Pure and Applied Functional Analysis
Volume 5, Number 5, 2020, 1217–1236

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

POSITIVE SOLUTIONS FOR THE ROBIN p-LAPLACIAN PLUS AN INDEFINITE POTENTIAL

NIKOLAOS S. PAPAGEORGIOU, VICENŢIU D. RĂDULESCU*, AND DUŠAN D. REPOVŠ

ABSTRACT. We consider a nonlinear elliptic equation driven by the Robin p-Laplacian plus an indefinite potential. In the reaction we have the competing effects of a strictly $(p-1)$-sublinear parametric term and of a $(p-1)$-linear and nonuniformly nonresonant term. We study the set of positive solutions as the parameter $\lambda > 0$ varies. We prove a bifurcation-type result for large values of the positive parameter λ. Also, we show that for all admissible $\lambda > 0$, the problem has a smallest positive solution $\overline{u}_\lambda$ and we study the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear parametric Robin problem:

$$(P_\lambda) \qquad \left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = f(z,u(z),\lambda) + g(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u(z)^{p-1} = 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega, \ \lambda > 0. \end{array} \right\}$$

In this problem, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega), \ 1 < p < \infty.$$

The potential function $\xi \in L^\infty(\Omega)$ is in general indefinite (that is, sign-changing). Therefore the differential operator (the left-hand side of (P_λ)) need not be coercive. In the reaction (the right-hand side of (P_λ)), we have the competing effects of two terms. The first is a parametric function which is strictly $(p-1)$-sublinear near $+\infty$. The second function (the perturbation of the parametric term), is $(p-1)$-linear near $+\infty$. Both functions are Carathéodory (that is, for all $x \in \mathbb{R}$ the mappings $z \mapsto f(z,x,\lambda)$ and $z \mapsto g(z,x)$ are measurable and for all $z \in \Omega$ the functions $x \mapsto f(z,x,\lambda)$ and $x \mapsto g(z,x)$ are continuous). In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u, defined by extension (according to the nonlinear Green's identity) of the map

$$C^1(\overline{\Omega}) \ni u \mapsto |Du|^{p-2}(Du,n)_{\mathbb{R}^N} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

2010 *Mathematics Subject Classification.* 35J20, 35J60.

Key words and phrases. Local minimizers, p-Laplacian, strong comparison, positive solutions, nonlinear regularity, minimal solution, indefinite potential.

This research was supported by the Slovenian Research Agency Grants P1-0292, J1-8131, J1-7025, N1-0064, and N1-0083.

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. This map is uniformly continuous from $C^1(\overline{\Omega})$ into $L^p(\partial\Omega)$ (in fact, locally Lipschitz if $p \geq 2$ and Hölder continuous if $1 < p < 2$). Also, $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. So, this map admits a unique extension to the whole Sobolev space. We refer for details to Lemma 3 and Theorem 1 in Casas & Fernández [5] (see also Papageorgiou, Rădulescu & Repovš [20, p. 28] for the classical case).

Our aim in this paper is to study the nonexistence, existence and multiplicity of positive solutions for problem (P_λ) as the parameter λ moves on the positive semi-axis $(0, +\infty)$. We prove a bifurcation-type result for large values of the parameter. More precisely, we show that there is a critical parameter value $\lambda^* > 0$ such that
(i) for all $\lambda > \lambda^*$, problem (P_λ) has at least two positive solutions;
(ii) for all $\lambda = \lambda^*$, problem (P_λ) has at least one positive solution;
(iii) for all $0 < \lambda < \lambda^*$, problem (P_λ) has no positive solutions.

Moreover, we show that for every admissible parameter $\lambda \in [\lambda^*, +\infty)$, problem (P_λ) has a smallest positive solution $\overline{u}_\lambda$ and we examine the continuity and monotocicity properties of the map $\lambda \mapsto \overline{u}_\lambda$.

The first such bifurcation-type result for parametric elliptic equations with competing nonlinearities was proved by Ambrosetti, Brezis & Cerami [2] (semilinear Dirichlet problems with concave-convex reaction). Their work was extended to Dirichlet p-Laplace equations by Garcia Azorero, Manfredi & Peral Alonso [7], Guo & Zhang [10], Hu & Papageorgiou [12]. For equations of logistic type there are the works of Rădulescu & Repovš [21] (semilinear Dirichlet problems) and Cardinali, Papageorgiou & Rubbioni [4] (nonlinear Neumann problems). For Robin problems, we mention the work of Papageorgiou & Rădulescu [16]. In all aforementioned works the differential operator is coercive and the reaction has a different pair of competing nonlinearities. In the present paper we distinguish a new class of competition phenomena, which lead to bifurcation-type results. In fact, the behaviour of the set of positive solutions as the parameter $\lambda > 0$ varies, is similar to that of superdiffusive logistic equations, since the "bifurcation" occurs for large values of $\lambda > 0$.

Our method of proof uses variational tools from critical point theory together with suitable truncation, perturbation and comparison arguments.

2. Mathematical background and hypotheses

Suppose that X is a Banach space. We denote by X^* the topological dual of X and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X).

Given $\varphi \in C^1(X, \mathbb{R})$ we say that φ satisfies the "Palais-Smale condition" (the "PS-condition" for short) if the following property holds:

$$\text{“Every sequence } \{u_n\}_{n\geq 1} \subseteq X \text{such that}$$
$$\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R} \text{ is bounded and } \varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$
$$\text{admits a strongly convergent subsequence”.}$$

This is a compactness-type condition on the functional φ. Using this condition, one can prove a deformation theorem from which follows the minimax theory for the critical values of φ. Prominent in this theory is the so-called "mountain pass theorem", which we recall here because we will use it in the sequel.

Theorem 2.1. *Assume that* $\varphi \in C^1(X, \mathbb{R})$ *satisfies the PS-condition,* u_0*,* $u_1 \in X$*,* $||u_1 - u_0|| > \rho > 0$*,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : ||u - u_0|| = \rho\} = m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$*, where*

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}.$$

Then $c \geq m_\rho$ *and* c *is a critical value of* φ *(that is, we can find* $\hat{u} \in X$ *such that* $\varphi'(\hat{u}) = 0$ *and* $\varphi(\hat{u}) = c$*).*

Remark 2.2. We mention that if $\varphi' = A + K$, with $A : X \to X^*$ a continuous map of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in X) and $K : X \to X^*$ is completely continuous (that is, if $u_n \xrightarrow{w} u$ in X, then $K(u_n) \to K(u)$ in X^*), then φ satisfies the PS-condition (see Marano & Papageorgiou [14, Proposition 2.2]). This is the case in our setting.

The analysis of problem (P_λ) involves the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue space $L^p(\partial\Omega)$.

We denote by $||\cdot||$ the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$||u|| = \left(||u||_p^p + ||Du||_p^p\right)^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).$$

The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we introduce the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using $\sigma(\cdot)$ we can define in the usual way the boundary Lebesgue spaces $L^q(\partial\Omega), 1 \leq q \leq \infty$. From the theory of Sobolev spaces we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map gives meaning to the notion of "boundary values" for any Sobolev function. The trace map is not surjective (in fact, $\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$) and $\ker\gamma_0 = W_0^{1,p}(\Omega)$. Moreover, γ_0 is compact into $L^q(\partial\Omega)$ for all $q \in [1, \frac{(N-1)p}{N-p})$ if $p < N$ and into $L^p(\partial\Omega)$ for all $1 \leq q < \infty$ if $N \leq p$. In the sequel, for the sake of notational simplicity, we will drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,p}(\Omega).$$

In the next proposition, we have collected the main properties of this map (see Gasinski & Papageorgiou [9, p. 279]).

Proposition 2.3. *The map $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (thus, maximal monotone, too) and of type $(S)_+$.*

Now we introduce our conditions on the potential function $\xi(\cdot)$ and on the boundary coefficient $\beta(\cdot)$.

$H(\xi)$: $\xi \in L^\infty(\Omega)$

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ for some $0 < \alpha < 1$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark 2.4. When $\beta \equiv 0$, we have the Neumann problem.

