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ESTIMATES OF SUB AND SUPER SOLUTIONS OF SCHRÖDINGER EQUATIONS WITH VERY SINGULAR POTENTIALS

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ABSTRACT. Consider operators $L_V := \Delta + V$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Assume that $V \in C^1(\Omega)$ satisfies $V(x) \le \bar{a}\,\mathrm{dist}\,(x, \partial\Omega)^{-2}$ in Ω and that L_V has a ground state Φ_V in Ω. Under these assumptions and some conditions on the ground state, we derive sharp, two-sided estimates of weighted integrals of positive L_V superharmonic and L_V subharmonic functions. We show that these conditions hold for large classes of operators.

1. Introduction

Let Ω be a bounded Lip domain in $\mathbb{R}^N$, $N \ge 3$. We study the operator

$$L_V := \Delta + V$$

where $V \in C^1(\Omega)$. We assume that the potential V satisfies the conditions:

$$\text{(A1)} \qquad \exists \bar{a} > 0 : \quad |V(x)| \le \bar{a}\delta(x)^{-2} \quad \forall x \in \Omega$$

$$\delta(x) = \delta_{\partial\Omega}(x) := \mathrm{dist}\,(x, \partial\Omega).$$

and, with $\gamma_\pm$ as defined below,

$$\text{(A2)} \qquad \gamma_- < 1 < \gamma_+.$$

The definition of $\gamma_\pm$:

$$\text{(1.1)} \qquad \begin{aligned} \gamma_+ &:= \sup\{\gamma : \exists u_\gamma > 0 \text{ such that } L^{\gamma V} u_\gamma = 0\}, \\ \gamma_- &:= \inf\{\gamma : \exists u_\gamma > 0 \text{ such that } L^{\gamma V} u_\gamma = 0\}. \end{aligned}$$

By a theorem of Allegretto and Piepenbrink [25] or [23, Theorem 2.3], (1.1) is equivalent to,

$$\text{(1.2)} \qquad \begin{aligned} \gamma_+ &= \sup\{\gamma : \int_\Omega |\nabla\phi|^2\,dx \ge \gamma \int_\Omega \phi^2 V\,dx \quad \forall \phi \in H_0^1(\Omega)\}, \\ \gamma_- &= \inf\{\gamma : \int_\Omega |\nabla\phi|^2\,dx \ge \gamma \int_\Omega \phi^2 V\,dx \quad \forall \phi \in H_0^1(\Omega)\}. \end{aligned}$$

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Condition (A1) and Hardy's inequality imply that $\gamma_+ > 0$ and $\gamma_- < 0$. If V is positive then $\gamma_- = -\infty$ and γ_+ is *the Hardy constant relative to* V in Ω, denoted by $c_H(V)$. If V is negative then $\gamma_+ = \infty$.

Finally, for every $\gamma \in (\gamma_-, \gamma_+)$ there exists a Green function for $L_{\gamma V}$ in Ω. Thus condition (A2) implies that L_V has a Green function G_V in Ω. If D is a Lipschitz subdomain of Ω, the Green function of L_V in D is denoted by G_V^D.

Conditions (A1) and (A2) imply:

(i) L_V has a *ground state* in the sense of Agmon [1]. The ground state Φ_V is normalized by the condition $\Phi_V(x_0) = 1$ where x_0 is a fixed reference point in Ω.

(ii) L_V is weakly coercive in the sense of Ancona [3]. A proof, due to [22], is provided in [16, Lemma 1.1].

Consequently the results of Ancona [3] apply to the operators under consideration. In particular:

= L_V possesses a Martin kernel K_V such that, for every $y \in \partial\Omega$, $x \mapsto K_V(x, y)$ is positive L_V harmonic in Ω and vanishes on $\partial\Omega \setminus \{y\}$ and the following holds:

Representation Theorem. *For every positive L_V-harmonic function u there exists a measure $\nu \in \mathfrak{M}_+(\partial\Omega)$ (= the space of positive, bounded Borel measures) such that*

$$u(x) = \int_{\partial\Omega} K_V(x, y)d\nu(y) =: \mathbb{K}_V[\nu] \quad x \in \Omega. \tag{1.3}$$

Conversely, for every such measure ν, the function u above is L_V harmonic.

= The *Boundary Harnack Principle* (briefly BHP). (See its statement in the next section.)

In addition, sharp two-sided estimates of Green and Martin kernels of the operator L_V, recently obtained by the author [16] are valid under conditions (A1), (A2), (in Lipschitz domains). A statement of these estimates is provided in the next section.

Previously these estimates have been obtained in two special cases: (a) $V = 0$ in Lipshitz domains, Bogdan [7], (b) $V = \gamma/\delta^2$, $\gamma < C_H(\Omega)$ in smooth domains, Filippas, Moschini and Tertikas [11].

The Martin kernel is similar to the Poisson kernel. However, unlike the Poisson kernel, the mass of $K_V(\cdot, y)$ at y need not be finite. For instance, if $V = \gamma/\delta(x)^2$ with $\gamma < C_H(\Omega)$ (= the Hardy constant in Ω) then the mass is zero when $\mu > 0$ and infinity when $\mu < 0$. Therefore, in these cases, L_V has no Poisson kernel but possesses a Martin kernel.

Definition 1.1. (i) A function $u > 0$ is *local L_V harmonic* (respectively superharmonic) if it is defined and L_V harmonic (respectively superharmonic) in a one-sided neighborhood of $\partial\Omega$.

(ii) A positive local L_V harmonic function u, has *minimal growth* at $\partial\Omega$ if, for every positive local L_V superharmonic function v,

$$\limsup_{\delta(x)\to 0} \frac{u}{v}(x) < \infty.$$

(iii) A positive L_V superharmonic is called an *L_V potential* if it does not dominate any positive L_V harmonic function. It is known [2] that u is an L_V potential if and only

$$u = \mathbb{G}_V[\tau] := \int_\Omega G_V(x,y)d\tau(y)$$

where τ is a positive Radon measure on Ω such that $\mathbb{G}_V[\tau] < \infty$. (See also (1.6) below).

The Green kernel G_V is uniquely determined by the following conditions, [1].
(a) For every $y \in \Omega$, $-L_V G_V(x,y) = \delta_y$ (the Dirac measure at y) and
(b) the function $x \mapsto G_V(x,y)$ is of minimal growth in $\Omega \setminus \{y\}$.

The BHP and a result of [4] imply,

There exists a constant $C > 0$ such that for every $y_0 \in \Omega$,

$$C^{-1}G_V(x,y_0) \le \Phi_V(x) \le CG_V(x,y_0) \quad \textit{when} \;\; \delta(x) < \delta(y_0)/2. \tag{1.4}$$

A proof - based on [5] - is provided in [16, Lemma 1.2].

It is well-known [14] that for any compact set $E \subset \Omega$ there exists a constant $c(E)$ such that

$$\frac{1}{c(E)}|x-y|^{2-N} \le G_V(x,y) \le c(E)|x-y|^{2-N} \quad \forall (x,y) \in E \times E. \tag{1.5}$$

This inequality and (1.4) imply that,

$$\mathbb{G}_V[\tau] := \int_\Omega G_V(x,y)d\tau(y) < \infty \Longleftrightarrow \tau \in \mathfrak{M}_+(\Omega;\Phi_V). \tag{1.6}$$

In fact if τ is a positive Radon measure but $\tau \notin \mathfrak{M}_+(\Omega;\Phi_V)$ then $\mathbb{G}_V[\tau] \equiv \infty$.

It follows that, in the present context, *a positive L_V superharmonic w is an L_V potential if and only if $w = \mathbb{G}_V[\tau]$ for some $\tau \in \mathfrak{M}_+(\Omega;\Phi_V)$.*

In this paper we study operators L_V such that V *is strongly singular on* $\partial\Omega$, i.e. V satisfies (A1) and

$$\limsup_{\delta\to 0} V\delta^2 \ne 0.$$

Of special interest are potentials of the form

$$\begin{aligned} &V = \gamma V_F, \quad F \subset \partial\Omega \text{ compact}, \ \gamma \in \mathbb{R} \\ &V_F = \frac{1}{\delta_F^2}, \quad \delta_F(x) = \mathrm{dist}\,(x,F), \quad \gamma < C_H(V_F). \end{aligned} \tag{1.7}$$

The main part of our study is devoted to the derivation of weighted integral estimates of positive L_V superharmonic and L_V subharmonic functions. The weight W is given by,

$$W := \frac{\Phi_V}{\Phi_0}. \tag{1.8}$$

The estimates are sharp and two sided (see Theorems 3.7 and 3.8 below). This indicates that in the present context the weight W is optimal.

