Pure and Applied Functional Analysis
Volume 5, Number 5, 2020, 1189–1197

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

EXISTENCE FOR A FREE BOUNDARY PROBLEM DESCRIBING A PROPAGATING DISTURBANCE

GABRIELA MARINOSCHI

ABSTRACT. We prove the existence of a solution to an 1-D free boundary problem which describes the propagation of disturbances of shock type, modeled by a non standard variational inequality.

1. Introduction

This paper concerns the existence of a solution to the differential inclusion

$$\begin{aligned} \frac{dL}{dt}(t) + \partial I_K(\Gamma(t, L(t))) &\ni U(t, L(t)), \text{ a.e. } t \in (0, T), \\ L(0) &= L_0 \geq 0, \end{aligned} \tag{1.1}$$

where Γ and U depending on t and x are given functions, K is the set $\{z \in \mathbb{R}; z \geq \Gamma^* > 0\}$ and ∂I_K is the subdifferential of I_K, the indicator function of K. We note that (1.1) can be equivalently written as

$$\frac{dL}{dt}(t) = U(t, L(t)), \quad \text{in } \{t \geq 0;\ \Gamma(t, L(t)) > \Gamma^*\}, \tag{1.2}$$

$$\frac{dL}{dt}(t) \geq U(t, L(t)), \quad \text{in } \{t \geq 0;\ \Gamma(t, L(t)) = \Gamma^*\}. \tag{1.3}$$

The variational inequality (1.1) can describe a discontinuity occurring at the surface $L(t)$ of a system of particles moving with the velocity $U(t, x)$, when an intrinsic constraint forces the particles lying on the surface $L(t)$ to advance with a velocity greater than $U(t, L(t))$. More precisely, a particle on the surface $x = L(t)$ moves at each time with the surface velocity $U(t, L(t))$ as long as the function $\Gamma(t, L(t))$ exceeds a prescribed value Γ^*, but it is pushed out from the surface with a velocity greater than $U(t, L(t))$ if $\Gamma(t, L(t))$ decreases up to Γ^*, or below it, at a moment t. In other words, $L(t)$ remains a material surface advancing with the group velocity as well as $\Gamma(t, L(t))$ is larger than Γ^* and exhibits a discontinuous behavior if $\Gamma(t, L(t))$ is equal or lower than Γ^*.

The starting point for the study of such a variational inequality was an 1-D model of epidermis cell growth introduced in [6] for the stationary case, and developed in [7] for the dynamical case. We stress that the current approach (1.1) of the behavior of the 1-D epidermis free boundary is different by that treated in [6] and [7], where another process modeled by a reversed inequality was studied.

2010 *Mathematics Subject Classification.* 35R35, 35R37, 35R70.
Key words and phrases. Free boundary problem, differential inclusions, variational inequalities.

Since the study of (1.1) is related to this 1-D model, the analysis was restricted here to the 1-D case. In few words, the growth of the epidermis, viewed as a body of different type of cells, takes place with the group velocity $U(t,x)$. The bonds between cells maintain the tissue firm if the cohesion between cells represented by the function $\Gamma(t,x)$ has values greater than the critical threshold Γ^*. Thus, at each time t, the cells found at the position $L(t)$, representing the free surface, advance with the velocity $U(t,L(t))$, according to (1.2). In this case, the boundary is material and this assumption was often used in the literature for a cell system advance (see e.g. [5]). By cell ageing, the cohesion decreases and it is lost when Γ reaches Γ^* at $L(t)$, producing a shock-type action for the surface velocity, which is a process described by our current model. The particles lying on the surface $L(t)$ at that moment are detached from the system and are thrown outside it in the direction of movement, the surface velocity exhibiting a jump described by (1.3).

Another interpretation of the variational inequality (1.1) may be related to hysteresis processes, because (1.1) is similar to the differential inclusions arising in mathematical models of hysteresis of stop type used in the study of rheological models (see e.g., [9], p. 25).