Let $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma_p(u) = ||Du||_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

Also, let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies

$$|f_0(z, x)| \leq a(z)(1 + |x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega), 1 < r \leq p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases}$ (the critical Sobolev exponent).

We set $F_0(z, x) = \int_0^x f_0(z, s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\gamma_p(u) - \int_\Omega F_0(z, u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

In the framework of variational methods, the local minimizers of φ_0 play an important role. As we will see in the sequel, solutions of the problem are often generated by minimizing φ_0 on a constrained set defined by using the usual pointwise order on $W^{1,p}(\Omega)$ (this is done, via truncation of $f_0(z, \cdot)$). It is well-known that the order cone

$$W_+ = \{u \in W^{1,p}(\Omega) : u(z) \geq 0 \text{ for almost all } z \in \Omega\}$$

of $W^{1,p}(\Omega)$ has an empty interior. So, it is not clear if the constrained minimizer is in fact an unconstrained local minimizer of φ_0 over all of $W^{1,p}(\Omega)$.

The next result is helpful in this direction. It is a particular case of a more general result that can be found in Papageorgiou & Rădulescu [17]. The first to prove this relation between Hölder and Sobolev local minimizers were Brezis & Nirenberg [3].

Proposition 2.5. *Assume that $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in C^1(\overline{\Omega}) \textit{ with } ||h||_{C^1(\overline{\Omega})} \leq \rho_0.$$

Then $u_0 \in C^{1,\vartheta}(\overline{\Omega})$ with $\vartheta \in (0, 1)$ and u_0 is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \textit{ for all } h \in W^{1,p}(\Omega) \textit{with } ||h|| \leq \rho_1.$$

As we already mentioned in the first section of this paper, our approach involves also comparison arguments. The next proposition will be helpful in this direction. It is a special case of a more general result of Papageorgiou, Rădulescu & Repovš [19].

Proposition 2.6. *Assume that* $h_1, h_2, \vartheta \in L^\infty(\Omega), \vartheta(z) \geq 0$ *for almost all* $z \in \Omega$

$$0 < \eta \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega$$

and $u_1, u_2 \in C^{1,\mu}(\overline{\Omega})$ *with* $0 < \mu \leq 1$ *are such that* $u_1 \leq u_2$ *and*

$$\begin{aligned} &-\Delta_p u_1 + \vartheta(z)|u_1|^{p-2}u_1 = h_1, \\ &-\Delta_p u_2 + \vartheta(z)|u_2|^{p-2}u_2 = h_2 \text{ for almost all } z \in \Omega. \end{aligned}$$

Then $u_2 - u_1 \in \operatorname{int} \widehat{C_+} = \left\{u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0\right\}$.

Next, we consider the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{2.1}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue" if problem (2.1) admits a nontrivial solution $\hat{u}$, which is known as an "eigenfunction" corresponding to $\hat{\lambda}$. We denote by $\hat{\sigma}(p)$ the set of eigenvalues of problem (2.1). It is easy to see that $\hat{\sigma}(p) \subseteq \mathbb{R}$ is closed and has a smallest element $\hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta) \in \mathbb{R}$ (first eigenvalue), which has the following properties (for details, we refer to Papageorgiou & Rădulescu [16] and Fragnelli, Mugnai & Papageorgiou [6]).

Proposition 2.7. *If hypotheses* $H(\xi), H(\beta)$ *are satisfied, then problem (2.1) has a smallest eigenvalue* $\hat{\lambda}_1 \in \mathbb{R}$ *such that*

(a) $\hat{\lambda}_1$ *is isolated in* $\hat{\sigma}(p)$ *(that is, there exists* $\epsilon > 0$ *such that* $(\hat{\lambda}_1, \hat{\lambda}, +\epsilon) \cap \hat{\sigma}(p) = \emptyset$*);*

(b) $\hat{\lambda}_1$ *is simple (that is, if* $\hat{u}, \hat{v}$ *are eigenfunctions corresponding to* $\hat{\lambda}_1$*, then* $\hat{u} = \eta\hat{v}$ *for some* $\eta \in \mathbb{R}\backslash\{0\}$*);*

(2.2)

(*c*)
$$\hat{\lambda}_1 = \inf \left\{ \frac{\gamma_0(u)}{||u||_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right\}.$$

Remark 2.8. The infimum in (2.2) is realized on the corresponding one-dimensional eigenspace.

It follows from (2.2) that the elements of this eigenspace have fixed sign. We denote by $\hat{u}_1$ the positive, L^p-normalized (that is, $||\hat{u}_1||_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. We know that $\hat{u}_1 \in D_+$ (see [16], [6]). Also, every eigenvalue different from $\hat{\lambda}_1$ has eigenfunctions in $C^1(\overline{\Omega})$ which are nodal (that is, sign-changing). Finally, if $\xi \in L^\infty(\Omega), \xi(z) \geq 0$ for almost all $z \in \Omega$ and either $\xi \not\equiv 0$ or $\beta \not\equiv 0$, then $\hat{\lambda}_1 > 0$.

An easy consequence of the above properties is the following lemma (see Mugnai & Papageorgiou [15, Lemma 4.11]).

Lemma 2.9. *If hypotheses* $H(\xi), H(\beta)$ *hold,* $\eta \in L^\infty(\Omega), \eta(z) \leq \hat{\lambda}_1$ *for almost all* $z \in \Omega$ *and the inequality is strict on a set of positive measure, then there exists* $c_0 > 0$ *such that*

$$c_0||u||^p \leq \gamma_p(u) - \int_\Omega \eta(z)|u|^p dz \text{ for all } u \in W^{1,p}(\Omega).$$

The hypotheses on the two terms of the reaction of (P_λ) are the following.

$H(f)$ $f : \Omega \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a Carathéodory function such that for all $\lambda > 0$, $f(z, x, \lambda) \geq 0$ for almost all $z \in \Omega$, all $x \geq 0$, $f(z, 0, \lambda) = 0$ for almost all $z \in \Omega$, and

(i) for every $\rho > 0$ and every $\lambda_0 > 0$, there exists $a_{\rho,\lambda_0} \in L^\infty(\Omega)$ such that

$$0 \leq f(z, x, \lambda) \leq a_{\rho,\lambda_0}(z) \text{ for almost all } z \in \Omega, \text{ all } 0 \leq x \leq \rho,\ 0 < \lambda \leq \lambda_0;$$

(ii) for every $\lambda > 0$, we have

$$\lim_{x \to +\infty} \frac{f(z, x, \lambda)}{x^{p-1}} = \lim_{x \to 0^+} \frac{f(z, x, \lambda)}{x^{p-1}} = 0 \text{ uniformly for almost all } z \in \Omega;$$

(iii) if $F(z, x, \lambda) = \int_0^x f(z, s, \lambda) ds$, then there exist $v_0 \in L^p(\Omega)$ and $\tilde{\lambda} > 0$ such that $\int_\Omega F(z, v_0(z), \lambda) dz > 0$ for all $\lambda > \tilde{\lambda}$;

(iv) • we have $f(z, x, \lambda) \to 0^+$ as $\lambda \to 0^+$ uniformly for almost all $z \in \Omega$, all $x \in C \subseteq \mathbb{R}_+$ bounded, $f(z, x, \lambda) \to +\infty$ as $\lambda \to +\infty$ for almost all $z \in \Omega$, all $x > 0$;
• for every $s > 0$, we can find $\tilde{\eta}_s > 0$ such that

$$0 < \tilde{\eta}_s \leq f(z, x, \mu) - f(z, x, \lambda) \text{ for almost all } z \in \Omega, \text{all } x \geq s,\ \text{all } 0 < \lambda < \mu.$$

Remark 2.10. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality that

$$f(z, \cdot, \lambda)|_{(-\infty,0]} = 0 \text{ for almost all } z \in \Omega, \text{all } \lambda > 0. \tag{2.3}$$

Note that hypothesis $H(f)(ii)$ implies that $f(z, \cdot, \lambda)$ is strictly $(p-1)$-sublinear near $+\infty$ and also near 0^+. Hypothesis $H(f)(iii)$ is satisfied if there exists $\tilde{\lambda} > 0$ such that $L(z) = \{x \in \mathbb{R} : f(z, x, \lambda) > 0\}$ is nonempty for almost all $z \in \Omega$, all $\lambda > \tilde{\lambda}$. Finally, note that hypothesis $H(f)(iv)$ implies that for almost all $z \in \Omega$, all $x > 0$, the mapping $\lambda \mapsto f(z, x, \lambda)$ is strictly increasing.

