Pure and Applied Functional Analysis
Volume 5, Number 5, 2020, 1205–1215

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

NON-VARIATIONAL ELLIPTIC EQUATIONS INVOLVING (p,q)-LAPLACIAN, CONVECTION AND CONVOLUTION

DUMITRU MOTREANU AND VIORICA VENERA MOTREANU

Abstract. The paper is devoted to a nonlinear elliptic Dirichlet problem whose leading operator is the (p,q)-Laplacian and with a reaction term involving convection (i.e., it depends on the solution and its gradient) and the convolution of the solution with an integrable function considered as a parameter. The results presented here establish existence, uniqueness and continuous dependence of the solution with respect to the parameter.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\partial\Omega$ be its boundary. Fix real numbers p, q, μ satisfying $1 < q < p < N$ and $\mu \geq 0$. (The assumption $p < N$ is made for simplicity of the exposition; the complementary situation $p \geq N$ can be handled along the same lines.) In the sequel, corresponding to any real number $r \in (1, +\infty)$, the notation r' will stand for the Hölder conjugate of r, that is, $r' = r/(r-1)$.

By $W_0^{1,p}(\Omega)$ we denote the usual Sobolev space, which will be equipped with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx \Big)^{\frac{1}{p}}. \tag{1.1}$$

The notation $|\nabla u|$ means the Euclidean norm of the gradient ∇u. Accordingly, the space $W_0^{1,q}(\Omega)$ will be also used. We recall that the negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x)\, dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

Similarly, $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ acts as

$$\langle -\Delta_q u, v\rangle = \int_\Omega |\nabla u(x)|^{q-2} \nabla u(x) \cdot \nabla v(x)\, dx, \quad \forall v \in W_0^{1,q}(\Omega).$$

Since Ω is bounded and $p > q > 1$, the operator $-\Delta_p - \mu\Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is well defined. Important special cases are the p-Laplacian (for $\mu = 0$) and the (p,q)-Laplacian (for $\mu = 1$).

2010 *Mathematics Subject Classification.* 35J92 (primary); 47H30 (secondary).
Key words and phrases. (p,q)-Laplacian, Dirichlet problem, convection term, convolution, parameter.

The Dirichlet problems driven by the operator $-\Delta_p - \mu\Delta_q$ and depending on a parameter λ stated as

$$-\Delta_p u - \mu\Delta_q u = F(x, u, \nabla u, \lambda) \tag{1.2}$$

are fundamental. They include for instance the eigenvalue problem for the p-Laplacian

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for which we refer to [6]. Here $\lambda \in \mathbb{R}$ and $\mu = 0$. A more general problem is that of the Fučik spectrum for the p-Laplacian

$$\begin{cases} -\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see, e.g., [4]), where u^+ and u^- stand for the positive and negative parts of u, respectively, whereas the parameter is $(a, b) \in \mathbb{R}^2$.

In the present paper we focus on a problem of type (1.2) with a convection term (i.e., depending on the solution u and its gradient ∇u) and a completely different choice of parameter, which is now a function $\rho \in L^1(\mathbb{R}^N)$ appearing through the convolution $\rho * u$ with the solution $u \in W_0^{1,p}(\Omega)$. In this respect it is convenient to consider the Sobolev space $W_0^{1,p}(\Omega)$ embedded in $W^{1,p}(\mathbb{R}^N)$ by identifying every $u \in W_0^{1,p}(\Omega)$ with its extension $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$ equal to zero outside Ω. Recall that given $\rho \in L^1(\mathbb{R}^N)$ and $u \in W_0^{1,p}(\Omega) \subset W^{1,p}(\mathbb{R}^N)$ the convolution $\rho * u$ is defined by

$$\rho * u(x) = \int_{\mathbb{R}^{\mathbb{N}}} \rho(x - y)u(y)dy \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Consequently, let us note that

$$\rho * u = \rho * \tilde{u} \in W^{1,p}(\mathbb{R}^N). \tag{1.3}$$

Notice also that

$$\operatorname{supp} \rho * u \subset \overline{\Omega + \operatorname{supp} \rho}. \tag{1.4}$$

The convection is described by a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ (i.e., $f(\cdot, s, \xi)$ is measurable on Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$) satisfying the following growth condition. Hereafter, p^* stands for the Sobolev critical exponent $p^* = Np/(N - p)$ (recall that we assume $p < N$).

