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EXPONENTIAL INTEGRABILITY IN THE SPIRIT OF MOSER-TRUDINGER'S INEQUALITIES OF FUNCTIONS WITH FINITE NON-LOCAL, NON-CONVEX ENERGY

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ABSTRACT. Let $d \geq 1$, $p \geq d$, and let Ω be a smooth bounded open subset of $\mathbb{R}^d$. We prove some exponential integrability in the spirit of Moser-Trudinger's inequalities for measurable functions u defined in Ω such that

$$\int_\Omega \int_{\substack{\Omega \\ |u(x)-u(y)|>\delta}} \frac{1}{|x-y|^{d+p}} \, dx \, dy < +\infty,$$

for some $\delta > 0$. This double integral appeared in characterizations of Sobolev spaces and involved in improvements of the Sobolev inequaliies, Poincaré inequalities, and Hardy inequalities.

1. INTRODUCTION

Let (ρ_n) be a sequence of non-negative radial functions satisfying
(1.1)
$$\lim_{n\to\infty} \int_\tau^\infty \rho_n(r) r^{N-1} \, dr = 0 \quad \forall \, \tau > 0, \quad \text{and} \quad \lim_{n\to\infty} \int_0^{+\infty} \rho_n(r) r^{N-1} \, dr = 1.$$

Set

(1.2)
$$K_{d,p} := \int_{\mathbb{S}^{d-1}} |\sigma \cdot e|^p \, d\sigma,$$

for some $e \in \mathbb{S}^{d-1}$, the unit sphere in $\mathbb{R}^d$.

Jean Bourgain, Haim Brezis, and Petru Mironescu [10, Theorems 1 and 2] (see also [11] and [8]) proved the following BBM formula:

Proposition 1.1. *Let $d \geq 1$, $p > 1$ and let Ω be a smooth bounded open subset in $\mathbb{R}^d$ or $\Omega = \mathbb{R}^d$. Assume that $g \in L^p(\Omega)$ and let (ρ_n) satisfy (1.1). Then $g \in W^{1,p}(\mathbb{R}^d)$ if and only if*

$$\liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|) \, dx \, dy < +\infty.$$

Moreover, for $g \in W^{1,p}(\Omega)$,

$$\lim_{n \to \infty} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dx \, dy = K_{d,p} \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx.$$

A variant of Propposition 1.1 for $p = 1$ involving functions of bounded variations was obtained by Jean Bourgain, Haim Brezis, and Petru Mironescu [10] and Juan Davila [21]. Further studies related to this characterizations can be founded in [2, 6, 16, 17, 20, 25, 35, 38, 39, 42, 43, 44].

We next discuss another characterization of Sobolev spaces in the spirit of the BBM formula. To this end, for $d \geq 1$, $p \geq 1$, and $\delta > 0$, for a measurable subset O of $\mathbb{R}^d$, and for a measurable function u defined in O, set

$$(1.3) \qquad I_{\delta,p}(u, O) = \int_O \int_{\substack{O \\ |u(x) - u(y)| > \delta}} \frac{\delta^p}{|x - y|^{d+p}} \, dx \, dy.$$

This quantity has its root in estimates for the topological degree in [13, 12, 29, 34, 45] which has the motivation from the study of the Ginzburg Landau equation [9].

It was shown [28, Theorems 2 and 5] and [7, Theorem 1] that

Proposition 1.2. *Let $d \geq 1$ and Ω be a smooth bounded open subset in $\mathbb{R}^d$ or $\Omega = \mathbb{R}^d$ and let $p > 1$ and $g \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$ if and only if*

$$\liminf_{\delta \to 0} I_{\delta,p}(u, \Omega) < +\infty.$$

Moreover, for $g \in W^{1,p}(\Omega)$,

$$(1.4) \qquad \lim_{\delta \to 0} I_{\delta,p}(u, \Omega) = \frac{1}{p} K_{d,p} \int_\Omega |\nabla u(x)|^p \, dx,$$

where $K_{d,p}$ is defined by (1.2). We also have, for all $\delta > 0$,

$$(1.5) \qquad I_{\delta,p}(u, \Omega) \leq C_{d,p} \int_{\mathbb{R}^N} |\nabla u(x)|^p \, dx \quad \forall u \in W^{1,p}(\Omega),$$

for some positive constant $C_{d,p}$ depending only on d and p.

The case $p = 1$ is more delicate. One has [28, Theorem 8] (see also [18, Proposition 2]), for $u \in W^{1,1}(\Omega)$,

$$\liminf_{\delta \to 0} I_{\delta,1}(u, \Omega) \geq K_{d,1} \int_\Omega |\nabla u| \, dx$$

and (see [28, Theorem 8] and [7, Theorem 1]) that $u \in BV(\Omega)$ provided that $u \in L^1(\Omega)$ and $\liminf_{\delta \to 0} I_{\delta,1}(u, \Omega) < +\infty$. Let B_r denote the ball centered at 0 and of radius r. An example due to Augusto Ponce presented in [28] showed that there exists $u \in W^{1,1}(B_1)$ such that $\lim_{\delta \to 0} I_{\delta,1}(u, B_1) = +\infty$. When $d = 1$, there exists $u \in W^{1,1}(0, 1)$ [18, Pathology 2] such that

$$K_{1,1} \int_0^1 |\nabla u| \, dx = \liminf_{\delta \to 0} I_{\delta,1}(u, (0, 1)) < \limsup_{\delta \to 0} I_{\delta,1}(u, (0, 1)) = +\infty.$$

It turns out that the concept of Γ-convergence fits very well this setting. It was shown [30, 32] that the Γ-limit exists for $p \geq 1$. Surprisingly, the Γ-limit, which is positive, is strictly less than the pointwise limit [32, 30]. The quantity $I_{\delta,1}$ has a

similar form with non-local filters using in denoising process [19], in particular with Yaroslavsky's ones [47, 48]. A discussion on a connection between nonlocal filters using $I_{\delta,1}$ and local ones involving the total variations via the Γ-convergence theory is given in [18, Section 5.2]. Further interesting investigations related to the Γ-limit of $I_{\delta,p}$ are given in [3, 4, 5, 18].