The derivation is based on assumptions (A1), (A2) and two conditions on the behavior of the ground state, (see (B1) and (B2) in section 3). In the case that V is positive and Ω is a C^2 domain these conditions are satisfied if there exists $\alpha > 1/2$ and $C > 0$ such that

$$\Phi_V(x) \le C\delta(x)^\alpha \quad \forall x \in \Omega. \tag{1.9}$$

Thus our approach *does not require the availability of sharp estimates of the ground state.*

Linear and nonlinear boundary value problems for operators L_V with V as in (1.7) have been investigated by many authors. But, to our knowledge, these were restricted to potentials where the ground state is known, i.e. sharp two-sided estimates of the ground state are available. Usually, the analysis depends on these sharp estimates in an essential way.

In the case of strongly singular potentials (1.7), such estimates are available only in a few instances, in smooth domains:

(i) F a singleton, (also in cones)

(ii) $F = \partial\Omega$,

(iii) $F = F_k$ a smooth k-dim. manifold without boundary.

The estimates of the ground state in (i) and (ii) are classical. For the estimate in case (iii) see Fall and Mahmoudi [10]. For (i) in cones see Devyver, Pinchover and Psaradakis [8].

The interest in problems involving operators L_V with strongly singular potentials increased considerably in the last decade. Following is a list of a few works in the area:

Bandle, Moroz and Reichel [6], Marcus and P.T. Nguyen [18], [19], Gkikas and Veron [13], P.T. Nguyen [21], Y. Du and L. Wei [9], [24], Marcus and Moroz [17], Chen and Veron [15].

In most of these, the authors deal with (positive) solutions of nonlinear equations such as $-L_V u + f(u) = 0$ under various conditions on the nonlinear term and with potential $V = \gamma\delta^{-2}$ or $V = \gamma/|x|^2$ $(0 \in \partial\Omega)$ or a combination thereof.

The estimates established in the present paper provide a basis for the study of positive solutions of boundary value problems for equations as above for a large family of potentials, including for instance, *any positive potential satisfying conditions (A1), (A2) and* (1.9) in smooth domains.

Examples of specific classes of potentials that satisfy these conditions are presented in Section 8.

The main tools used in the paper are: (a) potential theoretic results (mentioned above) and (b) estimates of the Green and Martin kernels (described in the next section).

These tools are valid in bounded Lipschitz domains and the methods employed in the present paper can be adapted to the case of Lipschits domains. However, for the sake of clarity, in this paper we present our results for the case of bounded C^2 domains.

The adaptation of our results to the case of Lipschitz domains involves modifications that require careful technical treatment. This will be presented in a separate note.

The paper is organized as follows.

Section 2 provides statements of some results from the literature that are frequently used in this paper.

In Section 3 we state the main results of the present paper, in which it is assumed that the domains are bounded of class C^2.

In Section 4 we establish estimates of positive L_V harmonic functions: Theorems 3.1 and 3.2.

In section 5 we derive estimates of L_V potentials: Theorems 3.3 and 3.4.

In Section 6 we establish sharp, two-sided estimates of L_V superharmonic and L_V subharmonic functions: Theorems 3.7 and 3.8.

Section 7 is devoted to a discussion of boundary trace in terms of harmonic measures $d\omega_V^{x_0,D}$ of L_V in Lipschitz domains $D \Subset \Omega$ relative to a point $x_0 \in D$. In particular we establish an *equivalence relation*, on $\Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}$, $0 < \beta < \beta_0$, between the weight measure WdS and the harmonic measure $d\omega_V^{x_0,D_\beta}$ where $D_\beta = \{x \in \Omega : \delta(x) > \beta\}$, (Theorem 7.5). Here β_0 is a number depending on Ω (see Section 2 for details) and $x_0 \in D_{\beta_0}$ is a fixed reference point.

Section 8 provides examples of families of strongly singular potentials which satisfy the assumptions of this paper.

2. Preliminaries

Denote,

$$T(r,\rho) = \{\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{N-1} : |\xi_1| < \rho,\ |\xi'| < r\}.$$

Assuming that Ω is a bounded Lipschitz domain, there exist positive numbers r_0, κ such that, for every $y \in \partial\Omega$, there exist: (i) a set of Euclidean coordinates $\xi = \xi_y$ centered at y with the positive ξ_1 axis pointing in the direction of $\mathbf{n}_y$[1] and (ii) a function F_y uniformly Lipschitz in $\mathbb{R}^{N-1}$ with Lipschitz constant $\leq \kappa$ such that

$$\begin{aligned} Q_y(r_0,\rho_0) &:= \Omega \cap T_y(r_0,\rho_0) \\ &= \{\xi = (\xi_1,\xi') : F_y(\xi') < \xi_1 < \rho_0,\ |\xi'| < r_0\}, \end{aligned} \tag{2.1}$$

where $T_y(r_0,\rho_0) = y + T(r_0,\rho_0)$ in coordinates $\xi = \xi_y$ and $\rho_0 = 10\kappa r_0$. Without loss of generality, we assume that $\kappa > 1$.

The set of coordinates ξ_y is called a standard set of coordinates at y and $T_y(r,\rho)$ with $0 < r \leq r_0$ and $\rho = c\kappa r$, $2 < c \leq 10$ is called a standard cylinder at y.

If Ω is a bounded C^2 domain there exists $\beta_0 > 0$ such that for every $x \in \Omega_{\beta_0}$ there is a unique point $\sigma(x) \in \partial\Omega$ such that

$$|x - \sigma(x)| = \delta(x)$$

and $x \mapsto \delta(x)$ is in $C^2(\Omega_{\beta_0})$ while $x \mapsto \sigma(x)$ is in $C^1(\Omega_{\beta_0})$. The set of coordinates (δ, σ) defined in this way in Ω_{β_0} is called *the flow coordinates set.* We denote

$$\begin{aligned} D_\beta = \{x \in \Omega : \delta(x) > \beta\}, \quad \Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \\ \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}. \end{aligned} \tag{2.2}$$

[1]If Ω is smooth, $\mathbf{n}_y$ denotes the inward normal at y. If Ω is Lipschitz, $\mathbf{n}_y$ denotes an approximate normal.

Notation. Let f_i, $i = 1, 2$, be positive functions on some domain X. Then the notation $f_1 \sim f_2$ in X means: there exists $C > 0$ such that

$$\frac{1}{C} f_1 \leq f_2 \leq C f_1 \quad \text{in } X.$$

The notation $f_1 \lesssim f_2$ means: there exists $C > 0$ such that $f_1 \leq C f_2$ in X. The constant C will be called a *similarity constant.*

The BHP and estimates of the Green and Martin kernels will be frequently used in the sequel. Therefore, for the convenience of the reader, we state them here. These results are valid in *bounded Lipschitz domains.*

The Boundary Harnack Principle:

Theorem 2.1 (Ancona [3])**.** *Let $P \in \partial\Omega$ and let $T_P(r, \rho)$ be a standard cylinder at P. There exists a constant c depending only on N, $\bar{a}$ and $\frac{\rho}{r}$ such that whenever u is a positive L_V harmonic function in $Q_P(r, \rho)$ that vanishes continuously on $\partial\Omega \cap T_P(r, \rho)$ then*

$$c^{-1} r^{N-2}\, \mathbb{G}^V_\Omega(x, A') \leq \frac{u(x)}{u(A)} \leq c\, r^{N-2}\, \mathbb{G}^V_\Omega(x, A'), \quad \forall x \in \Omega \cap \overline{T}^P(\frac{r}{2}; \frac{\rho}{2}) \tag{2.3}$$

where $A = (\rho/2)(1, 0, ..., 0)$, $A' = (2\rho/3)(1, 0..., 0)$ in the corresponding set of local coordinates ξ_P.

In particular, for any pair u, v of positive L_V harmonic functions in $Q_P(r, \rho)$ that vanish on $\partial\Omega \cap T^P(r, \rho)$:

$$u(x)/v(x) \leq C u(A)/v(A), \qquad \forall x \in \Omega \cap \overline{T}^P(r/2, \rho/2)) \tag{2.4}$$

where $C = c^2$.

Estimates of the Green and Martin kernels (Marcus [16]):

Theorem 2.2. *Assume (A1), (A2) and $N \geq 3$.*

Then, for every $b > 0$ there exists a constant $C(b)$, depending also on $N, r_0, \kappa, \bar{a}$, such that: if $x, y \in \Omega$ and

$$|x - y| \leq \frac{1}{b} \min(\delta(x), \delta(y)) \tag{2.5}$$

then

$$\frac{1}{C(b)} |x - y|^{2-N} \leq G_V(x, y) \leq C(b) |x - y|^{2-N}. \tag{2.6}$$

In the next theorems, C stands for a constant depending only on $r_0, \kappa, \bar{a}$ and N.