$H(g)$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)$ and $p \leq r < p^*$ such that

$$(g(z, x)) \leq a(1)(1 + x^{r-1}) \text{ for almost all } z \in \Omega,\ \text{all } x \geq 0;$$

(ii) there exists a function $\eta_0 \in L^\infty(\Omega)$ such that $\eta_0(z) \leq \hat{\lambda}_1$ for almost all $z \in \Omega$, $\eta_0 \not\equiv \hat{\lambda}_1$, $\limsup_{x \to +\infty} \frac{g(z,x)}{x^{p-1}} \leq \eta_0(z)$ and $\limsup_{x \to 0^+} \frac{g(z,x)}{x^{p-1}} \leq \eta_0(z)$ uniformly for almost all $z \in \Omega$;

(iii) for almost all $z \in \Omega$ the mapping $x \mapsto \frac{g(z,x)}{x^{p-1}}$ is nondecreasing on $(0, +\infty)$.

Remark 2.11. As we did for $f(z, \cdot, \lambda)$, without any loss of generality, we may assume that

$$g(z, \cdot)|_{(-\infty,0]} = 0 \text{ for almost all } z \in \Omega. \tag{2.4}$$

Hypothesis $H(g)(ii)$ says that asymptotically at $+\infty$ and at 0^+ we have nonuniform nonresonance with respect to $\hat{\lambda}_1$ from the left.

H_0 : for every $\rho > 0$ and every $\tilde{\lambda} > 0$, we can find $\hat{\xi}_0^{\tilde{\lambda}} > 0$ such that for almost all $z \in \Omega$ and all $0 < \lambda \leq \lambda_0$, the function $x \mapsto f(z,x,\lambda) + g(z,x) + \hat{\xi}_\rho^{\hat{\lambda}} x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 2.12. This hypothesis is satisfied if, for example, for almost all $z \in \Omega$ and every $\lambda > 0$, the functions $f(z,\cdot,\lambda)$ and $g(z,\cdot)$ are differentiable and for every $\rho > 0$ and $\hat{\lambda} > 0$, there exists $\hat{\xi}_\rho^{\tilde{\lambda}} > 0$ such that

$$(f'(z,x,\lambda) + g'_x(z,x))x \geq -\hat{\xi}_\rho^{\tilde{\lambda}} x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

Examples. The following pairs of functions f and g satisfy hypotheses $H(f)$, $H(g)$, H_0. For the sake of simplicity we drop the z-dependence. Also recall (2.3) and (2.4).

$$f_1(x,\lambda) = \begin{cases} \lambda x^{p-1}\ln(1+x) & \text{if } 0 \leq x \leq 1 \\ \lambda x^{q-1} & \text{if } 1 < x \end{cases} \qquad 1 < q < p$$

$$g_1(x) = \eta x^{p-1} \qquad \text{for } x \geq 0,\ \eta < \hat{\lambda}_1,$$

$$f_2(x,\lambda) = \begin{cases} \lambda x^{r-1} & \text{if } 0 \leq x \leq 1 \\ \lambda x^{q-1} & \text{if } 1 < x \end{cases} \qquad 1 < q < p < r,$$

$$g_2(x) = \begin{cases} c x^{\tau-1} - x^{q-1} & \text{if } 0 \leq x \leq 1 \\ \eta x^{p-1} + (c-1-\eta) & \text{if } 1 < x \end{cases} \qquad 1 < q < p \leq \tau, \eta < \hat{\lambda}_1, \quad c > \max\{\eta+1, 0\},$$

$$f_3(x,\lambda) = \begin{cases} \lambda(x^{\tau-1} - x^{r-1}) & \text{if } 0 \leq x \leq 1 \\ \lambda x^{q-1}\ln x & \text{if } 1 < x \end{cases} \qquad 1 < q < p < \tau < r,$$

$$g_3(x) = \begin{cases} \eta(x^{p-1} + x^{r-1}) & \text{if } 0 \leq x \leq 1 \\ \eta(x^{p-1} + x^{q-1}) & \text{if } 1 < x \end{cases} \qquad 1 < q < p < r,\ \eta < \hat{\lambda}_1,$$

$$f_4(x,\lambda) = \begin{cases} x^{\tau-1} & \text{if } 0 \leq x \leq \rho(\lambda) \\ x^{q-1} + \mu(\lambda) & \text{if } \rho(\lambda) < x \end{cases}$$

$$g_4(x) = \eta x^{p-1}$$

with $\rho : (0,+\infty) \to (0,+\infty)$ strictly increasing, continuous, $\rho(\lambda) \to 0^+$ as $\lambda \to 0^+$, $\rho(\lambda) \to +\infty$ as $\lambda \to +\infty$, $\mu(\lambda) = [\rho(\lambda)^{\tau-1} - 1]\rho(\lambda)^{q-1}$, $1 < q < p < \tau$ and $\eta < \hat{\lambda}_1$.

Finally, we fix some basic notation which we will use throughout this work. Let $x \in \mathbb{R}$ and set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Also, if $u, \hat{u} \in W^{1,p}(\Omega)$ and $u \leq \hat{u}$, then

$$[u,\hat{u}] = \{v \in W^{1,p}(\Omega) : u(z) \leq v(z) \leq \hat{u}(z) \text{ for almost all } z \in \Omega\}.$$

We denote by $\mathrm{int}_{C^1(\overline{\Omega})}[u,\hat{u}]$ the interior in $C^1(\overline{\Omega})$ of $[u,\hat{u}] \cap C^1(\overline{\Omega})$.

Under the hypotheses on the data of problem (P_λ), the main result of this paper is the following bifurcation-type theorem.

Theorem. *Assume that hypotheses $H(\xi), H(\beta), H(f), H(g), H_0$ hold. Then there exists $\lambda^* > 0$ such that*

(a) *for all $\lambda > \lambda^*$ problem (P_λ) has at least two positive solutions*

$$u_0,\ \hat{u} \in D_+;$$

(b) *for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution*

$$u_{\lambda^*} \in D_+;$$

(c) *for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has no positive solution.*

Finally, if $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

3. Positive solutions

Throughout the rest of the paper we assume that hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, $H(g)$, H_0 are fulfilled.

We introduce the two following two sets:

$$\mathcal{L} = \{\lambda > 0 : \text{ problem } (P_\lambda) \text{ admits a positive solution}\},$$
$$S(\lambda) = \text{ the set of positive solutions for problem } (P_\lambda).$$

We set $\lambda^* = \inf \mathcal{L}$ with the usual convention that $\inf \emptyset = +\infty$.

Proposition 3.1. *We have $\mathcal{L} \neq \emptyset$ and so $0 \leq \lambda^* < +\infty$.*

Proof. From hypotheses $H(f)(i), (ii)$, we see that given $\epsilon > 0$ and $\lambda > 0$, we can find $c_1 = c_1(\epsilon, \lambda) > 0$ such that

$$F(z, x, \lambda) \leq \frac{\epsilon}{p} x^p + c_1 \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.1}$$

Similarly, hypotheses $H(g)(i), (ii)$ imply that we can find $c_2 = c_2(\epsilon) > 0$ such that

$$G(z, x) \leq (\eta_0(z) + \epsilon) x^p + c_2 \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.2}$$

Let $\mu > ||\xi||_\infty$ (see hypothesis $H(\xi)$) and consider the Carathéodory function $k_\lambda(z, x)$ defined by

$$k_\lambda(z, x) = f(z, x, \lambda) + g(z, x) \text{ for all } (z, x) \in \Omega \times \mathbb{R}, \ \lambda > 0 \text{ (see (2.3), (2.4))}.$$

We set $K_\lambda(z, x) = \int_0^x k_\lambda(z, s) ds$ and consider the C^1-functional $\Psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\Psi_\lambda(u) = \frac{1}{p} \gamma_p(u) + \frac{\mu}{p} ||u^-||_p^p - \int_\Omega K_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (3.1) and (3.2), we have for all $u \in W^{1,p}(\Omega)$.

