermore, near any point $x_0 \in \Sigma$ such that $\sup_{B_r(x_0)} u_n \to +\infty$ as $n \to +\infty$ there exists $x_n \to x_0$, $L_n \to +\infty$ and $r_n \to$ satisfying

$$v_n(x) := u_n(x_n + r_n x) + \log r_n \leq 0 \leq \log 2 + v_n(0); \qquad |x| \leq L_n. \tag{3.2}$$

Either $v_v \to v$ in $C^{3,\alpha}_{loc}(\mathbb{R}^4)$, where v solves

$$\Delta^2 v = e^{4v} \qquad \textit{in } \mathbb{R}^4,$$

or $v_n \to -\infty$ a.e. and there exists $\gamma_n \to +\infty$ such that, up to a subsequence

$$\frac{v_n}{\gamma_n} \to \psi \qquad \textit{in } C^{3,\alpha}_{loc}(\mathbb{R}^4),$$

where ψ is a non-positive quadratic polynomial.

For the proof, we begin stating the following result obtained in [27], [43], extending the one in [6] to higher dimensions.

Proposition 3.2. *Suppose v solves*

$$\begin{cases} \Delta^2 v = f & \text{in } B_R(x_0) \subseteq \mathbb{R}^4; \\ v = \Delta v = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

If f satisfies $\|f\|_{L^1(B_R(x_0))} = \alpha < 8\pi^2$, then for any $p < \frac{8\pi^2}{\alpha}$ one has that

$$\int_{B_R(x_0)} e^{4p|v|} dx \leq C(p) R^4.$$

Then one has the following result from [36], which follows from an integration by parts and the mean value theorem.

Lemma 3.3. *Suppose h solves*

$$\Delta^2 h = 0 \qquad \text{in } B_R(y) \subseteq \mathbb{R}^n.$$

Then h satisfies

$$h(y) - \fint_{B_R(y)} h(z) dz = \frac{R^2}{2(n+2)} \Delta h(y).$$

This result allows to prove a Liouville type theorem for the bi-harmonic equation, namely the following result.

Theorem 3.4. *Assume h is bi-harmonic in $\mathbb{R}^n$ and such that $h(x) \leq C(1 + |x|^2)$ for some $C > 0$. Then the Laplacian of h is constant and h is a polynomial of degree less or equal to 2.*

Proof. From Lemma 3.3 and the growth condition on h one finds that for any $x \in \mathbb{R}^n$ there holds

$$\Delta h(x) = 2(n+2) \lim_{R \to +\infty} R^{-2} \fint_{B_R(y)} |h(y)| dy = \Delta h(0) =: 2na,$$

with $a \geq 0$. But then the function $H(x) = h(x) + a|x|^2$ is harmonic and satisfies $H(x) \leq C(1 + |x|^2)$, so the claim follows from the well-known fact that harmonic functions with at most quadratic growth are polynomials. ☐

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Let us choose a maximal number of points $x^{(i)} \in \Omega$, $1 \leq i \leq I$ such that for all indices i and all $R > 0$

$$\liminf_n \int_{B_R(x^{(i)})} V_n e^{4u_n} dx \geq 8\pi^2.$$

From the upper bound on the exponential integrals one has that $I \leq C\Lambda$. Given $x \in \Omega \setminus \cup_{i=1}^{l} \{x^{(i)}\}$ it is possible to find $R > 0$ such that

$$\limsup_n \int_{B_R(x_0)} V_n e^{4u_n} dx < 8\pi^2. \tag{3.3}$$

Let us write $u_n = v_n + h_n$ on $B_R(x_0)$, where

$$\begin{cases} \Delta^2 v_n = V_n e^{4u_n} & \text{in } B_R(x_0); \\ v_n = \Delta v_n = 0 & \text{on } \partial B_R(x_0), \end{cases}$$

and where $\Delta^2 h_n = 0$ in $B_R(x_0)$. By Proposition 3.2 and the upper bound on the exponential integrals we have that

$$\|h_n^+\|_{L^1(B_R(x_0))} \leq \|u_n^+\|_{L^1(B_R(x_0))} + \|v_n\|_{L^1(B_R(x_0))} \leq C. \tag{3.4}$$

Next, we consider the following cases.