Theorem 2.3. *Assume (A1), (A2) and $N \geq 3$.*

If $x, y \in \Omega$ and

$$\max(\delta(x), \delta(y)) \leq r_0/10\kappa \tag{2.7}$$

$$\min(\delta(x), \delta(y)) \leq \frac{|x - y|}{16(1 + \kappa)^2} \tag{2.8}$$

then

$$\begin{aligned}\frac{1}{C}|x-y|^{2-N}\frac{\varphi_{\gamma V}(x)\varphi_{\gamma V}(y)}{\varphi_{\gamma V}(x_y)^2} &\leq G_V(x,y)\\ &\leq C|x-y|^{2-N}\frac{\varphi_{\gamma V}(x)\varphi_{\gamma V}(y)}{\varphi_{\gamma V}(x_y)^2}.\end{aligned} \tag{2.9}$$

The point x_y *depends on the* pair (x,y). *If*

$$\hat{r}(x,y) := |x-y| \vee \delta(x) \vee \delta(y) \leq r_0/10\kappa$$

x_y *can be chosen arbitrarily in the set*

$$A(x,y) := \{z \in \Omega : \frac{1}{2}\hat{r}(x,y) \leq \delta(z) \leq 2\hat{r}(x,y)\} \cap B_{4\hat{r}(x,y)}(\frac{x+y}{2})\}. \tag{2.10}$$

Otherwise set $x_y = x_0$ *where* x_0 *is a fixed reference point.*

Theorem 2.4. *Assume (A1), (A2) and* $N \geq 3$.
If $x \in \Omega$, $y \in \partial\Omega$ *and* $|x-y| < \frac{r_0}{10\kappa}$ *then*

$$\frac{1}{C}\frac{\varphi_{\gamma V}(x)}{\varphi_{\gamma V}(x_y)^2}|x-y|^{2-N} \leq K_\Omega^{\gamma V}(x,y) \leq C\frac{\varphi_{\gamma V}(x)}{\varphi_{\gamma V}(x_y)^2}|x-y|^{2-N}, \tag{2.11}$$

where x_y *is an arbitrary point in* $A(x,y)$.

Finally we recall two well-known results that will be used later on. These results apply to a general class of operators that includes in particular, operators L_V satisfying (A1) and (A2).

I. Riesz decomposition lemma. *Let* u *be a positive* L_V *superharmonic function. Then* u *has a unique representation of the form:*

$$u = p + w \quad \text{where } p \text{ is an } L_V \text{ potential, } w \text{ is } L_V \text{ harmonic.} \tag{2.12}$$

II. Characterization of L_V potentials. *A positive* L_V *superharmonic function* p *is an* L_V *potential if and only if it is a Green potential, i.e.,* $p = \mathbb{G}_V[\tau]$ *for some* $\tau \in \mathfrak{M}(\Omega;\Phi_V)$. (See [2, Theorem 12]).

3. Main results

In the results stated below Ω is a bounded C^2 domain in $\mathbb{R}^N$. As mentioned in the introduction, the results will be extended, in a separate note, to the case of bounded Lipschits domains.

The first result provides a sharp estimate of positive L_V harmonic functions.

Theorem 3.1. *Assume conditions (A1), (A2). In addition assume that,*

$$lim_{\beta\to 0}\int_{\Sigma_\beta}\frac{\Phi_V^2}{\delta}dS = 0. \tag{B1}$$

Then

$$\frac{1}{C}\|\nu\| \leq \int_{\Sigma_\beta}\frac{\Phi_V}{\Phi_0}\mathbb{K}_V[\nu]dS \leq C\|\nu\| \quad \forall \nu \in \mathcal{M}_+(\partial\Omega), \tag{3.1}$$

where C *depends on* $\bar{a}, \Omega$ *and the rate of convergence in (B1). Condition (B1) is also necessary.*

Notation Let u be a positive L_V harmonic function. Put

$$\sigma_\beta(u) := \sigma(u, D_\beta) = \begin{cases} W u\, \mathbf{1}_{\Sigma_\beta} dH_{N-1} & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.2}$$

where $W := \frac{\Phi_V}{\Phi_0}$. Let

$$M(u) := \{\sigma_\beta(u) : 0 < \beta < \beta_0\}.$$

Theorem 3.1 implies that $M(u)$ is a bounded set of measures on $\mathbb{R}^N$. Therefore, for any sequence $\beta_n \to 0$, there is a subsequence $\{\beta_{n'}\}$ such that $\{\sigma_{\beta_{n'}}(u)\}$ converges weakly to a measure in $\mathfrak{M}(\partial\Omega)$. The set of weak limit points of $M(u)$ as $\beta \to 0$ (with respect to weak convergence) is denoted by $T(u)$.

A measure $\nu' \in T(u)$ is called an approximate trace of u.

Theorem 3.2. *Assume conditions (A1), (A2), (B1) and let $u = \mathbb{K}_V[\nu]$, $\nu \in \mathcal{M}(\partial\Omega)$ positive. If ν' is an approximate trace of u then ν and ν' are mutually absolutely continuous and*

$$\frac{1}{C} \le h := \frac{d\nu'}{d\nu} \le C$$

where C is the constant in (3.1).

In the following theorems we present estimates of L_V potentials. Recall that w is an L_V potential iff $w = \mathbb{G}_V[\tau]$ for some positive measure $\tau \in \mathcal{M}(\Omega; \Phi_V)$. Alternatively we refer to such a function as a 'Green potential'.

Theorem 3.3. *Assume (A1) and (A2). Then there exists a constant c depending on $\bar{a}$, r_0 and κ such that, for every $\tau \in \mathfrak{M}_+(\Omega; \Phi_V)$,*

$$\frac{1}{c}\int_{\Omega_{r_0/4}} \Phi_V d\tau \le \int_{\Omega_{r_0/12\kappa}} \frac{\Phi_V}{\Phi_0}\mathbb{G}_V[\tau]dx \tag{3.3}$$

and

$$\frac{1}{c}\int_{\Omega} \Phi_V d\tau \le \int_{\Omega} \frac{\Phi_V}{\Phi_0}\mathbb{G}_V[\tau]dx \tag{3.4}$$

Remark. See also Lemmas 5.1 and 5.2 below for more specific estimates concerning surface integrals on manifolds Σ_β.

Theorem 3.4. *Assume (A1), (A2). In addition assume that there exist α, α^* such that:*

$$0 < \alpha - 1/2 < \alpha^* \le \alpha, \quad \frac{1}{c}\delta^\alpha \le \Phi_V \le c\delta^{\alpha^*} \quad \textit{in } \Omega_{r_0}. \tag{B2}$$

Then there exists $c' > 0$, depending on a, $\bar{a}$, α^*, α and Ω such that for every $\tau \in \mathfrak{M}_+(\Omega; \Phi_V)$

$$\int_{\Omega} \frac{\Phi_V}{\Phi_0}\mathbb{G}_V[\tau]dx \le c' \int_{\Omega} \Phi_V d\tau. \tag{3.5}$$

Corollary 3.5. *Assume (A1), (A2), (B1) and (B2). Then*

$$\int_{\Sigma_\beta} \frac{\Phi_V}{\Phi_0}\mathbb{G}_V[\tau]dS_x \to 0 \quad \textit{as } \beta \to 0. \tag{3.6}$$

Using these facts and the Riesz decomposition lemma we obtain,

Proposition 3.6. *Assume (A1), (A2).*
(i) *If u is a positive L_V superharmonic function then*

$$-L_V u = \tau \in \mathfrak{M}(\Omega; \Phi_V) \tag{3.7}$$

and there exists a non-negative measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that

$$u = \mathbb{G}_V[\tau] + \mathbb{K}_V[\nu]. \tag{3.8}$$

(ii) *Let u be a non-negative L_V subharmonic function and $\tau := L_V u$. Then, $\tau \in \mathfrak{M}(\Omega; \Phi_V)$ if and only if u is dominated by an L_V superharmonic function. If the above condition holds then there exists a non-negative measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that*

$$u + \mathbb{G}_V[\tau] = \mathbb{K}_V[\nu]. \tag{3.9}$$

Combining Proposition 3.6 with Theorems 3.1, 3.3 and 3.4 we obtain the folowing two sided estimates.