(H) There are constants $a_1, a_2 \geq 0$, $\alpha, \beta \in [0, p - 1)$, $r \in [1, p^*)$, and a function $\sigma \in L^{r'}(\Omega)$ such that

$$|f(x, s, \xi)| \leq \sigma(x) + a_1|s|^\alpha + a_2|\xi|^\beta$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

In the sequel, we will actually identify f with the function $\tilde{f} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ obtained by extending $f(\cdot, s, \xi)$ by 0 outside Ω.

Corresponding to $\rho \in L^1(\mathbb{R}^N)$, we formulate the Dirichlet problem

$$\begin{cases} -\Delta_p u - \mu\Delta_q u = f(x, \rho * u, \nabla(\rho * u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

By a weak solution of problem (1.5) we mean any $u \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega (|\nabla u|^{p-2} + \mu |\nabla u|^{q-2}) \nabla u \cdot \nabla v \, dx - \int_\Omega f(x, \rho * u, \nabla(\rho * u)) v \, dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$. This makes sense under the growth condition (H).

A relevant feature of problem (1.5) is the combined effect of convection and convolution. We emphasize that the right-hand side of (1.5) depends not only on the solution u but also on its gradient ∇u. Hence, generally, this problem does not have a variational structure, which makes the variational methods not applicable. For different methods that can be employed in studying problems with convection we cite [5, 8, 9]. Notice that in problem (1.5) in addition to convection we also have convolution.

In Section 3, we establish the existence of a weak solution to problem (1.5) under hypothesis (H). Our approach relies on the surjectivity theorem for pseudomonotone operators whose basic prerequisites are discussed in Section 2. Then, in Section 4, we show the uniqueness of solution to (1.5) in two situations: $p \geq 2$ and $\|\rho\|_{L^1(\mathbb{R}^N)}$ is sufficiently small or $q \geq 2$ and $\|\rho\|_{L^1(\mathbb{R}^N)}^{q-1}/\mu$ is sufficiently small. Finally, in Section 5, a continuity property of the solution u to (1.5) with respect to $\rho \in L^1(\mathbb{R}^N)$ is proved entailing the upper semicontinuity of the solution set upon ρ.

2. Preliminary tools

Given $p \in (1, N)$, in view of Rellich-Kondrachov theorem, the Sobolev space $W_0^{1,p}(\Omega)$ is compactly embedded into $L^\theta(\Omega)$ if $1 \leq \theta < p^* (= \frac{Np}{N-p})$ and continuously embedded for $\theta = p^*$. Thus for every $r \in [1, p^*]$ there exists a positive constant S_r such that

$$\|u\|_r \leq S_r \|u\|, \quad \forall u \in W_0^{1,p}(\Omega), \tag{2.1}$$

where $\|u\|_r := \|u\|_{L^r(\Omega)}$ denotes the norm on $L^r(\Omega)$. Likewise, there exists a constant $S > 0$ with

$$\|u\|_q \leq S \||\nabla u|\|_q, \quad \forall u \in W_0^{1,q}(\Omega). \tag{2.2}$$

Let us recall that for every $r \geq 2$ we have

$$\langle -\Delta_r u_1 + \Delta_r u_2, u_1 - u_2 \rangle \geq 2^{2-r} \||\nabla u_1 - \nabla u_2|\|_r^r, \quad \forall\, u_1, u_2 \in W_0^{1,r}(\Omega) \tag{2.3}$$

(see [7, §12]).

If $\rho \in L^1(\mathbb{R}^N)$ and $u \in W_0^{1,p}(\Omega)$, the weak partial derivatives of the convolution $\rho * u$ are expressed by

$$\frac{\partial}{\partial x_i}(\rho * u) = \rho * \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N), \quad \forall\ i = 1, \dots, N. \tag{2.4}$$

Based mainly on Tonelli's and Fubini's theorems as well as on Hölder's inequality there hold the estimates

$$\|\rho * u\|_{L^r(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \|u\|_r \tag{2.5}$$

whenever $r \in [1, p^*]$ and

$$\left\| \rho * \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^N)} \leq \|\rho\|_{L^1(\mathbb{R}^N)} \left\| \frac{\partial u}{\partial x_i} \right\|_p, \quad \forall i = 1, \dots, N \tag{2.6}$$