$$\begin{aligned} \Psi_\lambda(u) \quad & \geq c_3 ||u^-||^p + \frac{1}{p} \gamma_p(u^+) - \frac{1}{p} \int_\Omega (\eta_0(z) + 2\epsilon)\, (u^+)^p dz - c_4 \\ & \text{for some } c_3, c_4 > 0 \text{ (recall that } \mu > ||\xi||_\infty) \\ & \geq c_3 ||u^-||^p + (c_0 - 2\epsilon)\, ||u^+||^p - c_4. \end{aligned}$$

Choosing $\epsilon \in (0, \frac{c_0}{2})$, we obtain

$$\Psi_\lambda(u) \geq c_5 ||u||^p - c_4 \text{ for some } c_5 > 0, \text{ all } u \in W^{1,p}(\Omega),$$

$$\Rightarrow \quad \Psi_\lambda(\cdot) \text{ is coercive.}$$

Also, using the Sobolev embedding theorem and the compactness of the trace map, we see that

$$\Psi_\lambda(\cdot) \text{ is sequentially weakly lower semicontinuous.}$$

By the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\Psi_\lambda(u_\lambda) = \inf\left\{\Psi_\lambda(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.3}$$

Hypotheses $H(f)(i),(ii)$ imply that for every $\lambda > 0$, we can find $c_6 = c_6(\lambda) > 0$ such that

$$0 \le F(z,x,\lambda) \le c_6 x^p \text{ for almost all } z \in \Omega, \text{ all } x \ge 0.$$

Evidently, in hypothesis $H(f)(iii)$ we can have $v_0 \ge 0$ (see (2.3)). Consider the continuous integral functional $i_\lambda : L^p(\Omega) \to \mathbb{R}$ defined by

$$i_\lambda(v) = \int_\Omega F(z,v(z),\lambda)dz \text{ for all } v \in L^p(\Omega),$$

$$\Rightarrow \quad i_\lambda(v_0) > 0 \text{ for all } \lambda > \tilde{\lambda} > 0 \text{ (see hypothesis } H(f)(iii)).$$

Exploiting the density of $W^{1,p}(\Omega)$ in $L^p(\Omega)$, we can find $\tilde{v}_0 \in W^{1,p}(\Omega)$, $\tilde{v}_0 \ge 0$, $\tilde{v}_0 \ne 0$ such that

$$i_\lambda(\tilde{v}_0) > 0 \text{ for all } \lambda > \tilde{\lambda}.$$

Then using hypothesis $H(f)(iv)$ and Fatou's lemma, we infer that

$$\lim_{\lambda\to+\infty} \int_\Omega F(z,\tilde{v}_0,\lambda)dz = +\infty. \tag{3.4}$$

On the other hand, hypothesis $H(g)(i)$ implies that if $G(z,x) = \int_0^x g(z,s)ds$, then

$$\left|\int_\Omega G(z,\tilde{v}_0)dz\right| \le c_7 \text{ for some } c_7 > 0. \tag{3.5}$$

Then from (3.4) and (3.5) we see that for large enough $\lambda > \tilde{\lambda}$, we have

$$\Psi_\lambda(\tilde{v}_0) < 0,$$

$$\Rightarrow \quad \Psi_\lambda(u_\lambda) < 0 = \Psi_\lambda(0) \text{ (see (3.3))}$$

$$\Rightarrow \quad u_\lambda \ne 0.$$

From (3.3) we have

$$\Psi'_\lambda(u_\lambda) = 0,$$

$$\Rightarrow \quad \langle A(u_\lambda),h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma - \int_\Omega \mu(u_\lambda^-)^{p-1} h d\sigma$$

$$\int_\Omega [f(z,u_\lambda,\lambda) + g(z,u_\lambda)]h dz \text{ for all } h \in W^{1,p}(\Omega). \tag{3.6}$$

In (3.6) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\gamma_p(u_\lambda^-) + \mu||u_\lambda^-||_p^p = 0 \text{ (see (2.3), (2.4))},$$

$$\Rightarrow \quad c_8||u_\lambda^-||^p \le 0 \text{ for some } c_8 > 0 \text{ (recall that } \mu > ||\xi||_\infty),$$

$$\Rightarrow \quad u_\lambda \ge 0, u_\lambda \ne 0.$$

Then it follows from (3.6) that $u_\lambda \in S_\lambda \subseteq D_+$ and so for large enough $\lambda > \tilde{\lambda}$, we have $\lambda \in \mathcal{L}$, hence $\mathcal{L} \neq \emptyset$. □

Proposition 3.2. *For every $\lambda \in \mathcal{L}$ we have $S(\lambda) \subseteq D_+$ and $\lambda^* > 0$.*

Proof. Let $\lambda \in \mathcal{L}$ and let $u \in S(\lambda)$. Reasoning as in Papageorgiou & Rădulescu [16] using the nonlinear Green identity, we have

$$\left\{ \begin{array}{l} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = f(z,u(z),\lambda) + g(z,u(z)) \text{ for almost all } z \in \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{3.7}$$

By (3.7) and Papageorgiou & Rădulescu [17] (see Proposition 2.10) we have

$$u \in L^\infty(\Omega).$$

Invoking Theorem 2 of Lieberman [13], we infer that

$$u \in C_+ \backslash \{0\}.$$

Let $\rho = ||u||_\infty$ and let $\hat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis H_0. Then

$$\Delta_p u(z) \leq \left(||\xi||_\infty + \hat{\xi}_\rho^\lambda \right) u(z)^{p-1} \text{ for almost all } z \in \Omega. \tag{3.8}$$

From (3.8) and the nonlinear maximum principle (see, for example, Gasinski & Papageorgiou [8, p. 738]), we have

$$\begin{array}{rl} & u \in D_+, \\ \Rightarrow & S(\lambda) \subseteq D_+ \text{ for all } \lambda > 0. \end{array}$$

Next, we show that $\lambda^* = \inf \mathcal{L} > 0$. Hypotheses $H(f)(i), (ii), (iv)$ imply that given $\epsilon > 0$, we can find $\overline{\lambda} > 0$ such that

$$0 \leq f(z,x,\overline{\lambda}) \leq \epsilon x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.9}$$

Hypothesis $H(g)(ii)$ implies that we can find $M, \delta > 0$ such that

$$g(z,x) \leq (\eta_0(z) + \epsilon)x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geq M, \ 0 \leq x \leq \delta. \tag{3.10}$$

On the other hand, by hypothesis $H(g)(iii)$, we have for almost all $z \in \Omega$ and all $\delta \leq x \leq M$

$$\begin{array}{rll} & \frac{g(z,x)}{x^{p-1}} & \leq \dfrac{g(z,M)}{M^{p-1}}, \\ \Rightarrow & g(z,x) & \leq \dfrac{g(z,M)}{M^{p-1}} x^{p-1} \\ & & \leq (\eta_0(z) + \epsilon)x^{p-1} \text{ (see (3.10))}. \end{array} \tag{3.11}$$

So, by (3.10) and (3.11), we infer that

$$g(z,x) \leq (\eta_0(z) + \epsilon)x^{p-1} \text{ for almost all } z \in \Omega, \text{ all } x \geq 0. \tag{3.12}$$

Let $\lambda \in (0, \overline{\lambda})$ (see (3.9)) and assume that $\lambda \in \mathcal{L}$. Then from the first part of the proof, we know that we can find $u_\lambda \in S(\lambda) \subseteq D_+$. For every $h \in W^{1,p}(\Omega)$, $h \geq 0$ we have

$$\langle A(u_\lambda), h \rangle + \int_\Omega \xi(z) u_\lambda^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h d\sigma$$

$$= \int_\Omega [f(z,u_\lambda,\lambda)+g(z,u_\lambda)]hdz$$

$$\leq \int_\Omega (\eta_0(z)+2\epsilon)u_\lambda^{p-1}hdz \text{ (see (3.9), (3.12) and hypothesis } H(f)(iv)). \tag{3.13}$$

In (3.13) we choose $h=u_\lambda\in W^{1,p}(\Omega), u_\lambda\geq 0$. Then

$$\begin{aligned} &\gamma_p(u_\lambda)-\textstyle\int_\Omega \eta_0(z)u_\lambda^{p-1}dz \leq 2\epsilon||u_\lambda||^p,\\ \Rightarrow\ & c_0\leq 2\epsilon \text{ (see Lemma 2.9)}, \end{aligned}$$

Choosing $\epsilon\in(0,\frac{c_0}{2})$, we get a contradiction. Therefore $\lambda\notin\mathcal{L}$ and so

$$0<\overline{\lambda}\leq\lambda^*.$$

The proof is now complete. □

Next, we show that $\mathcal{L}$ is half-line.