Theorem 3.7. *Assume (A1), (A2), (B1) and (B2). Let u be a positive L_V superharmonic function and let τ, ν be as in Proposition 3.6. Then there exists a constant C depending only on $\bar{a}$, α^*, α, r_0, Ω such that*

$$\frac{1}{C}\left(\int_\Omega \Phi_V \, d\tau + \|\nu\|\right) \leq \int_\Omega \frac{\Phi_V}{\Phi_0} u dx \leq C\left(\int_\Omega \Phi_V \, d\tau + \|\nu\|\right). \tag{3.10}$$

Theorem 3.8. *Assume (A1), (A2), (B1) and (B2). Let u be a positive L_V subharmonic function and assume that*

$$\tau := L_V u \in \mathfrak{M}(\Omega; \Phi_V). \tag{3.11}$$

Put $\nu := \mathrm{tr}_V(u)$. Then there exists a constant C depending only on $\bar{a}$, α^*, α, r_0, Ω such that

$$\frac{1}{C} \|\nu\| \leq \int_\Omega \frac{\Phi_V}{\Phi_0} u dx + \int_{\Omega_{r_0/4}} \Phi_V \, d\tau \leq C \|\nu\| . \tag{3.12}$$

In Section 7 we establish an equivalence relation between the measure WdS on Σ_β and the harmonic measure of L_V in D_β (see Theorem 7.5). This relation provides further indication to the effect that the weight $W = \frac{\Phi_V}{\Phi_0}$ is optimal in the present context.

Estimates (3.1) and (3.5) and a version of Proposition 3.6 have been proved in [18] in the special case $V = \gamma/\delta^2$ and in [19] in the case $V = \gamma/\delta_{F_k}^2$ where F_k is a smooth k-dimensional manifold without boundary. In both papers the proofs dependended, in an essential way, on the fact that, in those cases, the precise behavior of the ground state is known. In [18] the estimates have been applied to a study of boundary value problem for the equation $-L_V u + u^p = 0$. These results have been extended in [17] by a considerable relaxation of the conditions on γ. In [19] the above estimates have been applied to the study of a more general family of nonlinear equations with absorption of the form $-L_V u + f(x, u) = 0$.

Theorems 3.2, 3.3, the related Lemmas 5.1 and 5.2 and consequently the lower estimates in Theorems 3.7 nd 3.8 are new even in the model case $V = \gamma/\delta^2$.

4. Estimates of L_V harmonic functions and approximate boundary trace

Proof of Theorem 3.1. For $y \in \partial\Omega$ denote

$$\mathcal{C}_y = \{x \in \Omega : \langle x - y, \mathbf{n}_y\rangle > \frac{1}{2}|x-y|,\ \delta(x) < \beta_0/2\}$$

It is known [3] that there exists $t_0 > 0$ such that

$$K_V(y + t\mathbf{n}_y, y)G_V(y + t\mathbf{n}_y, x_0) \sim t^{2-N} \quad t \in (0, t_0) \tag{4.1}$$

with similarity constant dependent on $\bar{a}, \Omega$ but independent of y. We assume that $\epsilon_b < t_0$. Hence, by the strong Harnack inequality,

$$\frac{K_0(x,y)}{K_V(x,y)} \sim \frac{G_V(x,x_0)}{G_0(x,x_0)} \sim \frac{\Phi_V(x)}{\Phi_0(x)} \quad \text{in } \mathcal{C}_b(y).$$

Thus,

$$\int_{\mathcal{C}_b(y)\cap\Sigma_\beta} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x,y)dS_x \sim \int_{\mathcal{C}_b(y)\cap\Sigma_\beta} K_0(x,y)dS_x, \tag{4.2}$$

with similarity constants independent of $y \in \partial\Omega$ and $\beta \in (0, \epsilon_b)$.

By BHP $K_V(\cdot, y) \sim G_V(\cdot, x_0)$ in $\Omega_{\epsilon_b} \setminus \mathcal{C}_b(y)$, uniformly with respect to $y \in \partial\Omega$. Therefore condition (B1) and (4.2) imply that there exists a constant $C = C(V, \Omega)$ such that

$$\int_{\Sigma_\beta} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x,y)dS_x \sim \int_{\Sigma_\beta} K_0(x,y)dS_x, \tag{4.3}$$

with similarity constants independent of $y \in \partial\Omega$ and $\beta \in (0, \epsilon_b)$.

By Fubini's theorem (4.3) implies (3.1).

The last part of the proof also shows that condition (B1) is necessary. □

Proof of Theorem 3.2. By assumption, ν' is an approximate trace of u, i.e., there exists a sequence of positive numbers $\{\beta_n\}$ converging to zero such that

$$\sigma_{\beta_n}(u) \rightharpoonup \nu'.$$

Let $E \subset \partial\Omega$ be a compact set and denote by ν_E the measure on $\partial\Omega$ given by $\nu_E(A) = \nu(E \cap A)$. We similarly define the measure $(\nu')_E$: $(\nu')_E(A) = \nu'(E \cap A)$.

Put $u_E := \mathbb{K}_V[\nu_E]$. Taking a further subsequence if necessary, we may assume that $\{\sigma_{\beta_n}(u_E)\}$ is also weakly convergent. The weak limit of this sequence is denoted by $(\nu_E)'$. Note that $(\nu_E)'$ and $(\nu')_E$ are *different measures.*

We have to show that

$$\frac{1}{C}\nu(E) \le \nu'(E) \le C\nu(E) \tag{\#}$$

with C as in (3.1). First observe that

$$(\nu_E)'(\partial\Omega) \le \nu'(E)$$

and, by (3.1),

$$\frac{1}{C}\nu(E) \le (\nu_E)'(\partial\Omega).$$

This proves the left inequality in (#).

Given $\epsilon > 0$, let $E \subset O \subset \partial\Omega$, O relatively open and $\nu(O \setminus E) < \epsilon$. Further put $\tilde{\nu}_O := \nu_{\partial\Omega\setminus O}$. Then - taking a subsequence if necessary - we may assume that both sequences $\{\sigma_{\beta_n}(\mathbb{K}_V[\nu_O])\}$ and $\{\sigma_{\beta_n}([\tilde{\nu}_O])\}$ converge weakly. We denote the limits by $(\nu_O)'$ and $(\tilde{\nu}_O)'$ respectively. Obviously

$$(\nu_O)' + (\tilde{\nu}_O)' = \nu'$$

and $(\tilde{\nu}_O)'(E) = 0$. Therefore, by (3.1),

$$\nu'(E) \leq (\nu_O)'(\partial\Omega) \leq C\nu(O) \leq C(\nu(E) + \epsilon)$$

This implies the right inequality in (#). □

5. Estimates of L_V potentials

In this section we prove Theorems 3.3 and 3.4 and Proposition 3.5. The proofs are based on two lemmas.

Lemma 5.1. *Assume that (A1) and (A2) hold. Let $\tau \in \mathfrak{M}_+(\Omega; \Phi_V)$ and denote*

$$I_1(\beta) := \frac{1}{\beta}\int_{\Sigma_\beta} \Phi_V(x) \int_\Omega G_V(x,y)\chi_{a\beta}(|x-y|)d\tau(y)dS_x$$

where $\chi_s(t) = \mathbf{1}_{(0,s)}(t)$ and $a \geq 16\kappa^2$. Then there exists a constant c depending only on a, $\bar{a}$ and Ω such that,

$$\frac{1}{c}\int_{\Omega_{3a\beta/2}} \Phi_V d\tau \leq I_1(\beta) \leq c\int_\Omega \Phi_V d\tau \quad \forall \beta \in (0, r_0/3a). \tag{5.1}$$

Proof. The domain of integration in $I_1(\beta)$ is $\{(x,y) \in \Sigma_\beta \times \Omega : |x-y| < a\beta\}$. We partition the domain of integration into three parts and estimate each of the resulting integrals separately. Accordingly we denote:

$$I_{1,1}(\beta) := \frac{1}{\beta}\int_{\Sigma_\beta} \Phi_V(x) \int_{\beta/a\leq\delta(y)\leq\beta} G_V(x,y)\chi_{a\beta}(|x-y|)d\tau(y)dS_x,$$
$$I_{1,2}(\beta) := \frac{1}{\beta}\int_{\Sigma_\beta} \Phi_V(x) \int_{\delta(y)\leq\beta/a} G_V(x,y)\chi_{a\beta}(|x-y|)d\tau(y)dS_x,$$
$$I_{1,3}(\beta) := \frac{1}{\beta}\int_{\Sigma_\beta} \Phi_V(x) \int_{\beta\leq\delta(y)} G_V(x,y)\chi_{a\beta}(|x-y|)d\tau(y)dS_x$$

so that $I_1 = I_{1,1} + I_{1,2} + I_{1,3}$.