One can obtain new and improved variants of Poincaré's inequality, Sobolev's inequality and Rellich-Kondarachov's compactness criterion using the information of $I_{\delta,p}$ instead of the one of the gradient [33, Theorems 1, 2, and 3]. Concerning the Sobolev inequality, one has

Proposition 1.3. *Let $1 < p < d$ and set $q = dp/(d - p)$ and fix $\delta > 0$ arbitrary. We have, for $u \in L^p(\mathbb{R}^d)$,*

$$(1.6) \qquad \left(\int_{|u| > \lambda \delta} |u|^q \right)^{1/q} \leq C \left(I_{\delta,p}(u, \mathbb{R}^d) \right)^{1/p},$$

for some positive constants λ and C independent of u.

Concerning the Poincaré inequality, one obtains

Proposition 1.4. *Let $d \geq 1$, $p \geq 1$, $\delta > 0$, let B be an open ball of $\mathbb{R}^d$, and let $u \in L^p(B)$. There exists a positive constant $C_{d,p}$ depending only on d and p such that*

$$(1.7) \qquad \int_B \int_B |u(x) - u(y)|^p dx dy \leq C_{d,p} \left(|B|^{\frac{d+p}{d}} I_{\delta,p}(u, B) + \delta^p |B|^2 \right).$$

The proof of Sobolev's inequality (1.6) is based on the one of Poincaré's inequality (1.7) and uses the theory of sharp functions due to Charles Fefferman and Elias Stein [23] and the method of truncation due to Vladimir Mazya [26]. The proof of Poincaré's inequality (1.7) has its roots in [7] and uses John-Nirenberg's inequality [24].

Remark 1.1. For a measurable function defined in B, by applying (1.7) for u_k with
$u_k = \min \left\{ k, \max\{u, -k\} \right\}$ and letting $k \to +\infty$, one also obtains (1.7) for measurable functions.

With Marco Squassina, the second author also established new and improved variants of Hardy and Caffarelli, Kohn, Nirenberg's inequality [35] using the quantity $I_{\delta,p}$. The approach used in [35] does not involve the integration-by-parts arguments and can be extended for the fractional Sobolev spaces [36]. Other investigations related to $I_{\delta,p}$ can be found in [14, 18, 31, 33, 36, 37, 38].

Let Ω be a smooth bounded open subset of $\mathbb{R}^d$ and $p \geq d$. It follows from (1.7) that $u \in BMO(\Omega)$ provided that $u \in L^1(\Omega)$ and $I_{\delta,p}(u, \Omega) < +\infty$. More precisely, one has, for $p \geq d$,

$$\|u\|_{BMO(\Omega)} \leq C_\Omega \left(I_{\delta,p}(u, \Omega) + \delta^d \right),$$

where

$$(1.8) \qquad \|u\|_{BMO(\Omega)} := \sup_{\text{ball } B \subset \Omega} \frac{1}{|B|} \int_B |u - u_B| \, dx.$$

Here, for a given a measurable set O of $\mathbb{R}^d$ and a function $u \in L^1(O)$, one sets

$$(1.9) \qquad |O| := \mathrm{meas}(O) \quad \text{and} \quad u_O = \fint_O u\,dx \text{ with } \fint_O u\,dx := \frac{1}{|O|}\int_O u\,dx.$$

One can then derive the exponential integrability of u from John-Nirenberg's inequality:

$$(1.10) \qquad \fint_B e^{c|u-u_B|/\|u\|_{BMO(B)}} \leq C,$$

for some positive constant c and C depending only on d and for any open ball B.

Using the Poincaré inequality, one can prove that $u \in W^{1,p}(\Omega)$ then $u \in BMO(\Omega)$, this yields the exponential integrability of u in (1.10). In fact, for $u \in W^{1,p}(\Omega)$ with $p \geq d$, one can improve (1.10). First, Morrey's inequality (see, e.g., [15]) states that $u \in C^\alpha(\Omega)$ with $\alpha = 1 - d/p$ if $u \in W_0^{1,p}(\Omega)$ for $p > d$. Second, Moser-Trudinger's inequality [27, 46, 40, 41] confirms that

$$\sup_{\|u\|_{W_0^{1,d}(\Omega)} \leq 1} \int_\Omega e^{\alpha|u|^{d/(d-1)}} \leq C,$$

for some positive constants α and C depending only on Ω.

The goal of this paper is to understand whether or not a better integrability property of u than (1.10) inequality holds when $u \in L^p(\Omega)$ and $I_{\delta,p}(u,\Omega) < +\infty$. It is worth noting that, for all $\delta > 0$, there exists $u \in L^\infty(\Omega) \setminus C(\bar{\Omega})$ such that $I_{\delta,p}(u,\Omega) = 0$ for all $p \geq 1$. A simple example is the function $u = \delta \mathbb{1}_B$ in Ω, for some ball $B \Subset \Omega$, where $\mathbb{1}_O$ denotes the characteristic function of a subset O of $\mathbb{R}^d$. One can also show that there exists a function u such that $I_\delta(u,\Omega) < +\infty$ and $u \notin L^\infty(\Omega)$. An example for this is the function $u(x) = (\ln \lambda)^{-1} \ln \ln |x|^{-1}$ for $x \in B_{1/e}$ and $\lambda > p/d$ (the verification is given in Section 3).