Case 1. Assume $\|h_n\|_{L^1(B_{R/2}(x_0))} \leq C$ for all n. Then by Lemma 3.3 we have that, for all n and $x \in B_{R/8}(x_0)$

$$|\Delta h_n(x)| = \left| \fint_{B_{R/8}(x)} \Delta h_n(y) dy \right| \leq C R^{-2} \fint_{B_{R/2}(x_0)} |h_n(z)| dx \leq C;$$

with $(h_n)_n$ locally bounded in $C^4(B_{R/8}(x_0))$. From Lemma 3.3 and the above inequality it follows that

$$\fint_{B_R(x_0)} |h_n(x)| dx \leq C - \fint_{B_R(x_0)} h_n(x) dx = C + \frac{1}{12} R^2 \Delta h_n(x_0) - h_n(x_0) \leq C.$$

Repeating the first step on every ball contained in $B_R(x_0)$ one finds that $(h_n)_n$ is locally bounded in $C^4(B_R(x_0))$. From Proposition 3.2 and (3.3) one deduces

$$\Delta^2 v_n = V_n e^{4u_n} = (V_n e^{4h_n} e^{4v_n})$$

is uniformly bounded in L^p for some $p > 1$. By Proposition 3.2 and by elliptic regularity theory it follows that $(v_n)_n$ is locally bounded in $C^{3,\alpha}(B_R(x_0))$ for any $\alpha \in (0,1)$, so the same holds for $(u_n)_n$.

Case 2. Suppose now that $\beta_n := \|h_n\|_{L^1(B_{R/2}(x_0))} \to +\infty$. Let us consider the normalized function

$$\varphi_n = \frac{h_n}{\|h_n\|_{L^1(B_{R/2}(x_0))}}.$$

Reasoning as for the previous case one deduces that $(\varphi_n)_n$ is locally bounded in $C^4(B_R(x_0))$, and hence up to a subsequence there is convergence in $C^{3,\alpha}_{loc}(B_R(x_0))$ to some biharmonic and normalized (in L^1) φ, which cannot vanish identically.

From (3.4) it follows that $\varphi \leq 0$. By Lemma 3.3 then one finds that $\Delta\varphi \neq 0$ whenever $\varphi = 0$. Consider then the set

$$S_0 = \{x \in B_R(x_0) \mid \varphi(x) = 0\} :$$

this has codimension greater or equal to 1, and hence also vanishing Lebesgue measure; it is also closed and nowhere dense. We deduce that $\varphi < 0$ a.e. and therefore $h_n \to -\infty$ a.e. and locally uniformly away from S_0. Notice also that

$$\Delta^2 v_n = (V_n e^{4h_n} e^{4v_n})$$

is locally bounded in L^p on $B_R(x_0)$ and away from S_0 for some $p > 1$. This implies the boundedness of $(v_n)_n$ in $C^{3,\alpha}$ for any $\alpha \in (0,1)$ on $B_R(x_0)$ and away from S_0.

As a consequence, $u_n \to -\infty$ a.e. and locally uniformly on $\Omega \setminus S_0$, together with the fact that $\frac{u_n}{\beta_n} \to \varphi$.

Since only one among the two cases occurs, using a covering argument and the connectedness of $\Omega \setminus \cup_{i=1}^{I}\{x^{(i)}\}$, we obtain one of the following alternatives. The first is that, up to a subsequence, $(u_n)_n$ is locally bounded in $C^{3,\alpha}$ away from $\cup_{i=1}^{I}\{x^{(i)}\}$, and hence is relatively compact here. The second is that $u_n \to -\infty$ a.e. and locally uniformly on $\Omega \setminus (S_0 \cup \cup_{i=1}^{I}\{x^{(i)}\})$ and $\frac{u_n}{\beta_n}$ converges to φ biharmonic and non-positive.