Estimate of $I_{1,1}(\beta)$.

By the Hardy (chain) inequality (see e.g. [16, Lemma 3.2]) , there exists $C(a) > 0$ such that, if

$$\beta/a \leq \delta(y) \leq \beta, \quad x \in \Sigma_\beta, \quad |x-y| \leq a\beta \tag{*}$$

then

$$\frac{1}{C(a)}\Phi_V(x) \leq \Phi_V(y) \leq C(a)\Phi_V(x). \tag{5.2}$$

By Theorem 2.2, if (*) holds then

$$\frac{1}{c}|x-y|^{2-N} \leq G_V(x,y) \leq c|x-y|^{2-N}$$

for some constant $c = c(a)$. Hence,

$$I_{1,1}(\beta) \sim \frac{1}{\beta}\int_{\Sigma_\beta}\int_{\beta/a\le\delta(y)\le\beta} |x-y|^{2-N}\chi_{a\beta}(|x-y|)\Phi_V(y)d\tau(y)dS_x \tag{5.3}$$
$$= \frac{1}{\beta}\int_{\beta/a\le\delta(y)\le\beta}\int_{\Sigma_\beta} |x-y|^{2-N}\chi_{a\beta}(|x-y|)dS_x\Phi_V(y)d\tau(y)$$

By (*),

$$\int_{\Sigma_\beta} |x-y|^{2-N}\chi_{a\beta}(|x-y|)dS_x \lesssim \int_0^{a\beta} dr = a\beta, \tag{5.4}$$
$$(a-1)\beta \le \int_{[\beta-\delta(y)]}^{a\beta} dr \lesssim \int_{\Sigma_\beta} |x-y|^{2-N}\chi_{a\beta}(|x-y|)dS_x.$$

Hence by (5.3),

$$I_{1,1}(\beta) \sim \int_{\beta/a\le\delta(y)\le\beta} \Phi_V d\tau \tag{5.5}$$

with similarity constant depending on a and $\partial\Omega$.

Estimate of $I_{1,2}(\beta)$. Here we assume that $\beta < r_0/3a$. Since $\delta(y) < \beta/a$ it follows that in the domain of integration of $I_{1,2}$,

$$(a-1)\delta(y) \le \beta - \delta(y) \le |x-y| < a\beta. \tag{5.6}$$

Thus the pair (x, y) satisfies the conditions of Theorem 2.3 and consequently,

$$\Phi_V(x)G_V(x,y) \sim \frac{\Phi_V(x)^2}{\Phi_V(x_y)^2}\Phi_V(y)|x-y|^{2-N}, \tag{5.7}$$

where x_y may be chosen as follows: $x_y := \eta + |x-y|\mathbf{n}_\eta$ with $\eta \in \partial\Omega$ the closest point to y.

Then, by (5.6), $\delta(x_y) = |x-y| \sim \beta$ and $|x_y - x| \le a\beta$. Hence, by the strong Hardy inequality, there exists $c'(a) > 0$ such that

$$\frac{1}{c'}\Phi_V(x) \sim \Phi_V(x_y) \le c'\,\Phi_V(x).$$

Therefore,

$$\Phi_V(x)G_V(x,y) \sim \frac{\Phi_V(x)^2}{\Phi_V(x_y)^2}\Phi_V(y)|x-y|^{2-N} \tag{5.8}$$
$$\sim |x-y|^{2-N}\Phi_V(y).$$

with similarity constant depending on a. It follows that,

$$I_{1,2}(\beta) \sim \frac{1}{\beta}\int_{\Sigma_\beta}\int_{\delta(y)\le\beta/a} |x-y|^{2-N}\chi_{a\beta}(|x-y|)\Phi_V(y)d\tau(y)dS_x. \tag{5.9}$$

By (5.4) and (5.9) we conclude that,

$$I_{1,2}(\beta) \sim \int_{\delta(y)\le\beta/a} \Phi_V(y)d\tau(y). \tag{5.10}$$

Estimate of $I_{1,3}(\beta)$.

In this case, as $|x-y| < a\beta$ and $\delta(y) > \beta$ inequality (5.2) holds. Moreover, by Theorem 2.2 the inequality below holds:

$$\frac{1}{c}|x-y|^{2-N} \leq G_V(x,y) \leq c|x-y|^{2-N}.$$

Therefore, as in (5.3), we obtain

$$\begin{aligned} I_{1,3}(\beta) &\sim \frac{1}{\beta}\int_{\Sigma_\beta}\int_{\beta\leq\delta(y)} |x-y|^{2-N}\chi_{a\beta}(|x-y|)\Phi_V(y)d\tau(y)dS_x \\ &\sim \frac{1}{\beta}\int_{\beta\leq\delta(y)}\int_{\Sigma_\beta} |x-y|^{2-N}\chi_{a\beta}(|x-y|)dS_x\Phi_V(y)d\tau(y) \\ &\lesssim \int_{\beta\leq\delta(y)} \Phi_V d\tau \end{aligned} \tag{5.11}$$

We also have a (partial) estimate from below.

If y is a point such that $\beta \leq \delta(y) < \frac{3a}{2}\beta$ then $B_{a\beta}(y)\cap\Sigma_\beta$ contains an $(N-1)$ dimensional ball of radius $\beta/2$ and consequently there exists a constant $c_3(a) > 0$ such that

$$\int_{\Sigma_\beta} |x-y|^{2-N}\chi_{a\beta}(|x-y|)dS_x > c_3.$$

Therefore

$$c_3\int_{\beta\leq\delta(y)<\frac{3a}{2}\beta} \Phi_V d\tau \leq I_{1,3}(\beta) \tag{5.12}$$

In conclusion, there exists a constant $c > 0$ such that (5.1) holds. □

Lemma 5.2. *Assume (A1), (A2). In addition assume that there exist* α, α^* *such that:*

$$0 < \alpha - 1/2 < \alpha^* \leq \alpha, \quad \frac{1}{c}\delta^\alpha \leq \Phi_V \leq c\delta^{\alpha^*} \quad in\ \ \Omega_{r_0}. \tag{B2}$$

Then there exists $C > 0$ *such that for every* $\tau \in \mathfrak{M}_+(\Omega;\Phi_V)$ *and* $\beta \in (0, r_0)$,

$$\begin{aligned} I_{2,\lambda}(\beta) :=& \frac{1}{\beta^\lambda}\int_{\Sigma_\beta}\Phi_V(x)\int_\Omega G_V(x,y)(1-\chi_{a\beta}(|x-y|))d\tau(y)dS_x \\ \leq& C\int_\Omega \Phi_V d\tau, \end{aligned} \tag{5.13}$$

where $\lambda := 2(\alpha^* - \alpha) + 1 > 0$.

Proof. Since $\delta(x) = \beta$ and $|x-y| \geq a\beta$,

$$a\inf(\delta(x),\delta(y)) \leq |x-y|.$$

Therefore, as before, we can estimate $G_V(x,y)$ by (5.7). Here we choose $x_y = \eta' + |x-y|\mathbf{n}_{\eta'}$ where η' is the nearest point to x on $\partial\Omega$. Thus x and x_y are on a normal to $\partial\Omega$ and $|x-y| = \delta(x_y) \geq a\delta(x)$.