(see [2]). From (2.4)–(2.6) it follows that the linear mapping $u \in W_0^{1,p}(\Omega) \mapsto \rho * u \in W^{1,p}(\mathbb{R}^N)$ is continuous. Moreover, using (1.1), Minkowski's inequality, the convexity of the function $t \mapsto t^p$ on $(0, +\infty)$, (2.4) and (2.6), observe that

$$\begin{aligned} \||\nabla(\rho * u)|\|^p_{L^p(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\nabla(\rho * u)|^p \, dx = \int_{\mathbb{R}^N} \left(\sum_{i=1}^N (\rho * \frac{\partial u}{\partial x_i})^2 \right)^{\frac{p}{2}} dx \\ &\leq \int_{\mathbb{R}^N} \left(\sum_{i=1}^N \left| \rho * \frac{\partial u}{\partial x_i} \right| \right)^p dx \leq N^{p-1} \sum_{i=1}^N \left\| \rho * \frac{\partial u}{\partial x_i} \right\|^p_{L^p(\mathbb{R}^N)} \\ &\leq N^{p-1} \|\rho\|^p_{L^1(\mathbb{R}^N)} \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^p_p \leq N^p \|\rho\|^p_{L^1(\mathbb{R}^N)} \|u\|^p. \end{aligned} \tag{2.7}$$

For easy reference we recall a few things about the pseudomonotone operators that will be used later on. Let X be a reflexive Banach space with the norm $\|\cdot\|$, its dual X^* and the duality pairing $\langle \cdot, \cdot \rangle$ between X and X^*. The norm convergence in X and X^* is denoted by $\to$, while the weak convergence is denoted by $\rightharpoonup$. A map $A : X \to X^*$ is called bounded if it maps bounded sets to bounded sets. It is said to be coercive if

$$\lim_{\|u\| \to +\infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty.$$

A map $A : X \to X^*$ is called pseudomonotone if $u_n \rightharpoonup u$ in X and

$$\limsup_{n \to +\infty} \langle Au_n, u_n - u \rangle \leq 0 \tag{2.8}$$

imply

$$\langle Au, u - w \rangle \leq \liminf_{n \to +\infty} \langle Au_n, u_n - w \rangle, \ \forall \, w \in X.$$

We state the surjectivity theorem for pseudomonotone operators. More details can be found in [1, 10].

Theorem 2.1. *Let X be a reflexive Banach space, let $A : X \to X^*$ be a pseudomonotone, bounded and coercive operator, and let $g \in X^*$. Then there exists at least a solution $u \in X$ of the equation $Au = g$.*

A map $A : X \to X^*$ satisfies the (S_+)-property if $u_n \rightharpoonup u$ in X and (2.8) ensure the strong convergence $u_n \to u$ in X.

A multivalued map $T : X \to 2^Y$ with nonempty values between the topological spaces X and Y is called upper semicontinuous at the point $x_0 \in X$ if for every neighborhood V of the set $S(x_0)$ in Y there exists a neighborhood U of x_0 in X such that $S(x) \subset V$ for all $x \in U$. The multivalued map $S : X \to 2^Y$ is called upper semicontinuous if it is upper semicontinuous at each point of X.

3. Existence of solutions

Now we state our existence result on problem (1.5).

Theorem 3.1. *Under hypothesis* (H)*, problem* (1.5) *admits at least a (weak) solution.*

Proof. Since $q < p$ and the domain Ω is bounded, one has the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, so it follows that the operator $-\Delta_p - \mu\Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous (see, e.g., [3, Lemma 2.111]). Moreover, this operator is bounded.

Due to the growth condition postulated in assumption (H), the Nemytskii operator $N_f : W^{1,p}(\mathbb{R}^N) \to L^{(p^*)'}(\mathbb{R}^N) \subset W^{-1,p'}(\mathbb{R}^N)$ associated with the function $f(x, s, \xi)$, that is

$$N_f(u) = f(x, u, \nabla u),$$

is well defined, continuous and bounded. The notation $(p^*)'$ and p' is related to p^* and p, respectively, complying with the convention in Section 1.

Consider the inclusion map $E : W_0^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ given by the extension outside Ω with 0, so $E(u) = \tilde{u}$ (see Section 1). Denoting by $E^* : W^{-1,p'}(\mathbb{R}^N) \to W^{-1,p'}(\Omega)$ the adjoint map of E, we introduce the nonlinear operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ as

$$Au = -\Delta_p u - \mu\Delta_q u - E^* N_f(\rho * Eu), \ \forall u \in W_0^{1,p}(\Omega). \tag{3.1}$$

The above discussion (together with (2.7)) shows that the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous and bounded.

By (3.1) it is seen that $u \in W_0^{1,p}(\Omega)$ is a weak solution for problem (1.5) if and only if it holds

$$\langle Au, v\rangle = 0, \ \forall\, v \in W_0^{1,p}(\Omega). \tag{3.2}$$

Therefore if we can prove the surjectivity of the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, the existence of a weak solution ensues. In turn, to show that A is surjective, we apply Theorem 2.1.