In this work, we address the gap between the exponential integrability (1.10) and the boundedness for functions u with $I_{\delta,p}(u,\Omega) < +\infty$ for some $\delta > 0$ and $p \geq d$. Our first result is

Theorem 1.1. *Let $p > d \geq 1$, $\delta > 0$, and let B be a an open ball of $\mathbb{R}^d$. We have,*
   i) for $M > 0$ and $\alpha > 0$, there exists a constant $0 \leq \beta = \beta(\alpha, M) \leq 1$ depending only on M and α such that

$$(1.11) \qquad \sup_{|B|^{\frac{p-d}{d}} \delta^{-p} I_{\delta,p}(u,B) \leq M} \fint_B e^{\alpha\left(\frac{p}{d}\right)^{\beta\delta^{-1}|u-u_B|}} dx \leq C,$$

   ii) given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α, d, and p such that

$$(1.12) \qquad \sup_{|B|^{\frac{p-d}{d}} \delta^{-p} I_{\delta,p}(u,B) \leq M_0} \fint_B e^{\alpha\left(\frac{p}{d}\right)^{\delta^{-1}|u-u_B|}} dx \leq C.$$

Here C denotes a positive constant depending only on d, p, and α.

As a consequence of Theorem 1.1, we obtain

Proposition 1.5. *Let $p > d \geq 1$, $\delta > 0$, and let Ω be a smooth bounded open subset of $\mathbb{R}^d$. We have,*

i) for $M > 0$ and $\alpha > 0$, there exists a constant $0 \leq \beta = \beta(\alpha, M) \leq 1$ depending only on α and M such that

$$\sup_{\delta^{-p} I_{\delta,p}(u,\Omega) \leq M} \int_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\beta\delta^{-1}|u|}} dx \leq C_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\beta\delta^{-1}\|u\|_{L^1(\Omega)}}},$$

ii) given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α, d, p, and Ω such that

$$\sup_{\delta^{-p} I_{\delta,p}(u,\Omega) \leq M_0} \int_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\delta^{-1}|u|}} dx \leq C_\Omega e^{\alpha \left(\frac{p}{d}\right)^{\delta^{-1}\|u\|_{L^1(\Omega)}}}.$$

Here C_Ω denotes a positive constant depending only on d, p, α, and Ω.

Here is a variant of *ii)* of Theorem 1.1.

Theorem 1.2. *Let $p = d \geq 1$, $\delta > 0$, and let B be a an open ball of $\mathbb{R}^d$. Given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α and d such that*

$$(1.13) \qquad \sup_{\delta^{-d} I_{\delta,p}(u,B) \leq M_0} \fint_B e^{\alpha \delta^{-1}|u - u_B|} dx \leq C,$$

for some positive constant C depending only on d and α.

Remark 1.2. Inequality (1.13) shares some similarities with John-Nirenberg's inequality but is different. In fact, fixing $\delta > 0$, as a consequence of (1.7), we have

$$\|u\|_{BMO(B)} \leq C(M + \delta),$$

if $I_{\delta,p}(u) \leq M$. Hence $\|u\|_{BMO(B)}$ does not generally converge to 0 and (1.13) cannot be derived from (1.10).

As a consequence of Theorem 1.2, we have

Proposition 1.6. *Let $p = d \geq 1$, $\delta > 0$, and let Ω be a smooth bounded open subset of $\mathbb{R}^d$. Given $\alpha > 0$, there exists a positive constant M_0 (small) depending only on α, d, and Ω such that*

$$\sup_{\delta^{-p} I_{\delta,p}(u,\Omega) \leq M_0} \int_\Omega e^{\alpha \delta^{-1}|u|} dx \leq C_\Omega e^{\alpha \delta^{-1}\|u\|_{L^1(\Omega)}},$$

for some positive constant C_Ω depending only on d, α, and Ω.

The exponential growths in (1.12) and (1.13) are optimal. In fact, we have

Proposition 1.7. *Let $p \geq d \geq 1$, $\gamma > p/d$, and $\alpha > 0$, and let B be a an open ball of $\mathbb{R}^d$.*

i) If $p > d$ then for any $M > 0$ there exists $u \in L^1(B)$ such that

$$(1.14) \qquad I_{\delta,p}(u, B) \leq M \quad and \quad \int_B e^{\alpha \gamma^{\delta^{-1}|u - u_B|}} dx = +\infty.$$

i) If $p = d$ then there exists a bounded sequence $(u_n) \subset L^1(B)$ such that

$$(1.15) \qquad \lim_{n \to +\infty} I_{\delta,p}(u_n, B) = 0 \quad and \quad \lim_{n \to +\infty} \int_B e^{\alpha \left(\delta^{-1}|u_n - u_{nB}|\right)^\gamma} dx = +\infty.$$

2. Proofs of Theorems 1.1 and 1.2

This section contains the proof of the Theorems 1.1 and 1.2. We first establish two lemmas used in the proof of (1.11), (1.12), and (1.13) and then establish Theorems 1.1 and 1.2.