By assumption (B2),

$$\frac{\Phi_V(x)}{\Phi_V(x_y)} \leq c(a)\frac{\beta^{\alpha^*}}{|x-y|^\alpha}.$$

Hence, by the first relation in (5.8)

$$
\begin{aligned}
I_{2,0} &\lesssim \int_{\Sigma_\beta\cap[|x-y|>a\beta]} \frac{\Phi_V(x)^2}{\Phi_V(x_y)^2}|x-y|^{2-N}\int_\Omega \Phi_V(y)d\tau(y)\,dS_x \\
&\lesssim \beta^{2\alpha^*}\int_{\Sigma_\beta\cap[|x-y|>a\beta]} |x-y|^{2-N-2\alpha}\int_\Omega \Phi_V(y)d\tau(y)\,dS_x \\
&\lesssim \beta^{2\alpha^*}\int_{a\beta}^1 r^{-2\alpha}dr\int_\Omega \Phi_V(y)d\tau(y) \\
&\lesssim \beta^{2\alpha^*-2\alpha+1}\int_\Omega \Phi_V(y)d\tau(y).
\end{aligned} \tag{5.14}
$$

Therefore (5.13) holds with $\lambda = 2\alpha^* - 2\alpha + 1$. □

Proof of Theorem 3.3. By Lemma 5.1,

$$\frac{1}{c}\int_{\Omega_{3a\beta/2}} \Phi_V d\tau \le I_1(\beta)$$

for every $\beta < r_0/3a$. Therefore for every $r_0/6a < \beta < r_0/3a$,

$$\frac{1}{c}\int_{\Omega_{r_0/4}} \Phi_V d\tau \le I_1(\beta).$$

This implies (3.3). More precisely, there exists a constant c^* depending only on a, $\bar{a}$, r_0, κ such that

$$c^*\int_{\Omega_{r_0/4}} \Phi_V\,d\tau \le \int_{[\frac{r_0}{6a}<\delta<\frac{r_0}{3a}]} \frac{\Phi_V}{\Phi_0}\mathbb{G}_V[\tau]dx. \tag{5.15}$$

A suitable constant is given by $c^* = (\inf_{\Omega_{r_0/4}} H)^{-1}\frac{r_0}{6ac}$ where H is the Jacobian of the transformation from Euclidean coordinates to flow coordinates (δ,σ). It is known that $H(x)\to 1$ as $\delta(x)\to 0$.

Put $\tau' = \tau\mathbf{1}_{[\delta\ge r_0/4]}$. Then

$$
\begin{aligned}
&\int_\Omega \frac{\Phi_V}{\Phi_0}\int_\Omega G_V(x,y)d\tau'(y)dx = \int_\Omega\int_\Omega \frac{\Phi_V}{\Phi_0}G_V(x,y)dxd\tau'(y) \ge \\
&\int_{[\delta(y)\ge r_0/4]}\int_{|x-y|<r_0/8} \frac{\Phi_V}{\Phi_0}(x)G_V(x,y)dxd\tau'(y) \ge \\
&c_1\int_{[\delta(y)\ge r_0/4]}\int_{|x-y|<r_0/8} \frac{\Phi_V}{\Phi_0}(x)|x-y|^{2-N}dxd\tau'(y) \ge \\
&c_2\int_{[\delta(y)\ge r_0/4]}\int_{|x-y|<r_0/8} |x-y|^{2-N}dx\Phi_V(y)d\tau'(y) \ge \\
&c_3\int_{[\delta\ge r_0/4]} \Phi_V d\tau' \ge c_4\int_\Omega \Phi_V d\tau',
\end{aligned} \tag{5.16}
$$

the constants depending only on a, $\bar{a}$, r_0, κ. (By Harnack: $\Phi_V(x)\ge c\Phi_V(y)$ in the domain of integration above.) Combining (5.15) and (5.16) we obtain (3.4). □

Proof of Theorem 3.4. By (5.1) and (5.13)

$$I_1(\beta) \le c_1 \int_\Omega \Phi_V \, d\tau, \quad \text{and} \quad I_{2,\lambda}(\beta) \le c_2 \int_\Omega \Phi_V \, d\tau$$

for every $\beta \in (0, r_1)$ where $r_1 := r_0/12\kappa$. Therefore

$$\int_{\Sigma_\beta} \frac{\Phi_V}{\beta} \mathbb{G}_V[\tau] dx \le I_1(\beta) + \beta^{\lambda-1} I_{2,\lambda}(\beta) \le c \max(1, \beta^{\lambda-1}) \int_\Omega \Phi_V \, d\tau$$

where λ is a positive number. Consequently, integrating over β in $(0, r_1)$,

$$\int_{\Omega_{r_1}} \frac{\Phi_V}{\Phi_0} \mathbb{G}_V[\tau] dx \le C_1 \int_\Omega \Phi_V \, d\tau \tag{5.17}$$

where C_1 depends on $\bar{a}$, r_0, κ, α^*.

Therefore, to obtain (3.5), it remains to show that

$$\int_{\mathcal{D}_{r_1}} \frac{\Phi_V}{\Phi_0} \mathbb{G}_V[\tau] dx \le C_2 \int_\Omega \Phi_V \, d\tau \tag{5.18}$$

with C_2 depending on the parameters mentioned above.

Recall that the ground state is normalized by $\Phi_V(x_0) = 1$. Therefore, by Harnack's inequality, it follows that $\Phi_V \sim 1$ in D_{r_1}, i.e, Φ_V is bounded and bounded away from zero in D_{r_1} by constants depending only on $\bar{a}, r_0, \kappa$.

Let $r_2 = r_1/2$ and write,

$$\begin{aligned} \int_{\mathcal{D}_{r_1}} \frac{\Phi_V}{\Phi_0} \mathbb{G}_V[\tau] dx = & \int_{\mathcal{D}_{r_1}} \frac{\Phi_V}{\Phi_0} \int_{D_{r_2}} G_V(x,y) d\tau(y) dx \\ & + \int_{\mathcal{D}_{r_1}} \frac{\Phi_V}{\Phi_0} \int_{\Omega \setminus D_{r_2}} G_V(x,y) d\tau(y) dx =: J_1 + J_2. \end{aligned}$$

In J_2, $x \in \mathcal{D}_{r_1}$ and $y \in \Omega \setminus \mathcal{D}_{r_2}$. Therefore $G_V(x,y) \sim \Phi_V(y)$. Consequently

$$J_2 \lesssim \int_{\Omega \setminus D_{r_2}} \Phi_V \, d\tau. \tag{5.19}$$

In J_1 $x, y \in D_{r_2}$ and therefore, by Theorem 2.2,

$$J_1 \lesssim \int_{D_{r_1}} \int_{D_{r_2}} |x-y|^{2-N} d\tau(y) dx \lesssim \tau(D_{r_2}) \lesssim \int_{D_{r_2}} \Phi_V d\tau. \tag{5.20}$$

Combining these inequalities we obtain (5.18). □

Proof of Corollary 3.5. Given $\epsilon > 0$ choose $\beta_\epsilon > 0$ sufficiently small so that

$$\int_{\bar{\Omega}_{\beta_\epsilon}} \Phi_V \, d\tau < \epsilon.$$

Put

$$\tau_1 = \tau \mathbf{1}_{\bar{\Omega}_{\beta'}}, \quad \tau_2 = \tau - \tau_1 \quad u_i = \mathbb{G}_V[\tau_i].$$

By Theorem 3.4

$$J_1 := \int_{\Sigma_\beta} \frac{\Phi_V}{\delta} u_1 \, dS \le c' \epsilon \quad \forall \beta \in (0, \beta_\epsilon). \tag{5.21}$$

If E is a compact subset of Ω, there exists a constant c_E such that, for every $y \in E$,

$$\frac{1}{c_E}\Phi_V(x) \leq G_V(x,y) \leq c_E \Phi_V(x) \quad \text{when} \quad \delta(x) < \frac{1}{2}\mathrm{dist}\,(E, \partial\Omega).$$

Consequently, for every $y \in \Omega$ such that $\delta(y) \geq \beta_\epsilon$

$$G_V(x,y) \lesssim \Phi_V(x) \quad \forall x \in \Omega_{\beta_\epsilon/2} \tag{5.22}$$

with similarity constant independent of y. Hence

$$u_2 \sim \Phi_V \|\tau_2\| \quad \text{in } \Omega_{\beta_\epsilon/2}. \tag{5.23}$$

Therefore, by condition (B1),

$$J_2 := \int_{\Sigma_\beta} \frac{\Phi_V}{\delta} u_2 dS_x \to 0 \quad \text{as} \quad \beta \to 0.$$

This fact and (5.21) imply (3.6). □

6. L_V superharmonic and subharmonic functions

Assume that V satisfies conditions (A1), (A2).

Let $D \Subset \Omega$ be a Lipschitz domain, denote by P_V^D the Poisson kernel of L_V in D and by $\omega_V^{x_0,D}$ the harmonic measure of L_V on ∂D relative to a fixed reference point $x_0 \in D$. Then,

$$d\omega_V^{x_0,D} = P_V^D(x_0, \cdot)dS. \tag{6.1}$$

Lemma 6.1. (i) *Let u be positive L_V superharmonic and denote:*

$$w := \sup\{v \leq u : v \text{ is } L_V \text{ harmonic}\}.$$

Then w is L_V harmonic.

(ii) *Let u be positive L_V subharmonic. Assume that there exists a positive L_V superharmonic function dominating u. Then*

$$w := \inf\{v \geq u : v \text{ is } L_V \text{ superharmonic}\}$$

is L_V harmonic.

