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A UNIFORM CONTINUITY PROPERTY OF THE WINDING NUMBER OF SELF-MAPPINGS OF THE CIRCLE

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Abstract. Let $u : \mathbb{S}^1 \to \mathbb{S}^1$. When u is continuous, it has a winding number $\deg u$, which satisfies $\deg u = \deg v$ if $u, v \in C^0(\mathbb{S}^1; \mathbb{S}^1)$ and $\|u - v\|_{L^\infty} < 2$. In particular, $u \mapsto \deg u$ is uniformly continuous for the sup norm.

The winding number $\deg u$ can be naturally defined, by density, when u is merely *VMO*. For such u's, the winding number deg is continuous with respect to the *BMO* norm.

Let $1 < p < \infty$. In view of the above and of the embedding $W^{1/p,p}(\mathbb{S}^1) \hookrightarrow VMO$, maps in $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ have a well-defined winding number, continuous with respect to the $W^{1/p,p}$ norm. However, an example due to Brezis and Nirenberg yields sequences $(u_n), (v_n) \subset W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ such that $\|u_n - v_n\|_{W^{1/p,p}} \to 0$ as $n \to \infty$ and $\deg u_n \neq \deg v_n$, $\forall\, n$. Thus deg is not uniformly continuous with respect to the $W^{1/p,p}$ norm (and, a fortiori, with respect to the *BMO* norm).

The above sequences satisfy $\|u_n\|_{W^{1/p,p}} \to \infty$, $\|v_n\|_{W^{1/p,p}} \to \infty$. We prove that a similar phenomenon cannot occur for bounded sequences. More specifically, we prove the following uniform continuity result. Given $1 < p < \infty$ and $M > 0$, there exists some $\delta = \delta(p, M) > 0$ such that

$$[\|u\|_{W^{1/p,p}} \le M,\ \|u - v\|_{W^{1/p,p}} \le \delta] \implies \deg u = \deg v.$$

1. Introduction

If $u \in C^0(\mathbb{S}^1; \mathbb{S}^1)$, then u has a winding number $\deg u$, which is continuous with respect to the uniform convergence, and in particular is a homotopic invariant. In fact, deg is "better than just continuous": it is uniformly continuous, since

$$[u, v \in C^0(\mathbb{S}^1; \mathbb{S}^1),\ \|u - v\|_{L^\infty} < 2] \implies \deg u = \deg v. \tag{1.1}$$

The winding number can still be "naturally" defined (i.e., by density, starting from smooth maps) when u is slightly less than continuous, more specifically when $u \in VMO(\mathbb{S}^1; \mathbb{S}^1)$. This has been first noticed by Boutet de Monvel and Gabber [3, Appendix] for the space $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$, and then extended and investigated in depth by Brezis and Nirenberg for maps $u \in VMO(\mathbb{S}^1; \mathbb{S}^1)$ [5]. While, in this setting, the winding number is still continuous with respect to norm convergence, and thus provides a homotopic invariant, there is no *global* analogue of (1.1), even if we replace the *BMO* norm by one of the *stronger* norms $W^{1/p,p}$, $1 < p < \infty$. More specifically,

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a construction from [5] yields two sequences of smooth maps u_n, $v_n : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\begin{aligned} &\|u_n - v_n\|_{W^{1/p,p}} \to 0 \text{ as } n \to \infty, \forall\, 1 < p < \infty, \\ &\text{and } \deg u_n \neq \deg v_n, \forall\, n \end{aligned} \tag{1.2}$$

(see [4, Lemma 6.4]). Here, $\| \ \|_{W^{1/p,p}} = \| \ \|_{L^p} + | \ |_{W^{1/p,p}}$, where $| \ |_{W^{s,p}}$ stands for the Gagliardo seminorm, given for $0 < s < 1$, $1 \leq p < \infty$ and Ω, an N-dimensional Lipschitz bounded domain or compact embedded manifold, by

$$|u|^p_{W^{s,p}} = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}\, dxdy.$$

(The fact that the $W^{1/p,p}$ norm is stronger than the BMO one on $\mathbb{S}^1$ follows from the embedding $W^{1/p,p} \hookrightarrow BMO$, valid in 1D.)