If there is concentration of volume, only the second alternative can happen. Assuming indeed that there is some concentration point and that $u_n \to u$ in $C^{3,\alpha}_{loc}(\Omega \setminus \cup_{i=1}^{I}\{x^{(i)}\})$, by a result in [39] one would have the distributional convergence

$$V_n e^{4u_n} \rightharpoonup e^{4u} + \sum_{i=1}^{I} m_i \delta_{x^{(i)}}; \qquad m_i \geq 16\pi^2.$$

From the logarithmic asymptotic behaviour of the Green's function it can be shown via representation formulas that

$$u(x) \geq 2\log\left(\frac{1}{|x - x^{(i)}|}\right) - C \qquad \text{near } x^{(i)},$$

which would imply $e^{4u} \notin L^1$, contradicting the integrability assumptions on the exponential functions.

The behaviour of solutions near the concentration points can be studied using some arguments in [40]. Let $x_0 \in S$ be such that $\sup_{B_r(x_0)} u_n \to +\infty$ for every $r > 0$. Given $R < dist(x_0, \partial\Omega)$ let $r_n \in [0, R)$ and $x_n \in \bar{B}_{r_n}(x_0)$ be such that

$$\begin{aligned}
(R - r_n)e^{u_n(x_n)} &= (R - r_n) \sup_{\bar{B}_{r_n}(x_0)} e^{u_n} \\
&= \max_{0 \leq r < R}\left((R - r) \sup_{\bar{B}_r(x_0)} e^{u_n}\right) =: L_n \to +\infty.
\end{aligned}$$

Setting

$$v_n(x) = u_n(x_n + s_n x) + \log s_n; \qquad s_n = \frac{(R - r_n)}{2L_n},$$

one has that

$$\begin{aligned}
\sup_{\bar{B}_{L_n}(x_0)} e^{v_n} &= s_n \sup_{\bar{B}_{(R-r_n)/2}(x_n)} e^{u_n} \\
&\leq s_n \sup_{\bar{B}_{(R+r_n)/2}(x_0)} e^{u_n} = L_n^{-1}\left(R - \frac{R + r_n}{2}\right) \sup_{\bar{B}_{(R+r_n)/2}(x_0)} e^{u_n} \\
&\leq L_n^{-1}(R - r_n)e^{u_n(x_n)} = 1 = 2e^{v_n(0)},
\end{aligned}$$

which is equivalent to (3.2).

Since v_n solves $\Delta^2 v_n = W_n e^{4v_n}$ in domains exhausting $\mathbb{R}^4$ and since

$$W_n(x) = V_n(x_n + s_n x) \to 1 \qquad \text{locally uniformly in } \mathbb{R}^4,$$

and since we have $\int W_n e^{4v_n} dx \leq \Lambda$, applying the previous result on the functions v_n's, we obtain the claimed theorem. □

In [1] an example of blow-up without quantization, even in the radial setting, was shown. More examples of blow-ups were found in [21], to which we refer for oher more recent results in this direction.

We also would like to mention some geometric applications to the study of (3.1), concerning Branson's *Q-curvature*, defined on a four-manifold by

$$Q_g = \frac{1}{12}(-\Delta_g R_g + R_g^2 - 3|Ric_g|_g^2),$$

where R_g is the scalar curvature and Ric_g the Ricci tensor. The latter quantity is a natural higher-order conformal counterpart of the Gaussian curvature, and transforms conformally via the Paneitz operator P_g

$$P_g \psi = \Delta_g^2 \psi - \operatorname{div}\left(\frac{2}{3} R_g \nabla \psi - 2Ric_g(\cdot, \nabla \psi)\right) \tag{3.5}$$

by the law

$$P_g w + 2Q_g = 2Q_{\tilde{g}} e^{4w}, \quad \tilde{g} = e^{2w} g.$$

The principal terms in this formula are analogous to (3.1), and we notice the similarity to the second formula in (1.1).