We note that (1.2)-(1.3) is a free boundary problem, because the set $\{t;\Gamma(t,L(t))=\Gamma^*\}$ and the domain $\{t;\ \Gamma(t,L(t))>\Gamma^*\}$ are unknown. We shall prove below the existence of a solution to (1.1) in a certain generalized (distributional) sense. To this purpose, first we introduce some notation and recall a few definitions. We shall use results from the monographs [3] and [4], but as they already belong to a classical knowledge, we do no longer provide the specific citations. The main existence result is provided in Section 2.

1.1. **Preliminaries.** The indicator function $I_K : K \to (-\infty,\infty]$ equals zero at a point of K and $+\infty$ otherwise. Then, $\partial I_K : K \to 2^{\mathbb{R}}$ is defined by

$$\partial I_K(\zeta) = \{\chi \in \mathbb{R};\ \chi(\zeta-\overline{\zeta}) \geq 0,\ \forall \overline{\zeta} \in K\}.$$

We recall that $\partial I_K(\zeta) = N_K(\zeta) \subset \mathbb{R}$, the normal cone to K at ζ. We have

$$N_K(\zeta) = \{\chi \in \mathbb{R};\ \chi \leq 0 \text{ if } \zeta = \Gamma^* \text{ and } \chi = 0 \text{ if } \zeta > \Gamma^*\}.$$

Also, we denote by $\mathcal{K}$ the set

$$\mathcal{K} = \{z \in L^\infty(0,T);\ z(t) \geq \Gamma^* \text{ a.e. } t \in (0,T)\} \tag{1.4}$$

and by $\mathcal{N}_{\mathcal{K}}(z) \subset (L^\infty(0,T))^*$ the corresponding normal cone at $z \in \mathcal{K}$, that is

$$\mathcal{N}_{\mathcal{K}}(z) = \{\eta \in (L^\infty(0,T))^*;\ \eta(z-y) \geq 0,\ \forall y \in \mathcal{K}\}, \tag{1.5}$$

where $\eta(z-y)$ is the value of the functional $\eta \in (L^\infty(0,T))^*$ at $(z-y) \in L^\infty(0,T)$ (see [2], p. 242-244). We note that $v \in \mathcal{N}_{\mathcal{K}}(z) \cap L^1(0,T)$ iff $v(t) \in N_K(z(t))$, a.e. $t \in (0,T)$.

Here, $(L^\infty(0,T))^*$ is the dual of the space $L^\infty(0,T)$ of essentially bounded functions on $(0,T)$. The space $(L^\infty(0,T))^*$ is a linear subspace of $\mathcal{M}([0,T]) = (C[0,T])^*$, the space of bounded Radon measures on $[0,T]$. We note that by the Lebesgue decomposition theorem (see e.g. [8]), every $\mu \in (L^\infty(0,T))^*$ can be uniquely written as $\mu = \mu_a + \mu_s$, where $\mu_a \in L^1(0,T)$ and μ_s is a singular measure (that is there exists a Lebesgue measurable set $S \subset [0,T]$ with $\text{meas}([0,T]\backslash S) = 0$ and $\mu_s(\varphi) = 0$

for all $\varphi \in L^\infty(S)$). This means that μ_s has the support on a set of zero measure $([0,T]\backslash S)$.

We denote by $BV([0,T])$ the space of functions $v : [0,T] \to \mathbb{R}$ with bounded variation, that is

$$\|v\|_{BV([0,T])} = \sup\left\{\sum_{i=0}^{N-1} |v(t_{i+1}) - v(t_i)|\,;\; 0 = t_0 < t_1 < ... < t_N = T\right\} < \infty.$$

Every $v \in BV([0,T])$ has a unique decomposition $v = v^a + v^s$, where $v^a \in AC[0,T]$ and $v^s \in BV([0,T])$, (see e.g., [1]). Here, $AC[0,T]$ is the space of absolutely continuous functions on $[0,T]$ and v^s is a singular part (for instance it can be a jump function with bounded variation or a function with bounded variation with a.e. zero derivative).