2.1. **Two useful lemmas.** For $x \in \mathbb{R}^d$ and $\rho > 0$, let $B_\rho(x)$ denote the ball in $\mathbb{R}^d$ centered at x and of radius ρ. We have

Lemma 2.1. *Let $d \geq 1$, $\lambda > 0$, and let $E \subset F \subset \mathbb{R}^d$ be measurable subsets of $\mathbb{R}^d$ with $0 < |E| < |F| < \infty$. Let $\rho > 0$ be such that $|E| = |B_\rho|$ and let $x \in \mathbb{R}^d$ be such that $B_{2\rho}(x) \subset F$. Then*

$$(2.1) \qquad \int_{F \setminus E} \frac{dy}{|x - y|^\lambda} \geq C_{d,\lambda} |E|^{1 - \frac{\lambda}{d}},$$

for some positive constant $C_{d,\lambda}$ depending only on d and λ. As a consequence, if $p \geq 1$, $|E| = |B_\rho|$ for some $\rho > 0$ and, for $p \geq 1$, D is measurable subset of F such that for almost every $x \in D$, the ball $B_{2\rho}(x) \subset F$, then

$$(2.2) \qquad \int_D \int_{F \setminus E} \frac{dy\,dx}{|x - y|^{d+p}} \geq C_{d,p}|D|\,|E|^{-\frac{p}{d}},$$

for some positive constant $C_{d,p}$ depending only on d and p.

Proof. For $y \in \mathbb{R}^d$, we have

$$B_\rho(y) = \big(B_\rho(y) \setminus E\big) \cup \big(B_\rho(y) \cap E\big) \text{ and } E = \big(E \setminus B_\rho(y)\big) \cup \big(E \cap B_\rho(y)\big).$$

Since $|E| = |B_\rho(y)|$, it follows that

$$(2.3) \qquad |E \setminus B_\rho(y)| = |B_\rho(y) \setminus E| \text{ for } y \in \mathbb{R}^d.$$

Fix x such that $B_{2\rho}(x) \subset F$. We have

$$\int_{F \setminus E} \frac{dy}{|y - x|^\lambda} = \int_{(F \setminus E) \cap B_\rho(x)} \frac{dy}{|y - x|^\lambda} + \int_{(F \setminus E) \cap \big(F \setminus B_\rho(x)\big)} \frac{dy}{|y - x|^\lambda}$$

$$(2.4) \qquad \geq \frac{1}{\rho^\lambda}\,|(F \setminus E) \cap B_\rho(x)| + \int_{(F \setminus E) \cap \big(F \setminus B_\rho(x)\big)} \frac{dy}{|y - x|^\lambda}.$$

Then

$$(2.5) \quad |(F \setminus E) \cap B_\rho(x)| \overset{B_\rho(x) \subset F}{=} |B_\rho(x) \setminus E| \overset{(2.3)}{=} |E \setminus B_\rho(x)| \overset{E \subseteq F}{=} |(F \setminus B_\rho(x)) \cap E|.$$

Combining (2.4) and (2.5) yields

$$\int_{F \setminus E} \frac{dy}{|y - x|^\lambda} \geq \frac{1}{\rho^\lambda}\,|(F \setminus B_\rho(x)) \cap E| + \int_{(F \setminus E) \cap \big(F \setminus B_\rho(x)\big)} \frac{dy}{|y - x|^\lambda}.$$

This yields

$$\int_{F \setminus E} \frac{dy}{|y - x|^\lambda} \geq \int_{\big(F \setminus B_\rho(x)\big) \cap E} \frac{dy}{|y - x|^\lambda} + \int_{(F \setminus E) \cap \big(F \setminus B_\rho(x)\big)} \frac{dy}{|y - x|^\lambda}$$

$$\overset{E \subseteq F}{\geq} \int_{F \setminus B_\rho} \frac{dy}{|y - x|^\lambda} \overset{B_{2\rho}(x) \subset F}{\geq} \int_{B_{2\rho} \setminus B_\rho} \frac{dy}{|y - x|^\lambda} \geq C_{d,p}|E|^{1 - \lambda/d},$$

which is (2.1).

Integrating (2.1) w.r.t. x in D, we obtain (2.2). $\qquad\square$

Remark 2.1. A similar version of inequality (2.1) has played crucial roles in deriving fractional versions of Sobolev [22] and Hardy [1] inequalities.

The following simple lemma is also used in the proof of Theorem 1.1.

Lemma 2.2. *Let* $d \geq 1$, $p > 1$, $\delta > 0$, *and let* O *be a ball in* $\mathbb{R}^d$. *Let* $g \in L^1_{loc}(O)$. *We have,* $k \in \mathbb{N}_+$,

$$(2.6) \qquad \int_O \int_O \frac{\delta^p}{|x-y|^{d+p}} \leq 2^{-k(p-1)} \int_O \int_O \frac{\delta^p}{|x-y|^{d+p}}.$$
$$\scriptstyle |u(x)-u(y)|>2^k\delta \qquad\qquad\qquad\qquad |u(x)-u(y)|>\delta$$

Proof. By considering the function u/δ and by the recurrence, it suffices to consider the case $\delta = 1$ and $k = 1$. We have

$$\iint_{O\times O} \frac{dx\,dy}{|x-y|^{d+p}} = \iint_{O\times O} \frac{dx\,dy}{|x-y|^{d+p}}$$
$$\scriptstyle |u(x)-u(y)|>2 \qquad\qquad |u(x)-u(x/2+y/2)+u(x/2+y/2)-u(y)|>1$$

$$\leq \iint_{O\times O} \frac{dx\,dy}{|x-y|^{d+p}} + \iint_{O\times O} \frac{dx\,dy}{|x-y|^{d+p}}.$$
$$\scriptstyle |u(x)-u(x/2+y/2)|>1 \qquad\qquad |u(x/2+y/2)-u(y)|>1$$