In [17] and [29] some quantization results for the last equation were proven on closed manifolds. Applications to existence of solutions were given via min-max theory in [15], extending some previous result from [8]. We refer to [30] for a more detailed review of the background and methods.

4. Log-determinant functionals

We discuss next another problem from spectral theory and conformal geometry where Liouville equations appear. Consider a compact closed Riemannian manifold (M^n, g): by *Weyl's asymptotic formula* the eigenvalues λ_j of $-\Delta_g$ behave asymptotically as $\lambda_j \sim j^{2/n}$ for $j \to \infty$. The *determinant* of $-\Delta_g$ is formally the product of all its eigenvalues, with a rigorous definition that can be obtained via holomorphic extension of the *zeta function*

$$\zeta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}.$$

Weyl's asymptotic law implies that $\zeta(s)$ is analytic for $\mathrm{Re}(s) > n/2$: one can anyway meromorphically extend ζ so that it becomes regular near $s = 0$ (see e.g. [38]). From the formal calculation $\zeta'(0) = -\sum_{j=1}^{\infty} \log \lambda_j = -\log \det(-\Delta_g)$ one then defines $\zeta'(0)$ as

$$\det(-\Delta_g) = e^{-\zeta'(0)}.$$

The transformation laws in (1.1) allowed Polyakov ([37]) to compute the logarithm of the ratio of determinants of two conformally-equivalent metrics of the same volume by the following expression

$$\log \frac{\det(-\Delta_{\tilde{g}})}{\det(-\Delta_g)} = -\frac{1}{12\pi} \int_\Sigma (|\nabla w|_g^2 + 2K_g w)\ dv_g. \tag{4.1}$$

By this formula, critical points of the regularized determinant in a given conformal class produce constant Gaussian curvature metrics. In [35, 34] Osgood, Phillips and Sarnak proved variationally existence of conformal extremals for all given genuses, with uniqueness holding for non-positive Euler characteristic and up to Möbius transformations on the sphere. Still in [35, 34], the authors used formula (4.1) in order to prove compactness of isospectral metrics on every closed surface.

In four dimension formulas similar to (4.1) were obtained for determinants of *conformally covariant operators*, enjoying covariance properties analogous to the first equation in (1.1). A differential operator A_g is *conformally covariant of bi-degree* (a, b) if

$$A_{\tilde{g}}\psi = e^{-bw} A_g(e^{aw}\psi), \quad \tilde{g} = e^{2w} g, \tag{4.2}$$

for each smooth function ψ. One well-known example is the *conformal Laplacian* in dimension $n \geq 3$

$$L_g = -\Delta_g + \frac{(n-2)}{4(n-1)} R_g :$$

this operator satisfies (4.2) with $a = \frac{n-2}{2}$ and $b = \frac{n+2}{2}$. Other examples include the *Dirac operator* $\not{D}_g$, which satisfies (4.2) with $a = \frac{n-1}{2}$, $b = \frac{n+1}{2}$, and the *Paneitz operator* in four dimensions, with $a = 0$ and $b = 4$.

Branson and Oersted generalized in [4] Polyakov's formula to 4-manifolds (M, g): they showed that the logarithmic ratio of two determinants for general conformally covariant operators is the linear combination of three universal functionals, with coefficients depending on the specific operator. More precisely, if $A = A_g$ is conformally covariant and has no kernel (otherwise, the formula is more involved), then one has

$$F_A[w] = \log \frac{\det A_{\tilde{g}}}{\det A_g} = \gamma_1(A) I[w] + \gamma_2(A) II[w] + \gamma_3(A) III[w], \quad \tilde{g} = e^{2w} g, \tag{4.3}$$

where $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$ and I, II, III are defined by