We note that if $v \in BV([0,T])$, then its distributional derivative $\frac{dv}{dt} := \mu$ belongs to $(L^\infty(0,T))^*$, and in virtue of the Lebesgue decomposition, we have the following representation, as the sum of the absolutely continuous part (in the sense of measure) and the singular part

$$\frac{dv}{dt} = \mu_a + \mu_s = \frac{dv^a}{dt} + \frac{dv^s}{dt} \in \mathcal{D}'(0,T), \tag{1.6}$$

where $\mathcal{D}'(0,T)$ is the space of Schwartz distributions on $(0,T)$.

2. The main result

We shall assume that the following hypotheses hold for the functions occurring in (1.1):

$$\begin{aligned} &U \in C([0,\infty) \times [0,\infty)),\\ &x \to U(t,x) \text{ is Lipschitz with the Lipschitz constant } U_{Lip},\\ &0 \le U(t,x),\ \text{for all } (t,x) \in [0,\infty) \times [0,\infty),\\ &0 \le U(t,0) \le U_{0\max},\ \text{for all } t \ge 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &\Gamma \in C([0,\infty) \times [0,\infty)),\ \frac{\partial\Gamma}{\partial t} \in L^\infty((0,\infty) \times (0,\infty)),\\ &0 \le \Gamma(t,x) \le \Gamma_{\max},\ \text{for all } (t,x) \in [0,\infty) \times [0,\infty),\\ &\left|\frac{\partial\Gamma}{\partial t}(t,x)\right| \le C_\Gamma,\ \text{for all } (t,x) \in (0,\infty) \times [0,\infty), \end{aligned} \tag{2.2}$$

and

$$\eta_* := \inf_{y \ge L_0}\left\{\frac{1}{y}\int_{L_0}^{y} \Gamma(t,\sigma)d\sigma\right\},\ \eta_* > \Gamma^* > 0,\ L_0 \ge 0. \tag{2.3}$$

An example of a function Γ complying with these hypotheses is of the form $\Gamma(t,\sigma) = a(t) - 2\gamma_0\sigma e^{-\sigma^2}$, where $a(t) > \eta_*$ and $\gamma_0 > 0$. Then,

$$\inf\left\{\frac{1}{y}\int_0^y \Gamma(t,\sigma)d\sigma\right\} = \inf\left\{a(t) - \gamma_0\frac{1 - e^{-y^2}}{y}\right\} = a(t) = \eta_* > \Gamma^*.$$

In particular, for a a positive constant, the graphic of such a function may appropriately describe the behavior of the cohesion function in the cell growth model discussed in the introduction.

Definition 2.1. The function $L : [0, T] \to \mathbb{R}$ is called a solution to the variational inequality (1.1) if the following conditions hold:

$$L \in BV([0,T]),\ L(0) = L_0,\ L = L^a + L^s, \tag{2.4}$$

$$\begin{aligned} \frac{dL^a}{dt} + \mu_a(t) &= U(t, L(t)),\ \text{a.e. } t \in (0,T), \\ \frac{dL^s}{dt} + \mu_s &= 0, \text{ in } \mathcal{D}'(0,T), \end{aligned} \tag{2.5}$$

$$\begin{aligned} \mu_a &\in L^1(0,T),\ \mu_s \in (L^\infty(0,T))^*, \\ \mu_a(t) &\in N_K(\Gamma(t, L(t))),\ \text{a.e. } t \in (0,T),\ \mu_s \in \mathcal{N}_{\mathcal{K}}(\Gamma(\cdot, L(\cdot))). \end{aligned} \tag{2.6}$$

Here, $L^a \in AC[0, T]$ is the absolutely continuous part of L and L^s is the singular part, while μ_a and μ_s are the absolutely continuous and singular parts, respectively of $\mu \in (L^\infty(0,T))^*$.