Our starting point is the following estimate

$$\begin{aligned}
&\left|\int_\Omega f(x, \rho * u, \nabla(\rho * u)) w \, dx\right| \\
\le\ & \|\sigma\|_{r'}\|w\|_r + a_1 \,\|\rho * u\|^\alpha_{L^{p^*}(\mathbb{R}^N)}\|w\|_{\frac{p^*}{p^*-\alpha}} + a_2 \||\nabla(\rho * u)|\|^\beta_{L^p(\mathbb{R}^N)}\|w\|_{\frac{p}{p-\beta}} \\
\le\ & \|\sigma\|_{r'}\|w\|_r + a_1\|\rho\|^\alpha_{L^1(\mathbb{R}^N)} S^\alpha_{p^*}\|u\|^\alpha\|w\|_{\frac{p^*}{p^*-\alpha}} \\
& + a_2 N^\beta\|\rho\|^\beta_{L^1(\mathbb{R}^N)}\|u\|^\beta\|w\|_{\frac{p}{p-\beta}}
\end{aligned} \tag{3.3}$$

for all u, $w \in W_0^{1,p}(\Omega)$, where as before $u \in W_0^{1,p}(\Omega)$ is identified with $E(u) = \tilde{u}$. The estimate in (3.3) is obtained from assumption (H) in conjunction with Hölder's inequality, (2.5), (2.7), and (2.1).

Next we claim that the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ in (3.1) is pseudomonotone. In order to show this, let a sequence (u_n) satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$

and (2.8). Since the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, we infer that there is a constant $C > 0$ such that

$$N\|\rho\|_{L^1(\mathbb{R}^N)}\|u_n\| \le C, \quad \forall\, n. \tag{3.4}$$

Then (3.3) and (3.4) yield

$$\begin{aligned} &\left|\int_\Omega f(x, \rho * u_n, \nabla(\rho * u_n))(u_n - u)\,dx\right| \\ \le\ & \|\sigma\|_{r'}\|u_n - u\|_r + a_1 S_{p^*}^\alpha C^\alpha \|u_n - u\|_{\frac{p^*}{p^*-\alpha}} + a_2 C^\beta \|u_n - u\|_{\frac{p}{p-\beta}},\ \forall\, n. \end{aligned}$$

Using that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and the compact embeddings of $W_0^{1,p}(\Omega)$ into $L^r(\Omega)$, $L^{p^*/(p^*-\alpha)}(\Omega)$ and $L^{p/(p-\beta)}(\Omega)$, it turns out that

$$\lim_{n\to+\infty} \int_\Omega f(x, \rho * u_n, \nabla(\rho * u_n))(u_n - u)\,dx = 0. \tag{3.5}$$

On the basis of (2.8), (3.1) and (3.5) we derive

$$\limsup_{n\to+\infty} \langle -\Delta_p u_n - \mu\Delta_q u_n, u_n - u\rangle \le 0.$$

At this point the (S_+)-property of the operator $-\Delta_p - \mu\Delta_q$ on $W_0^{1,p}(\Omega)$ (see, e.g., [3, Chapter 2]) implies the strong convergence $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then the continuity of $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ implies $Au_n \to Au$ in $W^{-1,p'}(\Omega)$. This enables us to confirm the claim that the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is pseudomonotone.

The next step in the proof is to check that the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined in (3.1) is coercive. By (3.3), Sobolev embedding theorem, and the inequalities $\alpha + 1 < p$ and $\beta + 1 < p$, we get

$$\begin{aligned} &\int_\Omega f(x, \rho * u, \nabla(\rho * u))u\,dx \\ \le\ & \|\sigma\|_{r'}\|u\|_r + a_1\|\rho\|_{L^1(\mathbb{R}^N)}^\alpha S_{p^*}^\alpha \|u\|^\alpha \|u\|_{\frac{p^*}{p^*-\alpha}} \\ & + a_2 N^\beta \|\rho\|_{L^1(\mathbb{R}^N)}^\beta \|u\|^\beta \|u\|_{\frac{p}{p-\beta}} \\ \le\ & \|\sigma\|_{r'}\|u\|_r + b_1\|u\|^{\alpha+1} + b_2\|u\|^{\beta+1} \\ \le\ & \delta\|u\|^p + c, \end{aligned} \tag{3.6}$$

for all $u \in W_0^{1,p}(\Omega)$, with constants $b_1, b_2, c > 0$ and $\delta \in (0,1)$. On account of (3.1) and (3.6), we are able to find

$$\begin{aligned} \langle Au,\, u\rangle &= \langle -\Delta_p u - \mu\Delta_q u,\, u\rangle - \int_\Omega f(x, \rho * u, \nabla(\rho * u))u\,dx \\ &\ge (1-\delta)\|u\|^p - c \end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$. We deduce that the operator A in (3.1) is coercive because $p > 1$.