We prove that the above phenomenon can occur only when u_n, v_n "escape to infinity", that is, we have uniform continuity of the degree on bounded sets:

Theorem 1.1. *Let $1 < p < \infty$ and $M > 0$. Then there exists some $\delta = \delta(p, M) > 0$ such that*

$$\begin{aligned} &[u, v \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1),\ |u|_{W^{1/p,p}} \leq M,\ |u - v|_{W^{1/p,p}} \leq \delta] \\ &\implies \deg u = \deg v. \end{aligned} \tag{1.3}$$

This provides a partial (positive) answer to [4, Open Problem 3].

Remark 1.2. The new contribution of this note concerns the case where $p > 2$. When $1 < p \leq 2$, Theorem 1.1 is a special case of [4, Proposition 7.9]. This result asserts that, when $N \geq 1$ and $1 < p \leq N + 1$, the degree of maps in $u \in W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$ is uniformly continuous on bounded sets.

Theorem 1.1 settles thus completely the case of the dimension one. We point out that its analogue for $W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$ maps, with $N \geq 2$ and $p > N + 1$, is widely open.

Remark 1.3. When $1 < p \leq 2$, the proof of [4, Proposition 7.9] yields an *explicit* δ (in terms of M and p) in (1.3). A similar explicit estimate eludes us when $p > 2$.

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2. Proof

We first recall some basic properties of the winding number. Given $p \in (1, \infty)$, $\deg u$ for $u \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ is defined as follows. To start with, $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ is dense in $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ [5, Lemmas A.11 and A.12], and the map $u \mapsto \deg u$, initially defined for smooth maps $u \in C^\infty(\mathbb{S}^1; \mathbb{S}^1)$, is continuous with respect to the BMO norm [5, Theorem 1]. In view of the embedding $W^{1/p,p} \hookrightarrow VMO$ in 1D, we find that deg has a unique continuous extension to $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$. In particular, it suffices to work in (1.3) with *smooth maps.*

For further use, let us note the property

$$\deg(uv) = \deg u + \deg v, \ \forall\, u, v \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1). \tag{2.1}$$

As a consequence of (2.1), we have

$$\deg \overline{u} = -\deg u, \ \forall\, u \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1). \tag{2.2}$$

These properties are well known if u and v are smooth. The general case is obtained by density, using the continuity of the degree with respect to the $W^{1/p,p}$ norm and the following easy and standard fact ("$W^{s,p} \cap L^\infty$ is an algebra").

Lemma 2.1. *Let $0 < s < 1$ and $1 \le p < \infty$. Let Ω be an N-dimensional Lipschitz bounded domain or compact embedded manifold. If $(u_n), (v_n) \subset W^{s,p}(\Omega; \mathbb{C})$ and $u, v \in W^{s,p}(\Omega; \mathbb{C})$ satisfy*

$$|u_n - u|_{W^{s,p}} \to 0, \ |v_n - v|_{W^{s,p}} \to 0 \ \text{as } n \to \infty, \tag{2.3}$$

$$u_n \to u \ \text{and } v_n \to v \ \text{a.e. as } n \to \infty, \tag{2.4}$$

$$\|u_n\|_{L^\infty} \le C, \ \|v_n\|_{L^\infty} \le C, \ \forall\, n, \tag{2.5}$$

then

$$|u_n v_n - uv|_{W^{s,p}} \to 0 \ \text{as } n \to \infty. \tag{2.6}$$

Proof. We have

$$\begin{aligned} |[(u_n - u)v](x) - [(u_n - u)v](y)|^p \lesssim & |(u_n - u)(x)|^p |v(x) - v(y)|^p \\ & + |(u_n - u)(x) - (u_n - u)(y)|^p |v(y)|^p, \end{aligned}$$

so that

$$\begin{aligned} |u_n v - uv|^p_{W^{s,p}} \lesssim & \int_\Omega \int_\Omega \frac{|(u_n - u)(x)|^p |v(x) - v(y)|^p}{|x - y|^{N+sp}}\, dxdy \\ & + \int_\Omega \int_\Omega \frac{|(u_n - u)(x) - (u_n - u)(y)|^p |v(y)|^p}{|x - y|^{N+sp}}\, dxdy \to 0 \\ & \text{as } n \to \infty; \end{aligned}$$

here, we use dominated convergence for the first integral, and the facts that v is bounded and $|u_n - u|_{W^{s,p}} \to 0$ as $n \to \infty$ for the second integral. Thus $|u_n v - uv|_{W^{s,p}} \to 0$ as $n \to \infty$. Similarly, $|uv_n - uv|_{W^{s,p}} \to 0$ as $n \to \infty$. In view of the identity