Theorem 2.2. *Under the assumptions (2.1)-(2.3), the variational inequality (1.1) has at least one solution, $L \in BV([0,T])$, that is $L = L^\alpha + L^s$, satisfying the equations*

$$\frac{dL^a}{dt}(t) + \mu_a(t) = U(t, L(t)),\ \textit{a.e. } t \in (0,T), \tag{2.7}$$

$$\frac{dL^s}{dt} + \mu_s = 0, \qquad \textit{in } \mathcal{D}'(0,T), \tag{2.8}$$

where

$$\mu_a(t) \in N_K(\Gamma(t, L(t)),\ \textit{a.e. } t \in (0,T), \tag{2.9}$$

and

$$\mu_s \in \mathcal{N}_{\mathcal{K}}(\Gamma(\cdot, L(\cdot))). \tag{2.10}$$

Proof. For $\varepsilon > 0$ we introduce the Yosida approximation of ∂I_K,

$$(\partial I_K)_\varepsilon(z) = \frac{1}{\varepsilon}(I - (I + \varepsilon \partial I_K)^{-1})z,$$

and denote by $P_K z$ the projection of $z \in \mathbb{R}$ on K, given by

$$P_K z = \begin{cases} z, & z > \Gamma^* \\ \Gamma^*, & z \le \Gamma^*. \end{cases}$$

We recall that

$$(I + \varepsilon \partial I_K)^{-1} z = P_K z,\ \text{for all } z \in \mathbb{R}.$$

This implies that

$$(\partial I_K)_\varepsilon(z) = \begin{cases} 0, & z > \Gamma^* \\ \frac{1}{\varepsilon}(z - \Gamma^*), & z \le \Gamma^* \end{cases} = -\frac{1}{\varepsilon}(z - \Gamma^*)^-,\ \forall z \in \mathbb{R}.$$

Here, $(\cdot)^-$ represents the negative part.

Let us consider the approximating problem

$$\begin{aligned}(2.11)\qquad \frac{dL_\varepsilon}{dt}(t) + (\partial I_K)_\varepsilon(\Gamma(t, L_\varepsilon(t))) &= U(t, L_\varepsilon(t)),\ \text{a.e. } t \in (0, T),\\ L_\varepsilon(0) &= L_0 \geq 0.\end{aligned}$$

It is obvious that (2.11) has a unique solution $L_\varepsilon \in C^1[0, T]$ satisfying

$$L_\varepsilon(t) = L_0 + \int_0^t (U(s, L_\varepsilon(s)) - \mu_\varepsilon(s))ds,\ \text{for all } t \in [0, T],$$

where,

$$(2.12)\qquad \mu_\varepsilon(t) := (\partial I_K)_\varepsilon(z_\varepsilon(t)),\quad z_\varepsilon(t) := \Gamma(t, L_\varepsilon(t)).$$

Moreover, since $U - \mu_\varepsilon$ is positive it follows that $L_\varepsilon(t) \geq L_0$, for all $t \in [0, T]$.

To continue the proof we need some estimates. We multiply (2.11) by $(\Gamma(s, L_\varepsilon(s)) - \alpha)$, where α is a positive constant which will be specified a little later, and integrate on $(0, t)$ to obtain

$$\begin{aligned}(2.13)\qquad &\int_0^t \frac{dL_\varepsilon}{ds}(s)\Gamma(s, L_\varepsilon(s))ds - \int_0^t \alpha\frac{dL_\varepsilon}{ds}(s)ds + \int_0^t \mu_\varepsilon(s)(z_\varepsilon(s) - \alpha)ds\\ &= \int_0^t U(s, L_\varepsilon(s))(\Gamma(s, L_\varepsilon(s)) - \alpha)ds.\end{aligned}$$

We denote by j the potential of the function $\sigma \to \Gamma(t, \sigma)$, that is

$$(2.14)\qquad j(t, v) := \int_{L_0}^v \Gamma(t, \sigma)d\sigma,\ v \in \mathbb{R},\ v \geq L_0,\ \text{for all } t \in [0, T],$$

and note that

$$\frac{d}{dt}\int_{L_0}^{L_\varepsilon(t)} \Gamma(t, \sigma)d\sigma = \frac{dL_\varepsilon}{dt}(t)\Gamma(t, L_\varepsilon(t)) + \int_{L_0}^{L_\varepsilon(t)} \frac{\partial\Gamma}{\partial t}(t, \sigma)d\sigma,\ t \in [0, T].$$