In view of the above arguments we conclude that the operator A in (3.1) fulfills all the hypotheses of Theorem 2.1. Therefore Theorem 2.1 guarantees that the operator A is surjective, whence (3.2) holds for some $u \in W_0^{1,p}(\Omega)$, which completes the proof. □

Remark 3.2. The proof of Theorem 3.1 reveals that the operator A in (3.1) has the (S_+)-property. Furthermore, the (S_+)-property and the pseudomonotonicity of A are valid under the growth condition

(H^*) There are $a_1, a_2 \geq 0$, $\alpha \in [0, p^*-1)$, $\beta \in [0, p-1]$, $r \in [1, p^*)$, and $\sigma \in L^{r'}(\Omega)$ such that

$$|f(x,s,\xi)| \leq \sigma(x) + a_1|s|^\alpha + a_2|\xi|^\beta$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

which is more general than (H).

Remark 3.3. Theorem 3.1 remains true if assumption (H) is replaced by the more general growth condition

$(\bar{H})$ There are $a_1, a_2 \geq 0$, $r \in [1, p^*)$, and $\sigma \in L^{r'}(\Omega)$ such that

$$|f(x,s,\xi)| \leq \sigma(x) + a_1|s|^{p-1} + a_2|\xi|^{p-1}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

and the parameter $\rho \in L^1(\mathbb{R}^N)$ is assumed to be *small enough*, in the sense that

$$\|\rho\|^{p-1}_{L^1(\mathbb{R}^N)}\big(a_1 S^{p-1}_{p^*} S_{\frac{p^*}{p^*-p+1}} + a_2 N^{p-1} S_p\big) < 1. \tag{3.7}$$

Indeed this condition ensures that the constants b_1, b_2 arising in (3.6) satisfy $b_1+b_2 < 1$, which is sufficient for guaranteeing the coercivity of A.

Remark 3.4. (a) Note that the conditions (H), (H^*), and $(\bar{H})$ imply in particular that $f(\cdot,0,0) \in L^{r'}(\Omega)$.
(b) The conclusions of Theorem 3.1 and Remarks 3.2 and 3.3 remain valid if the corresponding growth conditions ((H), (H^*), and $(\bar{H})$, respectively) are required only for a.e. $x \in \Omega \cap \overline{\Omega + \operatorname{supp}\rho}$ (see (1.4)) and if we assume independently that $f(\cdot,0,0) \in L^{r'}(\Omega)$ for some $r \in [1, p^*)$.

Remark 3.5. (a) In fact the result stated in Theorem 3.1 remains valid if the operator $u \mapsto \rho * u$ is replaced by a continuous (possibly nonlinear) operator $T : W^{1,p}_0(\Omega) \to W^{1,p}(\mathbb{R}^N)$ such that

$$\| |\nabla T(u)| \|_{L^p(\mathbb{R}^N)} \leq c_T \|u\|, \quad \forall u \in W^{1,p}_0(\Omega),$$

for some constant $c_T > 0$. The corresponding problem is

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, T(u), \nabla(T(u))) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof follows the same scheme. In particular, (3.3) holds with c_T in place of $\|\rho\|_{L^1(\mathbb{R}^N)}$ and where S_{p^*} (the best constant in (2.1) for the domain Ω) is replaced by the analogous constant $S_{p^*,\mathbb{R}^N}$ for the domain $\mathbb{R}^N$.
(b) Remark 3.3 also holds for the operator T, in which case (3.7) becomes

$$c_T^{p-1}\big(a_1 S^{p-1}_{p^*,\mathbb{R}^N} S_{\frac{p^*}{p^*-p+1}} + a_2 N^{p-1} S_p\big) < 1.$$

4. Uniqueness

Our uniqueness result on problem (1.5) is as follows.