By a change of variables $z = x/2 + y/2$, we obtain

$$\iint_{O\times O} \frac{dx\,dy}{|x-y|^{d+p}} \leq 2^{-(p-1)} \iint_{O\times O} \frac{dx\,dy}{|x-y|^{d+p}},$$
$$\scriptstyle |u(x)-u(y)|>2 \qquad\qquad\qquad |u(x)-u(y)|>1$$

which yields the conclusion for $\delta = 1$ and $k = 1$. $\qquad\square$

2.2. **Proof of part i) of Theorem 1.1.** In this proof, for notational ease, we denote $I_{\delta,p}$ by I_δ for $\delta > 0$. Without loss of generality we can assume $B = B_1$, $u_B = 0$, and $\delta = 1$. Define $\tilde{u}$ in $B_{3/2}$ by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in B_1 \\ u\left(\dfrac{(2-|x|)x}{|x|}\right) & \text{if } x \in B_{3/2} \setminus B_1. \end{cases}$$

We have, for all $\tau > 0$,

$$(2.7) \qquad \left|\left\{ x \in B_{3/2}; |\tilde{u}| \geq \tau \right\}\right| \leq C \left|\left\{ x \in B_1; |u| \geq \tau \right\}\right|$$

and, see e.g., [18, Lemma 17],

$$(2.8) \qquad I_1\left(\tilde{u}, B_{3/2}\right) \leq C I_1\left(u, B_1\right).$$

Using John-Nirenberg's inequality, we have

$$(2.9) \qquad \left|\left\{ x \in B_{3/2}; |\tilde{u}| \geq \ell \right\}\right| \leq 1/8^d,$$

if $\ell \geq c_1 M$ for some $c_1 > 0$.

We claim that, for $\ell \geq c_1 M$ and $\lambda > 2$,

$$(2.10) \quad \left|\left\{x \in B_1 : |u(x)| \geq \lambda\ell\right\}\right| \leq c_2 I_\ell(u, B_1) \left|\left\{x \in B_1 : |u(x)| \geq (\lambda - 1)\ell\right\}\right|^{\frac{p}{d}}.$$

In fact, fix an arbitrary $x \in B_{5/4}$ and let ρ be such that $|B_\rho(x)| = \left|\left\{x \in B_{3/2}; |u| \geq (\lambda - 1)\ell\right\}\right|$. Since $\lambda > 2$, it follows from (2.9) that $\rho < 1/8$, which yields $B_{2\rho}(x) \subset B_{3/2}$. Applying Lemma 2.1 with $D = \{x \in B_{5/4}; |\tilde{u}| \geq \lambda\ell\} \cap O$, $E = \{x \in B_{3/2}; |\tilde{u}| \geq (\lambda - 1)\ell\}$ and $F = B_{3/2}$, and using (2.7) and (2.8), we obtain (2.10).

Applying Lemma 2.2, we have, for $k \in \mathbb{N}$,

$$I_{2^k}(u, B_1) \leq 2^{-k(p-1)} I_1(u, B_1) \leq 2^{-k(p-1)} M.$$

Fix k_0 be such that for $k \geq k_0$, one has $c_2 2^{-k(p-1)} M \leq e^{-2\alpha}$, which yields

$$(2.11) \quad c_2 I_{2^k}(u, B_1) \leq e^{-\alpha(p/d)^3}.$$

Set

$$(2.12) \quad \ell_0 = \max\left\{c_1 M, \left(c_3 M e^{2\alpha}\right)^{1/(p-1)}\right\}.$$

Then, for some c_3 larger than c_2,

$$\ell_0 \geq \max\{c_1 M, 2^{k_0}\}.$$

Using (2.10), (2.11), and a standard iterative process, we have, for $\lambda \in \mathbb{N}$ and

$$(2.13) \quad \left|\left\{x \in B_1; |u| > \lambda\ell_0\right\}\right| \leq e^{-\alpha(p/d)^{\lambda+2}} \left|\left\{x \in B_1; |u| > \ell_0\right\}\right|.$$

This implies

$$\int_{B_1} e^{\alpha(p/d)^{|u|}} \, dx \leq \int_{\substack{B_1 \\ |u| \geq \ell_0}} e^{\alpha(p/d)^{|u|}} \, dx + \int_{\substack{B_1 \\ |u| \leq \ell_0}} e^{\alpha(p/d)^{\ell_0}} \, dx \leq C.$$

This implies the conclusion of part i) with $\beta(\alpha, M) = \ell_0^{-1}$ where ℓ_0 is given by (2.12). $\qquad\square$

2.3. **Proof of part ii) of Theorem 1.1.** The proof of part ii) is in the spirit of part i). In fact, noting that if M_0 is small enough then (2.13) holds with $\ell_0 = 1$. The conclusion then follows. $\qquad\square$

2.4. **Proof of** (1.13) **of Theorem 1.2.** The proof is similar to the one of part ii) of Theorem 1.1 and is omitted. $\qquad\square$

2.5. **Proof of Propositions 1.5 and 1.6.** Propositions 1.5 and 1.6 can be derived from Theorems 1.1 and 1.2 respectively after using local charts and appropriately extending u in a neighborhood of Ω (see, e.g., [18, Lemma 17]. The details are omitted. $\qquad\square$

3. Proof of Proposition 1.7

Without loss of generality, one might assume that $B = B_{1/e}$ and $\delta = 1$.