$$I[w] = 4 \int_M w |W_g|_g^2\ dv_g - \Big(\int_M |W_g|_g^2\ dv_g\Big) \log \fint_M e^{4w}\ dv_g$$
$$II[w] = \int_M w P_g w\ dv_g + 4 \int_M Q_g w\ dv_g - \Big(\int_M Q_g\ dv_g\Big) \log \fint_M e^{4w}\ dv_g$$
$$III[w] = 12 \int_M (\Delta_g w + |\nabla w|_g^2)^2\ dv_g - 4 \int_M (w \Delta_g R_g + R_g |\nabla w|_g^2)\ dv_g.$$

Here W_g is the Weyl curvature tensor, and Q_g the Q-curvature. The three functionals I, II, III have a geometric meaning, since their critical points yield metrics with constant norm of the Weyl curvature, Q-curvature and scalar curvature respectively.

The Euler-Lagrange equation for F_A, i.e. a linear combination of I, II, and III, yields constant U_g-*curvature*, which is defined as

$$U_g = \gamma_1 |W_g|_g^2 + \gamma_2 Q_g - \gamma_3 \Delta_g R_g. \tag{4.4}$$

The Euler-Lagrange equation for the conformal factor becomes

$$\begin{aligned} &\mathcal{N}_g(w) + U_g = \mu e^{4w}; \\ &\mathcal{N}(w) = \frac{\gamma_2}{2} P_g w + 6\gamma_3 \Delta_g (\Delta_g w + |\nabla w|_g^2) \\ &-12\gamma_3 \mathrm{div}\big[(\Delta_g w + |\nabla w|_g^2)\nabla w\big] + 2\gamma_3 \mathrm{div}(R_g \nabla w), \end{aligned} \tag{4.5}$$

where

$$\mu = -\frac{\kappa_A}{\int_M e^{4w} dv_g}; \qquad \kappa_A = -\gamma_1 \int_M |W_g|_g^2 \, dv_g - \gamma_2 \int_M Q_g \, dv_g.$$

We obtain therefore a Liouville type equation of higher and mixed order, but with some *principal terms* that have the same scaling law.

Our aim in this section is to describe a compactness result from [18], which is in the spirit of Theorem 1.3 and 2.1 but on closed manifolds. The result is the following.

Theorem 4.1 ([18])**.** *Suppose M is a compact 4-manifold and that $\gamma_2, \gamma_3 \neq 0$, with $\frac{\gamma_2}{\gamma_3} \geq 6$. Suppose also that $(w_n)_n$ is a sequence of smooth solutions of*

$$\mathcal{N}_g(w_n) + \tilde{U}_n = \mu_n e^{4w_n} \qquad \text{on } M,$$

where $\mathcal{N}_g$ is given by (4.6). Assume that $\int_M e^{4w_n} dv_g = 1$, $\mu_n = \int_M \tilde{U}_n dv_g$ and $\tilde{U}_n \to U_g$ C^1-uniformly in M as $n \to +\infty$. Up to a subsequence, we have one of the following two alternatives:

i) *$(w_n - \fint_M w_n \, dv_g)_n$ is uniformly bounded in $C^{4,\alpha}(M)$-norm;*

ii) *$(w_n)_n$ blows up, i.e. $\max_M w_n \to +\infty$, $\fint_M w_n \, dv_g \to -\infty$ and*

$$\mu_n e^{4w_n} \rightharpoonup \sum_{i=1}^{l} 8\pi^2 \gamma_2 \delta_{p_i}$$

in the weak sense of distributions for distinct points $p_1, \dots, p_l \in M$.

As a consequence, solutions stay compact if $\int_M U_g dv_g \notin 8\pi^2 \gamma_2 \mathbb{N}$.

Being the operator of mixed type, the proof of the above result is quite involved. Some basic tools from the previous sections such as the maximum principle or the Green's representation formula do not hold in this case, hence some new ideas need to be devised. We will not present detailed arguments as in the previous sections, but we will only list the main steps involved in the proof. A related quantization result was proved in [19] for a Liouville n-Laplace equation: being this of second order truncation techniques were possible for finding a-priori estimates.