$$u_n v_n - uv = (u_n v - uv) + (uv_n - uv) + (u_n - u)(v_n - v),$$

it remains to prove that $X_n := |(u_n - u)(v_n - v)|^p_{W^{s,p}} \to 0$ as $n \to \infty$. This follows, by dominated convergence, from

$$\begin{aligned} X_n \lesssim & \int_\Omega \int_\Omega \frac{|(u_n - u)(x)|^p |(v_n - v)(x) - (v_n - v)(y)|^p}{|x - y|^{N+sp}}\, dxdy \\ & + \int_\Omega \int_\Omega \frac{|(u_n - u)(x) - (u_n - u)(y)|^p |(v_n - v)(y)|^p}{|x - y|^{N+sp}}\, dxdy. \end{aligned}$$

□

Proof of Theorem 1.1. Given $u : \mathbb{S}^1 \to \mathbb{C}$ measurable and bounded, we let $T_u : \mathbb{D} \to \mathbb{C}$ denote its harmonic extension to the unit disc $\mathbb{D}$. We set

$$c'_p := \inf\{|u|^p_{W^{1/p,p}};\ u \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1),\ T_u(0) = 0\}. \tag{2.7}$$

(Clearly, the inf is achieved in (2.7), but we will not need this fact.) We have [6, Section 2, item 5]

$$c'_p > 0. \tag{2.8}$$

We will prove (1.3) by complete (strong) induction on the integer part L of M^p/c'_p.

Step 1. Proof of (1.3) when $L = 0$

Let $\delta > 0$ be such that $(M+\delta)^p := c < c'_p$. If u, v are smooth and as in (1.3), then $|u|^p_{W^{1/p,p}} \le c$ and $|v|^p_{W^{1/p,p}} \le c$, and therefore there exists some $\varepsilon > 0$ such that $|T_u(x)| \ge \varepsilon$, $|T_v(x)| \ge \varepsilon$, $\forall\, x \in \mathbb{D}$ [6, Section 2, items 8 and 9]. Thus $u, v : \mathbb{S}^1 \to \mathbb{S}^1$ are smooth functions with smooth non vanishing extensions to $\overline{\mathbb{D}}$. It follows that $\deg u = \deg v = 0$.

Step 2. Proof of (1.3) for $L \ge 1$ (assuming that (1.3) holds for $0, \dots, L-1$)

Argue by contradiction. Then there exist $c < (L+1)\, c'_p$ and sequences $(u_n), (v_n) \subset C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ such that $|u_n|^p_{W^{1/p,p}} \le c$, $|v_n|^p_{W^{1/p,p}} \le c$, $\deg u_n \ne \deg v_n$, $\forall\, n$, and $|u_n - v_n|_{W^{1/p,p}} \to 0$ as $n \to \infty$.

Since at least one of the integers $\deg u_n$, $\deg v_n$ is non zero, we may assume that $\deg u_n \ne 0$, $\forall\, n$. Therefore, T_{u_n} has to vanish at some point $a_n \in \mathbb{D}$.

Let $M_a(z) = \dfrac{a - z}{1 - \overline{a}\, z}$, $\forall\, a \in \mathbb{D}$, $\forall\, z \in \overline{\mathbb{D}}$, denote the (normalized) Möbius transformation vanishing at a. Let $N_a : \mathbb{S}^1 \to \mathbb{S}^1$ be the restriction of M_a to $\mathbb{S}^1$. We note the following properties, valid for each $a \in \mathbb{D}$ and each $u \in C^0(\mathbb{S}^1;\mathbb{S}^1)$ [6, Section 2, item 1].

$$\deg(u \circ N_a) = \deg u, \tag{2.9}$$

$$T_{u\circ N_a} = (T_u) \circ M_a, \tag{2.10}$$

$$T_{u\circ N_a}(0) = T_u(a). \tag{2.11}$$

In addition, we have [6, Section 2, item 2]

$$|f \circ N_a|_{W^{1/p,p}} = |f|_{W^{1/p,p}},\ \forall\, 1 < p < \infty,\ \forall\, f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{C}). \tag{2.12}$$