Proof of assertion (1.14). Fix $\gamma > \lambda > p/d > 1$, set, for $x \in B_{1/e}$,

$$u(x) = g(|x|) \quad \text{where} \quad g(r) = (\ln \lambda)^{-1} \ln \ln(1/r) \text{ for } r \in I := (0, 1/e).$$

It is clear that $g \in L^1(I)$. Using polar coordinates, we have

$$I_{1,p}(u, B_{1/e}) = \int_0^{1/e} \int_0^{1/e} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1} \atop |g(r_1)-g(r_2)|>1} \frac{r_1^{d-1} r_2^{d-1}}{|r_1\sigma_1 - r_2\sigma_2|^{p+d}} \, d\sigma_1 \, d\sigma_2 \, dr_1 \, dr_2.$$

$$(3.1) \qquad \leq C_d \int_0^{1/e} \int_{0 \atop |g(r_1)-g(r_2)|>1}^{1/e} \frac{r_1^{d-1} r_2^{d-1}}{|r_1 - r_2|^{p+d}} \, dr_1 \, dr_2.$$

We have, for $0 < r_1 < r_2 < e^{-1}$,

$$|g(r_1) - g(r_2)| > 1 \text{ if and only if } r_2 > r_1^{1/\lambda} \text{ and } 0 < r_1 < e^{-\lambda},$$

this yields

$$\frac{r_1 r_2}{(r_2 - r_1)^{1+p/d}} \leq \frac{Cr_1}{r_2^{p/d}} \leq C,$$

for some positive constant C depending only on d, p, and λ. It follows that, for $0 < r_1 < r_2 < e^{-1}$ and $|g(r_1) - g(r_2)| > 1$,

$$(3.2) \qquad \frac{r_1^{d-1} r_2^{d-1}}{|r_1 - r_2|^{p+d}} = \left(\frac{r_1 r_2}{(r_2 - r_1)^{\frac{p}{d}+1}}\right)^{d-1} \frac{1}{|r_1 - r_2|^{\frac{p}{d}+1}} \leq \frac{C}{|r_1 - r_2|^{\frac{p}{d}+1}}.$$

We derive from (3.1) and (3.2) that

$$(3.3) \qquad I_{1,p}(u, B_{1/e}) \leq C I_{1,\frac{p}{d}}(g, I).$$

We have

$$(3.4) \quad I_{1,p/d}(g, I) = 2 \iint_{I \times I \atop {|g(r_1)-g(r_2)|>1 \atop r_1 < r_2}} \frac{1}{|r_2 - r_1|^{1+\frac{p}{d}}} \, dr_1 \, dr_2$$

$$\leq C \int_0^{e^{-\lambda}} \left(\frac{1}{\left(r_1^{1/\lambda} - r_1\right)^{\frac{p}{d}}} - \frac{1}{\left(e^{-1} - r_1\right)^{\frac{p}{d}}}\right) dr_1 < +\infty,$$

since $r_1^{1/\lambda} - r_1 \geq Cr_1^{1/\lambda}$ and $e^{-1} - r_1 \geq C$ for $r_1 \in (0, e^{-\lambda})$.

On the other hand, for any $\tau \in I$, we have, with $\rho = \frac{\ln\gamma}{\ln\lambda} - 1$,

$$(3.5) \qquad \int_{B_\tau} e^{\alpha\gamma^g} \, dx = \int_0^\tau e^{\alpha\left(\log r^{-1}\right)^{(1+\rho)}} r^{d-1} \, dr = +\infty,$$

since $\lim_{r \to 0_+} \left(\log r^{-1}\right)^{1+\rho} / \log r^{-1} = +\infty$.

Set, for $0 < \tau < e^{-1}$,

$$u_\tau(x) = u(\tau x) \text{ for } x \in B_{e^{-1}}.$$

Then

$$(3.6) \qquad I_{1,p}(u_\tau, B_{e^{-1}}) = \tau^{p-d} I_{1,p}(u, B_{\tau e^{-1}}) \to 0 \text{ as } \tau \to 0.$$

Combining (3.5) and (3.6) yields the conclusion since for any $M > 0$ we can choose $\tau > 0$ small enough so that $I_{1,p}(u_\tau, B_{e^{-1}}) \leq M$. □

Proof of assertion (1.15). Let $n \in \mathbb{N}$ large and fix $1 < q < \gamma$ and denote $q' = q/(q-1)$. Define

$$u_n(x) = g_n(|x|) \text{ where } g_n(r) = \begin{cases} \ln^{\frac{1}{q}} n & \text{if } 0 \leq r \leq \dfrac{1}{n}, \\[2mm] \dfrac{\ln\left(\frac{1}{r}\right)}{\ln^{\frac{1}{q'}} n} & \text{if } \dfrac{1}{n} \leq r \leq \dfrac{1}{e}. \end{cases}$$

As in (3.3), we have

$$(3.7) \qquad I_{1,d}(u_n, B_{1/e}) \leq C I_{1,1}(g_n, I),$$

where $I = (0, 1/e)$.