This yields

$$\int_0^t \frac{dL_\varepsilon}{ds}(s)\Gamma(s, L_\varepsilon(s))ds = \int_{L_0}^{L_\varepsilon(t)} \Gamma(t, \sigma)d\sigma - \int_0^t \int_{L_0}^{L_\varepsilon(s)} \frac{\partial\Gamma}{\partial s}(s, \sigma)d\sigma ds.$$

Replacing the left-hand side term of the previous equality in (2.13) we get

$$\begin{aligned}(2.15)\qquad &j(t, L_\varepsilon(t)) - \int_0^t \int_{L_0}^{L_\varepsilon(s)} \frac{\partial\Gamma}{\partial s}(s, \sigma)d\sigma ds - L_\varepsilon(t)\alpha + L_0\alpha\\ &+ \int_0^t \mu_\varepsilon(s)(z_\varepsilon(s) - \alpha)ds = \int_0^t U(s, L_\varepsilon(s))(z_\varepsilon(s) - \alpha)ds.\end{aligned}$$

By (2.3) we have

$$(2.16)\qquad \eta_* = \inf_{v \geq L_0}\left\{\frac{1}{v}\int_{L_0}^v \Gamma(t, \sigma)d\sigma\right\} \leq \frac{1}{v}\int_{L_0}^v \Gamma(t, \sigma)d\sigma,\ \text{for } v \geq L_0,$$

and so for $v = L_\varepsilon(t)$ we get

$$\eta_* L_\varepsilon(t) \leq \int_{L_0}^{L_\varepsilon(t)} \Gamma(t, \sigma)d\sigma,\ \forall t \in [0, T].$$

Using (2.1)-(2.3) and (2.16) in (2.15) we obtain

$$
\begin{aligned}
&(\eta_* - \alpha)L_\varepsilon(t) + \int_0^t \mu_\varepsilon(s)(z_\varepsilon(s) - \alpha)ds \\
\leq\ & \int_0^t \int_{L_0}^{L_\varepsilon(s)} \left| \frac{\partial \Gamma}{\partial s}(s, \sigma) \right| d\sigma ds + \int_0^t |U(s, L_\varepsilon(s))|\, |z_\varepsilon(s) - \alpha|\, ds \\
\leq\ & C_\Gamma \int_0^t L_\varepsilon(s)ds - C_\Gamma L_0 T + (\Gamma_{\max} + \alpha) \int_0^t |U_0(s) + U_{Lip} L_\varepsilon(s))|\, ds,
\end{aligned}
$$

where $U_0(t) = U(t, 0)$ and U_{Lip} is the Lipschitz constant of U. Recall that $z_\varepsilon(s) = \Gamma(s, L_\varepsilon(s)) \leq \Gamma_{\max}$. This yields

$$
\begin{aligned}
&(\eta_* - \alpha)L_\varepsilon(t) + \int_0^t (\partial I_K)_\varepsilon(\Gamma(s, L_\varepsilon(s)))(\Gamma(s, L_\varepsilon(s)) - \alpha)ds \\
&\leq C_1 + C_2 \int_0^t L_\varepsilon(s)ds,
\end{aligned}
\tag{2.17}
$$

where $C_1(T) = (\Gamma_{\max} + \alpha)U_{0\,\max}T$, $C_2 = (\Gamma_{\max} + \alpha)U_{Lip} + C_\Gamma$.

Now, we can choose $\alpha < \eta_*$ and so, for

$$\eta_* > \alpha > \Gamma^* \tag{2.18}$$

the first term on the left-hand side is positive. Applying the Gronwall lemma we obtain

$$L_\varepsilon(t) \leq \frac{C_1}{\eta_* - \alpha} e^{\frac{C_2}{\eta_* - \alpha} t} \text{ for all } t \in [0, T]. \tag{2.19}$$