Theorem 4.1. (a) *Assume that $p \geq 2$ and there exist constants $c_1, d_1 > 0$ such that*

$$|f(x,s_1,\xi_1) - f(x,s_2,\xi_2)| \leq c_1|s_1 - s_2|^{p-1} + d_1|\xi_1 - \xi_2|^{p-1} \tag{4.1}$$

for a.e. $x \in \Omega$, all $s_1, s_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^N$. If $\rho \in L^1(\mathbb{R}^N)$ is such that

$$\|\rho\|_{L^1(\mathbb{R}^N)} < \left(\frac{2^{2-p}}{(c_1 S_p^{p-1} + d_1 N^{\frac{p-1}{2}})S_p}\right)^{\frac{1}{p-1}}, \tag{4.2}$$

then, for every $\mu \geq 0$, problem (1.5) *has at most one weak solution.*
(b) *Assume that $q \geq 2$ and there exist constants $c_2, d_2 > 0$ such that*

$$|f(x,s_1,\xi_1) - f(x,s_2,\xi_2)| \leq c_2|s_1 - s_2|^{q-1} + d_2|\xi_1 - \xi_2|^{q-1} \tag{4.3}$$

for a.e. $x \in \Omega$, all $s_1, s_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^N$. If $\mu > 0$ and $\rho \in L^1(\mathbb{R}^N)$ satisfy

$$\frac{\|\rho\|^{q-1}_{L^1(\mathbb{R}^N)}}{\mu} < \frac{2^{2-q}}{(c_2 S^{q-1} + d_2 N^{\frac{q-1}{2}})S} \tag{4.4}$$

(see (2.2)), then problem (1.5) *has at most one weak solution.*

Proof. (a) Let $u_1, u_2 \in W_0^{1,p}(\Omega)$ be weak solutions to problem (1.5). From (2.3) and (4.1) we can derive

$$\begin{aligned}
2^{2-p}\|u_1 - u_2\|^p &\leq \langle -\Delta_p u_1 + \Delta_p u_2, u_1 - u_2\rangle + \mu\,\langle -\Delta_q u_1 + \Delta_q u_2, u_1 - u_2\rangle \\
&= \int_\Omega (f(x,\rho * u_1, \nabla(\rho * u_1)) - f(x,\rho * u_2, \nabla(\rho * u_2)))(u_1 - u_2)\,dx \\
&\leq \int_\Omega \left(c_1|\rho * u_1 - \rho * u_2|^{p-1} + d_1|\nabla(\rho * u_1 - \rho * u_2)|^{p-1}\right)|u_1 - u_2|\,dx.
\end{aligned}$$

Then Hölder's inequality and (2.5) imply

$$\begin{aligned}
2^{2-p}\|u_1 - u_2\|^p \;&\leq\; c_1\|\rho\|^{p-1}_{L^1(\mathbb{R}^N)}\|u_1 - u_2\|_p^p \\
&\quad + d_1\||\nabla(\rho * (u_1 - u_2))|\|^{p-1}_{L^p(\mathbb{R}^N)}\|u_1 - u_2\|_p.
\end{aligned} \tag{4.5}$$

Through the convexity of the function $t \mapsto t^{\frac{p}{2}}$ on $(0, +\infty)$ (note $p \geq 2$) and (2.5) we see that

$$\begin{aligned}
\||\nabla(\rho * (u_1 - u_2))|\|^p_{L^p(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\nabla(\rho * (u_1 - u_2))|^p\,dx \\
&= \int_{\mathbb{R}^N}\left(\sum_{i=1}^N \left(\rho * \frac{\partial}{\partial x_i}(u_1 - u_2)\right)^2\right)^{\frac{p}{2}} dx \\
&\leq N^{\frac{p}{2}-1}\sum_{i=1}^N \int_{\mathbb{R}^N}\left|\rho * \frac{\partial}{\partial x_i}(u_1 - u_2)\right|^p dx \\
&\leq N^{\frac{p}{2}-1}\|\rho\|^p_{L^1(\mathbb{R}^N)}\sum_{i=1}^N \int_\Omega \left|\frac{\partial}{\partial x_i}(u_1 - u_2)\right|^p dx
\end{aligned} \tag{4.6}$$

$$\leq \quad N^{\frac{p}{2}}\|\rho\|^p_{L^1(\mathbb{R}^N)}\|u_1-u_2\|^p.$$

Combining (4.5), (4.6) and (2.1) results in

$$(2^{2-p}-c_1\|\rho\|^{p-1}_{L^1(\mathbb{R}^N)}S_p^p-d_1N^{\frac{p-1}{2}}\|\rho\|^{p-1}_{L^1(\mathbb{R}^N)}S_p)\|u_1-u_2\|^p\leq 0.$$

In view of (4.2), the desired conclusion ensues.