We now estimate $I_{1,1}(g_n, I)$. Denote $J_n = (0, 1/n)$, and $K_n = I \setminus J_n$. We have

$$(3.8) \qquad I_{1,1}(g_n, I) = 2\mathcal{I}_1 + \mathcal{I}_2.$$

where

$$\mathcal{I}_1 = \iint_{\substack{I_n \times J_n \\ |g_n(r_1) - g_n(r_2)| > 1}} \frac{1}{|r_1 - r_2|^2} \, dr_1 \, dr_2 \quad \text{and} \quad \mathcal{I}_2 = \iint_{\substack{J_n \times J_n \\ |g_n(r_1) - g_n(r_2)| > 1}} \frac{1}{|r_1 - r_2|^2} \, dr_1 \, dr_2.$$

We next estimate $\mathcal{I}_1$ and $\mathcal{I}_2$. We begin with $\mathcal{I}_1$. For $(r_1, r_2) \in J_n \times K_n$, we have $|g_n(r_1) - g_n(r_2)| > 1$ if and only if

$$\left| \ln^{\frac{1}{q}} n - \frac{\ln\left(\frac{1}{r_2}\right)}{\ln^{\frac{1}{q'}} n} \right| > 1, \text{ this implies } \frac{1}{e} \geq r_2 > a_n := \frac{e^{(\log n)^{\frac{1}{q'}}}}{n}.$$

It follows that

$$(3.9) \qquad \mathcal{I}_1 = \int_0^{\frac{1}{n}} \int_{a_n}^{\frac{1}{e}} \frac{dr_2 dr_1}{|r_1 - r_2|^2} \leq \ln\left(\frac{a_n}{a_n - 1/n}\right) \to 0 \text{ as } n \to +\infty.$$

We next deal with $\mathcal{I}_2$. For $(r_1, r_2) \in K_n \times K_n$ with $r_2 \geq r_1$, we have $|g_n(r_1) - g_n(r_2)| > 1$ if and only if

$$\left| \frac{\ln\left(\frac{1}{r_1}\right) - \ln\left(\frac{1}{r_2}\right)}{\ln^{\frac{1}{q'}} n} \right| > 1, \text{ this implies } \frac{1}{e} \geq r_2 > r_1 b_n \text{ and } \frac{1}{n} \leq r_1 < \frac{1}{e b_n} \text{ with } b_n = e^{\ln^{\frac{1}{q'}} n}.$$

We then have
$$(3.10)$$
$$\mathcal{I}_2 = 2 \int_{1/n}^{1/(eb_n)} \int_{r_1 b_n}^{\frac{1}{e}} \frac{dr_2}{(r_2 - r_1)^2} dr_1 \leq 2 \int_{1/n}^{1/(eb_n)} \frac{1}{r_1(b_n - 1)} \, dr_1 \to 0 \text{ as } n \to +\infty.$$

Combining (3.8), (3.9), and (3.10) yields

$$\lim_{n \to \infty} I_{1,1}(g_n, I) = 0.$$

which yields, by (3.7),

$$(3.11) \qquad \lim_{n\to\infty} I_{1,d}\left(u_n, B_{1/e}\right) = 0.$$

We have

$$\int_{\frac{1}{n}}^{\frac{1}{2}} \frac{\ln\left(\frac{1}{r}\right)}{\ln^{\frac{1}{q'}} n} dr = \frac{1}{(\log n)^{\frac{1}{q'}}} \left(\frac{1}{2}\ln(2e) - \frac{1}{n}\ln(ne)\right) \to 0 \text{ as } n \to \infty.$$

This implies

$$(3.12) \qquad 0 \le \lim_{n\to+\infty} \int_{B_{1/e}} u_n \, dx \le \lim_{n\to+\infty} \int_I g_n \, dx = 0.$$

On the other hand, since $\gamma > q$, we have

$$(3.13) \qquad \int_I r^{d-1} e^{\alpha g_n^r} \ge \int_0^{\frac{1}{n}} r^{d-1} e^{\alpha \ln^{\gamma/q} n} = \frac{1}{dn^d} e^{\alpha \ln^{\gamma/q} n} \to +\infty \text{ as } n \to +\infty.$$

The conclusion now follows from (3.11), (3.12), and (3.13). $\qquad\square$

References

[1] Adimurthi and A Mallick, *A Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18** (2018), 917–949.

[2] L. Ambrosio, G. De Philippis, and L. Martinazzi, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Math. **134** (2011), 377–403.

[3] C. Antonucci, M. Gobbino, M. Migliorini, and N. Picenni, *On the shape factor of interaction laws for a non-local approximation of the Sobolev norm and the total variation*. C. R. Math. Acad. Sci. Paris **356** (2018), 859–864.

[4] C. Antonucci, M. Gobbino, M. Migliorini, N. Picenni, Optimal constants for a non-local approximation of Sobolev norms and total variation, arXiv: 1708.01231.

[5] C. Antonucci, M. Gobbino, N. Picenni, On the gap between gamma-limit and pointwise limit for a non-local approximation of the total variation, arXiv:1712.04413.

[6] G. Aubert and P. Kornprobst, *Can the nonlocal characterization of Sobolev spaces by Bourgain et al. be useful for solving variational problems?*, SIAM J. Numer. Anal. **47** (2009), 844–860.

[7] J. Bourgain and H.-M. Nguyen, *A new characterization of Sobolev spaces*, C. R. Math. Acad. Sci. Paris **343** 1 (2006).

[8] H. Brezis, *How to recognize constant functions. Connections with Sobolev spaces*, Volume in honor of M. Vishik, Uspekhi Mat. Nauk **57** (2002), 59–74; English translation in Russian Math. Surveys **57** (2002), 693–708.

[9] F. Bethuel, H. Brezis and F. Heilein, *Gnzburg-Landau vortices* Reprint of the 1994 edition, Modern Birkhäuser Classics. Birkhäuser/Springer, Cham, 2017.