Proof. (i) Let $\{D_n\}$ be a smooth exhaustion of Ω and let

$$u_n(x) = \int_{\partial D_n} P_V^{D_n}(x,\xi)h_n(\xi)dS_\xi \quad \forall x \in D_n, \quad h_n = u\lfloor_{\partial D_n}. \tag{6.2}$$

where h_n is the trace (in the Sobolev sense) of u on $\partial\Omega_n$. Then u_n is L_V harmonic, $u_n \leq u$ and $\{u_n\}$ is decreasing. Therefore the limit $\underline{v} := \lim u_n$ is L_V harmonic. Clearly $\underline{v}$ is the largest L_V harmonic function dominated by u so that $w = \underline{v}$.

(ii) In this case $\{u_n\}$ is an increasing sequence which may tend to infinity. However, if u is dominated by an L_V superharmonic function, $\{u_n\}$ converges to an L_V harmonic function $\underline{w}$. If v is an L_V superharmonic function dominting u the $v \geq u_n$ for every n. Therefore $\underline{w}$ is the smallest L_V superharmonic function dominating u. Consequently $w = \underline{w}$. □

Proof of Proposition 3.6. (i) This statement is an immediate consequence of the Riesz decomposition lemma and the fact that u is an L_V potential if and only if $u = \mathbb{G}_V[\tau]$ for some $\tau \in \mathfrak{M}(\Omega; \Phi_V)$.
(ii) If $\tau \in \mathfrak{M}(\Omega; \Phi_V)$ then $u+\mathbb{G}_V[\tau]$ is positive L_V harmonic. By the Representation Theorem $\exists\ \nu \in \mathfrak{M}(\partial\Omega)$ such that $u + \mathbb{G}_V[\tau] = \mathbb{K}_V[\nu]$. In this case u is dominated by $\mathbb{K}_V[\nu]$.

Conversely, if u is dominated by an L_V superharmonic function v and u_n is defined as in Lemma 6.1 then $u_n < v$ and

$$u + G_V^{D_n}[\tau \mathbf{1}_{D_n}] = u_n \leq v \quad \text{in } D_n.$$

It follows that $\mathbb{G}_V[\tau] = \lim G_V^{D_n}[\tau \mathbf{1}_{D_n}] < \infty$, which implies $\tau \in \mathfrak{M}(\Omega; \Phi_V)$. □

Proof of Theorems 3.7 and 3.8. These theorems are a direct consequence of Theorems 3.1, 3.3 and 3.4 applied to (3.8) and (3.9). □

7. The harmonic measure and the measure $W\,dS$.

Assume that V satisfies conditions (A1), (A2).

Let $D \Subset \Omega$ be a Lipschitz domain, denote by P_V^D the Poisson kernel of L_V in D and by $\omega_V^{x_0,D}$ the harmonic measure of L_V in D relative to a fixed reference point $x_0 \in D$. Then,

$$d\omega_V^{x_0,D} = P_V^D(x_0,\cdot)dS \quad \text{on } \partial D. \tag{7.1}$$

Let $\{D_n\}$ be a uniformly Lipschitz exhaustion of Ω. It is well known that if u is a positive Δ-harmonic function then

$$u\lfloor_{\partial D_n} d\omega_0^{x_0,D_n} \rightharpoonup \nu \tag{7.2}$$

where $\nu \in \mathfrak{M}(\partial\Omega)$ is the boundary trace of u and $\rightharpoonup$ indicates weak convergence in measure. In [20, Definition 3.6], (7.2) was used as a definition of boundary trace for solutions of certain semilinear equations with absorption. In this spirit we define,

Definition 7.1. A non-negative Borel function u defined in Ω has an L_V *boundary trace* $\nu \in \mathfrak{M}(\partial\Omega)$ if

$$\lim_{n\to\infty} \int_{\partial D_n} hud\omega_V^{x_0,D_n} = \int_{\partial\Omega} hd\nu \quad \forall h \in C(\bar{\Omega}), \tag{7.3}$$

for every uniformly Lipschitz exhaustion $\{D_n\}$ of Ω. The L^V trace will be denoted by $\mathrm{tr}_V(u)$. Here we assume that $D_{\beta_0} \subset D_1$ and $x_0 \in D_{\beta_0}$.

In the present context a 'good' definition of trace should imply two basic statements: (a) if u is positive L_V harmonic then $\mathrm{tr}_V(u)$ should be defined and coincide with the measure in its Martin representation and (b) if u is an L_V potential its L_V trace should be zero. The next lemmas show that, with the above definition, these statements are valid.

Lemma 7.2. *If $\{D_n\}$ is a uniformly Lipschitz exhaustion of Ω then, for every positive L_V harmonic function $u = \mathbb{K}_V[\nu]$,*

$$\lim_{n\to\infty} \int_{\partial D_n} hu \, d\omega_V^{x_0,D_n} = \int_{\partial\Omega} h \, d\nu \quad \forall h \in C(\bar{\Omega}). \tag{7.4}$$

Proof. First observe that

$$u(x_0) = \int_{\partial D_n} u \, d\omega_V^{x_0,D_n} \tag{7.5}$$

and, as the Martin kernel is normalized by $K_V(x_0, y) = 1$ for every $y \in \partial\Omega$,

$$u(x_0) = \int_{\partial\Omega} K_V(x_0, y) \, d\nu(y) = \nu(\partial\Omega).$$

Thus (7.4) holds for $h \equiv 1$. By (7.1) the following sequence of measures is bounded:

$$\sigma_n = \begin{cases} u \, d\omega_V^{x_0,D_n} & \text{on } \partial D_n \\ 0 & \text{on } \bar{\Omega} \setminus \partial D_n \end{cases} \qquad n \in \mathbb{N}.$$

Let $\{\sigma_{n_k}\}$ be a weakly convergent subsequence with limit ν'. Then

$$\nu(\partial\Omega) = \nu'(\partial\Omega).$$

Let $F \subset \partial\Omega$ be a compact set and define

$$\nu^F = \nu \mathbf{1}_F, \quad u^F = \mathbb{K}_V[\nu^F].$$

Let σ_n^F be defined in the same way as σ_n with u replaced by u^F. Proceeding as before we obtain a weakly convergent subsequence of $\{\sigma_n^F\}$ with limit $\hat{\nu}_F$ supported in F. Furthermore,

$$\hat{\nu}_F(F) = \hat{\nu}_F(\partial\Omega) = u^F(x_0) = \nu^F(\partial\Omega) = \nu(F).$$

Since $u^F \le u$ it follows that $\hat{\nu}_F \le \nu'$. Consequently $\nu(F) \le \nu'(F)$. As this inequality holds for every compact subset of $\partial\Omega$ it follows that $nu \le \nu'$. As the measures are positive and $\nu(\partial\Omega) = \nu'(\partial\Omega)$ it follows that $\nu = \nu'$ and therefore the whole sequence $\{\sigma_n\}$ converges weakly to ν. □

Lemma 7.3. *Assume (A1) and (A2). Then*

$$\begin{aligned} &(a) \quad \mathrm{tr}_V(\mathbb{K}_V[\nu]) = \nu && \forall \nu \in \mathfrak{M}_+(\partial\Omega) \\ &(b) \quad \mathrm{tr}_V(\mathbb{G}_V[\tau]) = 0 && \forall \tau \in \mathfrak{M}_+(\Omega; \Phi_V). \end{aligned} \tag{7.6}$$

Proof. (a) is a restatement of Lemma 7.2. (b) follows from the fact that $\mathbb{G}_V[\tau]$ is an L_V potential , i.e., it does not dominate any positive L_V harmonic function (see [2]). Let u_n be defined as in (6.2), with $u = \mathbb{G}_V[\tau]$. Then $\lim u_n$ is the largest L_V harmonic function dominated by u, which in this case is the zero function. Thus,

$$u_n(x_0) = \int_{\partial D_n} \mathbb{G}_V[\tau]\lfloor_{\partial D_n} \, d\omega_V^{x_0,D_n} \to 0.$$

□

Lemma 7.4. *Let $u \in L^1_{loc}$ be a positive function such that $-L_V u =: \tau$ where $|\tau| \in \mathfrak{M}(\Omega; \Phi_V)$. Then u has an L_V boundary trace, say ν, and $u = \mathbb{G}_V[\tau] + \mathbb{K}_V[\nu]$.*

Proof. Put $U = u + G_V(|\tau| - \tau)$. Then U is positive, L_V superharmonic: $-L_V U = |\tau|$. By propostion 3.6(i), $\mathrm{tr}_V(U) =: \nu$ exists and $U = \mathbb{G}_V[\tau'] + \mathbb{K}_V[\nu]$. Thus $\tau' = |\tau|$ and

$$u = U - G_V(|\tau| - \tau) = \mathbb{G}_V[\tau] + \mathbb{K}_V[\nu].$$

Since τ is the difference of two positive measures in $\mathfrak{M}(\Omega; \Phi_V)$ it follows that $\mathrm{tr}_V(\mathbb{G}_V[\tau]) = 0$. □

A comparison of (7.6) (a) with Theorem 3.2 and of (7.6) (b) with Corrollary 3.5 indicates that the behavior of the measure WdS on Σ_β is similar to that of the harmonic measure near the boundary. The next result gives a precise meaning to this relation.