Step 1: preliminary estimates. First, one can prove estimates on *non-optimal* Sobolev norms on solutions via sublinear cut-offs functions, namely that the following result holds true.

Proposition 4.2. *Let $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. Assume $\overline{f} = 0$ and $\|f\|_1 \le C_0$ for some $C_0 > 0$. Then there exists $C > 0$ so that*

$$\int_M \frac{(\Delta w)^2 + |\nabla w|^4}{[1 + (w - \overline{w})^2]^{\frac{2}{3}}} dv_g \le C \tag{4.6}$$

for every smooth solution w of $\mathcal{N}(w) = f$ in M. Moreover, given $1 \le q < 2$ there exists $C > 0$ so that

$$\|w - \overline{w}\|_{W^{2,q}} \le C \tag{4.7}$$

for any such solution w.

The condition $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$ allows to achieve the above estimates via an integration by parts. Once these are established, one can also control the mean oscillation

$$[w]_{BMO} = \left(\sup_{0<r<i_0} \fint_{B_r} (w - \overline{w}^r)^4 dv_g \right)^{\frac{1}{4}}$$

of solutions. For proving the next result we crucially relied on ideas from [16], where Caccioppoli-type estimates were employed.

Proposition 4.3. *Let $\frac{\gamma_2}{\gamma_3} > \frac{3}{2}$. Assume $\overline{f} = 0$ and $\|f\|_1 \le C_0$ for some $C_0 > 0$. There exists $C > 0$ such that for any smooth solution w of $\mathcal{N}(w) = f$ in M one has*

$$[w]_{BMO} \le C. \tag{4.8}$$

Step 2: *Linear* theory. We assume from now on that $\frac{\gamma_2}{\gamma_3} \ge 6$: this condition yields a *formal convexity* for the principal terms in F_A, and was also used in [8] for the equality case to prove uniqueness in some cases.

Let $\mathcal{M} = \{\mu \text{ Radon measure in } M : \ \mu(M) = 0\}$. For $\mu \in \mathcal{M}$ we say that a distributional solution w of $\mathcal{N}(w) = \mu$ in M is a SOLA if $w = \lim\limits_{n \to +\infty} w_n$ a.e., where w_n are smooth solutions of $\mathcal{N}(w_n) = f_n$ with $f_n \in C^\infty(M)$, $\overline{w}_n = \overline{f}_n = 0$ and $f_n dv \rightharpoonup \mu$ as $n \to +\infty$.

Let $L^{\theta,q)}(M, TM)$ be the *grand Lebesgue space* of all vector fields $F \in \bigcup\limits_{1 \le \tilde{q} < q} L^{\tilde{q}}(M, TM)$ with

$$\|F\|_{\theta,q)} = \sup_{0<\epsilon\le\epsilon_0} \epsilon^{\frac{\theta}{q}} \|F\|_{q(1-\epsilon)} < +\infty$$

and $W^{\theta,2,2)}$ be the *grand Sobolev space*

$$W^{\theta,2,2)} = \{w \in W^{2,1}(M) : \overline{w} = 0, \|w\|_{W^{\theta,2,2)}} := \|\Delta w\|_{\theta,2)} + \|\nabla w\|_{\theta,4)} < +\infty\}. \tag{4.9}$$

With these definitions at hand, using a nonlinear Hodge decomposition from [20, 22, 23] one can prove the following result.