[10] J. Bourgain, H. Brezis and P. Mironescu, *Another look at Sobolev spaces*, Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.

[11] J. Bourgain, H. Brezis and P. Mironescu, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, J. Anal. Math. **87** (2002), 77–101.

[12] J. Bourgain, H. Brezis and P. Mironescu, *Lifting, degree, and distributional Jacobian revisited*, Comm. Pure Appl. Math. **58** (2005), 529–551.

[13] J. Bourgain, H. Brezis and H.-M. Nguyen, A new estimate for the topological degree. C. R. Math. Acad. Sci. Paris **340** (2005), 787–791.

[14] H. Brezis and H.-M. Nguyen, *On a new class of functions related to VMO*, C. R. Acad. Sci. Paris **349** (2011), 157–160.

[15] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext. Springer, New York, 2011.

[16] H. Brezis and H.-M. Nguyen, *Two subtle convex nonlocal approximation of the BV-norm*, Nonlinear Anal., **137** (2016), 222–245.

[17] H. Brezis and H.-M. Nguyen, *The BBM formula revisited*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., **27** (2016), 515–533.

[18] H. Brezis and H.-M. Nguyen, *Non-local functionals related to the total variation and connections with image processing*, Ann. PDE **1** 4 (2018).

[19] A. Buades, B. Coll and J. M. Morel, *Neighborhood filters and PDE's*, Numer. Math. **105** (2006), 1–34.

[20] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. PDE **41** (2011), 203–240.

[21] J. Davila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), 519–527.

[22] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[23] C. Fefferman and E. Stein, H^p *spaces of several variables*, Acta. Math. **129** (1972) 137–193.

[24] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961) 415–426.

[25] G. Leoni and D. Spector, *Characterization of Sobolev and BV Spaces*, J. Funct. Anal. **261** (2011), 2926–2958, *Corrigendum to "Characterization of Sobolev and BV Spaces"*, J. Funct. Anal. **266** (2014), 1106–1114.

[26] V. G. Mazya, *Sobolev Spaces*, Springer, Berlin, 1985.

[27] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** 11 (1971), 1077–1092.

[28] H.-M. Nguyen, *Some new characterizations of Sobolev spaces*, J. Funct. Anal. **237** (2006), 689–720.

[29] H.-M. Nguyen, *Optimal constant in a new estimate for the degree*, J. Anal. Math. **101** (2007), 367–395.

[30] H.-M. Nguyen, Γ-*convergence and Sobolev norms*, C. R. Acad. Sci. Paris **345** (2007), 679–684.

[31] H.-M. Nguyen, *Further characterizations of Sobolev spaces*, J. European Math. Soc. **10** (2008), 191–229.

[32] H.-M. Nguyen, Γ-*convergence, Sobolev norms, and BV functions*, Duke Math. J. **157** (2011), 495–533.

[33] H.-M. Nguyen, *Some inequalities related to Sobolev norms*, Calc. Var. Partial Differential Equations **41** (2011), 483–509.

[34] H.-M. Nguyen, *A refined estimate for the topological degree*, C. R. Math. Acad. Sci. Paris **355** (2017), 1046–1049.

[35] H.-M. Nguyen, A. Pinamonti, M. Squassina, and E. Vecchi, *New characterizations of magnetic Sobolev spaces*, Adv. Nonlinear Anal. **7** (2018), 227–245.

[36] H.-M. Nguyen and M. Squassina, Marco *Logarithmic Sobolev inequality revisited*, C. R. Math. Acad. Sci. Paris **355** (2017), 447–451.

[37] H.-M. Nguyen and M. Squassina, *On Hardy and Caffarelli-Kohn-Nirenberg inequalities*, Journal d'Analyse Mathematique. 2017, to appear.

[38] H.-M. Nguyen and M. Squassina, *On anisotropic Sobolev spaces*, Commun. Contemp. Math. **21** (2019), 1850017, 13 pp.

[39] M. Squassina and B. Volzone, *Bourgain-Brezis-Mironescu formula for magnetic operators*, C. R. Math. Acad. Sci.Paris **354** (2016), 825–831.

[40] J. Peetre, *Espaces d'interpolation et theoreme de Soboleff*, Ann. Inst. Fourier (Grenoble) **16** (1966), 279–317.

[41] S. I. Pohozhaev, *The Sobolev imbedding in the case pl = n*, in: Proc. Tech. Sci. Conf. on Adv. Sci. Research 1964-1965, Mathematics Section, Moskov. Energet. Inst., Moscow 1965, pp. 158–170.

[42] A. Ponce, *A new approach to Sobolev spaces and connections to* Γ-*convergence*, Calc. Var. Partial Differential Equations **19** (2004), 229–255.

[43] A. Ponce, *An estimate in the spirit of Poincaré's inequality*, J. Eur. Math. Soc. **6** (2004), 1–15.

[44] A. Ponce and S. Daniel *On formulae decoupling the total variation of BV functions.*, Nonlinear Anal. **154** (2017), 241–257.

[45] J. V. Schaftingen, *Estimates by gap potentials of free homotopy decompositions of critical Sobolev maps*, arXiv:1811.01706v2.

[46] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications.* J. Math. Mech. **17** (1967), 473–483.

[47] L. P. Yaroslavsky, *Digital Picture Processing. An introduction*, Springer, 1985.

[48] L. P. Yaroslavsky and M. Eden, *Fundamentals of Digital Optics*, Springer, 1996.

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