Theorem 7.5. *Assume (A1), (A2), (B1). For $y \in \partial\Omega$, denote by $\mathcal{C}_y$ the spherical cone with vertex y whose axis points in the direction of $\mathbf{n}_y$ and its opening angle is $\pi/4$. Then, there exists a positive constant C independent of $y \in \partial\Omega$ such that, for every $\beta \in (0, \beta_0)$,*

$$\frac{1}{C} W(y + \beta \mathbf{n}_y) \leq \frac{1}{\beta^{N-1}} \int_{\mathcal{C}_y \cap \Sigma_\beta} d\omega_V^{x_0, D_\beta} \leq C W(y + \beta \mathbf{n}_y). \tag{7.7}$$

Proof. We know that

$$J^\beta := \int_{\Sigma_\beta} K_V(x, y) d\omega_V^{x_0, D_\beta} = K_V(x_0, y) = 1 \quad \forall \beta \in (0, \beta_0). \tag{7.8}$$

We write J^β as a sum of two integrals:

$$J^\beta = \int_{\mathcal{C}_y \cap \Sigma_\beta} + \int_{\Sigma_\beta \setminus \mathcal{C}_y} K_V(x, y) d\omega_V^{x_0, D_\beta} =: J_1^\beta + J_2^\beta. \tag{7.9}$$

By BHP,

$$K_V(\cdot, y) \sim G_V(x, x_0) \quad \text{in } \Omega_\beta \setminus \mathcal{C}_y$$

uniformly with respect to y. Therefore, by (7.6) (b),

$$J_2^\beta \lesssim \int_{\Sigma_\beta} G_V(x, x_0) d\omega_V^{x_0, D_\beta} \to 0 \quad \text{as } \beta \to 0$$

uniformly with respect to y. Consequently, by (7.8) and (7.9),

$$J_1^\beta \to 1 \quad \text{as } \beta \to 0 \tag{7.10}$$

uniformly with respect to y. In addition, by the strong Harnack inequlity,

$$J_1^\beta \sim K_V(y + \beta \mathbf{n}_y, y) \int_{\mathcal{C}_y \cap \Sigma_\beta} d\omega_V^{x_0, D_\beta}$$

with similarity constants independent of y. Therefore there exists C_1 indpendent of y such that

$$\frac{1}{C_1} \leq K_V(y + \beta \mathbf{n}_y, y) \int_{\mathcal{C}_y \cap \Sigma_\beta} d\omega_V^{x_0, D_\beta} \leq C_1 \quad \forall \beta \in (0, \beta_0). \tag{EST1}$$

Similarly we denote,

$$\tilde{J}^\beta := \int_{\mathcal{C}_y \cap \Sigma_\beta} + \int_{\Sigma_\beta \setminus \mathcal{C}_y} K_V(x, y) W dS =: \tilde{J}_1^\beta + \tilde{J}_2^\beta.$$

By (3.1) $\tilde{J}_\beta$ is bounded above and below by positive constants independent of $y \in \partial\Omega$ and $\beta \in (0, \beta_0)$. By BHP and (1.4)

$$K_V(\cdot, y) \sim G_V(\cdot, x_0) \sim \Phi_V \quad \text{in } \Omega_\beta \setminus \mathcal{C}_y.$$

Therefore, by (B1),

$$\tilde{J}_2^\beta \lesssim \int_{\Sigma_\beta} \frac{\Phi_V^2}{\Phi_0} dS \to 0 \quad \text{as } \beta \to 0.$$

It follows that $\tilde{J}_1^\beta$ is bounded above and below by positive constants independent of y, β. By the strong Harnack inequality,

$$\tilde{J}_1^\beta = \int_{\Sigma_\beta \cap \mathcal{C}_y} K_V(x, y) W dS \sim K_V(y + \beta \mathbf{n}_y, y) W(y + \beta \mathbf{n}_y) \beta^{N-1},$$

with similarity constants independent of y, β. Therefore there exists a constant C_2 independent of y such that

$$\text{(EST2)} \qquad \frac{1}{C_2} \leq K_V(y + \beta \mathbf{n}_y, y) W(y + \beta \mathbf{n}_y) \beta^{N-1} \leq C_2 \quad \forall \beta \in (0, \beta_0).$$

Combining (EST1) and (EST2) we obtain (7.7). □

8. Examples

Let Ω be a smooth bounded domain, $F \subset \partial\Omega$ a compact set. Denote

$$V_F = \frac{1}{\delta_F^2}, \quad \delta_F(x) = \operatorname{dist}(x, F),$$

$$c_H(V_F) = \inf_{C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega V_F \varphi^2}.$$

1. The model case $V = \gamma/\delta^2$ with $\gamma < C_H^\Omega$ (= Hardy constant for the domain). In every smooth domain $C_H^\Omega \leq 1/4$.

Here $\Phi_V \sim \delta^\alpha$, $\alpha > 1/2$, $\quad \forall \gamma < C_H^\Omega$. Thus (B1), (B2) hold.

2. $V = \gamma/\delta_F^2$ where $\emptyset \neq F$ is an arbitrary compact subset of $\partial\Omega$ and $0 < \gamma < C_H^\Omega$. Then,

$$0 < V_F < \gamma/\delta^2 \Longrightarrow \delta \leq \Phi_{V_F} \leq \delta^{\alpha^*}$$

for some $\alpha^* > 1/2$. Therefore (B1), (B2) hold.

If $\gamma < 0$:

$$-\gamma/\delta^2 \leq V_F \leq 0 \Longleftrightarrow \delta^\alpha \lesssim \Phi_V \lesssim \delta, \quad \alpha = \frac{1}{2} + \sqrt{\frac{1}{4} - \gamma} > 1.$$

(B1) always holds. (B2) holds if $\alpha < 3/2$, i.e., $\gamma > -3/4$. In summation,

$$(*) \qquad \text{(B1) holds} \quad \forall \gamma < C_H^\Omega, \quad \text{(B2) holds} \quad \forall \gamma \in (-\frac{3}{4}, C_H^\Omega).$$

3. $V = \gamma/\delta^2_{F_k}$, $0 < k < N-1$ and $\gamma < \min(C_H(V_{F_k}), \frac{2(N-k)-1}{4}) =: \gamma_k$. $F_k \subset \partial\Omega$ is a smooth k-dimensional manifold without boundary and

$$C_H(V_{F_k}) = \inf_{\phi \in C^1_c(\Omega)} \frac{\int_\Omega |\nabla\phi|^2\, dx}{\int_\Omega \phi^2/\delta^2_{F_k} dx}. \tag{8.1}$$

In a neighborhood of F_k,

$$\Phi_V \sim \delta\delta^\alpha_{F_k} \quad \text{where} \quad \alpha = \frac{1}{2}(k - N + \sqrt{(k-N)^2 - 4\gamma}).$$

The restriction on γ implies $\alpha > -1/2$. Therefore, if $0 \le \gamma$ then, in a neighborhood of F_k, $\delta \lesssim \Phi_V \lesssim \delta^{\alpha'}$ for some $\alpha' > 1/2$. Thus (B1), (B2) hold.

When $\gamma < 0$, (B1) always holds and the argument in **2.** shows that (B2) holds when $\gamma > -3/4$. In summation,

$$\text{(B1) holds} \quad \forall\gamma < \gamma_k, \quad \text{(B2) holds} \quad \forall\gamma \in (-\frac{3}{4}, \gamma_k) \tag{**}$$

As expected the restriction is weaker then in **2.**

4. $V = \gamma/\delta^2_F$ where F is an arbitrary compact subset of a manifold F_k (defined in **3.**), $0 < k \le N-1$.
If $k = N-1$ then $F_k = \partial\Omega$ and $(*)$ holds.
If $0 < k < N-1$ then again (**) holds.

In these examples, the conditions imposed on γ are sufficient but, in most cases, far from optimal. Therefore there is need for further research concentrating on special (more restricted) families of potentials.

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