Also, it follows that

$$
\begin{aligned}
0 &\leq \int_0^t (\partial I_K)_\varepsilon(\Gamma(s, L_\varepsilon(s)))(\Gamma(s, L_\varepsilon(s)) - \alpha)ds \\
&\leq C_1 e^{\frac{C_2}{\eta_* - \alpha} t}, \text{ for all } t \in [0, T].
\end{aligned}
\tag{2.20}
$$

By the definition of the subdifferential, we are entitled to write

$$\mu_\varepsilon(t)(z_\varepsilon(t) - \alpha - \rho\theta) \geq 0, \text{ a.e. } t \in (0, T), \tag{2.21}$$

where α, ρ and θ are such that $\alpha + \rho\theta \geq \Gamma^*$, with $\alpha > \Gamma^*$, $\rho > 0$, $|\theta| = 1$, $0 < \rho \leq \alpha - \Gamma^*$. Now, if 1_A is the characteristic function of the set A, we set

$$\theta(t) := \frac{\mu_\varepsilon(t)}{|\mu_\varepsilon(t)|} 1_{\{t;\ \mu_\varepsilon(t) \neq 0\}}$$

and by integrating (2.21), we get

$$\rho \int_0^t |\mu_\varepsilon(s)|\, ds \leq \int_0^t \mu_\varepsilon(s)(z_\varepsilon(s) - \alpha)ds. \tag{2.22}$$

By (2.20) we deduce that

$$\int_0^t |(\partial I_K)_\varepsilon(\Gamma(s, L_\varepsilon(s)))|\, ds \leq \frac{C_1}{\rho} e^{\frac{C_2}{\eta_* - \alpha} t}, \text{ for all } t \in [0, T], \tag{2.23}$$

while by (2.11) we obtain

$$
\begin{aligned}
\int_0^t \left|\frac{dL_\varepsilon}{ds}(s)\right| ds &\le \int_0^t |U(s,L_\varepsilon(s))|\,ds + \int_0^t |(\partial I_K)_\varepsilon(\Gamma(s,L_\varepsilon(s)))|\,ds \\
&\le \int_0^t (U_{0\max} + U_{Lip}\,|L_\varepsilon(s)|)ds + \frac{C_1}{\rho}e^{\frac{C_2}{\eta_*-\alpha}t} \\
&\le U_{0\max}T + \frac{C_1}{\rho}e^{\frac{C_2}{\eta_*-\alpha}T} + U_{Lip}\frac{C_1}{C_2}\left(e^{\frac{C_2}{\eta_*-\alpha}t} - 1\right), \text{ for all } t \in [0,T].
\end{aligned}
\tag{2.24}
$$

Writing (2.11) as

$$\frac{dL_\varepsilon}{dt}(t) - \frac{1}{\varepsilon}(\Gamma(t,L_\varepsilon(t)) - \Gamma^*)^- = U(t,L_\varepsilon(t))$$

and multiplying by $(\Gamma(t,L_\varepsilon(t)) - \Gamma^*)$ we get

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left\|(\Gamma(\cdot,L_\varepsilon(\cdot)) - \Gamma^*)^-\right\|^2_{L^2(0,T)} \\
&= \int_0^t \left(U(s,L_\varepsilon(s)) - \frac{dL_\varepsilon}{ds}(s)\right)(\Gamma(s,L_\varepsilon(s)) - \Gamma^*)ds \le C_T.
\end{aligned}
$$

This immediately yields

$$(\Gamma(t,L_\varepsilon(t)) - \Gamma^*)^- \le C_T\varepsilon, \text{ for all } t \in [0,T]. \tag{2.25}$$

By C_T we denote several constants, which can differ from line to line. They depend on the data and T, but are independent of ε. We conclude that $\{L_\varepsilon\}_\varepsilon$ is bounded in $C[0,T]$, $\left\{\frac{dL_\varepsilon}{dt}\right\}_\varepsilon$ and $\{\mu_\varepsilon = (\partial I_K)_\varepsilon(\Gamma(\cdot,L_\varepsilon(\cdot))\}_\varepsilon$ are bounded in $L^1(0,T)$.