Proposition 3.3. *Assume that $\lambda\in\mathcal{L}$. Then $[\lambda,+\infty)\subseteq\mathcal{L}$.*

Proof. Since $\lambda\in\mathcal{L}$, we can find $u_\lambda\in S(\lambda)\subseteq D_+$ (see Proposition 3.2). Let $\vartheta>\lambda$ and consider the following truncation-perturbation of the reaction in problem (P_ϑ):

$$\hat{k}_\vartheta(z,x)=\begin{cases} f(z,u_\lambda(z),\vartheta)+g(z,u_\lambda(z))+\mu u_\lambda(z)^{p-1} & \text{if } x\leq u_\lambda(z)\\ f(z,x,\vartheta)+g(z,x)+\mu x^{p-1} & \text{if } u_\lambda(z)<x. \end{cases} \tag{3.14}$$

Recall that $\mu>||\xi||_\infty$. We set $\hat{K}_\vartheta(z,x)=\int_0^x \hat{k}_\vartheta(z,s)ds$ and consider the C^1-functional $\hat{\psi}_\vartheta: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\psi}_\vartheta(u)=\frac{1}{p}\gamma_p(u)+\frac{\mu}{p}||u||_p^p-\int_\Omega \hat{K}_\vartheta(z,u)dz \text{ for all } u\in W^{1,p}(\Omega).$$

Reasoning as in the proof of Proposition 3.1, we can show that

- $\hat{\psi}_\vartheta(\cdot)$ is coercive;
- $\hat{\psi}_\vartheta(\cdot)$ is sequentially weakly lower semicontinuous.

So, we can find $u_\vartheta\in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\psi}_\vartheta(u_\vartheta)=\inf\left\{\hat{\psi}_\vartheta(u): u\in W^{1,p}(\Omega)\right\},\\ \Rightarrow\ &\hat{\psi}'_\vartheta(u_\vartheta)=0, \end{aligned}$$

$$\Rightarrow\ \langle A(u_\vartheta),h\rangle+\int_\Omega(\xi(z)+\mu)|u_\vartheta|^{p-2}u_\vartheta hdz+\int_{\partial\Omega}\beta(z)|u_\vartheta|^{p-2}u_\vartheta hd\sigma=$$

$$\int_\Omega \hat{k}_\vartheta(z,u_\vartheta)hdz \text{ for all } W^{1,p}(\Omega). \tag{3.15}$$

In (3.15) we choose $h=(u_\lambda-u_\vartheta)^+\in W^{1,p}(\Omega)$. Then we have

$$\left\langle A(u_\vartheta),(u_\lambda-u_\vartheta)^+\right\rangle+\int_\Omega(\xi(z)+\mu)|u_\vartheta|^{p-2}u_\vartheta(u_\lambda-u_\vartheta)^+dz+$$

$$\int_{\partial\Omega}\beta(z)|u_\vartheta|^{p-2}u_\vartheta(u_\lambda-u_\vartheta)^+d\sigma$$

$$= \int_\Omega[f(z,u_\lambda,\vartheta)+g(z,u_\lambda)+\mu u_\lambda^{p-1}](u_\lambda-u_\vartheta)^+dz \text{ (see (3.14))}$$

$$
\begin{aligned}
&\geq \int_\Omega [f(z,u_\lambda,\lambda)+g(z,u_\lambda)+\mu_\lambda^{p-1}](u_\lambda-u_\vartheta)^+dz \text{ (since } \lambda<\vartheta,\\
&\quad \text{see hypothesis } H(f)(iv))\\
&= \langle A(u_\lambda),(u_\lambda-u_\vartheta)^+\rangle+\int_\Omega(\xi(z)+\mu)u_\lambda^{p-1}(u_\lambda-u_\vartheta)^+dz\\
&\quad +\int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\lambda-u_\vartheta)^+d\sigma\\
&\quad \text{(since } u_\lambda\in S(\lambda)),\\
&\Rightarrow u_\lambda\leq u_\vartheta \text{ (see Proposition 2.3 and recall that } \mu>||\xi||_\infty).
\end{aligned}
$$

Then equation (3.15) becomes

$$
\begin{aligned}
&\langle A(u_\vartheta),h\rangle+\int_\Omega\xi(z)u_\vartheta^{p-1}hdz+\int_{\partial\Omega}\beta(z)u_\vartheta^{p-1}hd\sigma\\
&= \int_\Omega[f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)]hdz\\
&\quad \text{for all } h\in W^{1,p}(\Omega),\\
&\Rightarrow u_\vartheta\in S(\vartheta)\subseteq D_+ \text{ and so } \vartheta\in\mathcal{L}.
\end{aligned}
$$

Therefore we conclude that

$$[\lambda,+\infty)\subseteq\mathcal{L}.$$

The proof is now complete. □

An interesting byproduct of the above proof is the following corollary.

Corollary 3.4. *If hypotheses $H(\xi), H(\beta), H(f), H(g), H_0$ hold, $\lambda\in\mathcal{L}, \vartheta>\lambda$ and $u_\lambda\in S(\lambda)\subseteq D_+$, then $\vartheta\in\mathcal{L}$ and we can find $u_\vartheta\in S(\vartheta)\subseteq D_+$ such that $u_\lambda\leq u_\vartheta, u_\vartheta\neq u_\lambda$.*

In fact, we can improve the conclusion of this corollary as follows.

Proposition 3.5. *Assume that $\lambda\in\mathcal{L}$, $\vartheta>\lambda$ and $u_\lambda\in S(\lambda)\subseteq D_+$. Then $\vartheta\in\mathcal{L}$ and we can find $u_\vartheta\in S(\vartheta)\subseteq D_+$ such that $u_\vartheta-u_\lambda\in\operatorname{int}\widehat{C_+}$.*

Proof. From Corollary 3.4 we already know that $\vartheta\in\mathcal{L}$ and that there exists $u_\vartheta\in S(\vartheta)\subseteq D_+$ such that

$$u_\vartheta-u_\lambda\in C_+\backslash\{0\}.$$

Let $\rho=||u_\vartheta||_\infty$ and $\hat{\xi}_\rho^\vartheta>0$ as in H_0. We can always assume that $\hat{\xi}_\rho^\vartheta>||\xi||_\infty$. We have

$$
\begin{aligned}
&-\Delta_p u_\lambda+(\xi(z)+\hat{\xi}_\rho^\vartheta)u_\lambda^{p-1}\\
&= f(z,u_\lambda,\lambda)+g(z,u_\lambda)+\hat{\xi}_\rho^\vartheta u_\lambda^{p-1}\\
&\leq f(z,u_\vartheta,\lambda)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1} \text{ (see hypothesis } H_0 \text{ and recall that } \lambda<\vartheta)\\
&= f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1}-[f(z,u_\vartheta,\vartheta)-f(z,u_\vartheta,\lambda)]\\
&\leq f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1}-\tilde{\eta}_s\\
&\quad \text{with } 0<s=\min_{\overline{\Omega}}u_\vartheta \text{ (recall that } u_\vartheta\in D_+ \text{ and see hypothesis } H(f)(iv))
\end{aligned}
$$

$$< \quad f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\hat{\xi}_\rho^\vartheta u_\vartheta^{p-1}$$

$$(3.16) \quad -\Delta_p u_\vartheta+(\xi(z)+\hat{\xi}_\rho^\vartheta)u_\vartheta^{p-1} \text{ for almost all } z\in\Omega \text{ (since } u_\vartheta\in S(\vartheta)).$$

Since $\tilde{\eta}_s>0$, from (3.16) and Proposition 2.6, we infer that

$$u_\vartheta-u_\lambda\in \operatorname{int}\widehat{C_+}.$$

The proof is complete. □

Now let $\lambda>\lambda^*$. By Proposition 3.3 we know that $\lambda\in\mathcal{L}$. We show that problem (P_λ) has at least two positive solutions.