We consider $U_n := u_n \circ N_{a_n}$ and $V_n := v_n \circ N_{a_n}$. By properties (2.9)–(2.12) above and by the assumptions on u_n and v_n, we have

$$|U_n|^p_{W^{1/p,p}} \le c \text{ and } |V_n|^p_{W^{1/p,p}} \le c, \tag{2.13}$$

$$T_{U_n}(0) = 0,\ \forall\, n, \tag{2.14}$$

$$\deg U_n \ne \deg V_n,\ \forall\, n, \tag{2.15}$$

$$|U_n - V_n|_{W^{1/p,p}} \to 0 \tag{2.16}$$

and, possibly up to subsequences still denoted (U_n) and (V_n),

$$U_n \to U \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1) \text{ a.e. and weakly in } W^{1/p,p} \text{ as } n \to \infty, \tag{2.17}$$

$$V_n \to V \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1) \text{ a.e. and weakly in } W^{1/p,p} \text{ as } n \to \infty, \tag{2.18}$$

$$U - V = C \in \mathbb{C}, \tag{2.19}$$

$$T_U(0) = 0. \tag{2.20}$$

The last property follows, by dominated convergence, from (2.14), (2.17) and the fact that $T_v(0) = ⨍_{\mathbb{S}^1} v(x)\,dx$, $\forall\, v \in L^1(\mathbb{S}^1; \mathbb{C})$.

Claim 1. We have $C = 0$, and thus

$$V_n \to U \text{ a.e. as } n \to \infty. \tag{2.21}$$

Granted Claim 1, we continue as follows. Since $U_n \to U$ and $V_n \to U$ a.e. as $n \to \infty$, we have [6, Section 2, item 10]

$$|U_n|^p_{W^{1/p,p}} = |U|^p_{W^{1/p,p}} + |U_n\,\overline{U}|^p_{W^{1/p,p}} + o(1) \text{ as } n \to \infty, \tag{2.22}$$

$$|V_n|^p_{W^{1/p,p}} = |U|^p_{W^{1/p,p}} + |V_n\,\overline{U}|^p_{W^{1/p,p}} + o(1) \text{ as } n \to \infty. \tag{2.23}$$

Combining (2.20), the fact that $U \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ and the definition (2.7) of c'_p, we obtain

$$|U|^p_{W^{1/p,p}} \ge c'_p. \tag{2.24}$$

From (2.22)–(2.24), (2.1) and (2.2), we find that $\widetilde{U}_n := U_n\,\overline{U}$ and $\widetilde{V}_n := V_n\,\overline{U}$ satisfy

$$|\widetilde{U}_n|^p_{W^{1/p,p}} \le c - c'_p + o(1) \text{ as } n \to \infty, \tag{2.25}$$

$$|\widetilde{V}_n|^p_{W^{1/p,p}} \le c - c'_p + o(1), \tag{2.26}$$

$$\deg \widetilde{U}_n = \deg U_n - \deg U \ne \deg V_n - \deg U = \deg \widetilde{V}_n, \ \forall\, n. \tag{2.27}$$

Claim 2. We have

$$|\widetilde{U}_n - \widetilde{V}_n|_{W^{1/p,p}} \to 0 \text{ as } n \to \infty. \tag{2.28}$$

Granted Claim 2 and using (2.25)–(2.27) together with the fact that $c - c'_p < L\,c'_p$, we find that the sequences $(\widetilde{U}_n)$, $(\widetilde{V}_n)$ contradict, for large n, the induction hypothesis.

In order to complete the proof of the theorem, it thus remains to justify Claims 1 and 2.

Proof of Claim 1. Since $|U| = 1$ and $|V| = 1$ a.e., (2.19) implies that U takes values, a.e., in the set $\mathbb{S}^1 \cap (C + \mathbb{S}^1)$. When $C \ne 0$, this set contains at most two points. Since the essential range of U is connected [5, Section I.5, Comment 2], we find that U is constant a.e. Thus T_U is a constant of modulus 1, which is impossible, by (2.20).

Proof of Claim 2. Let us note that $\overline{U} \in W^{1/p,p} \cap L^\infty$. Claim 2 is then a consequence of (2.21) and Lemma 2.1, applied with $u_n = U_n - V_n, v_n = \overline{U}, u = 0$ and $v = \overline{U}$. □

Remark 2.2. The idea of "extracting" information concerning a map $u : \mathbb{S}^1 \to \mathbb{S}^1$ from the behavior of an appropriate extension of u to $\mathbb{D}$ (which is at the heart of the asymptotic analysis in Step 2) originates in [1], where the harmonic extension is considered; see also [2] for an extension by averages.

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