Proposition 4.4. *Let* $\frac{\gamma_2}{\gamma_3} \geq 6$*, let* $\frac{2}{3} \leq \theta < \frac{4}{3}$ *and assume the inequality* $\eta = |\gamma_2 - 6\gamma_3| \sup_M(|R| + \|Ric\|)$*. There exists* $C > 0$ *such that*

$$\begin{aligned} \|w_1 - w_2\|_{W^{\theta,2,2)}} \quad \leq \quad & C\|F_1 - F_2\|_{\theta,\frac{4}{3})}^{\frac{4-3\theta}{6}} (\|F_1\|_{\theta,\frac{4}{3})} + \|F_2\|_{\theta,\frac{4}{3})} + 1)^{\frac{\theta}{2}} \\ & + C\|F_1 - F_2\|_{\theta,\frac{4}{3})}^{\frac{4-3\theta}{12}} (\|F_1\|_{\theta,\frac{4}{3})} + \|F_2\|_{\theta,\frac{4}{3})} + 1)^{\frac{4+3\theta}{12}} \\ & + \eta(\|F_1\|_{\theta,\frac{4}{3})} + \|F_2\|_{\theta,\frac{4}{3})} + 1)^{\frac{1}{3}} \\ & \times O(\|\nabla(w_1 - w_2)\|_2 + \|\nabla(w_1 - w_2)\|_2^{\frac{1}{4}}) \end{aligned} \tag{4.10}$$

for all SOLA's w_1*,* w_2 *of* $\mathcal{N}(w_1) = \mu_1 \in \mathcal{M}$*,* $\mathcal{N}(w_2) = \mu_2 \in \mathcal{M}$*, where* $F_1 = \nabla\Delta^{-1}(\mu_1)$ *and* $F_2 = \nabla\Delta^{-1}(\mu_2)$*.*

The above result applies also when one of the w_i's is a distributional solution which has a logarithmic behaviour near a finite number of points. Such profiles arise when the data on the right-hand side are finite sums of Dirac masses.

We next state an existence result, which is proved via approximation of the right-hand side by smooth functions: for such more regular data, existence follows from variational principles.

Theorem 4.5. *Let* $\frac{\gamma_2}{\gamma_3} \geq 6$*. For any* $\mu \in \mathcal{M}$ *there exists a SOLA* w *of* $\mathcal{N}(w) = \mu$ *in* M *such that* $w \in W^{1,2,2)}$*. When* $\gamma_2 = 6\gamma_3$ *such a SOLA is unique.*

Step 3: fundamental solutions. This step consists in constructing approximate solutions for right-hand sides that are linear combinations of Dirac masses, and to determine that *fundamental solutions* have a a prescribed logarithmic behaviour near the poles.

Let $\mu_s = \displaystyle\sum_{i=1}^{l} \beta_i \delta_{p_i}$ be a linear combination of Dirac masses centred at distinct points $p_1, \dots, p_l \in M$. Given U as in (4.4), the coefficients $\beta_1, \dots, \beta_l \neq 0$ are chosen to satisfy

$$\sum_{i=1}^{l} \beta_i = \int_M U dv_g. \tag{4.11}$$

In $\mathbb{R}^4$ it can be shown via an integration by parts that the function $w_\alpha = \alpha \log |x|$ satisfies $\mathcal{N} w_\alpha = \beta_i \delta_0$ if $\alpha = \alpha(\beta_i) \neq 0$ is the unique solution of

$$-4\pi^2[(\gamma_2 + 12\gamma_3)\alpha + 18\gamma_3\alpha^2 + 6\gamma_3\alpha^3] = \beta_i. \tag{4.12}$$

For this reason, the function

$$w_0(x) = \sum_{i=1}^{l} \alpha_i \log \tilde{d}(x, p_i) \tag{4.13}$$

is an approximate solution of $\mathcal{N}(w) = \sum_{i=1}^{l} \beta_i \delta_{p_i} - U$ in M, where $\tilde{d}(x, p_i)$ stands for the distance function on M, smoothed away from p_i. More precisely, the function w_0 in (4.13) is a distributional solution of

$$\mathcal{N}(w_0) = \sum_{i=1}^{l} \beta_i \delta_{p_i} + f_0 \tag{4.14}$$

with $f_0 - \gamma_2 \mathrm{div}[\mathrm{Ric}(\cdot, \nabla w_0)] - (2\gamma_3 - \frac{\gamma_2}{3}) \mathrm{div}(R \nabla w_0) \in L^\infty(M)$. By the comment after Proposition 4.4, we can obtain asymptotic uniqueness of solutions when the right-hand side of the equation is μ_s.