Proposition 3.6. *If $\lambda>\lambda^*$, then problem (P_λ) has at least two positive solutions*

$$u_0,\ \hat{u}\in D_+,\ u_0\neq\hat{u}.$$

Proof. As we have already mentioned, $\lambda\in\mathcal{L}$. Let $\lambda^*<\eta<\lambda<\vartheta$. We have $\eta,\vartheta\in\mathcal{L}$ (see Proposition 3.3). According to Proposition 3.5, there are $u_\vartheta\in S(\vartheta)\subseteq D_+$ and $u_\mu\in S(\mu)\subseteq D_+$ such that

$$u_\vartheta-u_\mu\in \operatorname{int}\widehat{C_+}.$$

We introduce the Carathéodory function $l_\lambda(z,x)$ defined by

$$(3.17)\ l_\lambda(z,x)=\begin{cases} f(z,u_\eta(z),\lambda)+g(z,u_\eta(z))+\mu u_\eta(z)^{p-1} & \text{if } x<u_\eta(z)\\ f(z,x,\lambda)+g(z,x)+\mu x^{p-1} & \text{if } u_\eta(z)\le x\le u_\vartheta(z)\\ f(z,u_\vartheta(z),\lambda)+g(z,x)+\mu u_\vartheta(z)^{p-1} & \text{if } u_\vartheta(z)<x.\end{cases}$$

Recall that $\mu>||\xi||_\infty$. We set $L_\lambda(z,x)=\int_0^x l_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\lambda: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\varphi}_\lambda(u)=\frac{1}{p}\gamma_p(u)+\frac{\mu}{p}||u||_p^p-\int_\Omega L_\lambda(z,u)dz \text{ for all } u\in W^{1,p}(\Omega).$$

Since $\mu>||\xi||_\infty$, it is clear from (3.17) that $\hat{\varphi}_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0\in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\varphi}_\lambda(u_0)=\inf\left\{\hat{\varphi}_\lambda(u): u\in W^{1,p}(\Omega)\right\},\\ \Rightarrow\ &\hat{\varphi}_\lambda'(u_0)=0,\\ \Rightarrow\ &\langle A(u_0),h\rangle+\int_\Omega(\xi(z)+\mu)|u_0|^{p-2}u_0hdz+\int_{\partial\Omega}\beta(z)|u_0|^{p-2}u_0hd\sigma=\\ (3.18)\qquad &\int_\Omega l_\lambda(z,u_0)hdz \text{ for all } h\in W^{1,p}(\Omega).\end{aligned}$$

In (3.18) we first choose $h=(u_0-u_\vartheta)^+\in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\left\langle A(u_0),(u_0-u_\vartheta)^+\right\rangle+\int_\Omega(\xi(z)+\mu)u_0^{p-1}(u_0-u_\vartheta)^+dz\\ &+\int_{\partial\Omega}\beta(z)u_0^{p-1}(u_0-u_\vartheta)^+d\sigma\\ =\ &\int_\Omega[f(z,u_\vartheta,\lambda)+g(z,u_\vartheta)+\mu u_\vartheta^{p-1}](u_0-u_\vartheta)^+dz \text{ (see (3.17))}\\ \le\ &\int_\Omega[f(z,u_\vartheta,\vartheta)+g(z,u_\vartheta)+\mu u_\vartheta^{p-1}](u_0-u_\vartheta)^+dz\end{aligned}$$

$$
\begin{aligned}
& \text{(see hypothesis } H(f)(iv) \text{ and recall that } \lambda < \vartheta) \\
= \ & \langle A(u_\vartheta), (u_0 - u_\vartheta)^+ \rangle + \int_\Omega (\xi(z) + \mu) u_\vartheta^{p-1} (u_0 - u_\vartheta)^+ dz \\
& + \int_{\partial\Omega} \beta(z) u_\vartheta^{p-1} (u_0 - u_\vartheta)^+ d\sigma \\
& \text{(since } u_\vartheta \in S(\vartheta)), \\
\Rightarrow \ & u_0 \le u_\vartheta \text{ (see Proposition 2.3 and recall that } \mu > ||\xi||_\infty).
\end{aligned}
$$

Similarly, if in (3.18) we choose $h = (u_\eta - u_0)^+ \in W^{1,p}(\Omega)$, we can show that

$$u_\eta \le u_0.$$

So, we have proved that

$$u_0 \in [u_\eta, u_\vartheta]. \tag{3.19}$$

Then it follows from (3.17), (3.18) and (3.19) that $u_0 \in S(\lambda) \subseteq D_+$. Moreover, arguing as in the proof of Proposition 3.5, via Proposition 2.6, we show that

$$
\begin{aligned}
& u_\vartheta - u_0 \in \operatorname{int} \widehat{C_+} \text{ and } u_0 - u_\eta \in \operatorname{int} \widehat{C_+}, \\
\Rightarrow \ & u_0 \in \operatorname{int}_{C^1(\overline{\Omega})} [u_\eta, u_\vartheta].
\end{aligned} \tag{3.20}
$$

Let $\psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional introduced in the proof of Proposition 3.1. From (3.17) it is clear that

$$\psi_\lambda|_{[u_\eta, u_\vartheta]} = \hat{\varphi}_\lambda|_{[u_\eta, u_\vartheta]} + \hat{k}_\lambda \text{ with } \hat{k}_\lambda \in \mathbb{R}. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$
\begin{aligned}
& u_0 \text{ is local } C^1(\overline{\Omega}) - \text{minimizer of } \psi_\lambda, \\
\Rightarrow \ & u_0 \text{ is local } W^{1,p}(\Omega) - \text{minimizer of } \psi_\lambda \text{ (see Proposition 2.5).}
\end{aligned} \tag{3.22}
$$

Hypotheses $H(f)(ii)$ and $H(g)(ii)$ imply that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$F(z, x, \lambda) \le \frac{\epsilon}{p} x^p, \ G(z, x) \le \frac{1}{p}(\eta_0(z) + \epsilon) x^p \text{ for almost all } z \in \Omega, \text{ all } 0 \le x \le \delta. \tag{3.23}$$

For all $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \le \delta$, we have

$$
\begin{aligned}
\psi_\lambda(u) & \ge \frac{1}{p}\gamma_p(u^-) + \frac{\mu}{p}||u^-||_p^p + \frac{1}{p}\gamma_p(u^+) - \frac{1}{p}\int_\Omega \eta_0(z)(u^+)^p dz - \frac{2\epsilon}{p}||u^+||_p^p \\
& \text{(see (3.23) and recall the definition of } \psi_\lambda \text{ in the proof of Proposition 3.1)} \\
& \ge c_9 ||u^-||^p + \frac{1}{p}(c_0 - 2\epsilon)\, ||u^+||^p \text{ for some } c_9 > 0 \\
& \text{(recall that } \mu > ||\xi||_\infty \text{ and use Lemma 2.9).}
\end{aligned}
$$

Choosing $\epsilon \in \left(0, \frac{c_0}{2}\right)$, we conclude that

$$
\begin{aligned}
& \psi_\lambda(u) \ge c_{10} ||u||^p \text{ for some } c_{10} > 0, \text{ all } u \in C^1(\overline{\Omega}) \text{ with } ||u||_{C^1(\overline{\Omega})} \le \delta, \\
\Rightarrow \ & u = 0 \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \psi_\lambda, \\
\Rightarrow \ & u = 0 \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \psi_\lambda \text{ (see Proposition 2.5).}
\end{aligned} \tag{3.24}
$$

Without any loss of generality, we may assume that

$$0 = \psi_\lambda(0) \leq \psi_\lambda(u_0).$$

The analysis is similar if the opposite inequality holds using (3.24) instead of (3.22). In addition, we may assume that K_{ψ_λ} is finite. Otherwise since $K_{\psi_\lambda} \subseteq D_+ \cup \{0\}$, we see that we already have an infinity of positive solutions for problem (P_λ) and so we are done. Then on account of (3.22), we can find $\rho \in (0,1)$ small such that

$$0 = \psi_\lambda(0) \leq \psi_\lambda(u_0) < \inf\{\psi_\lambda(u) : ||u - u_0|| = \rho\} = m_\lambda, \ ||u_0|| > \rho \tag{3.25}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 29).