(b) As in part (a), on the basis of (2.3) and (4.3) we get

$$\begin{aligned}&\mu 2^{2-q}\int_\Omega|\nabla(u_1-u_2)|^q\,dx\\ \leq\quad&\langle-\Delta_pu_1+\Delta_pu_2,u_1-u_2\rangle+\mu\,\langle-\Delta_qu_1+\Delta_qu_2,u_1-u_2\rangle\\ \leq\quad&\int_\Omega\left(c_2|\rho*u_1-\rho*u_2|^{q-1}+d_2|\nabla(\rho*u_1-\rho*u_2)|^{q-1}\right)|u_1-u_2|\,dx.\end{aligned}$$

Through Hölder's inequality and (2.5) this leads to

$$\begin{aligned}&\mu 2^{2-q}\int_\Omega|\nabla(u_1-u_2)|^q\,dx\\ \leq\quad&c_2\|\rho\|^{q-1}_{L^1(\mathbb{R}^N)}\|u_1-u_2\|_q^q+d_2\||\nabla(\rho*(u_1-u_2))|\|^{q-1}_{L^q(\mathbb{R}^N)}\|u_1-u_2\|_q.\end{aligned}\tag{4.7}$$

We can argue along the lines of (4.6) because $q\geq 2$, which leads to

$$\||\nabla(\rho*(u_1-u_2))|\|^q_{L^q(\mathbb{R}^N)}\leq N^{\frac{q}{2}}\|\rho\|^q_{L^1(\mathbb{R}^N)}\||\nabla(u_1-u_2)|\|_q^q.\tag{4.8}$$

Inserting (4.8) and (2.2) in (4.7) gives

$$(\mu 2^{2-q}-c_2\|\rho\|^{q-1}_{L^1(\mathbb{R}^N)}S^q-d_2N^{\frac{q-1}{2}}\|\rho\|^{q-1}_{L^1(\mathbb{R}^N)}S)\int_\Omega|\nabla(u_1-u_2)|^q\,dx\leq 0.$$

Then (4.4) renders $u_1=u_2$. The proof is thus complete. □

Corollary 4.2. *Assume that* $f(\cdot,0,0)\in L^{r'}(\Omega)$ *for some* $r\in[1,p^*)$.
(a) *Assume* $p\geq 2$ *and (4.1). Then, there is* $\lambda_1>0$ *such that problem (1.5) has a unique solution whenever* $\|\rho\|_{L^1(\mathbb{R}^N)}<\lambda_1$.
(b) *Assume* $q\geq 2$ *and (4.3). Then, there is* $\lambda_2>0$ *such that problem (1.5) has a unique solution whenever* $\|\rho\|^{q-1}_{L^1(\mathbb{R}^N)}/\mu<\lambda_2$.

Proof. (a) Combining the assumption on $f(\cdot,0,0)$ with (4.1), we get that f fulfills the growth condition $(\bar{H})$ of Remark 3.3. We choose $\lambda_1>0$ small enough so that (3.7) and (4.2) hold whenever $\|\rho\|_{L^1(\mathbb{R}^N)}<\lambda_1$. Then the conclusion follows from Theorem 4.1 and Remark 3.3.
(b) The assumption on $f(\cdot,0,0)$ and (4.3) imply that f fulfills the growth condition (H) (recall that $q<p$). Then the conclusion follows from Theorems 3.1 and 4.1. □

5. Dependence on $\rho\in L^1(\mathbb{R}^N)$

The following statement deals with the dependence on the parameter $\rho\in L^1(\mathbb{R}^N)$ in problem (1.5).

Theorem 5.1. *Assume that condition (H) is satisfied. If $\rho_n \to \rho$ in $L^1(\mathbb{R}^N)$ and $u_n \in W_0^{1,p}(\Omega)$ is a weak solution of the equation*

$$
\begin{cases} -\Delta_p u_n - \mu \Delta_q u_n = f(x, \rho_n * u_n, \nabla(\rho_n * u_n)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}
$$

then there is a subsequence of (u_n) still denoted (u_n) such that $u_n \to u$ in $W_0^{1,p}(\Omega)$, for some weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.5).