Theorem 4.6. *Let $\frac{\gamma_2}{\gamma_3} \geq 6$. Then any fundamental solution w_s with right-hand side equal to μ_s satisfies $w_s \in C^\infty(M \setminus \{p_1, \ldots, p_l\})$ and has the same asymptotics as in* (4.13)*, with α_i given by* (4.12).

Step 4: blow-up analysis. We next consider a sequence of solutions w_n as in Theorem 4.1, aiming to show that if concentration occurs then the right-hand sides converge to a purely atomic measure, supported at finitely-many points.

Assume without loss of generality that $\int_M e^{4w_n} dv_g = 1$ for all n. Since $e^{4\overline{w}_n} \leq \frac{1}{\mathrm{vol} M}$ by Jensen's inequality, up to a subsequence assume that $\overline{w}_n \to c \in [-\infty, +\infty)$ as $n \to +\infty$. Since $e^{4w_n} \to e^{4w_0 + 4c}$ locally uniformly in $M \setminus S$, we have that

$$e^{4w_n} \rightharpoonup e^{4w_0+4c} dv + \sum_{i=1}^{l} \tilde{\beta}_i \delta_{p_i} \qquad \text{as } n \to +\infty$$

weakly in the sense of measures, where $S = \{p_1, \ldots, p_l\}$ and $\tilde{\beta}_i \geq \frac{\epsilon_0}{|\mu_0|}$. The function w_0 turns out to be a SOLA of

$$\mathcal{N}(w_0) = \mu_0 e^{4w_0+4c} + \sum_{i=1}^{l} \beta_i \delta_{p_i} - U \qquad \text{in } M \tag{4.15}$$

for $\beta_i = \mu_0 \tilde{\beta}_i$. Via a Pohozaev-type identity one can show the next Lemma 4.7.

Consider a sequence of solutions u_n to

$$\mathcal{N}_n(u_n) + U_n = \mu_n e^{4u_n} \qquad \text{in } B_r. \tag{4.16}$$

We assume that $\mu_n \to \mu_0$, that for some $(c_n)_n$

$$\sup_n \int_{B_r} e^{4u_n} dv_{g_n} < +\infty, \qquad \sup_n \int_{B_r} (u_n - c_n)^4 dv_{g_n} < +\infty, \tag{4.17}$$

and

$$U_n \to U_\infty \text{ in } C^1(\overline{B_r}), \qquad g_n \to g_\infty \text{ in } C^4(\overline{B_r}) \tag{4.18}$$

for some $U_\infty \in C^\infty(\overline{B_r})$ and a metric g_∞ defined on B_r. The second inequality in (4.17) will be in particular guaranteed by Proposition 4.3.

Lemma 4.7. *Let u_n be a solution of* (4.16) *which satisfies* (4.17)-(4.18) *in B_r. Suppose that*

$$\mu_n e^{4u_n} dv_{g_n} \rightharpoonup \beta\, \delta_0 \tag{4.19}$$

weakly in the sense of measures in B_r as $n \to +\infty$, for some $\beta \neq 0$. Then $\beta = 8\pi^2 \gamma_2$.

Arguing somehow as in Step 3 in the proof of Theorem 1.3, one can prove the following result.

Lemma 4.8. *In the above notation, there holds $c = -\infty$.*

Once this is established, the proof of Theorem 4.1 follows from Lemma 4.7.

Theorem 4.1 was used in [18] to find existence results for functionals F_A in general form via min-max theory, including determinants of the conformal Laplacian and and the squared Dirac operator. These extended previous results in [8] where direct methods of the calculus of variations were used.

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