By (2.24) it follows that $\|L_\varepsilon\|_{BV([0,T])} \le C_T$ for all $\varepsilon > 0$, and so, by Helly's theorem (see e.g., [2], p. 47), we have $L \in BV([0,T])$ and

$$L_\varepsilon(t) \to L(t), \text{ for all } t \in [0,T]. \tag{2.26}$$

In particular, $L_\varepsilon(0) \to L(0) = L_0$. Moreover, by Egorov's theorem, for each $\delta > 0$, there exists a set $\Omega_\delta \subset [0,T]$, such that $\text{meas}(\Omega_\delta) < \delta$ and

$$L_\varepsilon \to L \text{ uniformly on } \Omega_\delta, \text{ as } \varepsilon \to 0. \tag{2.27}$$

Then, the sequences $\left\{\frac{dL_\varepsilon}{dt}\right\}_\varepsilon$ and $\{\mu_\varepsilon\}_\varepsilon$ are weak-* compact in $(L^\infty(0,T))^*$, as specified in the proof of Corollary 2B in [8]. We stress that this is not directly implied by the Alaoglu theorem, but can be deduced by the following argument. Let us consider the linear operator $\Phi : C[0,T] \to L^\infty(Q)$, $\Phi v = \widetilde{\Phi}$, which maps a continuous function into the corresponding class of equivalence $\widetilde{\Phi}$ (of all functions a.e. equal). Its adjoint $\Phi^* : (L^\infty(Q))^* \to \mathcal{M}([0,T])$ is defined by $(\Phi^*\mu)(v) := \mu(\Phi v)$ for any $v \in C[0,T]$. If $\{\mu_n\}_n$ is bounded in $(L^\infty(Q))^*$ and also in $\mathcal{M}([0,T])$, then $\{\Phi^*\mu_n\}_n$ is bounded in $\mathcal{M}([0,T])$ which is the dual of the separable space $C[0,T]$ and so by the Alaoglu theorem $\{\Phi^*\mu_n\}_n$ is weak-* sequentially compact in $\mathcal{M}([0,T])$. Also $\{\mu_n\}_n$ is weak-* sequentially compact in $\mathcal{M}([0,T])$. Passing to the limit in $\mu_n(\Phi v) = (\Phi^*\mu_n)(v)$ we get $\mu(\Phi v) = (\Phi^*\mu)(v) :=$ for any $\widetilde{\Phi} \in L^\infty(Q)$ which is of the form Φv with $v \in C[0,T]$. Then, μ can be extended by the Hahn-Banach theorem to all $L^\infty(0,T)$ and so we conclude that $\{\mu_n\}_n$ is weak-star sequentially compact in $(L^\infty(Q))^*$. Therefore, one can extract a subsequence such that

$$\frac{dL_\varepsilon}{dt} \to \frac{dL}{dt} \text{ weak-* in } (L^\infty(0,T))^* \subset \mathcal{M}([0,T]), \tag{2.28}$$

$$\mu_\varepsilon \to \mu \text{ weak-* in } (L^\infty(0,T))^* \subset \mathcal{M}([0,T]). \tag{2.29}$$

Since $\Gamma(t,x)$ and $U(t,x)$ are continuous with respect to x it follows that

$$\Gamma(t, L_\varepsilon(t)) \to \Gamma(t, L(t)), \text{ for all } t \in [0,T], \tag{2.30}$$

$$U(t, L_\varepsilon(t)) \to U(t, L(t)), \text{ for all } t \in [0,T], \tag{2.31}$$

and by (2.25) we get

$$\Gamma(t, L(t)) \geq \Gamma^*, \text{ for all } t \in [0,T]. \tag{2.32}$$

Moreover, by (2.11) we have at limit

$$\frac{dL}{dt} + \mu = U(\cdot, L(\cdot)) \text{ in } \mathcal{D}'(0,T). \tag{2.33}$$