From the proof of Proposition 3.1 we know that

$$\begin{aligned} & \psi_\lambda(\cdot) \text{ is coercive,} \\ \Rightarrow \ & \psi_\lambda(\cdot) \text{ satisfies the PS-condition (see Section 2).} \end{aligned} \tag{3.26}$$

From (3.25) and (3.26) it follows that we can use Theorem 2.1 (the mountain pass theorem). So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \hat{u} \in K_{\psi_\lambda} \subseteq D_+ \cup \{0\} \text{ and } 0 < m_\lambda \leq \psi_\lambda(\hat{u}), \\ \Rightarrow \ & \hat{u} \in S(\lambda) \subseteq D_+ \text{ and } \hat{u} \neq u_0 \text{ (see (3.25)).} \end{aligned}$$

The proof is now complete. ☐

Next, we show that the critical parameter value $\lambda^* > 0$ is also admissible (that is, $\lambda^* \in \mathcal{L}$).

Proposition 3.7. *We have that* $\lambda^* \in \mathcal{L}$.

Proof. Let $\{\lambda_n\}_{n\geq 1} \subseteq (\lambda^*, +\infty)$ be such that $\lambda_n \to (\lambda^*)^+$ as $n \to \infty$. From the proof of Proposition 3.5, we know that we can find $u_n \in S(\lambda_n) \subseteq D_+$ $(n \in \mathbb{N})$ decreasing. We have

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}, \tag{3.27}$$

$$\begin{aligned} & \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma \\ & \qquad = \int_\Omega [f(z, u_n, \lambda_n) + g(z, u_n)] h dz \\ & \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \end{aligned} \tag{3.28}$$

In (3.28) we choose $h = u_n \in W^{1,p}(\Omega)$. Using (3.27) and hypotheses $H(\xi), H(\beta)$, $H(f)(i)$, $H(g)(i)$, we see that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Therefore, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u_{\lambda^*} \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_{\lambda^*} \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.29}$$

For every $n \in \mathbb{N}$, we have
$-\Delta_p u_n(z) + \xi(z) u_n(z)^{p-1} = f(z, u_n(z), \lambda_n) + g(z, u_n(z))$ for almost all $z \in \Omega$,

$$\frac{\partial u}{\partial n_p} + \beta(z)u_n^{p-1} = 0 \text{ on } \partial\Omega \text{ (see Papageorgiou \& Rădulescu [16]).} \tag{3.30}$$

From Papageorgiou & Rădulescu [17, Proposition 7] and (3.30), we know that we can find $c_{11} > 0$ such that

$$||u_n||_\infty \leq c_{11} \text{ for all } n \in \mathbb{N}.$$

Then invoking Theorem 2 of Lieberman [13], we can find $\gamma \in (0,1)$ and $c_{12} > 0$ such that

$$u_n \in C^{1,\gamma}(\overline{\Omega}) \text{ and } ||u_n||_{C^{1,\gamma}(\overline{\Omega})} \leq c_{12} \text{ for all } n \in \mathbb{N}. \tag{3.31}$$

Since $C^{1,\gamma}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, from (3.29) and (3.31), we have

$$u_n \to u_{\lambda^*} \text{ in } C^1(\overline{\Omega}). \tag{3.32}$$

Passing to the limit as $n \to \infty$ in (3.28) and using (3.32), we obtain

$$\begin{aligned} &\langle A(u_{\lambda^*}), h\rangle + \int_\Omega \xi(z)u_{\lambda^*}^{p-1}hdz + \int_{\partial\Omega} \beta(z)u_{\lambda^*}^{p-1}hd\sigma = \\ &\int_\Omega [f(z, u_{\lambda^*}, \lambda^*) + g(z, u_{\lambda^*})]hdz \text{ for all } h \in W^{1,p}(\Omega), \qquad (3.33)\\ \Rightarrow\ & u_{\lambda^*} \text{ is a nonnegative solution of } (P_{\lambda^*}). \end{aligned}$$

We need to show that $u_{\lambda^*} \neq 0$. Then we will have $u_{\lambda^*} \in S(\lambda^*) \subseteq D_+$ and $\lambda^* \in \mathcal{L}$.

Arguing by contradiction, suppose that $u_{\lambda^*} = 0$. Then from (3.32) we have

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}). \tag{3.34}$$

Hypotheses $H(f)(ii)$ and $H(g)(ii)$ imply that given $\epsilon > 0$, we can find $\delta = \delta(\epsilon) > 0$ such that

$$f(z, x, \lambda_1)x \leq \epsilon x^p,\ g(z,x)x \leq (\eta_0(z) + \epsilon)x^p \text{ for almost all } z \in \Omega,\ \text{all } 0 \leq x \leq \delta. \tag{3.35}$$

In (3.33) we choose $h = u_n \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} \gamma_p(u_n) &= \int_\Omega [f(z, u_n, \lambda_n) + g(z, u_n)]u_n dz \\ &\leq \int_\Omega [f(z, u_n, \lambda_1) + g(z, u_n)]u_n dz \text{ for all } n \in \mathbb{N} \text{ (see hypothesis } H(f)(iv)). \qquad (3.36) \end{aligned}$$

From (3.34), we see that we can find $n_0 \in \mathbb{N}$ such that

$$u_n(z) \in (0, \delta] \text{ for all } z \in \overline{\Omega},\ \text{all } n \geq n_0. \tag{3.37}$$

Then from (3.35), (3.36), (3.37), we see that

$$\begin{aligned} &\gamma_p(u_n) - \int_\Omega \eta_0(z)u_n^p dz \leq 2\epsilon||u_n||_p^p \text{ for all } n \geq n_0, \\ \Rightarrow\ & c_0||u_n||^p \leq 2\epsilon||u_n||_p^p \text{ for all } n \geq n_0 \text{ (see Lemma 2.9)}, \\ \Rightarrow\ & c_0 \leq 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, choosing $\epsilon \in \left(0, \frac{c_0}{2}\right)$, we have a contradiction. Therefore $u_{\lambda^*} \neq 0$ and so $u_{\lambda^*} \in S(\lambda^*) \subseteq D_+$, hence $\lambda^* \in \mathcal{L}$. □

So, we conclude that

$$\mathcal{L} = [\lambda^*, +\infty)\,.$$

4. Minimal positive solutions

In this section we show that for every $\lambda \in \mathcal{L}$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in D_+$ and we study the monotonicity and continuity properties of the map $\lambda \mapsto \bar{u}_\lambda$.

From Papageorgiou, Rădulescu & Repovš [18] (see the proof of Proposition 7), we know that $S(\lambda)$ is downward directed, that is, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \le u_1, u \le u_2$.