Proof. The existence of a solution u_n to problem (5.1) is guaranteed by Theorem 3.1. Proceeding as in (3.6) and using the boundedness of the sequence (ρ_n) in $L^1(\mathbb{R}^N)$ entail that there exist constants $c > 0$ and $\delta \in (0,1)$ independent of n such that

$$
\int_\Omega f(x, \rho_n * v, \nabla(\rho_n * v)) v \, dx \le \delta \|v\|^p + c \quad \text{for all } v \in W_0^{1,p}(\Omega), \text{ all } n. \tag{5.2}
$$

Acting with u_n as test function in (5.1), by means of (5.2), we infer that the sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$. Along a relabeled subsequence we may suppose that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Then as for (3.5), using the boundedness of the sequence (ρ_n) in $L^1(\mathbb{R}^N)$, we can prove that

$$
\lim_{n \to +\infty} \int_\Omega f(x, \rho_n * u_n, \nabla(\rho_n * u_n))(u_n - u) \, dx = 0. \tag{5.3}
$$

Acting on (5.1) with $u_n - u$, by (5.3) we arrive at

$$
\lim_{n \to +\infty} \langle -\Delta_p u_n - \mu \Delta_q u_n, u_n - u \rangle = 0.
$$

Through the (S_+)-property of the operator $-\Delta_p - \mu \Delta_q$ on $W_0^{1,p}(\Omega)$ (see, e.g., [3]) it follows that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Then from (2.5) and (2.7) we see that $\rho_n * u_n \to \rho * u$ in $W^{1,p}(\mathbb{R}^N)$. Hence the continuity of the operators $-\Delta_p - \mu \Delta_q$ and N_f allows us to pass to the limit in equation (5.1) as $n \to +\infty$ getting that u is a solution of (1.5). The proof is complete. $\square$

Corollary 5.2. *Assume that condition (H) holds. Then the multivalued map $\mathcal{S} : L^1(\Omega) \to 2^{W_0^{1,p}(\Omega)}$ assigning to every $\rho \in L^1(\Omega)$ the solution set $\mathcal{S}(\rho)$ of problem (1.5) is upper semicontinuous.*

Proof. Arguing by contradiction suppose that one can find $\rho_0 \in L^1(\Omega)$ such that the multivalued map $\mathcal{S} : L^1(\Omega) \to 2^{W_0^{1,p}(\Omega)}$ is not upper semicontinuous at ρ_0. Then there exist a neighborhood V_0 of the set $\mathcal{S}(\rho_0)$ in $W_0^{1,p}(\Omega)$, a sequence $\rho_n \to \rho_0$ in $L^1(\Omega)$ and a sequence $u_n \in W_0^{1,p}(\Omega)$ with $u_n \in \mathcal{S}(\rho_n)$ and $u_n \notin V_0$ for every n. Theorem 5.1 ensures that a subsequence (ρ_{n_k}) of (ρ_n) can be found such that $u_{n_k} \to u_0$ in $W_0^{1,p}(\Omega)$ as $k \to +\infty$, with $u_0 \in \mathcal{S}(\rho_0)$. Therefore $u_{n_k} \in V_0$ provided k is sufficiently large, thus reaching a contradiction, which completes the proof. $\square$

Acknowledgments. The authors express their gratitude to Dr. Lucas Fresse for helpful comments.

References

[1] H. Brezis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier (Grenoble) **18** (1968), 115–175.

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[3] S. Carl, V. K. Le, and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*, Springer Monographs in Mathematics, Springer, New York, 2007.

[4] M. Cuesta, G. de Figueiredo, and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian.* J. Differential Equations **159** (1999), 212–238.

[5] F. Faraci, D. Motreanu, and D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var. Partial Differential Equations **54** (2015), 525–538.

[6] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal. **64** (2006), 1057–1099.

[7] P. Lindqvist, *Notes on the p-Laplace equation* (second edition), University of Jyväskylä, Department of Mathematics and Statistics, report 161, Jyväskylä, 2017.

[8] D. Motreanu, *Nonlinear Differential Problems with Smooth and Nonsmooth Constraints*, Mathematical Analysis and Its Applications, Academic Press, London, 2018.

[9] D. Motreanu, V. V. Motreanu, and A. Moussaoui, *Location of nodal solutions for quasilinear elliptic equations with gradient dependence*, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), 293–307.

[10] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol. II A/B, Springer-Verlag, Berlin 1990.

Manuscript received May 28 2019
revised June 17 2019

D. Motreanu
Département de Mathématiques, Université de Perpignan, 66860 Perpignan, France
E-mail address: motreanu@univ-perp.fr

V. V. Motreanu
Collège Jean Moulin, 14 rue Jean Moulin, 54510 Tomblaine, France
E-mail address: vmotreanu@gmail.com