Further, by writing

$$\int_{\Omega_\delta} \mu_\varepsilon(t)(\Gamma(t, L_\varepsilon(t)) - v(t))dt = \int_0^T \mu_\varepsilon(t)\left(1_{\Omega_\delta}(t)(\Gamma(t, L_\varepsilon(t)) - v(t))\right)dt \geq 0,$$

for all $v \in L^\infty(0,T)$, $v(t) \geq \Gamma^*$ a.e., we obtain at limit $\mu(1_{\Omega_\delta}(\Gamma(\cdot, L(\cdot)) - v)) \geq 0$ for all $v \in \mathcal{K}$. Since δ is positive arbitrary, we get as $\delta \to 0$

$$\mu(\Gamma(\cdot, L(\cdot)) - v)) \geq 0, \text{ for all } v \in \mathcal{K}, \tag{2.34}$$

and so, $\mu \in \mathcal{N}_{\mathcal{K}}(\Gamma(\cdot, L(\cdot)))$.

Since $\mu \in (L^\infty(0,T))^* \subset \mathcal{M}([0,T])$, μ can be written as $\mu = \mu_a + \mu_s$, where μ_a is the absolutely continuous part (in the sense of measure) and μ_s is the singular part and so (2.33) implies that

$$\frac{dL^a}{dt}(t) + \mu_a(t) = U(t, L(t)), \text{ a.e. } t \in (0,T),$$
$$\frac{dL^s}{dt} + \mu_s = 0, \qquad \text{in } \mathcal{D}'(0,T),$$

where

$$\mu_a(t) \in N_K(\Gamma(t, L(t)), \text{ a.e. } t \in (0,T), \ \mu_s \in \mathcal{N}_{\mathcal{K}}(\Gamma(\cdot, L(\cdot))),$$

as claimed. This means that

$$\begin{aligned} \mu_s(\varphi) &= 0 \text{ if } \varphi \in \mathring{\mathcal{K}} \\ \mu_s(\varphi) &\leq 0 \text{ if } \varphi \in \partial\mathcal{K}. \end{aligned}$$

Recalling that

$$\text{supp } \mu_s \subset \{\Sigma \subset [0,T]; \ \mu_s \neq 0 \text{ on } \Sigma, \text{ i.e., } \mu_s(\varphi) \neq 0, \text{ for } \varphi \in L^\infty(\Sigma)\}$$

it follows that

$$\text{supp } \mu_s \subset \{t \in [0,T]; \Gamma(t, L(t)) = \Gamma^*\}.$$

Therefore, L has an absolutely continuous part L^a and a BV part L^s, where $\Gamma(t, L(t)) = \Gamma^*$. In particular, we note that L^s can be represented as jump functions at t, e.g. $L^s(t) = \alpha_i$ on $[t_i, t_{i+1})$, meaning that these jump points are those at which $\Gamma(t_i, L(t_i)) = \Gamma^*$. This completes the proof of the solution existence. □

Remark 2.3. It should be noted that since $L \to \partial I_K(\Gamma(\cdot, L))$ is not monotone, the uniqueness remains open.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variations and Free Discontinuity Problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces*, Springer, 2012.

[3] H. Brezis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North Holland, 1973.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York Dordrecht Heidelberg London, 2011.

[5] S. Cui and A. Friedman, *A hyperbolic free boundary problem modeling tumor growth*, Interfaces Free Bound. **5** (2003), 159–181.

[6] A. Gandolfi, M. Iannelli and G. Marinoschi, *An age-structured model of epidermis growth*, J. Math. Biol. **62** (2011), 111–141.

[7] A. Gandolfi, M. Iannelli and G. Marinoschi, *Time evolution for a model of epidermis growth*, J. Evol. Equat. **13** (2013), 509–533.

[8] R. T. Rockafeller, *Integrals which are convex functionals, II,* Pac. J. Math. **39** (1971), 439–469.

[9] A. Visintin, *Mathematical models of hysteresis*, In: The Science of Hysteresis (G. Bertotti, I. Mayergoyz (eds.) Elsevier, chap. 1, 2006, pp. pp. 1–123.

Manuscript received May 17 2019
revised August 29 2019

G. Marinoschi
"Gheorghe Mihoc-Caius Iacob"Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Calea 13 Septembrie 13, Bucharest, Romania
E-mail address: gabriela.marinoschi@acad.ro