Proposition 4.1. *Assume that $\lambda \in \mathcal{L} = [\lambda^*, +\infty)$. Then problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in S(\lambda) \subseteq D_+$ (that is, $\bar{u}_\lambda \le u$ for all $u \in S(\lambda)$).*

Proof. According to Lemma 3.10 of Hu & Papageorgiou [11, p. 178] and since $S(\lambda)$ is downward directed, we can find $\{u_n\}_{n\ge1} \subseteq S(\lambda)$ decreasing such that

$$\inf S(\lambda) = \inf_{n\ge1} u_n.$$

We have

$$0 \le u_n \le u_1 \text{ for all } n \in \mathbb{N}, \tag{4.1}$$

$$\langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n^{p-1} h d\sigma = \int_\Omega [f(z, u_n.\lambda) + g(z, u_n)] h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{4.2}$$

Then reasoning as in the proof of Proposition 3.7 (see the part of the proof after (3.28)) and using (4.1) and (4.2), we obtain

$$\begin{aligned} & u_n \to \bar{u}_\lambda \text{ in } C^1(\overline{\Omega}) \text{ with } \bar{u}_\lambda \in S(\lambda), \\ \Rightarrow\ & \bar{u}_\lambda = \inf S(\lambda). \end{aligned}$$

The proof is complete. ☐

Proposition 4.2. *The map $\lambda \mapsto \bar{u}_\lambda$ from $\overset{o}{\mathcal{L}} = (\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$ has the following properties:*

- *is strictly monotone in the sense that*
$$\overset{o}{\mathcal{L}} \ni \lambda < \vartheta \Rightarrow \bar{u}_\vartheta - \bar{u}_\lambda \in \operatorname{int} \widehat{C_+};$$
- *it is left continuous.*

Proof. First, we show the strict monotonicity of the map. So, let $\lambda \in \overset{o}{\mathcal{L}}$ and $\vartheta > \lambda$. Then $\vartheta \in \mathcal{L}$ and let $\bar{u}_\vartheta \in S(\vartheta) \subseteq D_+$ be the minimal solution of problem (P_ϑ). From the proof of Proposition 3.6, we know that we can find $u_\lambda \in S(\lambda) \subseteq D_+$ such that

$$\begin{aligned} & \bar{u}_\vartheta - u_\lambda \in \operatorname{int} \widehat{C_+} \text{ (see (3.20))}, \\ \Rightarrow\ & \bar{u}_\vartheta - \bar{u}_\lambda \in \operatorname{int} \widehat{C_+} \text{ (since } \bar{u}_\lambda \le u_\lambda). \end{aligned}$$

This proves the strict monotonicity of the map $\lambda \mapsto \bar{u}_\lambda$ from $\overset{o}{\mathcal{L}} = (\lambda^*, +\infty)$ into $C^1(\overline{\Omega})$.

Next, we show the left continuity of this map. So, let $\{\lambda_n\}_{n\geq 1} \subseteq \overset{o}{\mathcal{L}}$ and assume that $\lambda_n \to \lambda^-$. From the first part of the proof, we have

$$0 \leq \bar{u}_{\lambda_n} \leq \bar{u}_\lambda \text{ for all } n \geq 1$$

Then as before (see the proof of Proposition 3.7), we can say that

$$\bar{u}_{\lambda_n} \to \tilde{u}_\lambda \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty \tag{4.3}$$

and

$$\tilde{u}_\lambda \in S(\lambda) \subseteq D_+.$$

Suppose that $\tilde{u}_\lambda \neq \bar{u}_\lambda$. Then we can find $z_0 \in \overline{\Omega}$ such that

$$\begin{aligned} & \bar{u}_\lambda(z_0) < \tilde{u}_\lambda(z_0), \\ \Rightarrow \quad & \bar{u}_\lambda(z_0) < \bar{u}_{\lambda_n}(z_0) \text{ for all } n \geq n_0, \end{aligned}$$

which contradicts the first part of the proposition. Therefore

$$\begin{aligned} & \tilde{u}_\lambda = \bar{u}_\lambda, \\ \Rightarrow \quad & \lambda \mapsto \bar{u}_\lambda \text{ is continuous from } \overset{o}{\mathcal{L}} \text{ into } C^1(\overline{\Omega}). \end{aligned}$$

The proof is now complete. □

Remark 4.3. In our setting the equation was nonuniformly nonresonant as $x \to +\infty$ (see hypotheses $H(f)(ii), H(g)(ii)$). Is it possible to treat also the resonant case, that is,

$$\limsup_{x\to+\infty} \frac{g(z,x)}{x^{p-1}} \leq \hat{\lambda}_1 \text{ uniformly for almost all } z \in \Omega.$$

Moreover, what is the situation of asymptotical behavior as $x \to +\infty$ we are nonresonant with respect to $\hat{\lambda}_1$, but from above the principal eigenvalue, that is,

$$\liminf_{x\to+\infty} \frac{g(z,x)}{x^{p-1}} \geq \hat{\eta} > \hat{\lambda}_1 \text{ uniformly for almost all } z \in \Omega.$$

A careful inspection of the arguments of this paper, reveals that for the nonresonant case but from above $\hat{\lambda}_1$, if a bifurcation-type result holds, then it will be for small values of $\lambda > 0$. This also suggests that if we want to extend the results of this paper to the resonant case, we must have resonance from the left of $\hat{\lambda}_1$, in the sense that

$$\hat{\lambda}_1 x^{p-1} - [f(z,x,\lambda) + g(z,x)] \to +\infty \text{ uniformly for almost all } z \in \Omega, \text{ as } x \to +\infty.$$

In this way we can preserve the coercivity of the energy functional and we hope to be able to extend the results of paper to the resonant case.

Acknowledgements

The authors wish to thank a knowledgeable referee for his/her corrections and remarks that improved the presentation.

References

[1] S. Aizicovici, N. S. Papageorgiou and V. Staicu, *Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints*, Mem. Amer. Math. Soc. **196** (2008), no. 915, 70 pp.

[2] A. Ambrosetti, H. Brezis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Functional Anal. **122** (1994), 519–543.

[3] H. Brezis and L. Nirenberg, *H^1 versus C^1 local minimizers*, C.R. Acad. Sci. Paris, Sér. I **317** (1993), 465–472.

[4] T. Cardinali, N. S. Papageorgiou and P. Rubbioni, *Bifurcation phenomena for nonlinear superdiffusive Neumann equations of logistic type*, Ann. Mat. Pura Appl. (4) **193** (2013), 1–21.

[5] E. Casas and L.A. Fernández, *A Green's formula for quasilinear elliptic operators*, J. Math. Anal. Appl. **142** (1989), 62–73.

[6] G. Fragnelli, D. Mugnai and N. S. Papageorgiou, *The Brezis-Oswald result for quasilinear Robin problems*, Adv. Nonlinear Studies **16** (2016), 403–422.

[7] J. Garcia Azorero, J. Manfredi and I. Peral Alonso, *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Comm. Contemp. Math. **2** (2000), 385–404.

[8] L. Gasinski and N. S. Papageorgiou, *Nonlinear Analysis*, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[9] L. Gasinski and N. S. Papageorgiou, *Exercises in Analysis. Part 2: Nonlinear Analysis*, Problem Books in Mathematics. Springer, Cham, 2016.

[10] Z. Guo and Z. Zhang, *$W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. **286** (2003), 32–50.

[11] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Volume I: Theory*, Kluwer Academic Publishers, Dordrecht, 1997.

[12] S. Hu and N. S. Papageorgiou, *Multiplicity of solutions for parametric p-Laplacian equations with nonlinearity concave near the origin*, Tohoku Math. J. **62** (2010), 137–162.

[13] G. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[14] S.A. Marano and N. S. Papageorgiou, *Constant sign and nodal solutions for a Neumann problem with p-Laplacian and equidiffusive reaction term*, Topol. Methods Nonlin. Anal. **38** (2011), 233–248.

[15] D. Mugnai and N. S. Papageorgiou, *Resonant nonlinear Neumann problems with indefinite weight*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), 729–788.

[16] N. S. Papageorgiou and V. D. Rădulescu, *Multiple solutions with precise sign for parametric Robin problems*, J. Differential Equations **256** (2014), 2449–2479.

[17] N. S. Papageorgiou and V. D. Rădulescu, *Nonlinear nonhomogeneous Robin problems with superlinear reaction term*, Adv. Nonlin. Studies **16** (2016), 737-764.

[18] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Positive solutions for perturbation of the Robin eigenvalue problem plus on indefinite potential*, Discr. Contin. Dyn. Systems A **37** (2017), 2589–2618.

[19] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Positive solutions for nonlinear nonhomogeneous parametric Robin problems*, Forum Math. **30** (2018), 553–580.

[20] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Nonlinear Analysis - Theory and Methods*, Springer Monographs in Mathematics, Springer, Cham, 2019.

[21] V .D. Rădulescu and D. D. Repovš, *Combined effects of nonlinear problems arising in the study of anisotropic continuous media*, Nonlinear Anal. **75** (2012), 1524–1530.

Manuscript received April 20 2018
revised September 8 2018

N. S. Papageorgiou
National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
E-mail address: npapg@math.ntua.gr

V. D. Rădulescu
Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia & Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania
E-mail address: vicentiu.radulescu@imfm.si

D. D. Repovš
Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
E-mail address: radulescu@agh.edu.pl